STABILITY AND APPROXIMATION OF SOLUTIONS IN NEW REPRODUCING KERNEL HILBERT SPACES ON A SEMI-INFINITE DOMAIN

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Abstract. We introduce new reproducing kernel Hilbert spaces on a trapezoidal semi-infinite domain B_∞ in the plane. We establish uniform approximation results in terms of the number of nodes on compact subsets of B_∞ for solutions to nonhomogeneous hyperbolic partial differential equations in one of these spaces, $\widetilde{W}(B_\infty)$. Furthermore, we demonstrate the stability of such solutions with respect to the driver. Finally, we give an example to illustrate the efficiency and accuracy of our results.

1. Introduction

Reproducing kernel Hilbert spaces were introduced in the early twentieth century by Zaremba to study boundary value problems for harmonic and biharmonic functions [17,18], and by mid-century a general theory of reproducing kernel Hilbert spaces was established by Aronszajn [3] and Bergman [5]. Since their inception, reproducing kernel Hilbert spaces have seen increasing use for solving not only partial differential, integral, and ordinary differential equations [2,4,14], but also problems in optimal control [10], dynamical systems [7,11], and statistics [12,13,19]. Recently, reproducing kernel Hilbert spaces have attracted the attention of several researchers (e.g. [1,9,15,16]) after Cui and Lin [6] developed reproducing Hilbert spaces with piecewise polynomial kernels on compact intervals, $W_2^n[a,b]$, and compact rectangles, $W_2^{(m,n)}\big([a,b]\times[c,d]\big)$, and used them to help solve a wide variety of linear and nonlinear problems.

The aim of this paper is to introduce and study a new reproducing kernel Hilbert space $W_2^{(3,3)}(B_\infty)$ on a semi-infinite trapezoidal plane region:

$$B_\infty = \{(x,t) : 0 \le t \le 1 \text{ and } t \le x < \infty\}.$$

We use this space to solve initial value problems for nonhomogeneous hyperbolic partial differential equations, we analyze the stability of such solutions with respect to the driver, and we discuss the local uniform approximation of solutions in this new space in terms of the density of nodes in

$$B_r = \{(x,t) : 0 \le t \le 1 \text{ and } t \le x \le r\}.$$

The semi-infinite trapezoid B_∞ arises naturally in the uniqueness theory for hyperbolic partial differential equations in Ω_∞ ([9], Theorems 4.2 and 4.5). The new reproducing kernel Hilbert spaces $W_2^{(3,3)}(B_\infty)$ and $\widetilde{W}(B_\infty)$, introduced in

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2010 *Mathematics Subject Classification.* 46E22,47B32, 47B38.

Key words and phrases. reproducing kernel Hilbert space, stability of solutions, telegraph equation.

section 2, contain the solutions to nonhomogeneous hyperbolic partial differential equations, provided the driver is sufficiently smooth and square-integrable. These new spaces have piecewise polynomial kernels and this makes them convenient for numerical approximations.

This article is organized as follows. A summary of fundamental definitions, related notation, and common facts about reproducing kernel Hilbert spaces is presented in section 2. Section 3 is devoted to the representation of solutions to $Lu = f$ where L is a one-to-one, bounded, linear transformation between reproducing kernel Hilbert spaces. The local stability and uniform approximation of solutions in $W_2^{(3,3)}(B_\infty)$ for telegraph problems is examined in section 4. We give an example to illustrate the theory in section 5, and our conclusions are presented in section 6.

2. RKHS Preliminaries

This section is devoted to statements of basic concepts and associated notation for reproducing kernel Hilbert spaces. We also introduce some well-known facts about reproducing kernel Hilbert spaces which will be used in this work. Suggested general references for reproducing kernel Hilbert spaces are [3, 5, 6] and specific details for material in this section are given in [9, 15].

Definition 2.1. Let E be a nonempty set, and let $\left(\mathcal{H}, \langle\cdot,\cdot\rangle\right)$ be a Hilbert space of continuous real-valued functions on E. A function $K : E \times E \to \mathbb{R}$ is a reproducing kernel of $\mathcal{H}$ if

i) $K(\cdot,\theta) \in \mathcal{H}$, for all $\theta \in E$ and
ii) $\left\langle h, K(\theta,\cdot)\right\rangle = h(\theta)$ for all $\theta \in E$ and for all $h \in \mathcal{H}$.

Such a Hilbert space possessing a reproducing kernel will be called a reproducing kernel Hilbert space (RKHS). We begin with RKHSs of functions defined on intervals in $\mathbb{R}$.

