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ARTICLE TYPE

Blow up and asymptotic behavior of solutions for a $p(x)$-Laplacian equation with delay term and variable exponents

Stanislav Antontsev[1] | Jorge Ferreira*[2] | Erhan Pişkin[3] | Hazal Yüksekkaya[3]

[1]Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russian
[2]Department of Exact Sciences, Federal Fluminense University, Rio de Janeiro, Brazil
[3]Department of Mathematics, Department of Mathematics, Diyarbak, Turkey

Correspondence
*Jorge Ferreira, Av. dos Trabalhadores, 420,Volta Redonda, RJ, Brazil, Email: ferreirajorge2012@gmail.com

Summary

In this paper, we consider a nonlinear $p(x)-$Laplacian equation with delay of time and variable exponents. Firstly, we prove the blow up of solutions. Then, by applying an integral inequality due to Komornik, we obtain the decay result. These results improve and extend earlier results in the literature.

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KEYWORDS:
Blow up, Decay, Variable exponent, Delay term.

Dedicated to Professor Sergey Shmarev on the occasion of his 60th anniversary.

1 | INTRODUCTION

In this work, we investigate the following wave equation

$$\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \mu_1 u_t(x,t)\left|u_t\right|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)\left|u_t\right|^{m(x)-2}(x,t-\tau), & \\ \qquad = bu\,|u|^{q(x)-2} & \text{in } \Omega\times R^+ \\ \quad u(x,t) = 0 & \text{in } \partial\Omega\times[0,\infty), \\ \quad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ \quad u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times\ (0,\tau), \end{cases} \tag{1}$$

with delay term. Here, $\Omega\subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\tau>0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number and $b\geq 0$ is a constant. The term $\Delta_{p(\cdot)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called $p(\cdot)$-Laplacian. The functions u_0, u_1, f_0 are the initial dates that will be specified later.

$p(\cdot)$, $q(\cdot)$ and $m(\cdot)$ are the variable exponents; these are given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{aligned} 2 &\leq p^- \leq p(x) \leq p^+ \leq p^*, \\ 2 &\leq q^- \leq q(x) \leq q^+ \leq q^*, \\ 2 &\leq m^- \leq m(x) \leq m^+ \leq m^* \end{aligned} \tag{2}$$

where

$$p^{-} = ess\inf_{x\in\Omega} p(x),\quad p^{+} = ess\sup_{x\in\Omega} p(x),$$
$$q^{-} = ess\inf_{x\in\Omega} q(x),\quad q^{+} = ess\sup_{x\in\Omega} q(x),$$
$$m^{-} = ess\inf_{x\in\Omega} m(x),\quad m^{+} = ess\sup_{x\in\Omega} m(x)$$

and

$$p^{*} = \begin{cases} \frac{Np(x)}{ess\sup_{x\in\Omega}(N-m(x))} \text{ if } p^{+} < n, \\ +\infty \quad \text{if } p^{+} \geq n. \end{cases}$$

Many phenomena in engineering and physics lead up to problems that deal with evolution equations, which are modelled by partial differential equations. Up to now, there are many results about partial differential equations with time delay effects. Our main goal in this work is to study the equation with $p(\cdot)$-Laplacian and the delay term $\mu_2 u_t(x, t-\tau)$ which make the problem more interesting than those studied in the literature. The equation (1) is a very general equation.

Time delay appears in many practical problems such as thermal, biological, economic, chemical, physical phenomena and it can be a source of instability [11]. Mathematically, these properties have practical and theoretical importance. On the other hand, the delay term is a source that may destabilize the asymptotic stability of solutions for an evolutionary system. This result is well justified in mathematical analysis and physics examples, such as non-instant transmission phenomena and biological models [33].

The problems with variable exponents arise in many branches of sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [4,5,30]. Many works about wave equation with constant delay or delay effects with time-varying have been published.

Constant exponent:

(Delay).

In [7], Feng and Li studied the following equation

$$\begin{aligned} & u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) - \sigma(t)\int_0^t g(t-s)\,\Delta^2 u(s)\,ds + \mu_1 \left|u_t\right|^{m-1} u_t \\ & +\mu_2 \left|u_t(x, t-\tau)\right|^{m-1} u_t(x, t-\tau) \\ = {} & 0, \end{aligned} \tag{3}$$

where $\Omega \subseteq R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. They proved the general rates of energy decay of the initial value problem and the boundary value problem by using the energy perturbation method.

Messaoudi and Kafini [11] considered the following equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = b\,|u|^{p-2} u. \tag{4}$$

Under suitable conditions, they proved the blow-up of solutions of the equation (4) in a finite time.

Nicaise and Pignotti [21] discussed the following wave equation with time delay,

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \tag{5}$$

and they established stability results under the assumption $0 < \mu_2 < \mu_1$.

In [22], Park treated the Kirchhoff models with time delay and perturbation of p-Laplacian type

$$u_{tt} + \Delta^2 u - \Delta_p u - a_0 \Delta u_t + a_1 u_t(x, t-\tau) + f(u) = g(x), \tag{6}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the usual p-Laplacian operator and $a_0 > 0$, $a_1 \in R$, $\tau > 0$ is time delay. He established the existence of global attractors and the finite dimensionality of the attractors by establishing some functionals.

(Without Delay).

In [28], Piskin studied the following quasilinear hyperbolic equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m}\nabla u\right) - \Delta u_t + \left|u_t\right|^{q-1} u_t = |u|^{p-1} u, \tag{7}$$

where $m > 0$, $p, q \geq 1$. He investigated the global existence, decay and blow up of solutions. He proved the decay estimates of the energy function by using Nakao's inequality and obtained the blow up of solutions and lifespan estimates in three different ranges of the initial energy.

Wu and Xue[32] considered the following quasi-linear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}\left(|\nabla u|^m \nabla u\right) + a\left|u_t\right|^{\alpha} u_t = b|u|^{p-1} u, \tag{8}$$

where $a, b, \alpha, m, p \geq 0$. By using the multiplier methods, they gave the precise uniform estimation of the decay rate, when the initial datas are in the potential well.

Variable exponent nonlinearity:

Recently, much attention has been given to the study of nonlinear mathematical models of hyperbolic, parabolic and elliptic equations with variable exponents as nonlinearity. Actually, only few work regarding hyperbolic problems with nonlinearities of variable-exponent type have appeared[17].

