

Parameter Estimation in Uncertain Heat Equations

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Abstract

Uncertain heat equations are aimed to model the variation of temperature in a given region over time under uncertain influence. Parameter estimation is an important and significant topic in uncertain heat equations because after we construct a uncertain heat equation according to the specific problem to model a dynamic system, it is natural that the uncertain heat equation contains unknown parameters such as the unknown thermal diffusivity and unknown parameters of strength of heat source. For that matter, this paper first employs the moment method to estimate unknown parameters in uncertain heat equations. To show the process of parameter estimation, two numerical examples are given.

Keywords: Uncertain heat equation; method of moments; uncertainty theory; parameter estimation

1 Introduction

Winner process, defined by Winner in 1923, is a continuous and stationary independent increment stochastic process whose increments are normal random variables. It is not only a rather important concept in probability theory, but also a vital mathematical tool to model time evolution of random noises. Based on Winner process, stochastic (partial) differential equations were established and applied in many fields such as biology [5], medicine [6], finance [2] and physics [16]. It is natural that a stochastic differential equation or stochastic partial differential equation contains unknown parameters, and thus estimating unknown parameters in the equation is a vital topic. Fortunately, there are many effective methods (e.g., likelihood-based method [15, 7] and methods of moments [3]) proposed for the task of parameter estimation. For the detail and systemic introduction, the book (Bishwal [1]) is worth read.

However, when modeling many time-varying systems, stochastic (partial) differential equations driven by Winner process may fail. For example, Liu [11] pointed out two paradoxes about applying stochastic differential equations in stock prices, and Yang and Yao [18] proposed a paradox about applying stochastic heat equations in modeling real heat conduction. To solve those issues, Liu process as a likeness of Wiener process was proposed by Liu [10] under the framework of uncertainty theory which was founded by Liu [8] and perfected by Liu [10] as a branch of mathematics based on normality, duality, subadditivity and product axioms. In uncertainty theory, Liu process is a Lipschitz continuous and stationary independent increment process whose increments are normal uncertain variables.

Based on Liu process, uncertain differential equations [9] were proposed. And then, Chen and Liu [4] proved the existence and uniqueness theorem. More importantly, Yao and Chen [21] constructed the numerical methods for uncertain differential equations. Nowadays, the applications of uncertain differential equations have been involved in many fields such as finance (Liu [11]), optimal control (Zhu [24]), differential

game (Yang and Gao [17]) and so on. Due to so many applications, it is particularly important to estimate the parameters. Yao and Liu [22] first proposed method of moments to estimate unknown parameters in uncertain differential equations. Following that, Liu [13] explored likelihood-based methods to estimate unknown parameters. In addition, Liu and Yang [14] discussed moment estimations for parameters in high-order uncertain differential equations.

Similarly, based on Liu process, uncertain partial differential equations were presented by Yang and Yao [18]. As a type of uncertain partial differential equations, the uncertain heat equations, describing the variation of temperature in a given region over time while the strength of heat source is affected by the interference of noise, were first proposed by Yang and Yao [18]. Then Yang and Ni [19] presented the existence and uniqueness theorem, and Yang [20] discussed how to solve uncertain heat equations via numerical methods. Lately, Ye and Yang [23] studied the three-dimensional uncertain heat equations as the extension of the one-dimensional uncertain heat equations [18].

When we construct the uncertain heat equations to study the variation of temperature, the first problem is to determine the unknown parameter in uncertain heat equations including the unknown thermal diffusivity and unknown parameters of strength of heat source. Thus, it is very significant to explore parameter estimation in uncertain heat equations. This paper is aimed to estimate unknown parameter in uncertain heat equations by method of moments. The rest of this paper is organized as follows. Section 2 presents the method of moments for parameter estimation in uncertain heat equations based on some given observations. Section 3 gives two numerical examples to illustrate how to estimate unknown parameters by method of moments. At last, some conclusions are made in Section 4.