Lemma 2.2 ([6]). *The space*

$$W_2^1[0,1] = \{g : [0,1] \to \mathbb{R} \mid g \in AC[0,1] \text{ and } g' \in L^2[0,1]\},$$

equipped with the inner product

$$\langle g, h\rangle_{W_2^1[0,1]} = g(0)h(0) + \int_0^1 g'(\tau)h'(\tau)d\tau,$$

is a RKHS with reproducing kernel function q given by

$$q(t,\tau) = 1 + \int_0^\tau \chi_{[0,t]}(s)\, ds.$$

Lemma 2.3 ([15]). *For any t and s in $[0,1]$,*

$$\left\|q(t,\cdot) - q(s,\cdot)\right\|^2_{W_2^1[0,1]} = |t-s|.$$

Lemma 2.4 ([9]). *The space*

$$W_2^1[0,\infty) = \{f : [0,\infty) \to \mathbb{R} \mid f \in AC[0,\sigma] \text{ for all } \sigma > 0 \text{ and } f' \in L^2[0,\infty)\},$$

equipped with the inner product

$$\langle f, g\rangle_{W_2^1[0,\infty)} = f(1)g(1) + \int_0^\infty f'(x)g'(x)dx,$$

is a RKHS with reproducing kernel function δ given by

$$\delta(x,\xi) = \begin{cases} r_1(x,\xi) & \text{if } x \in [0,1] \text{ and } \xi \in [0,\infty), \\ \\ r_2(x,\xi) & \text{if } x \in [1,\infty) \text{ and } \xi \in [0,\infty), \end{cases}$$

where

$$r_1(x,\xi) = \begin{cases} 2-x & \text{if} \quad 0 \le \xi < x \le 1, \\ 2-\xi & \text{if} \quad 0 \le x < \xi \le 1, \\ 1 & \text{if} \quad 1 < \xi \le \infty, \end{cases}$$

$$r_2(x,\xi) = \begin{cases} 1 & \text{if} \quad 0 \le \xi < 1, \\ \xi & \text{if} \quad 1 \le \xi < x < \infty, \\ x & \text{if} \quad 1 \le x < \xi < \infty. \end{cases}$$

Lemma 2.5. *For any x and y in $[0,\infty)$,*

$$\|\delta(x,\cdot) - \delta(y,\cdot)\|^2_{W_2^1[0,\infty)} = |x-y|.$$

Proof. The proof is a minor modification of the argument used for Lemma 2.3 and the details are omitted. □

We next turn to RKHSs of functions defined on regions in the plane.

Definition 2.6. A function $u : B_r \to \mathbb{R}$ is said to be absolutely continuous (in the sense of Carathéodory) on B_r if and only if there exist $\lambda \in \mathbb{R}$, $f \in L^1[0,r]$, $g \in L^1[0,1]$, $h \in L^1(B_r)$ such that

$$u(x,t) = \lambda + \int_1^x f(\xi)d\xi + \int_0^t g(\tau)d\tau + \int_0^t \int_1^x h(\xi,\tau)d\xi d\tau,$$

for all $(x,t) \in B_r$. In this case we write $u \in AC(B_r)$. If $u : B_\infty \to \mathbb{R}$ belongs to $AC(B_r)$ for every $r \ge 1$ then we say $u \in AC_{loc}(B_\infty)$.

Definition 2.7. A function $u : B_r \to \mathbb{R}$ belongs to $W_2^{(1,1)}(B_r)$ provided $u \in AC(B_r)$ and the following square-integrability conditions are satisfied:

i) $\frac{\partial}{\partial t}u(1,\cdot) \in L^2[0,1]$;
ii) $\frac{\partial}{\partial x}u(\cdot,0) \in L^2[0,r]$;
iii) $\frac{\partial^2}{\partial x \partial t}u \in L^2(B_r)$.

Definition 2.8. A function $u : B_\infty \to \mathbb{R}$ belongs to $W_2^{(1,1)}(B_\infty)$ provided $u \in AC_{loc}(B_\infty)$ and the following square-integrability conditions are satisfied:

i) $\frac{\partial}{\partial t}u(1,\cdot) \in L^2[0,1]$;
ii) $\frac{\partial}{\partial x}u(\cdot,0) \in L^2[0,\infty)$;
iii) $\frac{\partial^2}{\partial x \partial t}u \in L^2(B_\infty)$.

Lemma 2.9. *Fix $d = r$ or $d = \infty$. The space $W_2^{(1,1)}(B_d)$, equipped with the inner product*

$$\begin{aligned}\langle u,v\rangle_{W_2^{(1,1)}(B_d)} = u(1,0)v(1,0) &+ \int_0^1 \frac{\partial}{\partial\tau}u(1,\tau)\frac{\partial}{\partial\tau}v(1,\tau)d\tau \\ &+ \int_0^d \frac{\partial}{\partial\xi}u(\xi,0)\frac{\partial}{\partial\xi}v(\xi,0)d\xi \\ &+ \iint_{B_d} \frac{\partial^2}{\partial\xi\partial\tau}u(\xi,\tau)\frac{\partial^2}{\partial\xi\partial\tau}u(\xi,\tau)d\xi d\tau,\end{aligned}$$

is a RKHS with reproducing kernel function $K_1(x,t;\cdot,*) = \delta(x,\cdot)q(t,*)$ where q and δ are defined in Lemmas 2.2 and 2.4, respectively.

