Neimark-Sacker, flip and transcritical bifurcation in a symmetric system of difference equations with exponential terms

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Abstract

In this paper, we study the conditions under which the following symmetric system of difference equations with exponential terms:

$$x_{n+1} = a_1 \frac{y_n}{b_1 + y_n} + c_1 \frac{x_n e^{k_1 - d_1 x_n}}{1 + e^{k_1 - d_1 x_n}},$$

$$y_{n+1} = a_2 \frac{x_n}{b_2 + x_n} + c_2 \frac{y_n e^{k_2 - d_2 y_n}}{1 + e^{k_2 - d_2 y_n}}$$

where a_i, b_i, c_i, d_i, k_i, for $i = 1, 2$, are real constants and the initial values x_0, y_0 are real numbers, undergoes Neimark-Sacker, flip and transcritical bifurcation. The analysis is conducted applying center manifold theory and the normal form bifurcation analysis.

Keywords: Difference equations, Neimark-Sacker bifurcation, flip bifurcation, transcritical bifurcation, center manifold, discrete dynamics.

1 Introduction

In recent years, an attractive and far-reaching theory over systems of difference equations has emerged due to their promising potential applications in many fields, such us in mechanics, biology, economy and social sciences. Discrete dynamical systems are often suitable for modeling experimental data, implementable for computer simulations, that can represent abrupt changes in the systems states, and possibly chaotic dynamics.

Sometimes, a slight change in parameter values causes a drastic, qualitative change in the systems behavior, hence, bifurcations play an important role in many real-world systems as a switching mechanism. Bifurcations refers to the qualitative changes in the dynamics of a system, as parameters are varied. Some classical results on bifurcations can be found in [1, 2, 3, 4, 5]. Many researchers over the last years provide a plethora of interesting results in this field, see for example [6, 7, 8, 9, 10, 11, 12, 13].

A great interest in the last years was emerged for symmetric, two dimensional, systems of difference equations (see, e.g., [14, 15]) and their generalization, the close-to-symmetric systems (see, e.g., [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]). A natural extension of symmetric systems consist the cyclic systems, which are higher dimensional systems and their study was initiated by Iričanin and Stević in [42]. Moreover, there is a considerable attention in investigating solutions, stability or chaotic dynamics in the generalization of cyclic systems, the so called, close-to-cyclic systems of difference equations (see, e.g., [43, 44, 45, 46, 47]).

Usually, in the case where the characteristic polynomial of the linearization of systems of difference equations has roots belonging to the unit circle, a case for which Carr [48] gave some classical results, there exist an interesting dynamical behavior (e.g. [17, 27, 43, 44, 49, 50, 51, 52, 53]). The existence of specific types of solutions of difference equations of this case was investigated, for example, in [54, 55, 56, 57, 58]. In addition, the method used in the biological model in [53], can be modified and applied in a wide range of difference equations and systems (see, e.g., [31, 36, 37, 38, 39, 40, 41, 59, 60, 61, 62]).

In a system of difference equations, bifurcation is associated with the existence of non-hyperbolic fixed points, that is the characteristic polynomial of their linearization has zeros belonging to the unit circle. This means that there is at least one eigenvalue of the Jacobian matrix evaluated at the corresponding fixed point with modulus one. Fold (saddle-node), transcritical or pitchfork bifurcation are associated with the existence of an eigenvalue approaching 1, flip (or period-doubling) bifurcation is associated with the existence of an eigenvalue approaching -1 and, at last, Neimark-Sacker bifurcation is associated with the existence of a pair of complex conjugate eigenvalues with modulus approaching 1.

Difference equations and systems with exponential terms appear frequently in some models in biology (see, for example, [19, 27, 51, 52, 63, 64, 65]). In [66] was considered the following ecological model of grassland ecosystem incorporating plant inhibition by litter:

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \quad L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}$$

where B is the living biomass, L the litter mass, N the total soil nitrogen, t the time in years and constants $a, b, c, d > 0$ and $0 < k < 1$.