2 Parameter Estimation

Consider a Cauchy problem of an uncertain heat equation (Yang and Yao [18])

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}; \boldsymbol{\mu}) + g(t, x, U_{t,x}; \boldsymbol{\sigma}) \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \varphi(x), & x \in \mathbb{R} \end{cases} \quad (1)$$

where a^2 is the unknown constant thermal diffusivity to be estimated, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are unknown parameters to be estimated, $f(t, x, U_{t,x}; \boldsymbol{\mu})$ is a heat source depending on unknown parameters $\boldsymbol{\mu}$, $g(t, x, U_{t,x}; \boldsymbol{\sigma})$ is the diffusion term of heat source depending on unknown parameters $\boldsymbol{\sigma}$, \dot{C}_t denotes the time white noise, C_t is a Liu process [10] as a likeness of Wiener process to describe white noise, and $\varphi(x)$ is a given initial temperature at time $t = 0$. We are interested in estimating a , $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ when there are $m \times n$ observations

$$\begin{pmatrix} u_{t_1, x_1} & u_{t_2, x_1} & \cdots & u_{t_m, x_1} \\ u_{t_1, x_2} & u_{t_2, x_2} & \cdots & u_{t_m, x_2} \\ \vdots & \vdots & & \vdots \\ u_{t_1, x_n} & u_{t_2, x_n} & \cdots & u_{t_m, x_n} \end{pmatrix}$$

of the solution $U_{t,x}$ at the times t_1, t_2, \dots, t_m with $t_1 < t_2 < \dots < t_m$ and the positions x_1, x_2, \dots, x_n , respectively. For this purpose, we consider the difference form of the equation (1),

$$\begin{cases} \frac{U_{t_{i+1}, x_j} - U_{t_i, x_j}}{t_{i+1} - t_i} - a^2 \left(\frac{U_{t_i, x_{j+1}} - U_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{U_{t_i, x_j} - U_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) \\ \quad = f(t_i, x_j, U_{t_i, x_j}; \boldsymbol{\mu}) + g(t_i, x_j, U_{t_i, x_j}; \boldsymbol{\sigma}) \frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i} \\ U_{0, x_j} = \varphi(x_j). \end{cases} \quad (2a)$$

$$\begin{cases} \\ U_{0, x_j} = \varphi(x_j). \end{cases} \quad (2b)$$

The equation (2a) can be rewritten as

$$\frac{\frac{U_{t_{i+1},x_j}-U_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{U_{t_i,x_{j+1}}-U_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{U_{t_i,x_j}-U_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - f(t_i, x_j, U_{t_i,x_j}; \boldsymbol{\mu})}{g(t_i, x_j, U_{t_i,x_j}; \boldsymbol{\sigma})} = \frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i}.$$

According to definition of Liu process Liu [10],

$$\frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i} \sim \mathcal{N}(0, 1)$$

is a standard normal uncertain variable. Thus its uncertainty distribution is

$$\Phi(x) = \left(1 + \exp \left(\frac{-\pi x}{\sqrt{3}} \right) \right)^{-1},$$

and its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Therefore,

$$\frac{\frac{U_{t_{i+1},x_j}-U_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{U_{t_i,x_{j+1}}-U_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{U_{t_i,x_j}-U_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - f(t_i, x_j, U_{t_i,x_j}; \boldsymbol{\mu})}{g(t_i, x_j, U_{t_i,x_j}; \boldsymbol{\sigma})} \sim \mathcal{N}(0, 1). \quad (3)$$

By substituting the observations into the equation (3), we write

$$z_{ij}(a, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \frac{\frac{u_{t_{i+1},x_j}-u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}}-u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j}-u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - f(t_i, x_j, u_{t_i,x_j}; \boldsymbol{\mu})}{g(t_i, x_j, u_{t_i,x_j}; \boldsymbol{\sigma})},$$