Proof. The proof is completely analogous to that of Theorem 3.8 in [9]. □

Definition 2.10 ([9])**.** A function $u : B_r \to \mathbb{R}$ belongs to $W_2^{(3,3)}(B_r)$ provided

(a) u and its following derivatives belong to $AC(B_r)$:
u_t, u_x, u_{tt}, u_{tx}, u_{xt}, u_{xx},
u_{ttx}, u_{txt}, u_{xtt}, u_{xxt}, u_{xtx}, u_{txx},
u_{ttxx}, u_{txtx}, u_{txxt}, u_{xtxt}, u_{xttx}, u_{xxtt},

(b) and the following square-integrability conditions are satisfied:
 i) $\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}u(1,\cdot) \in L^2[0,1]$ for all $i = 0,1,2$;
 ii) $\frac{\partial^3}{\partial x^3}\frac{\partial^j}{\partial t^j}u(\cdot,0) \in L^2[0,r]$ for all $j = 0,1,2$;
 iii) $\frac{\partial^6}{\partial x^3\partial t^3}u \in L^2(B_r)$.

Definition 2.11 ([9])**.** A function $u : B_\infty \to \mathbb{R}$ belongs to $W_2^{(3,3)}(B_\infty)$ provided

(a) u and its following derivatives belong to $AC_{loc}(B_\infty)$:
u_t, u_x, u_{tt}, u_{tx}, u_{xt}, u_{xx},
u_{ttx}, u_{txt}, u_{xtt}, u_{xxt}, u_{xtx}, u_{txx},
u_{ttxx}, u_{txtx}, u_{txxt}, u_{xtxt}, u_{xttx}, u_{xxtt},

(b) and the following square-integrability conditions are satisfied:
 i) $\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}u(1,\cdot) \in L^2[0,1]$ for all $i = 0,1,2$;
 ii) $\frac{\partial^3}{\partial x^3}\frac{\partial^j}{\partial t^j}u(\cdot,0) \in L^2[0,\infty)$ for all $j = 0,1,2$;
 iii) $\frac{\partial^6}{\partial x^3\partial t^3}u \in L^2(B_\infty)$.

Definition 2.12. The subspace $\widetilde{W}(B_\infty)$ of $W_2^{(3,3)}(B_\infty)$ is defined by

$$\widetilde{W}(B_\infty) = \left\{u \in W_2^{(3,3)}(B_\infty) :\ u(x,0) = 0 = u_t(x,0) \text{ for all } x > 0\right\}.$$

Lemma 2.13. *The space $\widetilde{W}(B_\infty)$, equipped with the inner product*

$$\begin{aligned}\left\langle u,v\right\rangle_{\widetilde{W}(B_\infty)} &= \sum_{j=0}^{2}\frac{\partial^j}{\partial x^j}u_{tt}(1,0)v_{tt}(1,0) + \int_0^\infty \frac{\partial^3}{\partial x^3}u_{tt}(x,0)\frac{\partial^3}{\partial x^3}v_{tt}(x,0)dx \\ &+ \sum_{i=0}^{2}\int_0^1 \frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}u(1,t)\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}v(1,t)dt \\ &+ \iint_{B_\infty} \frac{\partial^6}{\partial x^3\partial t^3}u(x,t)\frac{\partial^6}{\partial x^3\partial t^3}v(x,t)dxdt,\end{aligned}$$

is a RKHS with reproducing kernel function given by the product $K(x,t;\xi,\tau) = R(x,\xi)\,Q(t,\tau)$ *where*

$$Q(t,\tau) = \begin{cases} \frac{1}{4}t^2\tau^2 + \frac{1}{12}t^2\tau^3 - \frac{1}{24}t\tau^4 + \frac{1}{120}\tau^5 & \text{if } 0 \le \tau < t \le 1, \\ \frac{1}{4}\tau^2 t^2 + \frac{1}{12}\tau^2 t^3 - \frac{1}{24}\tau t^4 + \frac{1}{120}t^5 & \text{if } 0 \le t \le \tau \le 1, \end{cases}$$

$$R(x,\xi) = \begin{cases} R_1(x,\xi) & \text{if } x \in [0,1] \text{ and } \xi \in [0,\infty), \\ R_2(x,\xi) & \text{if } x \in [1,\infty) \text{ and } \xi \in [0,\infty), \end{cases}$$