(Delay)

Messaoudi and Kafini[19] discussed the following equation

$$u_{tt} - \Delta u + \mu_1 u_t(x,t)\left|u_t\right|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)\left|u_t\right|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2}. \tag{9}$$

They studied the decay estimates and global nonexistence of the equation (9).

(Without Delay).

Antontsev[1,2] looked to the following equation

$$\partial_{tt}u - \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)}\nabla u\right) - \alpha\Delta u_t = b(x,t)u|u|^{\sigma(x,t)-2}, \tag{10}$$

in Ω, a bounded domain of R^n, where $\alpha > 0$ is a constant and a, b, p, σ are given by functions. For certain solutions with non-positive initial energy, he proved the blow-up results. In[1,3], Antontsev studied the same equation (10) and he proved the local and the global existence of some weak solutions.

Messaoudi et. al.[8] studied the following equation

$$u_{tt} - \Delta u + u_t\left|u_t\right|^{m(\cdot)-2} = u|u|^{p(\cdot)-2}. \tag{11}$$

They proved a global result and obtained the stability result by applying an integral inequality due to Komornik.

In[15], the authors considered the following equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) + \left|u_t\right|^{m(\cdot)-2}u_t = 0, \tag{12}$$

where the exponents $m(\cdot)$ and $r(\cdot)$ are given by measurable functions on Ω. They proved the decay results for the solution under suitable assumptions. Also, the authors gave two numerical applications to illustrate the theoretical results.

In the presence of strong damping term $-\Delta u_t$, the equation (12) takes the following form

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) - \Delta u_t + \left|u_t\right|^{m(\cdot)-2}u_t = 0, \tag{13}$$

where Ω is a bounded domain. In[20], Messaoudi studied the nonlinear wave equation (13) with variable exponents. He established several decay results depending of the range of the variable exponents m and r. In recent years, some other authors investigated hyperbolic type equation with variable exponents (see[10,23,24,25,26,27,31]).

Our purpose is to study the blow up of solutions with negative initial energy and the decay results for the nonlinear wave equation (1) with time-dependent delay and variable exponents, in this paper. Our result extends the equation (4), from constant-exponent nonlinearities to variable-exponent nonlinearities.

This paper is organized as follows: In Sect. 2, the definition of variable exponent in Sobolev and Lebesgue spaces is introduced. In Sect. 3, we prove the blow up of solutions. Finally, in Sect. 4, the decay results will be obtained.

2 | PRELIMINARIES

In this part, we begin by introducing some preliminary facts about Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see[1,5,6,9,12,19,29]).

Let $p : \Omega \to [1,\infty)$ be a measurable function. We define the variable-exponent in Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to R;\ \text{measurable in } \Omega\ :\ \int_{\Omega} |u|^{p(\cdot)}\, dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. (see[5])

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, in the same way that the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

We also suppose that $p(\cdot)$, $q(\cdot)$ and $m(\cdot)$ satisfy the log-Hölder continuity condition:

$$|q(x) - q(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{14}$$

$A > 0$ and $0 < \delta < 1$.

Lemma 1. [1] (Poincare inequality) Assume that $q(\cdot)$ satisfies (14) and let Ω be a bounded domain of R^n. Then,

$$\|u\|_{p(\cdot)} \le c\, \|\nabla u\|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c\left(q^-, q^+, |\Omega|\right) > 0$.

Lemma 2. [14] If $p(\cdot) \in C\left(\overline{\Omega}\right)$ and $q : \Omega \to [1,\infty)$ is a measurable function such that

$$es\sin f_{x\in\Omega}\left(p^*(x) - q(x)\right) > 0 \text{ with } p_*(x) = \begin{cases} \frac{np(x)}{ess\sup_{x\in\Omega}(n-p(x))} & \text{if } p^+ < n, \\ +\infty & \text{if } p^+ \ge n, \end{cases} \tag{15}$$

is satisfied, then the embedding $W_0^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$ is continuous and compact.

Lemma 3. [1] If $p^+ < \infty$ and $p : \Omega \to [1,\infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Lemma 4. [1] (Hölder's inequality) Let $p, q, s \ge 1$ be measurable functions defined on Ω and

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.$$

that is satisfied. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \le 2\, \|f\|_{p(\cdot)}\, \|g\|_{q(\cdot)}.$$

Lemma 5. [1] (Unit ball property) Let $p \ge 1$ be a measurable function on Ω. Then,

$$\|f\|_{p(\cdot)} \le 1 \text{ if and only if } \varrho_{p(\cdot)}(f) \le 1,$$

where

$$\varrho_{p(\cdot)}(f)=\int_{\Omega}|f(x)|^{p(x)}\,dx.$$

Lemma 6. [1] If $p\geq 1$ is a measurable function on Ω. Then,

$$\min\left\{\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right\}\leq\varrho_{p(\cdot)}(u)\leq\max\left\{\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right\}$$

for any $u\in L^{p(\cdot)}(\Omega)$ and for a.e. $x\in\Omega$.

3 | BLOW UP

In this part, we deal with the blow up of the solution for the problem (1) with negative initial energy, when $b>0$. Now, we introduce, similarly to[21], the new variable:

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\ x\in\Omega,\ \rho\in(0,1),\ t>0$$

which implies that

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Using the above transformation, the problem (1) can be written as an equivalent form:

$$\begin{cases}u_{tt}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+\mu_1 u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}\\ \qquad\qquad +\mu_2 z(x,1,t)\,|z(x,1,t)|^{m(x)-2}\\ \qquad\qquad =bu\,|u|^{q(x)-2}\ \text{ in }\Omega\times(0,\infty),\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad \text{in }\Omega\times(0,1)\times(0,\infty)\\ z(x,\rho,0)=f_0(x,-\rho\tau) \qquad \text{in }\Omega\times(0,1)\\ u(x,t)=0 \qquad\qquad\qquad \text{on }\partial\Omega\times[0,\infty)\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x)\ \text{ in }\Omega.\end{cases}\tag{16}$$

Similar to the work of[19] we can write the following definition:

Definition 1. Fix $T>0$. We call (u,z) a strong solution of (16) if

$$\begin{aligned}u\ &\in\ W^{2,\infty}\left([0,T);L^2(\Omega)\right)\cap W^{1,\infty}\left([0,T);H_0^1(\Omega)\right)\\ &\quad\cap L^{\infty}\left([0,T);H^2(\Omega)\cap H_0^1(\Omega)\right),\\ u_t\ &\in\ L^{m(\cdot)}(\Omega\times(0,T)),\\ z\ &\in\ W^{1,\infty}\left([0,1]\times[0,T);L^2(\Omega)\right)\cap L^{\infty}\left([0,1];L^{m(\cdot)}(\Omega)\cap[0,T)\right)\end{aligned}\tag{17}$$

and (u,z) satisfies the initial data and (16) in the following sense:

$$\begin{aligned}&\int_{\Omega}u_{tt}(\cdot,t)\,v\,dx-\int_{\Omega}\operatorname{div} div\left(|\nabla u(\cdot,t)|^{p(\cdot)-2}\nabla u(\cdot,t)\right)v\,dx\\ &+\mu_1\int_{\Omega}\left|u_t(\cdot,t)\right|^{m(\cdot)-2}u_t(\cdot,t)\,v\,dx+\mu_2\int_{\Omega}|z(\cdot,1,t)|^{m(\cdot)-2}z(\cdot,1,t)\,v\,dx\\ =\ &b\int_{\Omega}|u(\cdot,t)|^{q(\cdot)-2}u(\cdot,t)\,v\,dx\end{aligned}\tag{18}$$

and

$$\tau\int_{\Omega}z_t(\cdot,\rho,t)\,w\,dx+\int_{\Omega}z_\rho(\cdot,\rho,t)\,w\,dx=0,\tag{19}$$

for a.e. $t\in[0,T)$ and for $(v,w)\in H_0^1(\Omega)\cap L^2(\Omega)$.

The functional energy associated to problem (16) is defined as

$$E(t) = \frac{1}{2}\left\|u_t\right\|^2 + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho - b\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx \tag{20}$$

for $t \geq 0$, where ξ is a continuous function that satisfies

$$\tau\left|\mu_2\right|(m(x)-1) < \xi(x) < \tau\left(\mu_1 m(x) - \left|\mu_2\right|\right),\ x\in\overline{\Omega}. \tag{21}$$

The following lemma gives that, under the condition $\mu_1 > \left|\mu_2\right|$, $E(t)$ is nonincreasing.

Lemma 7. Let (u,z) be a solution of (16). Then there exists some $C_0 > 0$ such that

$$E'(t) \leq -C_0\int_\Omega\left(\left|u_t\right|^{m(x)} + |z(x,1,t)|^{m(x)}\right)dx \leq 0. \tag{22}$$

Proof. Multiplying the first eq. in (16) by u_t, integrating over Ω, then multiplying the second eq. of (16) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$ and integrating over $\Omega\times(0,1)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int_\Omega u_t^2dx + \int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho - b\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx\right]\\
&= -\mu_1\int_\Omega\left|u_t\right|^{m(x)}dx - \frac{1}{\tau}\int_\Omega\int_0^1\xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\
&\quad-\mu_2\int_\Omega u_tz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned} \tag{23}$$

The last two terms of the right-hand side of (23) can be estimated as follows,

$$\begin{aligned}
&-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\
&= -\frac{1}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx\\
&= \frac{1}{\tau}\int_\Omega\frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)} - |z(x,1,t)|^{m(x)}\right)dx\\
&= \int_\Omega\frac{\xi(x)}{\tau m(x)}\left|u_t\right|^{m(x)}dx - \int_\Omega\frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}.
\end{aligned}$$

We use Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term, and then we obtain

$$\left|u_t\right||z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)}\left|u_t\right|^{m(x)} + \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

As a result, we get

$$
\begin{aligned}
&-\mu_2 \int_\Omega u_t z\, |z(x,1,t)|^{m(x)-2}\, dx \\
&\le |\mu_2| \left(\int_\Omega \frac{1}{m(x)} |u_t(t)|^{m(x)}\, dx + \int_\Omega \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)}\, dx \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{dE(t)}{dt} \le & -\int_\Omega \left[\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) \right] |u_t(t)|^{m(x)}\, dx \\
& -\int_\Omega \left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} \right) |z(x,1,t)|^{m(x)}\, dx.
\end{aligned}
$$

Consequently, for all $x \in \overline{\Omega}$, the relation (21) gives,

$$
\begin{aligned}
f_1(x) &= \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0, \\
f_2(x) &= \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.
\end{aligned}
$$

Since that $m(x)$, and hence, $\xi(x)$ is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. Thus, if we define

$$
C_0(x) = \min\left\{ f_1(x),\ f_2(x) \right\} > 0 \text{ for any } x \in \overline{\Omega}
$$

and take $C_0(x) = \inf_{\overline{\Omega}} C_0(x)$, so $C_0(x) \ge C_0 > 0$. Therefore,

$$
E'(t) \le -C_0 \left[\int_\Omega |u_t(t)|^{m(x)}\, dx + \int_\Omega |z(x,1,t)|^{m(x)}\, dx \right] \le 0.
$$

□

To establish the blow up, we suppose that $E(0) < 0$ in addition to (2).

Doing

$$
H(t) = -E(t), \tag{24}
$$

therefore,

$$
H'(t) = -E'(t) \ge 0,
$$

$$
0 < H(0) \le H(t) \le b \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx \le \frac{b}{p^-} \varrho(u), \tag{25}
$$

where

$$
\varrho(u) = \varrho_{q(\cdot)}(u) = \int_\Omega |u|^{q(x)}\, dx.
$$

Lemma 8. [17] Assume that the conditions of Lemma 2 hold. Then, exists a constant $C > 1$, depending only of Ω, such that

$$
\varrho^{s/p^-}(u) \le C \left(\|\nabla u\|_{p(\cdot)}^{p^-} + \varrho(u) \right). \tag{26}
$$

Then, we have the following inequalities:

i)

$$
\|u\|_{p^-}^{s} \le C \left(\|\nabla u\|_{p(\cdot)}^{p^-} + \|u\|_{q^-}^{q^-} \right), \tag{27}
$$

ii)

$$\varrho^{s/q^-}(u) \leq C\left(|H(t)| + \|u_t\|_2^2 + \varrho(u) + \int_0^1 \int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right), \tag{28}$$

iii)

$$\|u\|_{q^-}^s \leq C\left(|H(t)| + \|u_t\|_2^2 + \|u\|_{q^-}^{q^-} + \int_0^1 \int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right), \tag{29}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$ and $p^- \leq s \leq q^-$. Let (u, z) be a solution of (16), then

iv)

$$\varrho(u) \geq C\,\|u\|_{q^-}^{q^-}, \tag{30}$$

v)

$$\int_\Omega |u|^{m(x)}\,dx \leq C\left(\varrho^{m^-/q^-}(u) + \varrho^{m^+/q^-}(u)\right). \tag{31}$$

The blow up of the problem (16) is given by the following theorem:

Theorem 1. Let $u_0 \in W_0^{1,p(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$. The conditions (2) and (14) are provided and we suppose that

$$E(0) < 0.$$

Then, the solution (16) blows up in finite time.