$i = 1, 2, \dots, m-1, j = 2, 3, \dots, n-1$. Therefore, depending on the equation (3), we obtain that $z_{ij}(a, \boldsymbol{\mu}, \boldsymbol{\sigma})$, $i = 1, 2, \dots, m-1, j = 2, 3, \dots, n-1$ can be considered to $(m-1) \times (n-2)$ samples of $\mathcal{N}(0, 1)$. Then, we are going to employ the method of moments to estimate the unknown parameters $a, \boldsymbol{\mu}$ and $\boldsymbol{\sigma}$. Note that the k -th sample moments are

$$\frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} (z_{ij}(a, \boldsymbol{\mu}, \boldsymbol{\sigma}))^k, \quad k = 1, 2, \dots$$

and k -th population moments are

$$\left(\frac{\sqrt{3}}{\pi} \right)^k \int_0^1 \left(\ln \frac{\alpha}{1-\alpha} \right)^k d\alpha, \quad k = 1, 2, \dots$$

It follows from the method of moments that we can approximate k -th population moment by k -th sample moment for each positive integer k . If we denote the number of unknown parameters by K , then we set

$$\frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} (z_{ij}(a, \boldsymbol{\mu}, \boldsymbol{\sigma}))^k = \left(\frac{\sqrt{3}}{\pi} \right)^k \int_0^1 \left(\ln \frac{\alpha}{1-\alpha} \right)^k d\alpha, \quad k = 1, 2, \dots, K. \quad (4)$$

Here, we suppose that there exists a set of solutions $a^*, \boldsymbol{\mu}^*, \boldsymbol{\sigma}^*$ to system (4), which are the estimations of the parameters $a, \boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, respectively. Indeed, the above method to estimate the parameters in uncertain heat equations is called the method of moments. As the supplementary knowledge of population moments, the k -th population moments of an uncertain variable ξ is [12]

$$E[\xi^k] = \int_0^1 (\Phi^{-1}(\alpha))^k d\alpha$$

where k is a positive integer if the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ of ξ exists. For example, if an uncertain variable $\xi \sim \mathcal{N}(0, 1)$, then

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

and

$$E[\xi^k] = \left(\frac{\sqrt{3}}{\pi} \right)^k \int_0^1 \left(\ln \frac{\alpha}{1-\alpha} \right)^k d\alpha.$$

Specially, we have $E[\xi^k] = 0$ for any positive odd number k , and

$$E[\xi^2] = 1, \quad E[\xi^4] = \frac{21}{5}, \quad E[\xi^6] = \frac{279}{7}.$$

Then let's summarize the above process with the following theorem.

Theorem 2.1 Consider a Cauchy problem of an uncertain heat equation

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}; \boldsymbol{\mu}) + g(t, x, U_{t,x}; \boldsymbol{\sigma}) \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \varphi(x), & x \in \mathbb{R} \end{cases} \quad (5)$$

where $a > 0$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are unknown parameters, and $\varphi(x)$ is a given initial temperature at time $t = 0$. Assume

$$\begin{pmatrix} u_{t_1, x_1} & u_{t_2, x_1} & \cdots & u_{t_m, x_1} \\ u_{t_1, x_2} & u_{t_2, x_2} & \cdots & u_{t_m, x_2} \\ \vdots & \vdots & & \vdots \\ u_{t_1, x_n} & u_{t_2, x_n} & \cdots & u_{t_m, x_n} \end{pmatrix}$$

are observations of the solution $U_{t,x}$ of the uncertain heat equation (5) at the times t_1, t_2, \dots, t_m with $t_1 < t_2 < \dots < t_m$ and the positions x_1, x_2, \dots, x_n , respectively. Then the moment estimates of a , $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the solutions of the system of equations,

$$\begin{aligned} & \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \left(\frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^2 \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - f(t_i, x_j, u_{t_i, x_j}; \boldsymbol{\mu})}{g(t_i, x_j, u_{t_i, x_j}; \boldsymbol{\sigma})} \right)^k \\ & = \left(\frac{\sqrt{3}}{\pi} \right)^k \int_0^1 \left(\ln \frac{\alpha}{1-\alpha} \right)^k d\alpha, \quad k = 1, 2, \dots, K, \end{aligned} \quad (6)$$

where K is the number of unknown parameters.

