Improving a method of constructing finite time blow-up solutions and its an application

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Abstract: We in this paper improve a method of establishing the existence of finite time blow-up solutions, and then apply it to study the finite time blow-up, the blow-up time and the blow-up rate of the weak solutions on the initial boundary problem of $u_t - \Delta u_t - \Delta u_t = |u|^{p-1}u$. By applying this improved method, we prove that $I(u_0) < 0$ is a sufficient condition of the existence of the finite time blow-up solutions and $\frac{2(p-1)^{-1}\|u_0\|^2_{H_0^1}}{(p-1)\|\nabla u_0\|_2^2 - 2(p+1)J(u_0)}$ is an upper bound for the blow-up time, which generalize the blow-up results of the predecessors in the sense of the variation. Moreover, we estimate the upper blow-up rate of the blow-up solutions, too.

Key words: Pseudo-parabolic equation; Improving; A new blow-up condition; Finite time blow-up; Blow-up time; Blow-up rate

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1 Introductions

There are many ways in which we can solve the finite time blow-up problem of the parabolic or pseudo parabolic equations, among which the method used in [6, 8] is one of the simpler ways, we can summarized it as follows:

(i) It is constructed to some auxiliary functions $f(t) = \|u\|_2^2$(or $\|u\|^2_{H_0^1}$) and $\psi(t) = -(4 + \mu)J(u)$;

(ii) It is verified to $f'(t) \geq (4 + \mu)\psi(t)$, where $\mu > 0$ is a constant, and $\frac{d}{dt}\psi(t) \geq 0$ and $f'(t) > 0$ on $[0, T)$ under $J(u_0) < 0$;

(iii) It is proved to $\psi'(t)f(t) \geq \frac{4+\mu}{4}\psi(t)f'(t)$;

(iv) It is established to $\|u\|_2^2$(or $\|u\|^2_{H_0^1}$) blow-up in a finite time, and it is estimated to the upper bound for the blow-up time.

Inspired by [3, 6, 7, 8], we modify the (i)-(iv) above as follows:

(I) It is constructed to some auxiliary functions $f(t) = \|u\|_2^2$ (or $\|u\|^2_{H_0^1}$) and $\psi(t) = -(4 + \mu)J(u)$;

(II) It is verified to $f'(t) \geq (4 + \mu)\psi(t) + \frac{\mu}{2}\|\nabla u\|_2^2$, and $f'(t) > 0$ on $[0, T)$ under $I(u_0) < 0$;

(III) It is proved to $\psi'(t)f(t) \geq \frac{4+\mu}{4}\psi(t)f'(t) + \frac{\mu(4+\mu)}{8}\|\nabla u\|_2^2 f'(t)$ and $f(0) > 0$;

(IV) It is established to $\|u\|^2_{H_0^1}$ blow-up in a finite time, and it is estimated to the upper

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bound for the blow-up time. Here

$$J(u)=\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega F(u)dx \qquad \text{and} \qquad I(u)=\|\nabla u\|_2^2-\int_\Omega uf(u)dx,$$

respectively, here $sf(s)\geq \frac{4+\mu}{2}F(s)$ and $F(u):=\int_0^u f(s)ds$.

We next are going to apply (I)-(IV) to study the finite time blow-up problems of the weak solutions on the initial boundary value problem

$$\begin{cases} u_t-\Delta u_t-\Delta u_t=|u|^{p-1}u, & x\in\Omega, t\in(0,T),\\ u=0, & x\in\partial\Omega, t\in(0,T),\\ u(x,0)=u_0, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with the sufficiently smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $T\in(0,+\infty]$ is the maximal existence time of the weak solutions and $p>1$.

The problem (1.1) has been applied to characterize many physical phenomena, such as the aggregation of population [4], the theory of seepage of homogeneous fluids through a fissured rock [9] and the unidirectional propagation of nonlinear, dispersive, long waves [10].