and

$$R_1(x,\xi) = \begin{cases} \frac{23}{10} - \frac{13}{8}\xi + \frac{1}{3}\xi^2 - \frac{13}{8}x + \frac{7}{3}x\xi - \frac{3}{4}x\xi^2 + \frac{1}{3}x^2 - \frac{3}{4}x^2\xi & \text{if } 0 \le \xi \le x \le 1, \\ \quad + \frac{1}{2}x^2\xi^2 - \frac{1}{12}x^3\xi^2 + \frac{1}{24}\xi x^4 - \frac{1}{120}x^5 & \\ \frac{23}{10} - \frac{13}{8}\xi + \frac{1}{3}\xi^2 - \frac{1}{120}\xi^5 - \frac{13}{8}x + \frac{7}{3}x\xi - \frac{3}{4}x\xi^2 + \frac{1}{24}x\xi^4 & \text{if } 0 \le x < \xi \le 1, \\ \quad + \frac{1}{3}x^2 - \frac{3}{4}x^2\xi + \frac{1}{2}x^2\xi^2 - \frac{1}{12}x^2\xi^3 & \\ 1 + (x-1)(\xi-1) + \frac{1}{4}(x-1)^2(\xi-1)^2 & \text{if } 1 < \xi < \infty, \end{cases}$$

$$R_2(x,\xi) = \begin{cases} 1 + (x-1)(\xi-1) + \frac{1}{4}(x-1)^2(\xi-1)^2 & \text{if } 0 \le \xi < 1, \\ \frac{11}{5} - \frac{11}{8}\xi + \frac{1}{6}\xi^2 - \frac{1}{24}x\xi^4 - \frac{1}{4}x\xi^2 + \frac{5}{3}x\xi - \frac{11}{8}x & \text{if } 1 \le \xi \le x < \infty, \\ \quad + \frac{1}{12}x^2\xi^3 - \frac{1}{4}x^2\xi + \frac{1}{6}x^2 + \frac{1}{120}\xi^5 & \\ \frac{11}{5} - \frac{11}{8}\xi + \frac{1}{6}\xi^2 - \frac{1}{4}x\xi^2 + \frac{5}{3}x\xi - \frac{11}{8}x + \frac{1}{6}x^2 & \text{if } 1 \le x < \xi < \infty. \\ \quad - \frac{1}{4}x^2\xi + \frac{1}{12}x^3\xi^2 - \frac{1}{24}x^4\xi + \frac{1}{120}x^5 & \end{cases}$$

Proof. This is demonstrated using an argument similar to that for Theorem 3.11 in [9]. □

3. Representation of solutions

Let $\langle \mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}} \rangle$ and $\langle \widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle_{\widetilde{\mathcal{H}}} \rangle$ be reproducing kernel Hilbert spaces of continuous functions on a set E in $\mathbb{R}^n$, with reproducing kernel functions k and $\widetilde{k}$, respectively, and let $L : \mathcal{H} \to \widetilde{\mathcal{H}}$ be a one-to-one, bounded, linear transformation. If $u \in \mathcal{H}$ is a solution to

$$Lu = g \tag{3.1}$$

for a given $g \in \widetilde{\mathcal{H}}$ then u may be expressed in terms of a complete orthonormal basis for $\mathcal{H}$ generated using L. For more detail and proofs of the results in this section, see [6], especially chapter 6.

Let $\{s_i\}_{i=1}^{\infty}$ be a countable set of distinct points in E, and define

$$\Psi_i = L^* \tilde{k}_{s_i}, \tag{3.2}$$

where $L^* : \widetilde{\mathcal{H}} \to \mathcal{H}$ is the adjoint of L.

Theorem 3.1. *If $\{s_i\}_{i=1}^{\infty}$ is dense in E, then $\{\Psi_i\}_{i=1}^{\infty}$ is a complete set in $\mathcal{H}$ and*

$$\Psi_i = Lk_{s_i}$$

for all $i \in \mathbb{N}$.

Discarding functions from $\left\{\Psi_i\right\}_{i=1}^{\infty}$ if necessary, we may assume that Ψ_{i+1} is not in the linear span of $\left\{\Psi_1, ..., \Psi_i\right\}$ for all $i \in \mathbb{N}$. An orthonormal basis $\{\widetilde{\Psi}_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ can then be derived by applying the Gram-Schmidt orthonormalization process to $\{\Psi_i\}_{i=1}^{\infty}$:

$$\widetilde{\Psi}_i = \sum_{k=1}^{i} \beta_{ik} \Psi_k, \tag{3.3}$$

where the β_{ik} are orthonormalization coefficients of $\{\Psi_i\}_{i=1}^{\infty}$.

Theorem 3.2. *Let $\{s_i\}_{i=1}^{\infty}$ be a countable dense set of points of E, let $g \in \widetilde{\mathcal{H}}$, and let $u \in \mathcal{H}$ be a solution of $Lu = g$. Then u has the following Hilbert space representation:*

$$u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(s_k) \widetilde{\Psi}_i. \tag{3.4}$$

We observe that the truncation

$$u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} g(s_k) \widetilde{\Psi}_i. \tag{3.5}$$

is an approximation of the exact solution u to $Lu = g$.