Proof. We define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\,dx \tag{32}$$

where the small ε will be chosen later and

$$0 \leq \alpha \leq \min\left\{\frac{q^- - 2}{2q^-}, \frac{q^- - m^+}{q^-(m^+ - 1)}\right\}. \tag{33}$$

A direct differentiation of (32), using the first equation in (16), gives

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{p(x)} \\ &+\varepsilon b \int_\Omega |u|^{q(x)}\,dx - \varepsilon \mu_1 \int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} dx \\ &-\varepsilon \mu_2 \int_\Omega u z(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dx. \end{aligned}$$

From the definition of $H(t)$ and for $0 < a < 1$, we obtain

$$\begin{aligned}
L'(t) \geq\ & C_0 (1-\alpha) H^{-\alpha}(t)\left[\int_\Omega \left|u_t(t)\right|^{m(x)} dx + \int_\Omega \left|z(x,1,t)\right|^{m(x)} dx\right] \\
&+\varepsilon\left((1-a) q^- H(t) + \frac{(1-a) q^-}{2}\left\|u_t\right\|^2\right) \\
&\varepsilon (1-a) q^- \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \varepsilon (1-a) q^- \int_0^1 \int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)} dx d\rho \\
&+\varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{p(x)} + \varepsilon ab \int_\Omega |u|^{q(x)} dx \\
&-\varepsilon \mu_1 \int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} dx \\
&-\varepsilon \mu_2 \int_\Omega u z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2} dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
L'(t) \geq\ & C_0 (1-\alpha) H^{-\alpha}(t)\left[\int_\Omega \left|u_t(t)\right|^{m(x)} dx + \int_\Omega \left|z(x,1,t)\right|^{m(x)} dx\right] \\
&+\varepsilon (1-a) q^- H(t) + \varepsilon \frac{(1-a) q^- + 2}{2}\left\|u_t\right\|^2 \\
&+\varepsilon\left(\frac{(1-a) q^-}{p^+} - 1\right) \int_\Omega |\nabla u|^{p(x)} dx \\
&+\varepsilon (1-a) q^- \int_0^1 \int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)} dx d\rho + \varepsilon ab \varrho(u) \\
&-\varepsilon \mu_1 \int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} dx \\
&-\varepsilon \mu_2 \int_\Omega u z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2} dx. \qquad (34)
\end{aligned}$$

From Young's inequality, we get

$$\begin{aligned}
\int_\Omega \left|u_t\right|^{m(x)-1} |u| dx \leq\ & \frac{1}{m^-} \int_\Omega \delta^{m(x)} |u|^{m(x)} dx \\
&+\frac{m^+ - 1}{m^+} \int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} \left|u_t\right|^{m(x)} dx \qquad (35)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \left|z(x,1,t)\right|^{m(x)-1} |u| dx \leq\ & \frac{1}{m^+} \int_\Omega \delta^{m(x)} |u|^{m(x)} dx \\
&+\frac{m^+ - 1}{m^+} \int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} \left|z(x,1,t)\right|^{m(x)} dx. \qquad (36)
\end{aligned}$$

As in [16], the estimates (35) and (36) remain if δ is time-dependent. Thus, taking δ such that

$$\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),$$

where $k \geq 1$ is specified later, we obtain

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)} dx = kH^{-\alpha}(t) \int_\Omega |u_t|^{m(x)} dx, \tag{37}$$

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} |z(x,1,t)|^{m(x)} dx = kH^{-\alpha}(t) |z(x,1,t)|^{m(x)} dx \tag{38}$$

and

$$\begin{aligned}\int_\Omega \delta^{m(x)} |u|^{m(x)} dx &= \int_\Omega k^{1-m(x)} H^{\alpha(m(x)-1)}(t) |u|^{m(x)} dx \\ &\leq \int_\Omega k^{1-m^-} H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx. \end{aligned}\tag{39}$$

From (30) and (31), we have

$$\begin{aligned}&H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx \\ &\leq C\left[(\varrho(u))^{m^-/q^-+\alpha(m^+-1)} + (\varrho(u))^{m^+/q^-+\alpha(m^+-1)}\right]. \end{aligned}\tag{40}$$

From (33), we conclude that

$$s = m^- + \alpha q^-\left(m^+ - 1\right) \leq q^- \text{ and } s = m^+ + \alpha q^-\left(m^+ - 1\right) \leq q^-.$$

Therefore, Lemma 8 satisfies

$$H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx \leq C\left(\|\nabla u\|_{r(\cdot)}^{r^-} + \varrho(u)\right). \tag{41}$$

Combining (35)-(41), we get

$$\begin{aligned}L'(t) \geq\ & (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |u_t(t)|^{m(x)} dx \\ &+ (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |z(x,1,t)|^{m(x)} dx \\ &+ \varepsilon\left(\frac{(1-a)q^- - p^+}{p^+} - \frac{C}{m^- k^{1-m^-}}\right)\int_\Omega |\nabla u|^{p(x)} dx \\ &+ \varepsilon(1-a)q^- H(t) + \varepsilon\frac{(1-a)q^- + 2}{2}\|u_t\|^2 \\ &+ \varepsilon\left(ab - \frac{C}{m^- k^{1-m^-}}\right)\varrho(u) \\ &+ \varepsilon(1-a)q^- \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned}\tag{42}$$

Let us choose a small enough such that

$$\frac{(1-a)q^- + 2}{2} > 0,$$

and k so large that

$$\frac{(1-a)\,q^- - p^+}{p^+} - \frac{C}{m^- k^{1-m^-}} > 0 \text{ and } ab - \frac{C}{m^- k^{1-m^-}} > 0.$$