The problem (1.1) comprehensively studied by Xu et al. in [1]. In [1], authors studied the global existence, finite time blow-up, asymptotic decay and uniqueness of the weak solutions at three different initial energy levels, i.e. subcritical initial energy level $J(u_0)<d$, critical initial energy level $J(u_0)=d$ and sup-critical initial energy level $J(u_0)>d$, where the depth of potential well d is defined as $d=\inf_{u\in\mathcal{N}}J(u)$, the Nehari manifold $\mathcal{N}$ is defined as $\mathcal{N}=\left\{u\in H_0^1|, I(u)=0, \|u\|_{H_0^1}\neq 0\right\}$, and the energy and Nehari functionals are defined as

$$J(u)=\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{1.2}$$

$$I(u)=\|\nabla u\|_2^2-\|u\|_{p+1}^{p+1}, \tag{1.3}$$

respectively. In addition, there are many articles on the finite time blow-up and blow-up time of problem (1.1), such as [2, 5, 6, 7]. Now let's review some of the results related to the upper bound for the blow-up time. Under $J(u_0)<0$, author [6] proved that the upper bound for the blow-up time is given by

$$T\leq T_0:=\frac{\|u_0\|_{H_0^1}^2}{(1-p^2)J(u_0)}.$$

Under $J(u_0)<\frac{(p-1)\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(1+\lambda_1)}$, authors [5] verified that the upper bound for the blow-up time is given by

$$T\leq T_1:=\frac{8(p+1)(1+\lambda_1)\|u_0\|_{H_0^1}^2}{(p-1)^2\left[\lambda_1(p-1)\|u_0\|_{H_0^1}^2-2(p+1)(1+\lambda_1)J(u_0)\right]},$$

and under $J(u_0)<0$, the lower bound for the blow-up time is refined by comparing T_0 and T_1. Under $0<J(u_0)<\frac{(p-1)\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(1+\lambda_1)}$, authors [7] derived that the upper bound for the blow-up time is given by

$$T\leq T_2:=\frac{2\varepsilon c}{(\alpha-1)\|u_0\|_{H_0^1}^4},$$

where

$$c > \frac{1}{4\varepsilon^2}\|u_0\|_{H_0^1}^4, \tag{1.4}$$

$$0 < \varepsilon < \frac{1}{2\alpha\|u_0\|_{H_0^1}^2}\left(\frac{2(p-1)\lambda_1\|u_0\|_{H_0^1}^2}{(p+1)(1+\lambda_1)} - 4\alpha J(u_0)\right) \tag{1.5}$$

and

$$1 < \alpha < \frac{(p-1)\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(1+\lambda_1)J(u_0)}. \tag{1.6}$$

From the above reviews, we note that

QUE1. the blow-up and lifespan are still unsolved when $\frac{(p-1)\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(1+\lambda_1)} \le J(u_0) < \frac{p-1}{2(p+1)}\|\nabla u_0\|_2^2$;

QUE2. the blow-up rate is still unsolved when $J(u_0) < \frac{p-1}{2(p+1)}\|\nabla u_0\|_2^2$.

We now explain the reasonableness of **QUE1**. According to the eigenvalue problem, it is well known that if λ_1 is the principal eigenvalue of $-\Delta w = \lambda_1 w$ in H_0^1, then there must exist $u \in H_0^1$ such that $\|\nabla u\|_2^2 > \lambda_1\|u\|_2^2$, which implies $\|\nabla u\|_2^2 > \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2$.

We in this article try to solve the two unknown problems mentioned above. Before we state the main results, we first introduce the definition of weak solutions.

Definition 1.1 *([1]) A function $u = (x,t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T)$, if $u \in L^\infty\left([0,T); H_0^1(\Omega)\right)$ with $u_t \in L^2\left([0,T); H_0^1\right)$ satisfying*

(1) $\forall v \in H_0^1, t \in (0,T)$

$$\int_\Omega u_t v - \Delta u_t v - \Delta u v dx = \int_\Omega |u|^{p-1} v dx;$$

(2) $u(x,0) = u_0(x) \in H_0^1$ *and for* $t \in [0,T)$,

$$\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + J(u) \le J(u_0).$$

The local existence, uniqueness and continuity of the solutions have been given by Xu et al. [7]. The main results of this paper can be stated as the following theorem.