4. Stability of solutions and an error estimate

Let T denote the telegraph operator:

$$Tu = u_{tt} - u_{xx} + 2au_t + b^2 u$$

for constants $a > b \geq 0$. Consider the nonhomogeneous telegraph problem

$$Tu = f \quad \text{in } \Omega_\infty \tag{4.1}$$

subject to

$$u(x, 0) = \varphi(x) \text{ and } u_t(x, 0) = \psi(x) \quad \text{if } x \in [0, \infty). \tag{4.2}$$

In this section we investigate the stability of reproducing kernel Hilbert space solutions u to $Tu = f$ with respect to the driver f and analyze the approximation error when the truncation u_n in (3.5) is used in place of u. Our focus in this section will be on solutions in $\widetilde{W}(B_\infty)$.

The following two reproducing kernel Hilbert space existence and uniqueness results were obtained in [9].

Theorem 4.1. *If $f \in Y = L^2(\Omega_\infty) \cap \left(\cap_{i=1}^{4} W_2^{(i,1)}(\Omega_\infty)\right)$ then there exists $u \in W_2^{(3,3)}(\Omega_\infty)$ to (4.1) satisfying*

$$u(x, 0) = 0 = u_t(x, 0) \quad \textit{for all } 0 \leq x < \infty. \tag{4.3}$$

Theorem 4.2. *Let u and v belong to $W_2^{(3,3)}(B_\infty)$. If u and v satisfy (4.3) and*

$$Tu(x, t) = Tv(x, t) \quad \textit{for all } (x, t) \in B_\infty, \tag{4.4}$$

then $u = v$ on B_∞.

Let $S : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$S(x,t) = \begin{cases} \frac{1}{2} p\left((\alpha^2 - \beta^2)(t^2 - x^2)\right) e^{-\alpha t} & \text{if } |x| \leq t, \\ 0 & \text{if } |x| > t, \end{cases}$$

where

$$p(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^n (n!)^2}.$$

The key idea in the proof of Theorem 4.1 is that, for a fixed $f \in Y$, the function

$$u(x,t) = \int_0^t \int_{x-t+\tau}^{x+t-\tau} S(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau, \tag{4.5}$$

belongs to $\widetilde{W}(B_\infty)$ and solves the nonhomogeneous telegraph equation

$$Tu = f \text{ on } B_\infty. \tag{4.6}$$

Theorem 4.2 shows that (4.5) is the unique solution in $\widetilde{W}(B_\infty)$ to (4.6). It is easy to see that the system (4.1) and (4.2) can be simply transformed into a nonhomogeneous telegraph equation subject to homogeneous initial conditions. For example, let $v(x,t) = u(x,t) - \varphi(x) - t\psi(x)$; then (4.1) and (4.2) become

$$Tv = g \text{ in } B_\infty,$$

subject to the homogeneous initial conditions

$$v(x,0) = 0 = v_t(x,0) \text{ for } \ 0 \leq x < \infty.$$

Here $g = f + \varphi'' + t\psi'' - 2\alpha\psi - \beta^2(\varphi + t\psi)$.

Routine application of the techniques of proof of Theorems 4.1 and 4.2 yield a stability result for the solution to (4.6) in $\widetilde{W}(B_\infty)$ with respect to the driver f in Y. To state this result we will introduce some notation. If ϕ is a continuous real function on B_r, let its uniform norm be denoted by

$$\|\phi\|_{u(B_r)} = \max\left\{|\phi(x,t)| : (x,t) \in B_r\right\}.$$

Theorem 4.3. *Let f_1, f_2 belong to Y and let u_1, u_2 be the unique solutions in $\widetilde{W}(B_\infty)$ to $Tu_i = f_i$ $(i = 1,2)$ on B_∞. Then there corresponds an absolute constant $C > 0$ with the property*

$$\|u_1 - u_2\|_{u(B_r)} \leq C\|f_1 - f_2\|_{u(B_r)},$$

for all $r \in [1,\infty)$.

Proof. For all $(x,t) \in B_\infty$,

$$\left|u_1(x,t) - u_2(x,t)\right| \leq \int_0^t \int_{x-t+\tau}^{x+t-\tau} S(x-\xi, t-\tau)\left|f_1(\xi,\tau) - f_2(\xi,\tau)\right| d\xi d\tau.$$

Since f_1 and f_2 are continuous functions on B_∞,

$$\left\|f_1 - f_2\right\|_{u(B_r)} = \max_{(x,t)\in B_r} \left|f_1(x,t) - f_2(x,t)\right| < \infty.$$

Therefore, for all $(x,t) \in B_r$ we have

$$\left|u_1(x,t) - u_2(x,t)\right| \leq \left\|f_1 - f_2\right\|_{u(B_r)} \int_0^t \int_{x-t+\tau}^{x+t-\tau} S(x-\xi, t-\tau) d\xi d\tau.$$

