Global structure and one-sign solutions for second-order Sturm-Liouville difference equation with sign-changing weight

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Abstract

This paper is devoted to study the discrete Sturm-Liouville problem

$$\begin{cases} -\Delta(p(k)\Delta u(k-1)) + q(k)u(k) = \lambda m(k)u(k) + f_1(k,u(k),\lambda) + f_2(k,u(k),\lambda), & k \in [1,T]_Z, \\ a_0u(0) + b_0\Delta u(0) = 0,\ a_1u(T) + b_1\Delta u(T) = 0, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a parameter, $f_1, f_2 \in C([1,T]_Z \times \mathbb{R}^2, \mathbb{R})$, f_1 is not differentiable at the origin and infinity. Under some suitable assumptions on nonlinear terms, we prove the existence of unbounded continua of positive and negative solutions of this problem which bifurcate from intervals of the line of trivial solutions or from infinity, respectively.

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1. Introduction

Let $T > 1$ be an integer, let Z and $\mathbb{R}$ denote the sets of all integers and real numbers, respectively. In this paper, we discuss the following second-order nonlinear discrete Strum-Liouville boundary value problem

$$\begin{cases} -\Delta(p(k)\Delta u(k-1)) + q(k)u(k) = \lambda m(k)u(k) + f_1(k,u(k),\lambda) + f_2(k,u(k),\lambda), & k \in [1,T]_Z, \\ a_0u(0) - b_0\Delta u(0) = 0,\ a_1u(T+1) + b_1\Delta u(T) = 0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $a_0, b_0, a_1, b_1 \in \mathbb{R}$ satisfy $a_0b_0 \geq 0, a_1b_1 \geq 0$ with $a_0^2 + b_0^2 \neq 0, a_1^2 + b_1^2 \neq 0$; $p : [0,T]_Z \to [0,\infty)$ with

$$p(k_0) > 0, \quad k_0 \in [0,T]_Z;$$

$q : [1,T]_Z \to [0,\infty)$; $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator; the weight function $m : [1,T]_Z \to \mathbb{R}$ satisfies $m(k) \neq 0$ on $[1,T]_Z$ and m changes its sign on $[1,T]_Z$, i.e., there exists a proper subset $P^+ \subset [1,T]_Z$ such that

$$m(k) > 0, \quad k \in P^+; \qquad m(k) < 0, \quad k \in [1,T]_Z \setminus P^+.$$

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Let p^+ be the number of elements in P^+ and let p^- be the number of elements in $[1,T]_Z \setminus P^+$. Hence, $p^+ + p^- = T$. In addition, $f_1, f_2 \in C([0,T+1]_Z \times \mathbb{R}^2, \mathbb{R})$.

In the differential case, Sturm-Liouville problem with a positive weight function has been considered by several authors [4, 7, 6, 20] under the differential case. In 1977, Berestycki [4] discussed the nonlinear Sturm-Liouville problem

$$\begin{cases} -(pu')' + qu = \lambda a u + F(t, u, u', \lambda), \qquad t \in (0,1), \\ a_0 u(0) + b_0 u'(0) = 0, \\ c_0 u(1) + d_0 u'(1) = 0, \end{cases} \tag{2}$$

where $p \in C^1([0,1], (0,+\infty)), q \in C([0,1]), a \in C([0,1], (0,+\infty))$ and a_0, b_0, c_0, d_0 are real numbers such that $|a_0| + |b_0| \neq 0, |c_0| + |d_0| \neq 0$. λ is a real parameter and the nonlinear term F has the form $F = f + g, f, g \in C([0,1] \times \mathbb{R}^3)$ and f is not necessarily differentiable at the origin with respect to u. Using the result of Rabinowitz [17], the author obtained that there are two unbounded connected branches of problem (2) with bifurcation from interval of the line of trivial solutions. In 2013, Ma and Dai [12] generalized Berestycki's result, they established the unilateral global bifurcation which bifurcates from interval of the line of trivial solutions or from infinity of (2). Moreover, the authors indicated the existence of nodal solutions for a class of half-linear eigenvalue problems. When the weight function changes signs, the existence and multiplicity of nontrivial solutions for second-order differential problems were studied in [1, 15, 16, 20].

In the difference case, when $m(k) \geq 0$, There are many authors have discussed the existence and multiplicity of solutions for discrete Sturm-Liouville problems (1), can be seen in [10, 11, 14] and the references therein. But up to now, for the case that $m(k)$ changes its sign, to the author's knowledge, there is no paper concerned with the unilateral global bifurcation of (1). For the above reasons, based on the spectral results of [13], this paper shall establish the global bifurcation results which bifurcating from intervals of the trivial solutions axis or infinity for a class of discrete second-order Sturm-Liouville problem, respectively.

It is the purpose of this paper to show that there are two distinct unbounded continua $(\mathcal{C}_1^v)^+$ of positive solution and $(\mathcal{C}_1^v)^-$ of negative solution, which emanate from the bifurcation interval $I_{1,0}^v \times \{\mathbf{0}\}$ (see Section 3) of the line of trivial solutions. In addition, there are two distinct unbounded continua $(\mathcal{D}_1^v)^+$ of positive solution and $(\mathcal{D}_1^v)^-$ of negative solution, which emanate from the bifurcation interval $I_{1,\infty}^v \times \{\infty\}$ (see Section 4).

Furthermore, we assume that the following conditions:

$(\mathbf{C_1})$ $sf_1(k, s, \lambda) < 0$ for all $s \neq 0$;

$(\mathbf{C_2})$ There exist $f_0, f^0 \in (-\infty, 0)$ with $f_0 \neq f^0$, where

$$f_0 = \liminf_{|s| \to 0^+} \frac{f_1(k, s, \lambda)}{s}, \qquad f^0 = \limsup_{|s| \to 0^+} \frac{f_1(k, s, \lambda)}{s}$$

uniformly for $k \in [1,T]_Z, 0 < |s| \leq 1$ and for all $\lambda \in \mathbb{R}$;

$(\mathbf{C_3})$ $f_2(k, s, \lambda) = o(|s|)$, near $s = 0$, uniformly for $k \in [1,T]_Z$ and in every bounded interval of λ;

$(\mathbf{C_4})$ There exist $f_\infty, f^\infty \in (-\infty, 0)$ with $f_\infty \neq f^\infty$, where

$$f_\infty = \liminf_{|s| \to +\infty} \frac{f_1(k, s, \lambda)}{s}, \qquad f^\infty = \limsup_{|s| \to +\infty} \frac{f_1(k, s, \lambda)}{s}$$

uniformly for $k \in [1,T]_Z, |s| \geq C$ for some positive constant C large enough and for all $\lambda \in \mathbb{R}$;

$(\mathbf{C_5})$ $f_2(k,s,\lambda)=o(|s|)$, near $s=\infty$, uniformly for $k\in[1,T]_Z$ and in every bounded interval of λ.

This paper is organized as follows: In Section 2, we state some notations and preliminary results. Section 3 and Section 4 are devoted to study the bifurcation phenomena from the line of trivial solution and from infinity for (1) which are not linearizable, respectively. The final section we study the intertwining of the branch bifurcating from the trivial solution and from infinity, showing the existence of positive and negative solutions for a class of second-order Sturm-Liouville boundary value problem.

2. Some preliminaries

In this section, we introduce some lemmas and well-known results which will be used in the subsequent section.