Proof: It follows from the definition of moments method and (4) that the moment estimates of a , $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the solutions of the system (6). The theorem is proved.

Corollary 2.1 Consider a special Cauchy problem of an uncertain heat equation

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = \mu + \sigma \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \varphi(x), & x \in \mathbb{R} \end{cases} \quad (7)$$

where $a > 0$, μ and σ are three unknown parameters, and $\varphi(x)$ is a given initial temperature at time $t = 0$. Assume

$$\begin{pmatrix} u_{t_1, x_1} & u_{t_2, x_1} & \cdots & u_{t_m, x_1} \\ u_{t_1, x_2} & u_{t_2, x_2} & \cdots & u_{t_m, x_2} \\ \vdots & \vdots & & \vdots \\ u_{t_1, x_n} & u_{t_2, x_n} & \cdots & u_{t_m, x_n} \end{pmatrix}$$

are observations of the solution $U_{t,x}$ of the uncertain heat equation (7) at the times t_1, t_2, \dots, t_m with $t_1 < t_2 < \dots < t_m$ and the positions x_1, x_2, \dots, x_n , respectively. Then the moment estimates of a , μ and σ are the solutions of the system of equations,

$$\begin{cases} \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu}{\sigma} = 0 \\ \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \left(\frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu}{\sigma} \right)^2 = 1 \\ \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \left(\frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu}{\sigma} \right)^3 = 0. \end{cases}$$

Proof: It follows from Theorem 2.1 immediately.

Corollary 2.2 Consider a special Cauchy problem of an uncertain heat equation

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = \mu e^{-t} + \sigma U_{t,x} \sin(x) e^{-t} \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \varphi(x), & x \in \mathbb{R} \end{cases} \quad (8)$$

where $a > 0$, μ and σ are three unknown parameters, and $\varphi(x)$ is a given initial temperature at time $t = 0$. Assume

$$\begin{pmatrix} u_{t_1,x_1} & u_{t_2,x_1} & \cdots & u_{t_m,x_1} \\ u_{t_1,x_2} & u_{t_2,x_2} & \cdots & u_{t_m,x_2} \\ \vdots & \vdots & & \vdots \\ u_{t_1,x_n} & u_{t_2,x_n} & \cdots & u_{t_m,x_n} \end{pmatrix}$$

are observations of the solution $U_{t,x}$ of the uncertain heat equation (8) at the times t_1, t_2, \dots, t_m with $t_1 < t_2 < \dots < t_m$ and the positions x_1, x_2, \dots, x_n , respectively. Then the moment estimates of a , μ and σ are the solutions of the system of equations,

$$\begin{cases} \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu e^{-t_i}}{\sigma u_{t_i,x_j} \sin(x_j) e^{-t_i}} = 0 \\ \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \left(\frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu e^{-t_i}}{\sigma u_{t_i,x_j} \sin(x_j) e^{-t_i}} \right)^2 = 1 \\ \frac{1}{(m-1)(n-2)} \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} \left(\frac{\frac{u_{t_{i+1},x_j} - u_{t_i,x_j}}{t_{i+1}-t_i} - a^2 \left(\frac{u_{t_i,x_{j+1}} - u_{t_i,x_j}}{(x_{j+1}-x_j)(x_j-x_{j-1})} - \frac{u_{t_i,x_j} - u_{t_i,x_{j-1}}}{(x_j-x_{j-1})^2} \right) - \mu e^{-t_i}}{\sigma u_{t_i,x_j} \sin(x_j) e^{-t_i}} \right)^3 = 0. \end{cases}$$

Proof: It follows from Theorem 2.1 immediately.