Define a subset $\triangle_{(x,t)}$ of B_∞ for fixed $(x,t) \in B_\infty$ by

$$\triangle_{(x,t)} = \left\{(\xi,\tau) \in \mathbb{R}^2 : |x-\xi| \leq t-\tau, 0 \leq \tau \leq t\right\}.$$

Since $S = S(x,t)$ is an analytic function on the cone $|x| < t$, and is continuous on $|x| \leq t$, there corresponds a constant C such that

$$\sup_{(\xi,\tau)\in\triangle_{(x,t)}} \left|S(x-\xi,t-\tau)\right| \leq \sup_{(\xi,\tau)\in\triangle_{(1,1)}} \left|S(\xi,\tau)\right| = C \leq \frac{1}{2}. \tag{4.7}$$

Hence, $\left\|u_1 - u_2\right\|_{u(B_r)} \leq C\left\|f_1 - f_2\right\|_{u(B_r)}$. □

To analyze the local uniform error when the truncation u_n is used to approximate the solution u in $\widetilde{W}(B_\infty)$ to $Tu = g$, we will need boundedness of the telegraph operator $T : W_2^{(3,3)}(B_r) \to W_2^{(1,1)}(B_r)$ for each $r \geq 1$. This is the content of the next result.

Lemma 4.4. *Let $1 \leq r < \infty$ and let $C_r = \max\{2r, r^2\}$. Then the telegraph operator T satisfies*

$$\|Tu\|_{W_2^{(1,1)}(B_r)} \leq (2 + C_r(1 + 2a\sqrt{2} + 2b^2))\|u\|_{W_2^{(3,3)}(B_r)}$$

for all $u \in W_2^{(3,3)}(B_r)$.

Proof. The proof is a routine application of the tools in [8] used to show boundedness of the telegraph operator $T : W_2^{(3,3)}([0,1]\times[0,1]) \to W_2^{(1,1)}([0,1]\times[0,1])$ and the details are omitted. □

It is important to mention here that the telegraph operator T is not globally bounded on RKHSs of functions defined on the semi-infinite domain B_∞. For instance, let $u(x,t) = x^2t^2$. Then it is easy to see that u belongs to $W_2^{(3,3)}(B_\infty)$, but Tu does not belong to $W_2^{(1,1)}(B_\infty)$.

Theorem 4.5. *Let $g \in W_2^{(1,1)}(B_\infty)$ and let r and N be positive integers. Let $(x_i,t_j) \in B_r$ where $t_j = \frac{j}{N}(j = 0,1,2,...,N)$, $x_i = \frac{i}{N}(i = j, j+1,...,rN)$, and $n = \frac{((2r-1)N+2)(N+1)}{2}$. Let $u = u(x,t)$ be a solution in $W_2^{(3,3)}(B_r)$ to*

$$\begin{aligned} &Tu = g \ \text{ in } B_r,\\ &u(x,0) = 0 = u_t(x,0) \ \text{ if } 0 \leq x \leq r, \end{aligned}$$

and let u_n be the truncation of u given by (3.5). Then there exists a positive constant $C_r = O(\sqrt{r})$ such that

$$|u(x,t) - u_n(x,t)| \leq C_r\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}$$

for all $(x,t) \in B_r$.

Proof. For any $(x,t) \in B_r$ there exists $(x_i,t_j) \in B_r$ satisfying $x_i \leq x$, $t_j \leq t$, and such that $|x - x_i| < \frac{1}{N}$ and $|t - t_j| < \frac{1}{N}$. Using $Tu(x_i,t_j) = Tu_n(x_i,t_j)$ for $0 \leq j \leq N$ and $j \leq i \leq rN$, it follows that

$$\begin{aligned} \left|Tu(x,t) - Tu_n(x,t)\right| &= \left|Tu(x,t) - Tu(x_i,t_j) + Tu_n(x_i,t_j) - Tu_n(x,t)\right| \\ &\leq \left|Tu(x,t) - Tu(x_i,t_j)\right| + \left|Tu_n(x,t) - Tu_n(x_i,t_j)\right|. \end{aligned}$$

Observe that

$$\begin{aligned}\left|Tu(x,t)-Tu(x_i,t_j)\right| &= \left|\left\langle Tu(\cdot,\cdot),K_{(x,t)}(\cdot,\cdot)\right\rangle - \left\langle Tu(\cdot,\cdot),K_{(x_i,t_j)}(\cdot,\cdot)\right\rangle\right| \\ &= \left|\left\langle Tu(\cdot,\cdot),K_{(x,t)}(\cdot,\cdot)-K_{(x_i,t_j)}(\cdot,\cdot)\right\rangle\right| \\ &= \left|\left\langle g(\cdot,\cdot),K_{(x,t)}(\cdot,\cdot)-K_{(x_i,t_j)}(\cdot,\cdot)\right\rangle\right| \\ &\le \|g\|_{W_2^{(1,1)}(B_\infty)}\left\|K_{(x,t)}(\cdot,\cdot)-K_{(x_i,t_j)}(\cdot,\cdot)\right\|_{W_2^{(1,1)}(B_\infty)}.\end{aligned}$$