Theorem 1.2 *Let $p > 1$ and $u_0 \in H_0^1$ such that $I(u_0) < 0$. If $u(x,t;u_0)$ is a weak solution of problem* (1.1)*, then $\|u\|_{H_0^1}^2$ blows up at finite time T, which satisfies*

$$T \le T_3 := \frac{2(p-1)^{-1}\|u_0\|_{H_0^1}^2}{(p-1)\|\nabla u_0\|_2^2 - 2(p+1)J(u_0)}. \tag{1.7}$$

Moreover, the upper blow-up rate is $(T-t)^{-\frac{1}{p-1}}$, with the $\|u\|_{H_0^1}$ of this solution satisfies

$$\|u\|_{H_0^1} \le C_1(T-t)^{-\frac{1}{p-1}}. \tag{1.8}$$

Where

$$C_1 = \frac{2^{\frac{1}{p-1}}\|u_0\|_{H_0^1}^{\frac{p+1}{p-1}}}{\left\{(p-1)\left[(p-1)\|\nabla u_0\|_2^2 - 2(p+1)J(u_0)\right]\right\}^{\frac{1}{p-1}}}.$$

Remark 1.3 **(a)** *Theorem 1.2 shows that, by this improved method, we can establish not only the blow-up in finite time of the solutions with non-positive initial energy but also the blow-up in finite time of the solutions with positive initial energy.*

(b) *By the comparison method, we easily get that the upper bound for the blow-up time obtained by the modified method is smaller than that obtained by the original method which is used in [6, 8].*

2 Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into two subsections. We in Subsection I prove $I(u) < 0$ by means of arguing by contradiction, taking full advantage of the monotonicity of energy functional and modified the method in [7] under $I(u_0) < 0$. We in Subsection II prove the weak solution blow-up in a finite time, and estimate the upper blow-up rate and the new upper bound for the blow-up time by using (I)-(IV).

2.1 The invariance of $I(u)$ under $I(u_0) < 0$

Before to prove Theorem 1.2, we first introduce the following two Lemmas.

We begin by describing the monotonicity of the energy functional $J(u)$ on the problem (1.1).

Lemma 2.1 *Let $u_0 \in H_0^1$. If $u(x,t;u_0) \in H_0^1$ is a weak solution of problem* (1.1)*, then the energy functional is decreasing on* $[0,\infty)$*. Moreover,*

$$\frac{d}{dt}J(u) = -\|u_t\|_{H_0^1}^2. \tag{2.1}$$

Proof. Let $u(x,t;u_0)$ is a weak solution of problem (1.1). Multiplying (1.1) by u_t, and integrating on Ω, it derives that

$$\int_\Omega u_t^2 + |\nabla u_t|^2 dx = -\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx\right),$$

together with (1.2), we easily obtain the conclusions of Lemma 2.1.

We now prove that the Nehari functional $I(u)$ is negative under the initial Nehari functional $I(u_0) < 0$.

Lemma 2.2 *Under the assumption of Theorem 1.2, one has $I(u) < 0$ on* $[0,T)$*.*

Proof. Let $u(x,t;u_0)$ is a weak solution of problem (1.1). Multiplying (1.1) by u, and integrating on Ω, we get

$$\frac{d}{2dt}\int_\Omega u^2 + |\nabla u|^2 dx = -\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+1} dx,$$

i.e.

$$\frac{d}{dt}\|u\|_{H_0^1}^2 = -2\|\nabla u\|^2 + 2\|u\|_{p+1}^{p+1}. \tag{2.2}$$

The rest of proof is similar to that of Lemma 2.3 in [7].

Since $\|u\|_{H_0^1}^2$ is equivalent to $\|\nabla u\|_2^2$ in H_0^1, by (2.2), (1.3) and Lemma 2.2, we get the following Corollary.