3 Numerical Examples

Example 3.1 Consider a special Cauchy problem of an uncertain heat equation

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = \mu + \sigma \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = 0, & x \in \mathbb{R} \end{cases} \quad (9)$$

where $a > 0$, μ and σ are three unknown parameters. Assume that we have 15×10 observed data u_{t_i, x_j} , $i = 1, 2, \dots, 15$, $j = 1, 2, \dots, 10$ (See Table 2 and Figure 1) of the solution $U_{t,x}$ of the uncertain heat equation (9) at the times t_1, t_2, \dots, t_{15} and the positions x_1, x_2, \dots, x_{10} (See Table 1), respectively.

Table 1: Times and Positions in Example 3.1

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_i	0	0.4	0.8	1	1.4	1.7	1.9	2.1	2.4	2.8	3.2	3.4	3.8	4.2	4.5
j	1	2	3	4	5	6	7	8	9	10					
x_j	-2	-1.4	-1	-0.5	0.1	0.7	1.3	1.8	2.2	2.8					

Table 2: Observed Data in Example 3.1

u_{t_i, x_j}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1.6964	1.4914	0.8729	0.7096	2.2402	3.327	3.6521	3.617	1.0689	2.5022	2.7996	2.7619	-0.4642	-1.1465
2	0	0.9694	0.5667	-0.0983	-0.4625	0.0028	0.3719	0.5007	0.4801	-0.8625	-0.5899	-0.8212	-1.2381	-3.4326	-5.1131
3	0	1.9687	2.6567	2.9397	4.0082	5.3554	5.9235	6.0546	5.6364	6.2064	7.8835	9.0361	9.8476	12.1705	14.2577
4	0	0.3667	1.0033	2.0431	2.6477	2.5516	2.7743	2.3109	2.9358	2.6828	2.3837	3.08	2.5241	0.625	-0.2357
5	0	0.2799	1.0071	0.958	1.3817	0.7571	1.1518	1.3169	0.2553	1.6058	2.9751	2.7163	4.1583	5.6173	6.7065
6	0	0.885	2.0654	2.4861	3.5536	3.4129	4.0981	4.1134	4.7132	6.4638	6.7149	7.4307	8.1599	9.7018	10.3835
7	0	-0.2656	-0.1317	0.2133	0.3533	1.1829	2.3913	2.6115	2.988	4.0607	4.984	6.0647	7.4491	8.6007	9.5788
8	0	0.6476	0.364	0.6436	-0.0846	0.564	0.2097	-0.2201	-0.2144	-1.8156	-0.8496	-0.8342	-2.6162	-3.0675	-4.4516
9	0	2.4442	2.8511	3.1657	3.1263	4.0948	4.4683	4.7886	6.3255	6.8537	9.4141	10.5428	10.4437	11.9212	12.6882
10	0	4.2774	4.0912	4.2963	4.2099	5.5604	6.0542	6.2507	8.5653	8.724	12.1401	13.318	12.5879	14.3321	14.7101

According to Corollary 2.2, the estimates a^* , μ^* and σ^* solving the system of equations

$$\begin{cases} \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^*}{\sigma^*} = 0 \\ \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \left(\frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^*}{\sigma^*} \right)^2 = 1 \\ \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \left(\frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^*}{\sigma^*} \right)^3 = 0, \end{cases}$$

are

$$a^* = 0.3308, \quad \mu^* = 1.366, \quad \sigma^* = 6.2586.$$

Therefore, the uncertain heat equation is

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - 0.1094 \frac{\partial^2 U_{t,x}}{\partial x^2} = 1.366 + 6.2586 \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = 0, & x \in \mathbb{R}. \end{cases}$$