Once that k and a are fixed, we choose ε small enough such that

$$C_0 - \varepsilon\left(\frac{m^+ - 1}{m^+}\right) ck > 0$$

and

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0(x)\, u_1(x)\, dx > 0.$$

Hence, (42) becomes

$$L'(t) \geq \varepsilon\eta\left[H(t) + \left\|u_t\right\|^2 + \|\nabla u\|_{p(\cdot)}^{p^-} + \varrho_{p(\cdot)}(u) + \int_0^1 \int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx\, d\rho\right] \tag{43}$$

for a constant $\eta > 0$. Eventually,

$$L(t) \geq L(0) > 0, \forall t \geq 0.$$

Now, we denote, for some constants $\sigma, \Gamma > 0$, that

$$L'(t) \geq \Gamma L^\sigma(t).$$

For this reason, we estimate

$$\left|\int_\Omega u u_t(x,t)\, dx\right| \leq \|u\|_2 \left\|u_t\right\|_2 \leq C\, \|u\|_{p^-} \left\|u_t\right\|_2,$$

which indicates

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\, \|u\|_{p^-}^{1/(1-\alpha)} \left\|u_t\right\|_2^{1/(1-\alpha)}$$

and Young's inequality satisfies

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p^-}^{\mu(1-\alpha)} + \left\|u_t\right\|_2^{\Theta/(1-\alpha)}\right],$$

where $1/\mu + 1/\Theta = 1$. The choice of $\Theta = 2(1-\alpha)$ will make $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p^-$. Thus,

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p^-}^{s} + \left\|u_t\right\|^2\right],$$

where $s = \mu/(1-\alpha)$. From (29), we get

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\left[|H(t)| + \left\|u_t(t)\right\|^2 + \varrho(u) + \int_0^1 \int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx\, d\rho\right]. \tag{44}$$

Moreover, we have

$$
\begin{aligned}
L^{1/(1-\alpha)}(t) &= \left[H^{(1-\alpha)}(t) + \varepsilon \int_{\Omega} u u_t(x,t)\,dx \right]^{1/(1-\alpha)} \\
&\leq 2^{\alpha/(1-\alpha)} \left[H(t) + \left| \int_{\Omega} u u_t dx \right|^{1/(1-\alpha)} \right] \\
&\leq C \left[|H(t)| + \left\| u_t(t) \right\|^2 + \varrho(u) + \int_0^1 \int_{\Omega} \frac{\xi(x) \left| z(x,\rho,t) \right|^{m(x)}}{m(x)} dx\,d\rho \right].
\end{aligned} \tag{45}
$$

Therefore, for some $\Psi > 0$, from (43), we obtain

$$L'(t) \geq \Psi L^{1/(1-\alpha)}(t).$$

A simple Integration over $(0,t)$ gives

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Psi \alpha t/(1-\alpha)}$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\alpha}{\Psi \alpha \left[L(0) \right]^{\alpha/(1-\alpha)}}.$$

The proof of the theorem was completed. □

4 | DECAY

In this part, we prove our decay result, when $b = 0$. Now, we introduce the following variable

$$z(x,\rho,t) = u_t(x, t - \tau\rho),\ x \in \Omega,\ \rho \in (0,1),\ t > 0;$$

thus,

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Consequently, problem (1) is transformed into:

$$
\begin{cases}
u_{tt} - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u \right) + \mu_1 u_t(x,t) \left| u_t(x,t) \right|^{m(x)-2} \\
\quad + \mu_2 z(x,1,t) \left| z(x,1,t) \right|^{m(x)-2} = 0 \text{ in } \Omega \times (0,\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } \Omega \times (0,1) \times (0,\infty) \\
z(x,\rho,0) = f_0(x,-\rho\tau) \quad \text{in } \Omega \times (0,1) \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,1) \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{in } \Omega.
\end{cases} \tag{46}
$$

We define the modified functional energy to the problem (46) by

$$
\begin{aligned}
E(t) = &\ \frac{1}{2} \left\| u_t \right\|^2 + \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\
&+ \int_0^1 \int_{\Omega} \frac{\xi(x) \left| z(x,\rho,t) \right|^{m(x)}}{m(x)} dx\,d\rho,
\end{aligned} \tag{47}
$$

where ξ is the continuous function given in (21) and $t \geq 0$.

Similar to Lemma 7, we easily establish, for $\mu_1 > \left| \mu_2 \right|$ and for some $C_0 > 0$, that

$$E'(t) \leq -C_0 \int_{\Omega} \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)} \right) dx \leq 0. \tag{48}$$

Lemma 9. (Komornik,[13]) Let $E : R^+ \to R^+$ be a nonincreasing function and assume that there are constants $\sigma, \omega > 0$ such that

$$\int_s^{\infty} E^{1+\sigma}(t)\, dt \leq \frac{1}{\Omega} E^{\sigma}(0)\, E(s) = cE(s), \forall s > 0.$$

Then, we have

$$\begin{cases} E(t) \leq cE(0) / (1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \leq cE(0)\, e^{-\omega t} & \text{if } \sigma = 0. \end{cases}$$

for all $t \geq 0$.

To prove our main result, we need of the following lemmas.

Lemma 10. [19] The functional

$$F(t) = \tau \int_0^1 \int_{\Omega} e^{-\rho\tau} \xi(x) |z(x,\rho,t)|^{m(x)}\, dx d\rho$$

satisfies

$$F'(t) \leq \int_{\Omega} \xi(x) |u_t|^{m(x)}\, dx - \tau e^{-\tau} \int_0^1 \int_{\Omega} \xi(x) |z(x,\rho,t)|^{m(x)}\, dx d\rho$$

along the solution of (46).