Corollary 2.3 *Under the assumption of Theorem 1.2, we get $\frac{d}{dt}\|u\|_{H_0^1}^2 > 0$ and $\frac{d}{dt}\|\nabla u\|_2^2 > 0$ on* $[0,T)$*.*

2.2 Upper bound for the blow-up time and upper blow-up rate

Proof of Theorem 1.2. We will apply the arguing by contradiction to prove the solutions blow-up in a finite time under $I(u_0) < 0$. Assume that $u(x,t;u_0)$ is a global solution of problem (1.1), then the maximal existence time T satisfies $T = \infty$. We now construct an auxiliary function

$$f(t) = \|u\|_{H_0^1}^2. \tag{2.3}$$

Then, by (2.2) and (1.3), one has

$$\frac{d}{dt}f(t) = \frac{d}{dt}\|u\|_{H_0^1}^2 = -2(p+1)J(u) + (p-1)\|\nabla u\|_2^2. \tag{2.4}$$

Denote $\varphi(t) = -2(p+1)J(u)$. Employing (2.1), (2.3) and the inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 \leq \|u_t\|_{H_0^1}\|u\|_{H_0^1},$$

we get

$$\frac{d}{dt}\varphi(t)f(t) = 2(p+1)\|u_t\|_{H_0^1}^2\|u\|_{H_0^1}^2 \geq \frac{p+1}{2}\frac{d}{dt}f(t)\frac{d}{dt}f(t). \tag{2.5}$$

Inserting (2.4) into (2.5), we can obtain

$$\frac{d}{dt}\varphi(t)f(t) \geq \frac{p+1}{2}\frac{d}{dt}f(t)\varphi(t) + \frac{(p+1)(p-1)}{2}\|\nabla u\|_2^2\frac{d}{dt}f(t). \tag{2.6}$$

Note that employing Lemma 2.2 and (2.3), we can conclude from (2.5) that $f(t) > 0$ for all $t \in [0,T)$. Hence, it follows from Corollary 2.3 and (2.6) that

$$\frac{d}{dt}\left(\frac{\varphi(t)}{f^{\frac{p+1}{2}}(t)}\right) \geq -(p-1)\|\nabla u_0\|_2^2\frac{d}{dt}\left(\frac{1}{f(t)^{\frac{p+1}{2}}}\right). \tag{2.7}$$

Integrating (2.7) over $[0,t]$, it follows that

$$\frac{\varphi(t)}{f^{\frac{p+1}{2}}(t)} \geq \frac{\varphi(0)}{f^{\frac{p+1}{2}}(0)} + \frac{p-1}{f^{\frac{p+1}{2}}(0)}\|\nabla u_0\|_2^2 - \frac{p-1}{f^{\frac{p+1}{2}}(t)}\|\nabla u_0\|_2^2. \tag{2.8}$$

Once again employing Corollary 2.3 and (2.4), it derives from (2.8) that

$$\left(\frac{1}{f^{\frac{p-1}{2}}(t)}\right)' < -\frac{(p+1)(p-1)}{f^{\frac{p+1}{2}}(0)}\left(-J(u_0) + \frac{p-1}{2(p+1)}\|\nabla u_0\|_2^2\right). \tag{2.9}$$

Integrating (2.9) over $[0,t]$, it follows that

$$\frac{1}{f^{\frac{p-1}{2}}(t)} \leq \frac{1}{f^{\frac{p+1}{2}}(0)}\left(f(0) - (p+1)(p-1)\left(-J(u_0) + \frac{p-1}{2(p+1)}\|\nabla u_0\|_2^2\right)t\right), \tag{2.10}$$

which implies that there exists a finite time $T > 0$ such that the right side of (2.10) is equal to zero at $t = T$, with T satisfies

$$T \leq \frac{2(p-1)^{-1}f(0)}{(p-1)\|\nabla u_0\|_2^2 - 2(p+1)J(u_0)}, \tag{2.11}$$

which contradicts $T = \infty$. Therefore, under the initial Nehari functional $I(u_0) < 0$, the solution $u(x,t;u_0)$ of problem (1.1) blows up at finite time T. Combining (2.11) and (2.3), it follows that (1.7) holds.

Integrating (2.10) on $[t,T)$ and once again employing (2.3), it easily follows that (1.8) holds indeed.

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