Lemma 2.1. *([13], Theorem 1) Suppose that $m:[1,T]_Z\to\mathbb{R}$ satisfies $m(k)\neq 0$ on $[1,T]_Z$, and there exists a proper subset $P^+\subset[1,T]_Z$ such that $m(k)>0, k\in P^+$ and $m(k)<0, k\in[1,T]_Z\backslash P^+$, $q(k)\not\equiv 0$ on $[1,T]_Z$ or $a_0^2+a_1^2\neq 0$. Then the following indefinite weight linear eigenvalue problem*

$$\begin{cases} -\Delta(p(k)\Delta u(k-1))+q(k)u(k)=\lambda m(k)u(k), \quad k\in[1,T]_Z,\\ \\ a_0u(0)-b_0\Delta u(0)=0,\ \ a_1u(T+1)+b_1\Delta u(T)=0,\end{cases} \tag{3}$$

has T real eigenvalues

$$\lambda^-_{p^-}<\lambda^-_{p^--1}<\cdots<\lambda^-_1<0<\lambda^+_1<\lambda^+_2<\cdots<\lambda^+_{p^+},$$

and all of which are simple. Every eigenfunction ϕ_i^v corresponding to eigenvalues λ_i^v has exactly $i-1$ simple zeros in $[1,T]_Z$, where $v\in\{+,-\}$.

Set $X:=\{\mathbf{u}:[0,T+1]_Z\to\mathbb{R}|\ a_0u(0)-b_0\Delta u(0)=0,\ a_1u(T+1)+b_1\Delta u(T)=0\}$, where $\mathbf{u}=\big(u(0),u(1),\cdots,u(T+1)\big)\in\mathbb{R}^{T+2}$, then X is a Banach space under the norm

$$\|\mathbf{u}\|_X=\max_{k\in[0,T+1]_Z}|u(k)|.$$

Let $Y:=\{\mathbf{u}:[0,T+1]_Z\to\mathbb{R}\}$. Then Y is a Banach space under the norm $\|\mathbf{u}\|_Y=\max\limits_{k\in[0,T+1]_Z}|u(k)|$.

We use the terminology of Rabinowitz [18]. Let us denote $S_i^+=\{\mathbf{u}\in X:\mathbf{u}$ has exactly $i-1$ simple zeros in $[1,T]_Z$ and $\mathbf{u}>0$ near $k=0\}$ and let $S_i^-=-S_i^+$ and $S_i=S_i^+\cup S_i^-$. They are disjoint and open in X. Furthermore, we use $\mathscr{C}$ to denote the closure in $\mathbb{R}\times X$ of the set of nontrivial solutions of (1). $\mathscr{C}_1^\pm$ denote the subset of $\mathscr{C}$ with $\mathbf{u}\in S_1^\pm$, and $\mathscr{C}_1=\mathscr{C}_1^+\cup\mathscr{C}_1^-$.

In addition, we use the terminology of Rynne [20]. For any $\lambda\in\mathbb{R}$, we say that a subset $\mathscr{C}'\subset\mathscr{C}$ meets $(\lambda,\mathbf{0})$ (similarly, (λ,∞)) if there is a sequence $(\lambda_n,\mathbf{u}_n)\in\mathscr{C}'(n=1,2,\cdots)$ such that $\lambda_n\to\lambda,\|\mathbf{u}_n\|_X\to\mathbf{0}$ (similarly, $\|\mathbf{u}_n\|\to\infty$) as $n\to+\infty$. Furthermore, we will say that $\mathscr{C}'\subset\mathscr{C}$ meets $(\lambda,\mathbf{0})$ throught $\mathbb{R}\times S_i^\sigma$ if the sequence $(\lambda_n,\mathbf{u}_n)\in\mathscr{C}'(n=1,2,\cdots)$ can be chosen such that $\mathbf{u}_n\in S_i^\sigma$ for all n. If $I\subset\mathbb{R}$ is a bounded interval we say that $\mathscr{C}'\subset\mathscr{C}$ meets $I\times\{\mathbf{0}\}$ (similarly, $I\times\{\infty\}$) if $\mathscr{C}'$ meets $(\lambda,\mathbf{0})$ (similarly, (λ,∞)) for some $\lambda\in I$. Similarly, we can define $\mathscr{C}'$ meets $I\times\{\mathbf{0}\}$ or $I\times\{\infty\}$ through $\mathbb{R}\times S_i^\sigma$, where $\sigma=+$ or $-$.

Lemma 2.2. *Let* $(\mathbf{C_2})$ *and* $(\mathbf{C_3})$ *hold. If* $(\lambda, \mathbf{u})$ *is a solution of (1) and there exists* $k_0 \in [1, T]_Z$ *such that one of the following cases holds:*

(i) $u(k_0) = 0, \Delta u(k_0) = 0$;

(ii) $u(k_0) = 0, u(k_0 - 1)u(k_0 + 1) \geq 0$.

then $\mathbf{u} \equiv \mathbf{0}$.

Proof. (i) From (1) we obtain that

$$p(k_0 - 1)\Delta u(k_0 - 1) - p(k_0)\Delta u(k_0) + q(k_0)u(k_0) = \lambda m(k_0)u(k_0) + f_1(k_0, u(k_0), \lambda) + f_2(k_0, u(k_0), \lambda),$$

Connecting $u(k_0) = 0, \Delta u(k_0) = 0$ with the assumptions $(\mathbf{C_2})$ and $(\mathbf{C_3})$, there is

$$p(k_0 - 1)\Delta u(k_0 - 1) = 0.$$

Hence $u(k_0 - 1) = 0$. Step by step, we conclude that $\mathbf{u} \equiv \mathbf{0}$ for $k \leq k_0, k \in [0, T+1]_Z$. Similarly, by virtue of

$$p(k_0)\Delta u(k_0) - p(k_0 + 1)\Delta u(k_0 + 1) = 0,$$

there is $u(k_0 + 2) = 0$. Step by step, we conclude that $\mathbf{u} \equiv \mathbf{0}$ for $k \geq k_0, k \in [0, T+1]_Z$.

(ii) Similar to the calculation in (i), we have

$$p(k_0)u(k_0 + 1) + p(k_0 - 1)u(k_0 - 1) = 0,$$

we deduce that $u(k_0 - 1) = u(k_0 + 1) = 0$. It is obvious that $\mathbf{u} \equiv 0$. □

Define the operator $L : X \to Y$ by

$$Lu(k) = -\Delta(p(k)\Delta u(k-1)) + q(k)u(k), \quad k \in [1, T]_Z.$$

It is well known that L is a self-adjoint operator.

For fixed $\lambda \in \mathbb{R}$, we consider the following eigenvalue problem

$$\begin{cases} -\Delta(p(k)\Delta u(k-1)) + q(k)u(k) - \lambda m(k)u(k) = \mu u(k), & k \in [1, T]_Z, \\ a_0 u(0) - b_0 \Delta u(0) = 0, \ a_1 u(T+1) + b_1 \Delta u(T) = 0. \end{cases} \tag{4}$$

(4) has T real eigenvalues

$$\mu_1(\lambda) < \mu_2(\lambda) < \cdots < \mu_T(\lambda).$$

Moreover, every eigenfunction $\psi_i(k, \lambda)$ corresponding to eigenvalues $\mu_i(\lambda)$ has exactly $i - 1$ simple generalized zeros in $[1, T]_Z$.

The principal eigenvalue $\mu_1(\lambda)$ of (4) is the minimum of the Rayleigh quotient, that is,

$$\mu_1(\lambda) = \inf\left\{ \frac{\sum_{k=0}^{T} p(k)|\Delta u(k)|^2 + \sum_{k=1}^{T} q(k)|u(k)|^2 - \lambda \sum_{k=1}^{T} m(k)|u(k)|^2}{\sum_{k=1}^{T} |u(k)|^2} : \mathbf{u} \in X \right\}.$$

The eigenfunction $\psi_1(k,\lambda)$ corresponding to eigenvalues $\mu_1(\lambda)$ does not vanish on $[1,T]_Z$. Thus, clearly, λ is a principal eigenvalue of (3) if and only if $\mu_1(\lambda)=0$. Applying the similar method of Alyev [2], for fixed $\mathbf{u}\in E$, we may obtain that $\lambda\to\mu_1(\lambda)$ is a concave function, $\mu_1(0)>0$. Moreover, $\psi_1(k,\lambda_1^+)=\phi_1^+(k)$ and $\psi_1(k,\lambda_1^-)=\phi_1^-(k)$.