Lemma 11. [18] Let u be a solution of (46). Then, for some $C > 0$,

$$\varrho_{p(x)}(\nabla u) \geq C \|\nabla u\|_{p^-}^{p^+}. \tag{49}$$

Theorem 2. Let $u_0 \in W_0^{1,p(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$ be given and assume that $m(\cdot), p(\cdot) \in C(\bar{\Omega})$. Suppose that the conditions (14) are satisfied and

$$2 \leq p^- \leq p(x) \leq p^+ \leq m^- \leq m(x) \leq m^+ \leq p^*, \forall x \in C\left(\overline{\Omega}\right).$$

Then, there exist two constants $c, \alpha > 0$ independent of t such that any global solution of (46) satisfies,

$$\begin{cases} E(t) \leq ce^{-\alpha t} & \text{if } m(x) = 2, \\ E(t) \leq cE(0) / (1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

Proof. We multiply the first equation of (46) by $uE^r(t)$, for $q > 0$ that will be specified later, and integrate over $\Omega \times (s,T)$, $s < T$. So, we get

$$\int_s^T E^r(t) \int_{\Omega} \left[uu_{tt} - u \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) + \mu_1 uu_t |u_t|^{m(x)-2} + \mu_2 uz(x,1,t) |z(x,1,t)|^{m(x)-2} \right] dxdt = 0,$$

which implies that

$$\int_s^T E^r(t) \int_{\Omega} \left(\begin{array}{c} \frac{d}{dt}(uu_t) - u_t^2 + |\nabla u|^{p(x)} + \mu_1 uu_t(x,t) |u_t(x,t)|^{m(x)-2} \\ +\mu_2 uz(x,1,t) |z(x,1,t)|^{m(x)-2} \end{array} \right) dxdt = 0. \tag{50}$$

By using the definition of $E(t)$, given in (47), and the relation

$$\frac{d}{dt}\left(E^{r}(t)\int_{\Omega}uu_{t}dx\right)=rE^{r-1}(t)E'(t)\int_{\Omega}uu_{t}dx+E^{r}(t)\frac{d}{dt}\int_{\Omega}uu_{t}dx,$$

the equation (50) becomes

$$\begin{aligned}2\int_{s}^{T}E^{r+1}(t)\,dt\leq&-\int_{s}^{T}\frac{d}{dt}\left(E^{r}(t)\int_{\Omega}uu_{t}dx\right)dt+r\int_{s}^{T}E^{r-1}(t)E'(t)\int_{\Omega}uu_{t}dxdt\\&+2\int_{s}^{T}E^{r}(t)\int_{\Omega}u_{t}^{2}dxdt-\mu_{1}\int_{s}^{T}E^{r}(t)\int_{\Omega}uu_{t}\left|u_{t}\right|^{m(x)-2}dxdt\\&-\mu_{2}\int_{s}^{T}E^{r}(t)\int_{\Omega}uz(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dxdt\\&+2\int_{s}^{T}E^{r}(t)\int_{0}^{1}\int_{\Omega}\frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho dt.\end{aligned}\tag{51}$$

Now, we estimate the right-hand side terms of equation (51), respectively.

The first term is estimated as follows:

$$\begin{aligned}&\left|-\int_{s}^{T}\frac{d}{dt}\left(E^{r}(t)\int_{\Omega}uu_{t}dx\right)dt\right|\\&=\left|E^{r}(s)\int_{\Omega}uu_{t}(x,s)\,dx-E^{r}(T)\int_{\Omega}uu_{t}(x,T)\,dx\right|\\&\leq\frac{1}{2}E^{r}(s)\left[\int_{\Omega}u^{2}(x,s)\,dx+\int_{\Omega}u_{t}^{2}(x,s)\,dx\right]\\&\quad+\frac{1}{2}E^{r}(T)\left[\int_{\Omega}u^{2}(x,T)\,dx+\int_{\Omega}u_{t}^{2}(x,T)\,dx\right]\\&\leq E^{r}(s)\left[\frac{1}{2}c_{*}\int_{\Omega}\left|\nabla u(x,s)\right|^{2}dx+E(s)\right]\\&\quad+E^{r}(T)\left[\frac{1}{2}c_{*}\int_{\Omega}\left|\nabla u(x,T)\right|^{2}dx+E(T)\right]\end{aligned}$$

where c_* is the embedding constant. So, we get

$$\begin{aligned}\left|-\int_{s}^{T}\frac{d}{dt}\left(E^{r}(t)\int_{\Omega}uu_{t}dx\right)dt\right|\leq&\ E^{r}(s)\left[c\left\|\nabla u(s)\right\|_{p^-}^{2}+E(s)\right]\\&+E^{r}(T)\left[c\left\|\nabla u(T)\right\|_{p^-}^{2}+E(T)\right]\\\leq&\ E^{r+1}(s)+cE^{r}(s)\left(\left\|\nabla u(s)\right\|_{p^-}^{p^+}\right)^{\frac{2}{p^+}}\\&+E^{r+1}(T)+cE^{r}(T)\left(\left\|\nabla u(T)\right\|_{p^-}^{p^+}\right)^{\frac{2}{p^+}},\end{aligned}$$

where c is a generic positive constant that may change their value from a line to another. Then, we use (49), and recalling that $E(t)$ is nonincreasing, we obtain

$$\left|-\int_s^T \frac{d}{dt}\left(E^r(t)\int_\Omega uu_t dx\right)dt\right| \leq E^{r+1}(s)+cE^r(s)\left(\varrho_{p(x)}(\nabla u(s))\right)^{\frac{2}{p^+}}$$
$$+E^{r+1}(T)+cE^r(T)\left(\varrho_{p(x)}(\nabla u(T))\right)^{\frac{2}{p^+}}$$
$$\leq E^{r+1}(s)+c(E(s))^{r+\frac{2}{p^+}}+E^{r+1}(T)+c(E(T))^{r+\frac{2}{p^+}}$$
$$\leq E^{r+1}(s)+c(E(s))^{r+\frac{2}{p^+}}. \tag{52}$$

In the last estimate, for $p^- > 2$, we applied the following Hölder inequality

$$\int_\Omega |\nabla u|^2 dx \leq |\Omega|^{\frac{p^- -2}{p^-}}\left(\int_\Omega |\nabla u|^{p^-} dx\right)^{\frac{2}{p^-}}.$$

The estimate (52), for the case $p^- = 2$, is true.