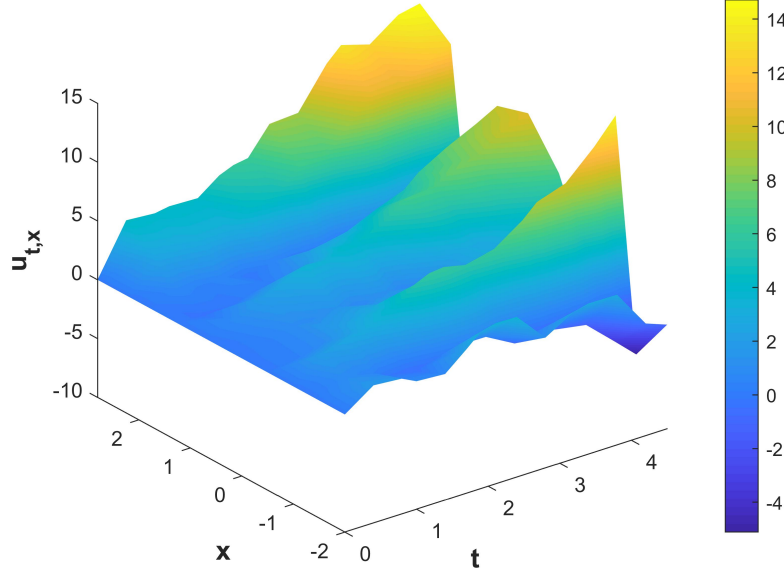


Figure 1: Observed Data in Example 3.1

Example 3.2 Consider a special Cauchy problem of an uncertain heat equation

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = \mu e^{-t} + \sigma U_{t,x} \sin(x) e^{-t} \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \sin(x), & x \in \mathbb{R} \end{cases} \quad (10)$$

where $a > 0$, μ and σ are three unknown parameters. Assume that we have 15×10 observed data u_{t_i, x_j} , $i = 1, 2, \dots, 15$, $j = 1, 2, \dots, 10$ (See Table 4 and Figure 2) of the solution $U_{t,x}$ of the uncertain heat equation (10) at the times t_1, t_2, \dots, t_{15} and the positions x_1, x_2, \dots, x_{10} (See Table 3), respectively.

Table 3: Times and Positions in Example 3.2

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_i	0	0.4	0.7	0.9	1.1	1.4	1.7	2	2.4	2.7	3	3.4	3.6	4	4.4
j	1	2	3	4	5	6	7	8	9	10					
x_j	0	0.5	0.9	1.3	1.8	2.2	2.8	3.4	3.8	4.3					

According to Corollary 2.2, the estimates a^* , μ^* and σ^* solving the system of equations

$$\begin{cases} \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^* e^{-t_i}}{\sigma^* u_{t_i, x_j} \sin(x_j) e^{-t_i}} = 0 \\ \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \left(\frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^* e^{-t_i}}{\sigma^* u_{t_i, x_j} \sin(x_j) e^{-t_i}} \right)^2 = 1 \\ \frac{1}{14 \times 8} \sum_{i=1}^{14} \sum_{j=2}^9 \left(\frac{\frac{u_{t_{i+1}, x_j} - u_{t_i, x_j}}{t_{i+1} - t_i} - a^{*2} \left(\frac{u_{t_i, x_{j+1}} - u_{t_i, x_j}}{(x_{j+1} - x_j)(x_j - x_{j-1})} - \frac{u_{t_i, x_j} - u_{t_i, x_{j-1}}}{(x_j - x_{j-1})^2} \right) - \mu^* e^{-t_i}}{\sigma^* u_{t_i, x_j} \sin(x_j) e^{-t_i}} \right)^3 = 0, \end{cases}$$