To prove the main results for (1), we need the following lemmas 2.3-2.5.

Lemma 2.3. *For every $v\in\{+,-\}$, there is*

$$\frac{d\mu_1(\lambda_1^v)}{d\lambda}=-\frac{\sum\limits_{k=1}^{T}m(k)|\phi_1^v(k)|^2}{\sum\limits_{k=1}^{T}|\phi_1^v(k)|^2}. \tag{5}$$

Proof. From (4), we have

$$\begin{cases} L\psi_1(k,\lambda)-\lambda m(k)\psi_1(k,\lambda)=\mu_1(\lambda)\psi_1(k,\lambda), & k\in[1,T]_Z,\\ a_0\psi_1(0,\lambda)-b_0\Delta\psi_1(0,\lambda)=0,\ a_1\psi_1(T+1,\lambda)+b_1\Delta\psi_1(T,\lambda)=0. \end{cases} \tag{6}$$

Take the derivative of both sides with respect to λ, one has

$$\frac{dL\psi_1(k,\lambda)}{d\lambda}-\lambda m(k)\frac{d\psi_1(k,\lambda)}{d\lambda}-\mu_1(\lambda)\frac{d\psi_1(k,\lambda)}{d\lambda}=m(k)\psi_1(k,\lambda)+\frac{d\mu_1(\lambda)}{d\lambda}\psi_1(k,\lambda). \tag{7}$$

Multiplying (7) by $\psi_1(k,\lambda)$ and summing from 1 to T, by virtue of the self-adjointness of the operator L, for every $v\in\{+,-\}$, we conclude that

$$-\mu_1(\lambda_1^v)\sum_{k=1}^{T}\frac{d(\phi_1^v(k))^2}{d\lambda}=\sum_{k=1}^{T}m(k)(\phi_1^v(k))^2+\frac{d\mu_1(\lambda_1^v)}{d\lambda}\sum_{k=1}^{T}(\phi_1^v(k))^2.$$

Since $\mu_1(\lambda_1^v)=0$, this implies that

$$\sum_{k=1}^{T}m(k)(\phi_1^v(k))^2+\frac{d\mu_1(\lambda_1^v)}{d\lambda}\sum_{k=1}^{T}(\phi_1^v(k))^2=0.$$

Therefore, (5) holds. □

Connecting (3) with (4), we consider the following problems

$$\begin{cases} Lu(k)+h(k)u(k)=\lambda m(k)u(k), & k\in[1,T]_Z,\\ a_0u(0)-b_0\Delta u(0)=0,\ a_1u(T+1)+b_1\Delta u(T)=0, \end{cases} \tag{8}$$

and

$$\begin{cases} Lu(k)-\lambda m(k)u(k)+h(k)u(k)=\mu u(k), & k\in[1,T]_Z,\\ a_0u(0)-b_0\Delta u(0)=0,\ a_1u(T+1)+b_1\Delta u(T)=0, \end{cases} \tag{9}$$

where $h(k)\geq 0, k\in[0,T+1]_Z$.

Lemma 2.4. *Let $\hat{\mu}_1(\lambda)$ is the smallest eigenvalue of problem (9), then*

$$h_0 \leq \hat{\mu}_1(\lambda) - \mu_1(\lambda) \leq h^0, \tag{10}$$

where $h_0 = \min\limits_{k\in[0,T+1]_Z} h(k)$, $h^0 = \max\limits_{k\in[0,T+1]_Z} h(k)$.

Proof. By virtue of the minimax property of eigenvalues, we have

$$\hat{\mu}_1(\lambda) = \inf\left\{\frac{\sum\limits_{k=0}^{T} p(k)|\Delta u(k)|^2 + \sum\limits_{k=1}^{T} q(k)|u(k)|^2 - \lambda \sum\limits_{k=1}^{T} m(k)|u(k)|^2 + \sum\limits_{k=1}^{T} h(k)|u(k)|^2}{\sum\limits_{k=1}^{T} |u(k)|^2} : \mathbf{u} \in X\right\}.$$

Combining $h(k) \geq 0$ with the definition of $\mu_1(\lambda)$, we obtain that $h_0 \leq \hat{\mu}_1(\lambda) - \mu_1(\lambda) \leq h^0$. The proof of this lemma is complete. □

Lemma 2.5. *If $\hat{\lambda}_1^+$ and $\hat{\lambda}_1^-$ are positive and negative principal eigenvalues of (8), respectively. We have the following conclusions:*

$$\lambda_1^+ + \frac{h_0 \sum\limits_{k=1}^{T} |\phi_1^+(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi_1^+(k)|^2} \leq \hat{\lambda}_1^+ \leq \lambda_1^+ + \frac{h^0 \sum\limits_{k=1}^{T} |\phi_1^+(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi_1^+(k)|^2}, \tag{11}$$

and

$$\lambda_1^- + \frac{h^0 \sum\limits_{k=1}^{T} |\phi_1^-(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi_1^-(k)|^2} \leq \hat{\lambda}_1^- \leq \lambda_1^- + \frac{h_0 \sum\limits_{k=1}^{T} |\phi_1^-(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi_1^-(k)|^2}. \tag{12}$$

Proof. We only prove (11) holds. In the plane rectangular coordinate system, we introduce the following points

$$P(\lambda_1^+, 0), \qquad Q(\hat{\lambda}_1^+, 0), \qquad S(\hat{\lambda}_1^+, \frac{d\mu_1(\lambda_1^+)}{d\lambda}(\lambda_1^+ - \hat{\lambda}_1^+)) \qquad \text{and} \qquad T(\hat{\lambda}_1^+, \mu_1(\hat{\lambda}_1^+)).$$

We notice that $|PQ| = \hat{\lambda}_1^+ - \lambda_1^+$, where $|PQ|$ is the distance between the points P and Q. Obviously, $|QS| < |QT|$.

Furthermore, we can see that

$$|PQ| = \frac{|QS|}{\tan\angle QPS} = |QS| \cdot \frac{\sum\limits_{k=1}^{T} |\phi_1^+(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi_1^+(k)|^2}.$$

Since $|QT| = -\mu_1(\hat{\lambda}_1^+)$, and $h_0 \leq -\mu_1(\hat{\lambda}_1^+) \leq h^0$. We can obtain easily the desired conclusions. □

3. Bifurcation from the line of trivial solutions

In this section, we assume that the hypotheses $(\mathbf{C_1})$-$(\mathbf{C_3})$ hold throughout, and $(\mathbf{C_4})$-$(\mathbf{C_5})$ do not, we shall study the unilateral global bifurcation phenomena of problem (1) which bifurcates from the line of trivial solution. In order to obtain the main result, the Dancer-type unilateral bifurcation theorem plays a key role.

In order to get the bifurcation of solutions of problem (1), we introduce the following approximate problem

$$\begin{cases} Lu(k) = \lambda m(k)u(k) + f_1(k, u(k)|u(k)|^{\varepsilon}, \lambda) + f_2(k, u(k), \lambda), \ k \in [1, T]_Z, \\ a_0 u(0) - b_0 \Delta u(0) = 0, \ a_1 u(T+1) + b_1 \Delta u(T) = 0. \end{cases} \tag{13}$$

The following lemma will be needed in our further consideration.