Similarly, we deal with the term

$$\left|r\int_s^T E^{r-1}(t)E'(t)\int_\Omega uu_t dxdt\right| \leq -c\int_s^T E^{r-1}(t)E'(t)\left[E(T)+cE^{\frac{2}{p^+}}(t)\right]dt$$
$$\leq -c\left[\int_s^T E^r(t)E'(t)+\int_s^T (E(t))^{r+\frac{2}{p^+}-1}E'(t)dt\right]$$
$$\leq c\left[E^{r+1}(s)+(E(s))^{r+\frac{2}{p^+}}\right]. \tag{53}$$

To treat the other term, we establish

$$\Omega_+ = \left\{x\in\Omega,\ \left|u_t(x,t)\right|\geq 1\right\} \text{ and } \Omega_- = \left\{x\in\Omega,\ \left|u_t(x,t)\right|<1\right\},$$

and we exploit the Hölder's and Young's inequalities, then we have

$$\left|2\int_s^T E^r(t)\int_\Omega u_t^2 dxdt\right| = \left|2\int_s^T E^r(t)\left[\int_{\Omega_+} u_t^2 dx+\int_{\Omega_-} u_t^2 dx\right]dt\right|$$
$$\leq c\int_s^T E^r(t)\left[\left(\int_{\Omega_+}\left|u_t\right|^{m^-}dx\right)^{2/m^-}+\left(\int_{\Omega_-}\left|u_t\right|^{m^+}dx\right)^{2/m^+}\right]dt$$
$$\leq c\int_s^T E^r(t)\left[\left(\int_\Omega\left|u_t\right|^{m(x)}dx\right)^{2/m^-}+\left(\int_\Omega\left|u_t\right|^{m(x)}dx\right)^{2/m^+}\right]dt$$
$$\leq c\int_s^T E^r(t)\left[\left(-E'(t)\right)^{2/m^-}+\left(-E'(t)\right)^{2/m^+}\right]dt$$
$$\leq c\varepsilon\int_s^T (E(t))^{rm^-/(m^- -2)}dt+c_\varepsilon\int_s^T\left(-E'(t)\right)dt$$
$$+c\varepsilon\int_s^T E^{r+1}(t)dt+c_\varepsilon\int_s^T\left(-E'(t)\right)^{2(r+1)/m^+}dt.$$

where $c_\varepsilon = \frac{1}{r+1}\left(\frac{\varepsilon(r+1)}{r}\right)^{-r}$.

Choose r such that $r = m^+/2 - 1$ will make $\frac{rm^-}{m^- - 2} = r + 1 + \frac{m^+ - m^-}{m^- - 2}$. Now, we consider two cases, $m^- > 2$ and $m^- = 2$.

For $m^- > 2$, we have

$$\begin{aligned}
&\left| 2\int_s^T E^r(t) \int_\Omega u_t^2 \, dx\,dt \right| \\
&\leq c\varepsilon \int_s^T E^{r+1}(t)\,dt + c\varepsilon \left(E(0)\right)^{\frac{m^+ - m^-}{m^- - 2}} \int_s^T E^{r+1}(t)\,dt + c_\varepsilon E(s) \\
&\leq \tilde{c}\varepsilon \int_s^T E^{r+1}(t)\,dt + c_\varepsilon E(s),
\end{aligned} \tag{54}$$

where $\tilde{c}$ is a positive constant.

For $m^- = 2$, we get

$$\begin{aligned}
\left| 2\int_s^T E^r(t) \int_\Omega u_t^2 \, dx\,dt \right| &= \left| 2\int_s^T E^r(t) \left[\int_{\Omega_+} u_t^2\,dx + \int_{\Omega_-} u_t^2\,dx \right] dt \right| \\
&\leq c \int_s^T E^r(t) \left[\int_{\Omega_+} |u_t|^{m(x)}\,dx + \left(\int_{\Omega_-} |u_t|^{m^+}\,dx \right)^{2/m^+} \right] dt \\
&\leq c \int_s^T E^r(t) \left[\int_{\Omega} |u_t|^{m(x)}\,dx + \left(\int_{\Omega} |u_t|^{m(x)}\,dx \right)^{2/m^+} \right] dt \\
&\leq c \int_s^T E^r(t)\left(-E'(t)\right) dt + c \int_s^T E^r(t)\left(-E'(t)\right)^{2/m^+} dt \\
&\leq c E^{r+1}(s) + c\varepsilon \int_s^T E^{r+1}(t)\,dt + c_\varepsilon \int_s^T \left(-E'(t)\right)^{2(r+1)/m^+} dt.
\end{aligned}$$

Therefore, with the choice of $r = m^+/2 - 1$, we obtain

$$\begin{aligned}
\left| 2\int_s^T E^r(t) \int_\Omega u_t^2 \, dx\,dt \right| &\leq c E^{r+1}(s) + c\varepsilon \int_s^T E^{r+1}(t)\,dt + c_\varepsilon E(s) \\
&\leq c\varepsilon \int_s^T E^{r+1}(t)\,dt + \left(c_\varepsilon + c E^r(0)\right) E(s) \\
&\leq c\varepsilon \int_s^T E^{r+1}(t)\,dt + \tilde{c}_\varepsilon E(s),
\end{aligned} \tag{55}$$

where $\tilde{c}_\varepsilon = c_\varepsilon + c E^r(0)$.

Due to $m^+ \geq p^+$ and $r = \frac{m^+}{2} - 1$, then $r + \frac{2}{p^+} - 1 \geq 0$. As a result, the estimates (52) and (53) become,

$$\left|-\int_s^T \frac{d}{dt}\left(E^r(t)\int_\Omega uu_t dx\right)dt\right| \le E^{r+1}(s) + c\,(E(s))^{r+\frac{2}{p^+}}$$
$$\le \left[E^r(0) + c\,(E(0))^{r+\frac{2}{p^+}-1}\right]E(s) = \tilde{c}E(s), \tag{56}$$

and

$$\left|r\int_s^T E^{r-1}(t)\,E'(t)\int_\Omega uu_t dxdt\right| \le c\left[E^{r+1}(s) + (E(s))^{r+\frac{2}{p^+}}\right]$$
$$\le c\left[E^r(0) + (E(0))^{r+\frac{2}{p^+}-1}\right]E(s) = \tilde{c}E(s), \tag{57}$$

respectively.