Table 4: Observed Data in Example 3.2

u_{t,x_j}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1.8095	2.7831	3.7697	3.776	3.8386	4.672	4.7752	4.5811	4.2402	4.5176	4.4445	4.4851	4.3671	4.335
2	0.4794	1.0201	1.3121	1.6269	1.6555	1.8337	2.2328	2.2931	2.2462	2.1512	2.1866	2.1353	2.1088	2.0181	1.9541
3	0.7833	2.3218	3.5642	3.7985	3.3734	3.9212	4.3306	4.1552	3.7353	4.08	4.1358	4.2517	4.2841	4.3372	4.5164
4	0.9636	2.3548	2.7374	2.41	2.7534	2.0303	2.2056	2.0973	2.196	2.1152	2.1845	2.1721	2.1868	2.1744	2.1323
5	0.9738	2.6113	1.6355	2.4353	2.2175	2.3801	2.3156	2.0558	2.2598	2.1887	2.2199	2.1673	2.1592	2.1574	2.1295
6	0.8085	1.7406	2.22	2.6434	2.9275	2.4503	2.6961	2.5493	2.6015	2.6312	2.648	2.6993	2.7331	2.7282	2.7665
7	0.335	1.0393	1.4744	1.7268	1.8817	1.9128	2.0875	2.1924	2.2956	2.2822	2.362	2.3993	2.4285	2.4576	2.5052
8	-0.2555	0.5164	0.8864	1.0785	1.2384	1.458	1.6292	1.7275	1.749	1.8033	1.8474	1.8408	1.8525	1.8531	1.8588
9	-0.6119	0.852	1.1152	1.1574	1.3356	1.5836	1.9519	1.9058	2.0595	2.0412	2.1639	2.1735	2.1871	2.1352	2.1314
10	-0.9162	1.5627	1.6444	1.7664	2.1308	2.5248	3.109	2.9356	3.2575	3.1617	3.3973	3.3738	3.3876	3.3024	3.3224

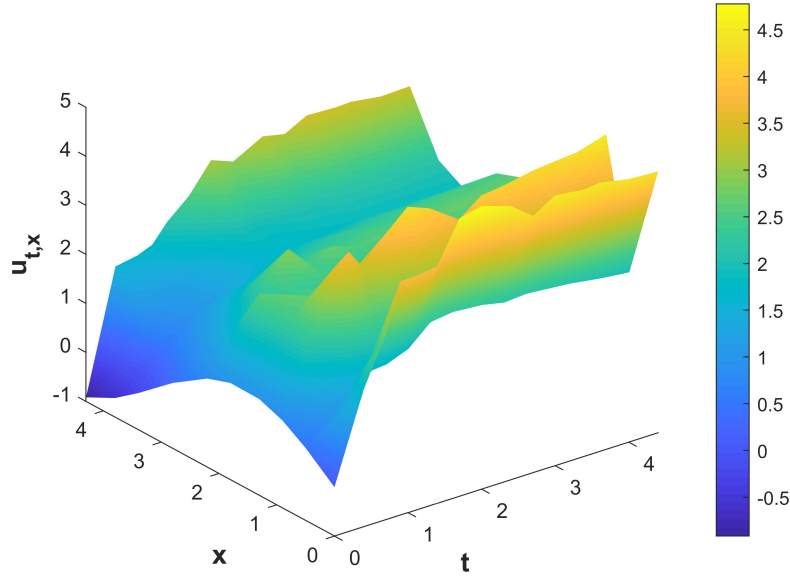


Figure 2: Observed Data in Example 3.2

are

$$a^* = 0.3321, \quad \mu^* = 0.6648, \quad \sigma^* = 1.7069.$$

Therefore, the uncertain heat equation is

$$\begin{cases} \frac{\partial U_{t,x}}{\partial t} - 0.1103 \frac{\partial^2 U_{t,x}}{\partial x^2} = 0.6648e^{-t} + 1.7069U_{t,x} \sin(x)e^{-t} \dot{C}_t, & t > 0, x \in \mathbb{R} \\ U_{0,x} = \sin(x), & x \in \mathbb{R}. \end{cases}$$

4 Conclusions

With the increasing application of the uncertain heat equations, the parameter estimation in uncertain heat equations becomes more and more important and significant. This paper employed the method of moments to estimate unknown parameters in the one-dimensional uncertain heat equations. In addition, for two special uncertain heat equations, numerical examples were given to illustrate how to estimate unknown parameters by method of moments. Future researches may consider parameter estimation and interval estimation in the three-dimensional uncertain differential equations.

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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