Lemma 3.1. *For every $v \in \{+, -\}$, let $d^v_{1,0} = \frac{f^0 \sum\limits_{k=1}^{T} |\phi^v_1(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi^v_1(k)|^2}, d^v_{2,0} = \frac{f_0 \sum\limits_{k=1}^{T} |\phi^v_1(k)|^2}{\sum\limits_{k=1}^{T} m(k)|\phi^v_1(k)|^2}$, and let $I^+_{1,0} = [\lambda^+_1 - d^+_{1,0}, \lambda^+_1 - d^+_{2,0}], I^-_{1,0} = [\lambda^-_1 - d^-_{2,0}, \lambda^-_1 - d^-_{1,0}]$. Let $\varepsilon_n \to 0,\ 0 \le \varepsilon_n \le 1$. If there exists a sequence $\{(\lambda_n, \mathbf{u}_n)\} \subset \mathbb{R} \times S^\sigma_1$ such that $(\lambda_n, \mathbf{u}_n)$ is a nontrivial solution of (13) corresponding to $\varepsilon = \varepsilon_n$, and $(\lambda_n, \mathbf{u}_n) \to (\lambda, \mathbf{0})$ in $\mathbb{R} \times X$. Then $\lambda \in I^v_{1,0}$, where $\sigma = +$ and $-$.*

Proof. Without loss of generality, let $\|\mathbf{u}_n\|_X \le 1$ and $\rho_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|_X}$. So ρ_n satisfies

$$\begin{cases} L\rho_n(k) = \lambda_n m(k)\rho_n(k) + f_{1,n}(k) + f_{2,n}(k), \qquad k \in [1, T]_Z, \\ a_0 \rho_n(0) - b_0 \Delta \rho_n(0) = 0, \ a_1 \rho_n(T+1) + b_1 \Delta \rho_n(T) = 0, \end{cases} \tag{14}$$

where $f_{1,n}(k) = \frac{f_1(k, u_n(k)|u_n(k)|^{\varepsilon_n}, \lambda_n)}{\|\mathbf{u}_n\|_X}$, $f_{2,n}(k) = \frac{f_2(k, u_n(k), \lambda_n)}{\|\mathbf{u}_n\|_X}$.

Setting $\overline{f}_2(k, u(k), \lambda) = \max\limits_{0 \le |s| \le \mathbf{u}} \left|f_2(k, s, \lambda)\right|$ for any $k \in [1, T]_Z$. According to $(\mathbf{C_3})$, $\overline{f_2}$ is nondecreasing with respect to $\mathbf{u}$ and

$$\lim_{\mathbf{u} \to 0} \frac{\overline{f}_2(k, u(k), \lambda)}{|u(k)|} = 0 \tag{15}$$

uniformly for $k \in [1, T]_Z$ and in every bounded interval of λ. By (15), it is easy to check that

$$\frac{|f_2(k, u(k), \lambda)|}{\|\mathbf{u}_n\|_X} \le \frac{\overline{f}_2(k, u(k), \lambda)}{\|\mathbf{u}_n\|_X} \le \frac{\overline{f}_2(k, \|\mathbf{u}\|_X, \lambda)}{\|\mathbf{u}_n\|_X} \to 0, \qquad \mathbf{u} \to \mathbf{0} \tag{16}$$

uniformly for $k \in [1, T]_Z$ and in every bounded interval of λ. Obviously, in view of $(\mathbf{C_2})$, there is

$$\begin{aligned} |f_{1,n}(k)| &= \left| \tfrac{f_1(k, u_n(k)|u_n(k)|^{\varepsilon_n}, \lambda_n)}{u_n(k)|u_n(k)|^{\varepsilon_n}} \tfrac{u_n(k)|u_n(k)|^{\varepsilon_n}}{\|\mathbf{u}_n\|_X} \right| \\ &\le -f_0 \|\mathbf{u}_n\|^{\varepsilon_n}_X \\ &\to -f_0, \quad n \to +\infty \end{aligned} \tag{17}$$

for any $k \in [1, T]_Z$. Connecting (14), (16) with (17), by the Arzela-Ascoli theorem, we may assume that $\rho_n \to \rho$ and $\|\rho\|_X = 1$. Therefore, ρ lies in the closure of S^σ_i.

Let us prove that in fact $\rho \in S_i^\sigma$. If $\rho \notin S_i^\sigma$, then $\rho \in \partial S_i^\sigma$. Hence ρ has at least one double zero in $[1,T]_Z$. We assume that there exists $k_0 \in [1,T]_Z$ such that either $\rho_n(k_0) \to 0, \Delta\rho_n(k_0) \to 0$ or $\rho_n(k_0) \to 0, \rho_n(k_0-1)\rho_n(k_0+1) \geq 0$ as $n \to +\infty$. By Lemma 2.2, we can see that $\rho_n \equiv \mathbf{0}$, which contradicts $\|\rho\|_X = 1$. Hence $\rho \in S_i^\sigma$.

Let $(\lambda_\varepsilon, \mathbf{u}_\varepsilon)$ be a solution of the following problem

$$\begin{cases} Lu(k) + h_\varepsilon(k,u(k))u(k) = \lambda m(k)u(k) + f_2(k,u(k),\lambda), & k \in [1,T]_Z, \\ a_0u(0) - b_0\Delta u(0) = 0,\ a_1u(T+1) + b_1\Delta u(T) = 0, \end{cases} \tag{18}$$

where

$$h_\varepsilon(k,u(k)) = \begin{cases} -\frac{f_1(k,u(k)|u(k)|^\varepsilon,\lambda)}{u(k)}, & u(k) \neq 0, \\ 0, & u(k) = 0. \end{cases} \tag{19}$$

By virtue of $(\mathbf{C_1})$ and $(\mathbf{C_2})$, we obtain

$$h_\varepsilon(k,u_\varepsilon(k)) \geq 0 \qquad \text{and} \qquad -f^0 \leq h_\varepsilon(k,u_\varepsilon(k)) \leq -f_0. \tag{20}$$

We know that the eigenvalue problem

$$\begin{cases} Lu_\varepsilon(k) + h_\varepsilon(k,u_\varepsilon)u_\varepsilon(k) = \lambda m(k)u_\varepsilon(k), & k \in [1,T]_Z, \\ a_0u_\varepsilon(0) - b_0\Delta u_\varepsilon(0) = 0,\ a_1u_\varepsilon(T+1) + b_1\Delta u_\varepsilon(T) = 0 \end{cases} \tag{21}$$

has two principal eigenvalues $\hat{\lambda}_\varepsilon^+$ and $\hat{\lambda}_\varepsilon^-$. Since $\mathbf{u}_\varepsilon$ does not vanish in $[1,T]_Z$. Applying the Lemma 2.5 to (21), it follows that $\hat{\lambda}_\varepsilon^+$ lies in $I_{1,0}^+$ and $\hat{\lambda}_\varepsilon^-$ lies in $I_{1,0}^-$, where

$$I_{1,0}^+ = [\lambda_1^+ - d_{1,0}^+, \lambda_1^+ - d_{2,0}^+], \qquad I_{1,0}^- = [\lambda_1^- - d_{2,0}^-, \lambda_1^- - d_{1,0}^-], \tag{22}$$

and

$$d_{1,0}^v = \frac{f^0 \sum_{k=1}^T |\phi_1^v(k)|^2}{\sum_{k=1}^T m(k)|\phi_1^v(k)|^2}, \qquad d_{2,0}^v = \frac{f_0 \sum_{k=1}^T |\phi_1^v(k)|^2}{\sum_{k=1}^T m(k)|\phi_1^v(k)|^2}.$$

Now let us prove that $\lambda \in I_{1,0}^v, v \in \{+,-\}$. Without loss of generality, we only prove that $\lambda \in I_{1,0}^+$. Suppose on the contrary that $\lambda \notin I_{1,0}^+$. Denote $\tau = \text{dist}\{\lambda, I_{1,0}^+\}$. Since $\lambda_n \to \lambda$, then there exists $n_\tau \in \mathbb{N}$ such that for all $n > n_\tau$, one has $|\lambda_n - \lambda| < \frac{\tau}{2}$. Therefore, $\text{dist}\{\lambda, I_{1,0}^+\} > \frac{\tau}{2}$ for $n > n_\tau$.