For the next term, by using Young's inequality, we show

$$\begin{aligned}
&\left|-\mu_1\int_s^T E^r(t)\int_\Omega u\left|u_t\right|^{m(x)-1} dxdt\right| \\
&\le \varepsilon\int_s^T E^r(t)\int_\Omega |u|^{m(x)}\,dxdt + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dxdt \\
&\le \varepsilon\int_s^T E^r(t)\left[\int_\Omega |u|^{m^-}\,dx + \int_\Omega |u|^{m^+}\,dx\right]dt \\
&+c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dxdt,
\end{aligned}$$

where we used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1} \text{ and } p'(x) = m(x),$$

and thus

$$c_\varepsilon(x) = (m(x)-1)\,m(x)^{m(x)/(1-m(x))}\,\varepsilon^{1/(1-m(x))}$$

Using the embedding, we get

$$\left|-\mu_1\int_s^T E^r(t)\int_\Omega u\left|u_t\right|^{m(x)-1} dxdt\right| \le \varepsilon\int_s^T E^r(t)\left[c_1\left\|\nabla u\right\|_{p^-}^{m^-} + c_2\left\|\nabla u\right\|_{p^-}^{m^+}\right]$$
$$+\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dx,$$

where c_1 and c_2 are positive constants independent of ε.

From (47) and (49), we have

$$\begin{aligned}
&\left|-\mu_1 \int_s^T E^r(t) \int_\Omega u\left|u_t\right|^{m(x)-1} dxdt\right| \\
&\leq \varepsilon \int_s^T E^r(t)\left[c_1\left(\varrho_{p(x)}(\nabla u)\right)^{\frac{m^-}{p^+}} + c_2\left(\varrho_{p(x)}(\nabla u)\right)^{\frac{m^+}{p^+}}\right] dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dxdt \\
&\leq \varepsilon c_1' \int_s^T E^{r+1}(t)\left(E(t)\right)^{\frac{m^-}{p^+}-1} dt + \varepsilon c_2' \int_s^T E^{r+1}(t)\left(E(t)\right)^{\frac{m^+}{p^+}-1} dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dxdt \\
&\leq c'\varepsilon\left(\left(E(0)\right)^{\frac{m^-}{p^+}-1} + \left(E(0)\right)^{\frac{m^+}{p^+}-1}\right)\int_s^T E^{r+1}(t)\,dt \\
&\quad + \int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|u_t\right|^{m(x)} dxdt,
\end{aligned} \tag{58}$$

where c_1', c_2' and c' are positive constants independent of ε.

The next term of (51) can be estimated in a similar way to obtain

$$\begin{aligned}
&\left|-\mu_2 \int_s^T E^r(t) \int_\Omega u\left|z(x,1,t)\right|^{m(x)-1} dxdt\right| \\
&\leq \varepsilon \int_s^T E^r(t)\left[c_1\left(\varrho_{p(x)}(\nabla u)\right)^{\frac{m^-}{p^+}} + c_2\left(\varrho_{p(x)}(\nabla u)\right)^{\frac{m^+}{p^+}}\right] dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|z(x,1,t)\right|^{m(x)} dxdt \\
&\leq c'\varepsilon\left(\left(E(0)\right)^{\frac{m^-}{p^+}-1} + \left(E(0)\right)^{\frac{m^+}{p^+}-1}\right)\int_s^T E^{r+1}(t)\,dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\left|z(x,1,t)\right|^{m(x)} dxdt.
\end{aligned} \tag{59}$$

For the last term of (51), by using Lemma 11, we have the following:

$$
\begin{aligned}
&2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\,dt\\
&\leq \frac{2}{m^-}\int_s^T E^r(t)\int_0^1\int_\Omega \xi(x)\,|z(x,\rho,t)|^{m(x)}\,dx\,d\rho\,dt\\
&\leq -\frac{2\tau}{m^-}\int_s^T E^r(t)\,\frac{d}{dt}\left(\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)\,|z|^{m(x)}\,dx\,d\rho\right)dt\\
&\quad+\frac{2}{m^-}\int_s^T E^r(t)\int_\Omega \xi(x)\,|u_t|^{m(x)}\,dx\,dt\\
&\leq -\frac{2\tau}{m^-}\left[E^r(t)\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)\,|z|^{m(x)}\,dx\,d\rho\right]_{t=s}^{t=T}\\
&\quad+\frac{2}{m^-}\int_s^T E^r(t)\int_\Omega \xi(x)\,|u_t|^{m(x)}\,dx\,dt.
\end{aligned}
$$

As $\xi(x)$ is bounded, by using (47), we get

$$
\begin{aligned}
&2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\,dt\\
&\leq \frac{2\tau e^{-\tau}}{m^-}E^r(s)\,E(s)+\frac{2c}{m^-}E^{r+1}(T)\\
&\leq \frac{2\tau e^{-\tau}}{m^-}E^r(0)\,E(s)+\frac{2c}{m^-}E^r(T)\,E(s)\leq c^*E(s),
\end{aligned}
\tag{60}
$$

for some $c^* > 0$.

By combining (51)-(60), we infer

$$
\begin{aligned}
\int_s^T E^{r+1}(t)\,dt \;\leq\; & c\varepsilon\left(\left(1+(E(0))^{\frac{m^-}{p^+}-1}+(E(0))^{\frac{m^+}{p^+}-1}\right)\right)\int_s^T E^{r+1}(t)\,dt\\
&+cE(s)+c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dx\,dt.
\end{aligned}
\tag{61}
$$

We choose $\varepsilon > 0$ so small such that

$$
c\varepsilon\left(1+(E(0))^{\frac{m^-}{p^+}-1}+(E(0))^{\frac{m^+}{p^+}-1}\right)<1.
$$

Then, we have

$$
\int_s^T E^{r+1}(t)\,dt\leq cE(s)+c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dx\,dt.
$$

Where ε is fixed, then $c_\varepsilon(x)\leq M$, since that $m(x)$ is bounded. So, we obtain

$$
\begin{aligned}
\int_{s}^{T} E^{r+1}(t)\,dt &\leq cE(s) + cM \int_{s}^{T} E^{r}(t) \int_{\Omega} |z(x,1,t)|^{m(x)}\,dxdt \\
&\leq cE(s) - C_0 M \int_{s}^{T} E^{r}(t)\,E'(t)\,dt \\
&\leq cE(s) + \frac{C_0 M}{r+1}\left(E^{r+1}(s) - E^{r+1}(T)\right) \\
&\leq cE(s).
\end{aligned}
\tag{62}
$$

Thus, by taking $T \to \infty$, we get

$$
\int_{s}^{\infty} E^{\frac{m^{+}}{2}}(t)\,dt \leq cE(s).
$$

Therefore, Komornik's Lemma (with $\sigma = r = m^{+}/2 - 1$) provides the desired result. □

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