Since $K_{(x,t)}(\cdot,\cdot)=R_x(\cdot)Q_t(\cdot)$, by using Lemmas 2.3 and 2.5 and the general properties of an inner product we obtain

$$\begin{aligned}\left|Tu(x,t)-Tu(x_i,t_j)\right| &\le \|g\|_{W_2^{(1,1)}(B_\infty)}\left\|\delta_x(\cdot)q_t(\cdot)-\delta_{x_i}(\cdot)q_{t_j}(\cdot)\right\|_{W_2^{(1,1)}(B_\infty)} \\ &= \|g\|_{W_2^{(1,1)}(B_\infty)}\left\|\delta_x(\cdot)q_t(\cdot)-\delta_x(\cdot)q_{t_j}(\cdot)+\delta_x(\cdot)q_{t_j}(\cdot)-\delta_{x_i}(\cdot)q_{t_j}(\cdot)\right\|_{W_2^{(1,1)}(B_\infty)} \\ &\le \|g\|_{W_2^{(1,1)}(B_\infty)}\left\|\delta_x(\cdot)\right\|_{W_2^1[0,\infty)}\left\|q_t(\cdot)-q_{t_j}(\cdot)\right\|_{W_2^1[0,1])} \\ &\quad + \|g\|_{W_2^{(1,1)}(B_\infty)}\left\|\delta_x(\cdot)-\delta_{x_i}(\cdot)\right\|_{W_2^1[0,\infty)}\left\|q_{t_j}(\cdot)\right\|_{W_2^1[0,1]} \\ &\le \sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\left(\sqrt{|t-t_j|}+\sqrt{|x-x_i|}\right) \\ &\le \sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\left(\frac{1}{\sqrt{N}}+\frac{1}{\sqrt{N}}\right) \\ &= 2\sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}.\end{aligned}$$

Proceeding in the same way as we did above, the following inequality holds:

$$\left|Tu_n(x,t)-Tu_n(x_i,t_j)\right| \le 2\sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}.$$

As a result we conclude that

$$\left|Tu(x,t)-Tu_n(x,t)\right| \le 4\sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}.$$

Therefore, it follows from (4.5) that

$$\begin{aligned}\left|u(x,t)-u_n(x,t)\right| &\le \int_0^t\int_{x-t+\tau}^{x+t-\tau} S(x-\xi,t-\tau)\left|g(\xi,\tau)-g_n(\xi,\tau)\right|d\xi d\tau \\ &= \int_0^t\int_{x-t+\tau}^{x+t-\tau} S(x-\xi,t-\tau)\left|Tu(\xi,\tau)-Tu_n(\xi,\tau)\right|d\xi d\tau \\ &\le \int_0^t\int_{x-t+\tau}^{x+t-\tau} S(x-\xi,t-\tau)\left(4\sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}\right)d\xi d\tau \\ &\le 4\sqrt{r+1}\;\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}}\sup_{(\xi,\tau)\in\triangle_{(1,1)}}\left|S(\xi,\tau)\right| \\ &= C_r\|g\|_{W_2^{(1,1)}(B_\infty)}\frac{1}{\sqrt{N}},\end{aligned}$$

where $C_r=4\sqrt{r+1}\sup_{(\xi,\tau)\in\triangle_{(1,1)}}\left|S(\xi,\tau)\right|\le 2\sqrt{r+1}$. $\square$

5. Application

Example 5.1. Solve the nonhomogeneous partial differential equation $u_{tt} - u_{xx} + 4u_t + u = f$ in B_∞ subject to the homogeneous initial conditions $u(x,0) = 0 = u_t(x,0)$ for $x \in (0,\infty)$; here

$$f(x,t) = \begin{cases} e^{-1}\big((2+8t+t^2)p(x) - t^2 p''(x)\big) & \text{if } 0 \le t \le x \le 1, \\ (2+8t)e^{-x} & \text{if } 0 \le t \le 1 \text{ and } 1 < x, \end{cases}$$

where $p(x) = -\frac{1}{6}x^3 + x^2 - \frac{5}{2}x + \frac{8}{3}$. Observe that an exact solution for this problem is

$$u(x,t) = \begin{cases} t^2 p(x) e^{-1} & \text{if } 0 \le t \le x \le 1, \\ t^2 e^{-x} & \text{if } 0 \le t \le 1 \text{ and } 1 < x, \end{cases}$$