We note that (λ_n, ρ_n) is a solution to the nonlinear problem (14) for $\varepsilon = \varepsilon_n$. Since $(\hat{\lambda}_{\varepsilon_n}^+, \mathbf{0}) \in \mathbb{R} \times S_1^+$ is the bifurcation point of problem (14), thus for every sufficiently large $n > n_\tau$, we can find an arbitrarily small $l_n^+ > 0$ such that $l_n^+ < \frac{\tau}{2}$ and $\lambda_n \in (\hat{\lambda}_{\varepsilon_n}^+ - l_n^+, \hat{\lambda}_{\varepsilon_n}^+ + l_n^+)$, where $\hat{\lambda}_{\varepsilon_n}^+$ is the positive principal eigenvalue of the eigenvalue problem

$$\begin{cases} L\rho_n(k) + h_\varepsilon(k, \rho_n(k))\rho_n(k) = \lambda m(k)\rho_n(k), & k \in [1, T]_Z, \\ a_0\rho_n(0) - b_0\Delta\rho_n(0) = 0, \ a_1\rho_n(T+1) + b_1\Delta\rho_n(T) = 0 \end{cases}$$

for $\varepsilon = \varepsilon_n$. Consequently,

$$\lambda_n \in (\hat{\lambda}_{\varepsilon_n}^+ - \frac{\tau}{2}, \hat{\lambda}_{\varepsilon_n}^+ + \frac{\tau}{2}).$$

By virtue of (22), we have $\hat{\lambda}_{\varepsilon_n}^+ \in [\lambda_1^+ - d_{1,0}^+, \lambda_1^+ - d_{2,0}^+]$. Hence $\operatorname{dist}\{\lambda, I_{1,0}^+\} < \frac{\tau}{2}$, which contradicts $\operatorname{dist}\{\lambda, I_{1,0}^+\} > \frac{\tau}{2}$. The proof of this lemma is complete. □

Based on the analysis above, we have the following interval bifurcation result for the problem (1).

Theorem 3.2. *For every $v \in \{+, -\}$, there exist continua $(\mathcal{C}_1^v)^+$ and $(\mathcal{C}_1^v)^-$, where $(\mathcal{C}_1^v)^+$ containing $I_{1,0}^v \times \{\mathbf{0}\}$ is unbounded and $(\mathcal{C}_1^v)^+ \subset (\mathbb{R} \times S_1^+) \bigcup (I_{1,0}^v \times \{\mathbf{0}\})$, $(\mathcal{C}_1^v)^-$ containing $I_{1,0}^v \times \{\mathbf{0}\}$ is unbounded and $(\mathcal{C}_1^v)^- \subset (\mathbb{R} \times S_1^-) \bigcup (I_{1,0}^v \times \{\mathbf{0}\})$.*

Proof. Without loss of generality, we only prove the case of $(\mathcal{C}_1^v)^-$. Let $(\mathcal{C}_1^v)^-$ be the component of $\mathscr{C}_1^- \cup (I_{1,0}^v \times \{\mathbf{0}\})$ containing $I_{1,0}^v \times \{\mathbf{0}\}$. We divide the proof into the following two steps.

First we show that $(\mathcal{C}_1^v)^- \subset (\mathbb{R} \times S_1^-) \cup (I_{1,0}^v \times \{\mathbf{0}\})$. For every $(\lambda, \mathbf{u}) \in (\mathcal{C}_1^v)^-$, there exist two situations: (i) $\mathbf{u} \in S_1^-$; (ii) $\mathbf{u} \in \partial S_1^-$.

If (i) holds, it is clear that $(\lambda, \mathbf{u}) \in \mathbb{R} \times S_1^-$. If (ii) holds, then u has at least one double zero in $[1, T]_Z$. In view of Lemma 2.2, it follows that $\mathbf{u} \equiv \mathbf{0}$. Hence, there exists a sequence $\{(\lambda_n, \mathbf{u}_n)\} \subset \mathbb{R} \times S_1^-$ such that $(\lambda_n, \mathbf{u}_n)$ is a solution of (13) corresponding to $\varepsilon = 0$, and $(\lambda_n, \mathbf{u}_n) \to (\lambda, \mathbf{0})$ in $\mathbb{R} \times S_1^-$. Lemma 3.1 implies that $\lambda \in I_{1,0}^v$. So $(\mathcal{C}_1^v)^- \cap (\mathbb{R} \times \{\mathbf{0}\}) \subset I_{1,0}^v \times \{\mathbf{0}\}$. Therefore, $(\mathcal{C}_1^v)^- \subset (\mathbb{R} \times S_1^-) \cup (I_{1,0}^v \times \{\mathbf{0}\})$. Similarly, $(\mathcal{C}_1^v)^+ \subset (\mathbb{R} \times S_1^+) \cup (I_{1,0}^v \times \{\mathbf{0}\})$.

We next to prove that $(\mathcal{C}_1^v)^-$ is unbounded.

Assume for contradiction that $(\mathcal{C}_1^v)^-$ is bounded. We know that $(\mathcal{C}_1^v)^-$ is compact in $\mathbb{R} \times E$. Following [4], we can find a neighborhood $\mathscr{O}$ of $(\mathcal{C}_1^v)^-$ such that $\partial\mathscr{O} \cap (\mathcal{C}_1^v)^- = \emptyset$. Consider the problem (13) for $\varepsilon > 0$. By virtue of Theorem 1.3 in [17], there exists an unbounded continuum $\mathrm{C}_{1,\varepsilon}^v$ of solutions of (13), which bifurcates from $(\lambda_1, \mathbf{0})$, and

$$\mathrm{C}_{1,\varepsilon}^v \subset (\mathbb{R} \times S_1 \cup \{(\lambda_1, \mathbf{0})\}).$$

Moreover, there are two continua $(\mathrm{C}_{1,\varepsilon}^{v})^{+}$ and $(\mathrm{C}_{1,\varepsilon}^{v})^{-}$, consisting of the bifurcation branch $\mathrm{C}_{1,\varepsilon}^{v}$. Furthermore, $(\mathrm{C}_{1,\varepsilon}^{v})^{+}$ and $(\mathrm{C}_{1,\varepsilon}^{v})^{-}$ are both unbounded.

Hence, for any $\varepsilon > 0$, there exists $(\lambda_\varepsilon, \mathbf{u}_\varepsilon) \in (\mathrm{C}_{1,\varepsilon}^{v})^{-} \cap \partial\mathscr{O}$. In view of the fact that $\mathscr{O}$ is bounded in $\mathbb{R} \times E$. Thus, taking a sequence $\varepsilon_n \to 0, n \to \infty$ such that $(\lambda_{\varepsilon_n}, \mathbf{u}_{\varepsilon_n}) \to (\lambda, \mathbf{u})$, where $(\lambda, \mathbf{u})$ is a solution of (1). Therefore, $\mathbf{u}$ lies in the closure of S_1^-.

If $\mathbf{u} \in \partial S_1^-$, it is easy to see from Lemma 2.2 that $\mathbf{u} \equiv \mathbf{0}$. By Lemma 3.1, we know that $\lambda \in I_{1,0}^{v}$, which is impossible, since $\mathscr{O}$ is a neighborhood of $I_{1,0}^{v} \times \{\mathbf{0}\}$. If $u \in S_1^-$, then $(\lambda, \mathbf{u}) \in \partial\mathscr{O} \cap \mathscr{C}^-$. Thus $\partial\mathscr{O} \cap \mathscr{C}^- \neq \emptyset$, which contradicts the assumption that $(\mathcal{C}_1^v)^-$ is bounded.

Consequently, $(\mathcal{C}_1^v)^+$ and $(\mathcal{C}_1^v)^-$ are both unbounded in $\mathbb{R} \times X$. □

Lemma 3.3. *If $f_2 \equiv 0$ and $(\lambda, \mathbf{u}) \in \mathbb{R} \times S_1$ is a solution of problem (1). Then $\lambda \in I_{1,0}^{+}$ or $\lambda \in I_{1,0}^{-}$.*

Proof. We suppose that $(\lambda, \mathbf{u}) \in \mathbb{R} \times S_1$, then

$$\begin{cases} Lu(k) + h(k, u(k))u(k) = \lambda m(k)u(k), \qquad k \in [1, T]_Z, \\ \\ a_0 u(0) - b_0 \Delta u(0) = 0,\ a_1 u(T+1) + b_1 \Delta u(T) = 0, \end{cases} \tag{23}$$

where

$$h(k, u(k)) = \begin{cases} -\frac{f_1(k, u(k), \lambda)}{u(k)}, & u(k) \neq 0, \\ \\ 0, & u(k) = 0. \end{cases}$$

By virtue of $(\mathbf{C_1})$ and $(\mathbf{C_2})$, we can see that

$$h(k, u(k)) \geq 0 \qquad \text{and} \qquad -f^0 \leq h(k, u(k)) \leq -f_0.$$

Thus λ is a principal eigenvalue of (23). Applying the Lemma 2.5 to (23), it can be easily seen that $\lambda \in I_{1,0}^{+}$ or $\lambda \in I_{1,0}^{-}$. □

Theorem 3.4. *If $f_2 \equiv 0$, then for every $v \in \{+, -\}$ and $\sigma \in \{+, -\}$, the continuum $(\mathcal{C}_1^v)^\sigma$ containing $I_{1,0}^{v} \times \{\mathbf{0}\}$ is unbounded and $(\mathcal{C}_1^v)^\sigma \subset (I_{1,0}^{v} \times S_1^\sigma) \bigcup (I_{1,0}^{v} \times \{\mathbf{0}\})$.*

Proof. Combining the facts of Lemma 3.3 with the proof of Theorem 3.2, the conclusions of the theorem hold. □

Remark 3.5. *Note that we can only construct the connected components of positive and negative solutions, but cannot construct connected components of other nodal solutions. The main reason is that when m changes sign, we cannot find a suitable interval of λ such that there exist sign-changing solutions for (1) under the hypotheses $(\mathbf{C_1})$, $(\mathbf{C_2})$ and $(\mathbf{C_3})$.*

4. Bifurcation from infinity

This section is mainly motivated by the Theorem 1.6 of [19], we shall study the unilateral global interval bifurcation phenomena of problem (1) which bifurcates from infinity. In this section, we assume that the hypotheses $(\mathbf{C_1})$, $(\mathbf{C_4})$-$(\mathbf{C_5})$ hold throughout, and $(\mathbf{C_2})$-$(\mathbf{C_3})$ do not. Let $\mathscr{D}$ to denote the set of nontrivial solutions of (1) under assumptions $(\mathbf{C_4})$ and $(\mathbf{C_5})$. Our second main result is the following theorem.

Theorem 4.1. *For every $v\in\{+,-\}$, let $d_{1,\infty}^v=\frac{f^{\infty}\sum_{k=1}^{T}|\phi_1^v(k)|^2}{\sum_{k=1}^{T}m(k)|\phi_1^v(k)|^2}$, $d_{2,\infty}^v=\frac{f_{\infty}\sum_{k=1}^{T}|\phi_1^v(k)|^2}{\sum_{k=1}^{T}m(k)|\phi_1^v(k)|^2}$, and let $I_{1,\infty}^+=[\lambda_1^+-d_{1,\infty}^+,\lambda_1^+-d_{2,\infty}^+]$, $I_{1,\infty}^-=[\lambda_1^--d_{2,\infty}^-,\lambda_1^--d_{1,\infty}^-]$. Then for every $\sigma=+$ and $-$, there exists a component $(\mathcal{D}_1^v)^\sigma$ of $\mathscr{D}\cup(I_{1,\infty}^v\times\{\infty\})$, containing $I_{1,\infty}^v\times\{\infty\}$. Moreover, if $\Lambda\subset\mathbb{R}$ is an interval such that $\Lambda\cap I_{1,\infty}^v=I_{1,\infty}^v$ and $\mathcal{M}$ is a neighborhood of $I_{1,\infty}^v\times\{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from $\mathbf{0}$, then either*

1°. $(\mathcal{D}_1^v)^\sigma-\mathcal{M}$ is bounded in $\mathbb{R}\times E$ and $(\mathcal{D}_1^v)^\sigma-\mathcal{M}$ meets $\mathcal{R}=\{(\lambda,\mathbf{0})|\lambda\in\mathbb{R}\}$ or

2°. $(\mathcal{D}_1^v)^\sigma-\mathcal{M}$ is unbounded.

Furthermore, if 2° occurs and $(\mathcal{D}_1^v)^\sigma-\mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $(\mathcal{D}_1^v)^\sigma-\mathcal{M}$ meets $I_{j,\infty}^v\times\{\infty\}$ for some $j\neq1$, where $I_{j,\infty}^+=[\lambda_j^+-d_{j,\infty}^1,\lambda_j^+-d_{j,\infty}^2]$, $I_{j,\infty}^-=[\lambda_j^--d_{j,\infty}^3,\lambda_j^--d_{j,\infty}^4]$, and $d_{j,\infty}^1,d_{j,\infty}^2,d_{j,\infty}^3,d_{j,\infty}^4$ are some constants.

Proof. If $(\lambda,\mathbf{u})\in\mathscr{D}$ and $\|\mathbf{u}\|_X\neq0$. Let $\omega=\frac{\mathbf{u}}{\|\mathbf{u}\|_X^2}$, dividing (1) by $\|\mathbf{u}\|_X^2$, we obtain

$$\begin{cases} L\omega(k)=\lambda m(k)\omega(k)+\frac{f_1(k,u(k),\lambda)}{\|\mathbf{u}\|_X^2}+\frac{f_2(k,u(k),\lambda)}{\|\mathbf{u}\|_X^2}, & k\in[1,T]_Z,\\ a_0\omega(0)-b_0\Delta\omega(0)=0,\ a_1\omega(T+1)+b_1\Delta\omega(T)=0. \end{cases}\tag{24}$$

Define

$$\widetilde{f}_1(k,\omega(k),\lambda)=\begin{cases}\|\omega\|_X^2f_1(k,\frac{\omega}{\|\omega\|_X^2},\lambda), & \omega\neq\mathbf{0},\\ 0, & \omega=\mathbf{0},\end{cases}$$

and

$$\widetilde{f}_2(k,\omega(k),\lambda)=\begin{cases}\|\omega\|_X^2f_2(k,\frac{\omega}{\|\omega\|_X^2},\lambda), & \omega\neq\mathbf{0},\\ 0, & \omega=\mathbf{0}.\end{cases}$$

Obviously, (24) is equivalent to

$$\begin{cases} L\omega(k)=\lambda m(k)\omega(k)+\widetilde{f}_1(k,\omega(k),\lambda)+\widetilde{f}_2(k,\omega(k),\lambda), & k\in[1,T]_Z,\\ a_0\omega(0)-b_0\Delta\omega(0)=0,\ a_1\omega(T+1)+b_1\Delta\omega(T)=0. \end{cases}\tag{25}$$

It is easily can be seen that $(\mathbf{C_4})$ and $(\mathbf{C_5})$ imply

$$\liminf_{|\omega|\to 0^+} \frac{\widetilde{f}_1(k,\omega(k),\lambda)}{\omega} = f_\infty, \qquad \limsup_{|\omega|\to 0^+} \frac{\widetilde{f}_1(k,\omega(k),\lambda)}{\omega} = f^\infty$$

and $\widetilde{f}_2(k,\omega(k),\lambda) = o(|\omega|)$, near $\omega = \mathbf{0}$, uniformly for $k \in [1,T]_Z$ and in every bounded interval of λ.

Applying Theorem 3.2 to the problem (25), which implies that there exists connected component $(\mathrm{D}_1^v)^\sigma$ of $\mathscr{C}_1^\sigma \bigcup (I_{1,\infty}^v \times \{\mathbf{0}\})$, containing $I_{1,\infty}^v \times \{\mathbf{0}\}$ is unbounded and

$$(\mathrm{D}_1^v)^\sigma \subset (\mathbb{R} \times S_1^\sigma \cup (I_{1,\infty}^v \times \{\mathbf{0}\})).$$

In view of $\omega \to \frac{\omega}{\|\omega\|_X^2} = \mathbf{u}$, it follows that $(\mathrm{D}_1^v)^\sigma \to (\mathcal{D}_1^v)^\sigma$. Furthermore, the conclusions in the theorem can be obtained. $\square$

Combining the facts of Theorem 3.2-3.4 with the proof of Theorem 4.1, we can obtain the following results.