and $u \in W_2^{(3,3)}(B_\infty)$. We truncate B_∞ to B_5 and approximate the solution numerically. The results of applying the reproducing kernel method with the reproducing kernel function in $\widetilde{W}(B_\infty)$ with 30 and 95 uniformly distributed nodes in B_5 are shown in Tables 1 and 2, respectively. We observe from Theorem 4.5 that the theoretical upper bound for the uniform error on B_5 when $N = 4$ and $r = 5$ is given by

$$\begin{aligned} |u(x,t) - u_n(x,t)| &\le \frac{4\sqrt{6}\ \sup_{(x,t)\in\triangle_{(1,1)}} |S(x,t)|\,\|f\|_{W_2^{(1,1)}(B_\infty)}}{2} \\ &\le \sqrt{6}\|f\|_{W_2^{(1,1)}(B_\infty)} \approx 0.48721. \end{aligned}$$

However, the much more accurate results displayed in Tables 1 and 2 reflect the extreme smoothness of the driver f at points in B_∞ off the line segment from $(1,0)$ to $(1,1)$.

t/x	**0**	**1/2**	**1**	**3/2**	**2**	**5/2**
0	**0**	**0**	**0**	**0**	**0**	**0**
1/2	**-**	76×10^{-8}	15×10^{-6}	18×10^{-6}	68×10^{-6}	14×10^{-5}
1	**-**	**-**	16×10^{-5}	10×10^{-5}	15×10^{-5}	24×10^{-5}

t/x	**3**	**7/2**	**4**	**9/2**	**5**
0	**0**	**0**	**0**	**0**	**0**
1/2	24×10^{-5}	38×10^{-5}	54×10^{-5}	74×10^{-5}	95×10^{-5}
1	43×10^{-5}	67×10^{-5}	98×10^{-5}	13×10^{-4}	16×10^{-4}

Table 1. The absolute error for Example 5.1 with 30 equally spaced nodes in B_5 in the space $\widetilde{W}(B_\infty)$.

t/x	**0**	**1/4**	**1/2**	**3/4**	**1**	**5/4**	**6/4**	**7/4**	**2**	**9/4**	**5/2**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1/4	**-**	17×10^{-7}	18×10^{-7}	36×10^{-8}	14×10^{-7}	83×10^{-8}	78×10^{-8}	14×10^{-7}	26×10^{-7}	40×10^{-7}	56×10^{-7}
1/2	**-**	**-**	12×10^{-6}	31×10^{-7}	12×10^{-6}	12×10^{-7}	81×10^{-7}	37×10^{-7}	42×10^{-7}	76×10^{-7}	12×10^{-6}
3/4	**-**	**-**	**-**	73×10^{-7}	42×10^{-6}	32×10^{-6}	13×10^{-6}	80×10^{-7}	12×10^{-6}	20×10^{-6}	29×10^{-6}
1	**-**	**-**	**-**	**-**	97×10^{-6}	81×10^{-6}	32×10^{-6}	14×10^{-6}	16×10^{-6}	24×10^{-6}	34×10^{-6}

t/x	**11/4**	**3**	**13/4**	**7/2**	**15/4**	**4**	**17/4**	**9/2**	**19/4**	**5**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1/4	74×10^{-7}	97×10^{-7}	12×10^{-6}	14×10^{-6}	17×10^{-6}	20×10^{-6}	24×10^{-6}	28×10^{-6}	32×10^{-6}	36×10^{-6}
1/2	22×10^{-6}	29×10^{-6}	36×10^{-6}	44×10^{-6}	53×10^{-6}	62×10^{-6}	73×10^{-6}	84×10^{-6}	96×10^{-6}	10×10^{-5}
3/4	38×10^{-6}	50×10^{-6}	63×10^{-6}	77×10^{-6}	93×10^{-6}	11×10^{-5}	12×10^{-5}	14×10^{-5}	17×10^{-5}	17×10^{-5}
1	46×10^{-6}	58×10^{-6}	74×10^{-6}	91×10^{-6}	11×10^{-5}	13×10^{-5}	15×10^{-5}	17×10^{-5}	19×10^{-5}	19×10^{-5}

TABLE 2. The absolute error for Example 5.1 with 95 equally spaced nodes in B_5 in the space $\widetilde{W}(B_\infty)$.

6. Conclusion

In this paper, we introduced new reproducing kernel Hilbert spaces, $W_2^{(3,3)}(B_\infty)$ and a closed subspace $\widetilde{W}(B_\infty)$, on a non-rectangular semi-infinite domain, B_∞. Despite the non-rectangular, non-compact nature of B_∞, the reproducing kernels for these spaces are piecewise polynomial functions. We established a uniform approximation result on B_r for solutions to the nonhomogeneous telegraph equation in $\widetilde{W}(B_\infty)$ in terms of the number of nodes. Finally, we illustrated the stability of such solutions with respect to the driver through a numerical example.

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