Theorem 4.2. *There exists a neighborhood $\mathcal{N} \subset \mathcal{M}$ of $I_{1,\infty}^v \times \{\infty\}$ such that $((\mathcal{D}_1^v)^\sigma \cap \mathcal{N}) \subset (\mathbb{R} \times S_1^\sigma \cup (I_{1,\infty}^v \times \{\infty\}))$ for $\sigma = +$ and $\sigma = -$.*

Theorem 4.3. *If $f_2 \equiv 0$, then for every $v \in \{+,-\}$ and $\sigma \in \{+,-\}$, the continuum $(\mathcal{D}_1^v)^\sigma$ containing $I_{1,\infty}^v \times \{\infty\}$ is unbounded and $((\mathcal{D}_1^v)^\sigma \cap \mathcal{N}) \subset (I_{1,\infty}^v \times S_1^\sigma) \cup (I_{1,\infty}^v \times \{\infty\})$.*

Remark 4.4. *Connecting Remark 3.5 with the above results, it is easy to see that we cannot construct the connected components of other nodal solutions for (1) under the hypotheses $(\mathbf{C_1})$, $(\mathbf{C_4})$ and $(\mathbf{C_5})$.*

5. Existence of one-sign solutions for nonlinear Sturm-Liouville problem

According to the bifurcation results in Section 3 and 4. The aim of this section is to discuss the existence of one-sign solutions for nonlinear Sturm-Liouville problem

$$\begin{cases} -\Delta(p(k)\Delta u(k-1)) + q(k)u(k) = \lambda m(k) g_1(u(k)) + g_2(u(k)), & k \in [1,T]_Z, \\ a_0 u(0) - b_0 \Delta u(0) = 0, \ a_1 u(T+1) + b_1 \Delta u(T) = 0, \end{cases} \tag{26}$$

where $m : [1,T]_Z \to \mathbb{R}$ satisfies $m(k) \neq 0$ on $[1,T]_Z$, $g_1, g_2 \in C(\mathbb{R},\mathbb{R})$ and g_1, g_2 satisfying the following conditions:

$(\mathbf{C_6})$ g_1 satisfies $s g_1(s) > 0$ for all $s \neq 0$ and there exist $g_*, g^* \in (0,\infty)$ such that

$$g_* = \lim_{|s|\to 0^+} \frac{g_1(s)}{s}, \qquad g^* = \lim_{|s|\to +\infty} \frac{g_1(s)}{s};$$

($\mathbf{C_7}$) g_2 satisfies $sg_2(s) < 0$ for all $s \neq 0$ and there exist $g_0, g^0, g_\infty, g^\infty \in (-\infty, 0)$ with $g_0 \neq g^0, g_\infty \neq g^\infty$, where

$$g_0 = \liminf_{|s|\to 0^+} \frac{g_2(s)}{s}, \qquad g^0 = \limsup_{|s|\to 0^+} \frac{g_2(s)}{s},$$

$$g_\infty = \liminf_{|s|\to +\infty} \frac{g_2(s)}{s}, \qquad g^\infty = \limsup_{|s|\to +\infty} \frac{g_2(s)}{s}.$$

Theorem 5.1. *Assume* ($\mathbf{C_6}$)-($\mathbf{C_7}$) *hold. If*

$$\frac{g_0 \sum_{k=1}^{T} |\phi_1^+(k)|^2}{(\lambda_1^+ - 1) g_* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2} < \lambda < \frac{g^\infty \sum_{k=1}^{T} |\phi_1^+(k)|^2}{(\lambda_1^+ - 1) g^* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2} \tag{27}$$

or

$$\frac{g_\infty \sum_{k=1}^{T} |\phi_1^+(k)|^2}{(\lambda_1^+ - 1) g^* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2} < \lambda < \frac{g^0 \sum_{k=1}^{T} |\phi_1^+(k)|^2}{(\lambda_1^+ - 1) g_* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2} \tag{28}$$

then problem (26) has at least two solutions $\mathbf{u}^+$ and $\mathbf{u}^-$, where $\mathbf{u}^+$ is positive in $[1,T]_Z$, and $\mathbf{u}^-$ is negative in $[1,T]_Z$.

Proof. We consider the following problem

$$\begin{cases} -\Delta(p(k)\Delta u(k-1)) + q(k)u(k) = \mu\lambda m(k) g_1(u(k)) + g_2(u(k)), \quad & k \in [1,T]_Z, \\ a_0 u(0) - b_0 \Delta u(0) = 0, \ a_1 u(T+1) + b_1 \Delta u(T) = 0, \end{cases} \tag{29}$$

where $\mu > 0$ is a bifurcation parameter.

By ($\mathbf{C_6}$), it is easy to see that there exists $\xi \in C(\mathbb{R},\mathbb{R})$ such that $g_1(s) = g_* s + \xi(s)$ and $\lim_{|s|\to 0^+} \frac{\xi(s)}{s} = 0$. Taking $\hat{\xi}(\mathbf{u}) = \max_{0\le |s| \le \mathbf{u}} |\xi(s)|$, then $\hat{\xi}$ is nondecreasing and $\lim_{|\mathbf{u}|\to 0^+} \frac{\hat{\xi}(\mathbf{u})}{|\mathbf{u}|} = 0$. This means that $\frac{\xi(\mathbf{u})}{\|\mathbf{u}\|_X} \le \frac{\hat{\xi}(|\mathbf{u}|)}{\|\mathbf{u}\|_X} \le \frac{\hat{\xi}(\|\mathbf{u}\|_X)}{\|\mathbf{u}\|_X} \to 0$, as $\|\mathbf{u}\|_X \to 0$. By simple calculation, we show that

$$I_{1,0}^+ := \Bigg[\lambda_1^+ - \frac{g^0 \sum_{k=1}^{T} |\phi_1^+(k)|^2}{\lambda g_* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2}, \lambda_1^+ - \frac{g_0 \sum_{k=1}^{T} |\phi_1^+(k)|^2}{\lambda g_* \sum_{k=1}^{T} m(k)|\phi_1^+(k)|^2}\Bigg].$$

Applying the Theorem 3.2, we obtain that there are two distinct unbounded continua $(\mathcal{C}_1^+)^+$ and $(\mathcal{C}_1^+)^-$. The connected component $(\mathcal{C}_1^+)^+$ of $\mathscr{C}_1^+ \cup (I_{1,0}^+ \times \{\mathbf{0}\})$, containing $I_{1,0}^+ \times \{\mathbf{0}\}$ and $(\mathcal{C}_1^+)^+ \subset \big(\mathbb{R} \times S_1^+ \cup (I_{1,0}^+ \times \{\mathbf{0}\})\big)$. Similarly, The connected component $(\mathcal{C}_1^+)^-$ of $\mathscr{C}_1^- \cup (I_{1,\mathbf{0}}^+ \times \{\mathbf{0}\})$, containing $I_{1,0}^+ \times \{\mathbf{0}\}$ and $(\mathcal{C}_1^+)^- \subset \big(\mathbb{R} \times S_1^- \cup (I_{1,0}^+ \times \{\mathbf{0}\})\big)$.

By $(\mathbf{C_6})$, it is easy to see that there exists $\eta \in C(\mathbb{R},\mathbb{R})$ such that $g_1(s) = g^* s + \eta(s)$ and $\lim_{|s|\to+\infty} \frac{\eta(s)}{s} = 0$. Taking $\hat{\eta}(\mathbf{u}) = \max_{\mathbf{u}\le|s|\le 2\mathbf{u}} |\eta(s)|$, then $\hat{\eta}$ is nondecreasing and $\lim_{|\mathbf{u}|\to+\infty} \frac{\hat{\eta}(\mathbf{u})}{|\mathbf{u}|} = 0$, which implies $\frac{\eta(\mathbf{u})}{\|\mathbf{u}\|_X} \le \frac{\hat{\eta}(|\mathbf{u}|)}{\|\mathbf{u}\|_X} \le \frac{\hat{\eta}(\|\mathbf{u}\|_X)}{\|\mathbf{u}\|_X} \to 0$, as $\|\mathbf{u}\|_X \to \infty$. Hence, it can be easily shown that

$$I_{1,\infty}^+ := \Big[\lambda_1^+ - \frac{g^\infty \sum_{k=1}^T |\phi_1^+(k)|^2}{\lambda g^* \sum_{k=1}^T m(k)|\phi_1^+(k)|^2}, \lambda_1^+ - \frac{g_\infty \sum_{k=1}^T |\phi_1^+(k)|^2}{\lambda g^* \sum_{k=1}^T m(k)|\phi_1^+(k)|^2}\Big].$$

Applying the Theorem 4.2, we see that there are two distinct unbounded continua $(\mathcal{D}_1^+)^+$ and $(\mathcal{D}_1^+)^-$ of $\mathscr{D} \cup (I_{1,\infty}^+ \times \{\infty\})$, containing $I_{1,\infty}^+ \times \{\infty\}$. In addition, by Theorem 4.2, we know that there exists a neighborhood $\mathcal{N} \subset \mathcal{M}$ of $I_{1,\infty}^+ \times \{\infty\}$ such that

$$((\mathcal{D}_1^+)^\sigma \cap \mathcal{N}) \subset (\mathbb{R} \times S_1^\sigma \cup (I_{1,\infty}^+ \times \{\infty\})),$$

where $\sigma = +$ and $\sigma = -$.

The remaining proof is similar to the proof of the Theorem 1 in [12]. This completes the proof of the theorem. $\square$

Theorem 5.2. *Assume* $(\mathbf{C_6})$-$(\mathbf{C_7})$ *hold. If*

$$\frac{g_\infty \sum_{k=1}^T |\phi_1^-(k)|^2}{(\lambda_1^- - 1)g^* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2} < \lambda < \frac{g^0 \sum_{k=1}^T |\phi_1^-(k)|^2}{(\lambda_1^- - 1)g_* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2} \tag{30}$$

or

$$\frac{g_0 \sum_{k=1}^T |\phi_1^-(k)|^2}{(\lambda_1^- - 1)g_* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2} < \lambda < \frac{g^\infty \sum_{k=1}^T |\phi_1^-(k)|^2}{(\lambda_1^- - 1)g^* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2} \tag{31}$$

holds, then problem (26) has at least two solutions $\mathbf{u}^+$ *and* $\mathbf{u}^-$*, where* $\mathbf{u}^+$ *is positive in* $[1,T]_Z$*, and* $\mathbf{u}^-$ *is negative in* $[1,T]_Z$*.*

Proof. Similar to the proof of Theorem 5.1, we have

$$I_{1,0}^- := \Big[\lambda_1^- - \frac{g_0 \sum_{k=1}^T |\phi_1^-(k)|^2}{\lambda g_* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2}, \lambda_1^- - \frac{g^0 \sum_{k=1}^T |\phi_1^-(k)|^2}{\lambda g_* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2}]\Big].$$

and

$$I_{1,\infty}^- := \Big[\lambda_1^- - \frac{g_\infty \sum_{k=1}^T |\phi_1^-(k)|^2}{\lambda g^* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2}, \lambda_1^- - \frac{g^\infty \sum_{k=1}^T |\phi_1^-(k)|^2}{\lambda g^* \sum_{k=1}^T m(k)|\phi_1^-(k)|^2}]\Big].$$

Applying the Theorem 3.2, we obtain that there are two distinct unbounded continua $(\mathcal{C}_1^-)^+$ and $(\mathcal{C}_1^-)^-$ such that $(\mathcal{C}_1^-)^\sigma \subset \left(\mathbb{R} \times S_1^\sigma \cup (I_{1,0}^- \times \{\mathbf{0}\})\right)$. Applying the Theorem 4.2, there are two distinct unbounded continua $(\mathcal{D}_1^-)^+$ and $(\mathcal{D}_1^-)^-$ of $\mathscr{D} \cup (I_{1,\infty}^- \times \{\infty\})$, containing $I_{1,\infty}^- \times \{\infty\}$. In addition, there exists a neighborhood $\mathcal{N}' \subset \mathcal{M}$ of $I_{1,\infty}^- \times \{\infty\}$ such that $((\mathcal{D}_1^-)^\sigma \cap \mathcal{N}') \subset (\mathbb{R} \times S_1^\sigma \cup (I_{1,\infty}^- \times \{\infty\}))$. Furthermore, $(\mathcal{C}_1^-)^+ = (\mathcal{D}_1^-)^+$ and $(\mathcal{C}_1^-)^- = (\mathcal{D}_1^-)^-$. □

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References

[1] Z.S. Aliyev, S.M. Hasanova, Global bifurcation of a second order nonlinear elliptic problem with an indefinite weight function, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* 42(1) (2016) 116–125.

[2] Z.S. Aliyev, R.A. Huseynova, Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight, *Electron. J. Qual. Theory Differ. Equ.* 92 (2017) 12 pp.

[3] Y. An, Exact multiplicity of sign-changing solutions for a class of second-order Dirichlet boundary value problem with weight function, *Abstr. Appl. Anal.* (2013) 9 pp.

[4] H. Berestycki, On some nonlinear Sturm-Liouville problems, *J. Differential Equations* 26(3) (1977) 375-390.

[5] K.J. Brown, S.S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function, *J. Math. Anal. Appl.* 75(1) (1980) 112–120.

[6] X. Cheng, G. Dai, Positive solutions of sub–superlinear Sturm–Liouville problems, *Appl. Math. Comput.* 261 (2015) 351–359.

[7] G. Dai, Global bifurcation from intervals for Sturm–Liouville problems which are not linearizable, *Electron. J. Qual. Theory Differ. Equ.* 65 (2013) 7 pp.

[8] L.C. Evans, Partial Differential Equations, *AMS, Providence, RI,* 1998.

[9] C. Gao, G. Dai, R. Ma, Existence of positive solutions to discrete second-order boundary value problems with indefinite weight, *Adv. Difference Equ.* 3 (2012) 10 pp.

[10] Y. Long, J. Chen, Existence of multiple solutions to second-order discrete Neumann boundary value problems, *Appl. Math. Lett.* 83 (2018), 7–14.

[11] R. Ma, Bifurcation from infinity and multiple solutions for some discrete Sturm-Liouville problems, *Comput. Math. Appl.* 54(4) (2007) 535–543.

[12] R. Ma, G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *Journal of Functional Analysis* 265(8) (2013) 1443–1459.

[13] R. Ma, C. Gao, Y. Lu, Spectrum theory of second-order difference equations with indefinite weight, *J. Spectr. Theory* 8(3) (2018) 971–985.

[14] R. Ma, C. Gao, Y. Xu, Bifurcation interval for positive solutions to discrete second-order boundary value problems, *J. Difference Equ. Appl.* 17(9) (2011) 1251–1265.

[15] R. Ma, X. Han, Existence of nodal solutions of a nonlinear eigenvalue problem with indefinite weight function, *Nonlinear Anal.* 71(5-6) (2009) 2119–2125.

[16] J.P. Pinasco, C. Scarola, A nodal inverse problem for second order Sturm-Liouville operators with indefinite weights, *Appl. Math. Comput.* 256 (2015) 819–830.

[17] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* 7 (1971) 487–513.

[18] P.H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, *Rocky Mountain J. Math.* 3(2) (1973) 161–202.

[19] P.H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* 14 (1973) 462–475.

[20] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.* 228(1) (1998) 141–156.