Motivated by this model, Papaschinopoulos *et al.* in [23] study the global asymptotic stability of the unique positive equilibrium, as well as the existence of periodic solutions on a permutation of the difference equation of litter mass L of the formentioned ecological model.

In addition, Papaschinopoulos *et al.* in [27] study the stability of zero equilibrium of the two dimensional system

$$\begin{aligned} x_{n+1} &= a_1\tfrac{y_n}{b_1+y_n} + c_1\tfrac{x_n e^{k_1-d_1x_n}}{1+e^{k_1-d_1x_n}}, \\ y_{n+1} &= a_2\tfrac{x_n}{b_2+x_n} + c_2\tfrac{y_n e^{k_2-d_2y_n}}{1+e^{k_2-d_2y_n}} \end{aligned} \tag{1.1}$$

where a_1, b_1, b_2, c_2, k_1 are real positive constants, a_2, c_1, k_2 are real negative constants, d_1, d_2 are real constants and x_0, y_0 are real numbers. Moreover, Mylona *et al.* in [49] investigate the stability of the origin and the occurrence of flip bifurcation on the three dimensional system that derives from (1.1).

Now, motivated by the above discrete time model, along with the recent studies on close-to-symmetric systems of difference equations and the potentials of difference equations systems with exponential terms, we study in this paper the conditions under which system (1.1) undergoes Neimark-Sacker, flip and transcritical bifurcation, that correspond to the three different cases in which there exist a non-hyperbolic fixed point in the system.

We investigate bifurcations in a neighborhood of the origin for the bifurcation parameter a_1 of (1.1), using normal form bifurcation analysis and center manifold reduction theorem, a method that is especially used in the case where the zero equilibrium is non-hyperbolic. The zero equilibrium usually corresponds to the physical situation where quantities x, y vanish.

2 Preliminaries

In this section we cite some preliminary results that are used for the bifurcation analysis in the case when there exist real eigenvalues of the Jacobian matrix of system (1.1). Those results are used in the case of flip and transcritical bifurcation.

Firstly, we can easily verify that $(0,0)$ is a fixed point of the system. Suppose that $a_1 = a_0 + \epsilon_0$, where ϵ_0 is a small number, is the bifurcation parameter. The Jacobian matrix of system (1.1) evaluated at the origin $(x_n, y_n, \epsilon_0) = (0,0,0)$ is:

$$J_0 = \begin{bmatrix} p_1 & \frac{a_0}{b_1} \\ \frac{a_2}{b_2} & p_2 \end{bmatrix} \tag{2.1}$$

where

$$p_1 = \frac{c_1 e^{k_1}}{1+e^{k_1}} \quad \text{and} \quad p_2 = \frac{c_2 e^{k_2}}{1+e^{k_2}} \tag{2.2}$$

and the corresponding characteristic polynomial is:

$$\lambda^2 - (p_1+p_2)\lambda + p_1 p_2 - \frac{a_0 a_2}{b_1 b_2} = 0 \tag{2.3}$$

with discriminant:

$$\Delta = (p_1 - p_2)^2 + 4\frac{a_0 a_2}{b_1 b_2}. \tag{2.4}$$

If a root of (2.3) is equal to 1 and the other is off the unit circle, then fold, transcritical or pitchfork bifurcation take place. In the case of a root of (2.3) is equal to -1 and the other is off the unit circle, then flip bifurcation occurs, while if equation (2.3) has two complex conjugate roots with modulus 1, then Neimark-Sacker bifurcation appears. In the first two cases, where the roots are real numbers, the discriminant (2.4) is positive, so we consider that:

$$-\frac{b_1 b_2 (p_1 - p_2)^2}{4a_0} < a_2 < 0, \tag{2.5}$$

and at the case of complex roots, the discriminant (2.4) is negative, so we consider that:

$$a_2 < -\frac{b_1 b_2 (p_1 - p_2)^2}{4a_0}. \tag{2.6}$$

The system (1.1) can be written as:

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array}\right] = J_0 \left[\begin{array}{c} x_n \\ y_n \end{array}\right] + \left[\begin{array}{c} f(x_n, y_n, \epsilon_0) \\ g(x_n, y_n, \epsilon_0) \end{array}\right], \tag{2.7}$$

where

$$f(x_n, y_n, \epsilon_0) = (a_0 + \epsilon_0)\frac{y_n}{b_1 + y_n} - \frac{a_0}{b_1} y_n + c_1 \frac{x_n e^{k_1 - d_1 x_n}}{1 + e^{k_1 - d_1 x_n}} - \frac{c_1 e^{k_1}}{1 + e^{k_1}} x_n,$$

$$g(x_n, y_n, \epsilon_0) = a_2 \frac{x_n}{b_2 + x_n} - \frac{a_2}{b_2} x_n + c_2 \frac{y_n e^{k_2 - d_2 y_n}}{1 + e^{k_2 - d_2 y_n}} - \frac{c_2 e^{k_2}}{1 + e^{k_2}} y_n$$

are smooth functions with Taylor expansions in (x, y, ϵ_0) starting with at least quadratic terms. We apply the coordinate transformation:

$$\left[\begin{array}{c} x_n \\ y_n \end{array}\right] = T \left[\begin{array}{c} u_n \\ v_n \end{array}\right] \tag{2.8}$$

where T is the matrix that diagonalizes J_0, corresponding to the eigenvalues λ_1 and λ_2 of J_0, defined by:

$$T = \left[\begin{array}{cc} 1 & 1 \\ A & B \end{array}\right], \tag{2.9}$$

where

$$A = -\frac{b_1}{a_0}(p_1 - \lambda_1), \quad B = -\frac{b_1}{a_0}(p_1 - \lambda_2),$$

with the determinant

$$D = B - A \neq 0. \tag{2.10}$$

Applying the coordinate transformation, the system (1.1) can be written as:

$$\left[\begin{array}{c} u_{n+1} \\ v_{n+1} \end{array}\right] = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right] \left[\begin{array}{c} u_n \\ v_n \end{array}\right] + T^{-1} \left[\begin{array}{c} f(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0) \\ g(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0) \end{array}\right], \tag{2.11}$$

where λ_1, λ_2 are the two distinct eigenvalues of J_0 and we suppose that $|\lambda_1| = 1$ and $|\lambda_2| \neq 1$.

To apply the center manifold theorem depending on the parameter ϵ_0 we increase the number of equations by writing the system (2.11) in the form:

$$\left[\begin{array}{c} u_{n+1} \\ \epsilon_{0_{n+1}} \\ v_{n+1} \end{array}\right] = \left[\begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_2 \end{array}\right] \left[\begin{array}{c} u_n \\ \epsilon_{0_n} \\ v_n \end{array}\right] + \left[\begin{array}{c} \bar{f}(u_n, v_n, \epsilon_0) \\ 0 \\ \bar{g}(u_n, v_n, \epsilon_0) \end{array}\right], \tag{2.12}$$

where

$$\bar{f}(u_n, v_n, \epsilon_0) = \frac{B}{D} f(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0) - \frac{1}{D} g(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0),$$

$$\bar{g}(u_n, v_n, \epsilon_0) = -\frac{A}{D} f(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0) + \frac{1}{D} g(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0).$$

As $\bar{f}$, $\bar{g}$ are C^2 functions with $\bar{f}(0,0,0) = 0$, $\bar{g}(0,0,0) = 0$ and $D\bar{f}(0,0,0) = 0_3$, $D\bar{g}(0,0,0) = 0_3$, where 0_3 the 3-dimensional zero vector, according to Theorem 5.1 in [5], the system (2.12) can be represented locally by the center manifold M_c with the form:

$$M_c = \left\{ \begin{array}{l} (u, \epsilon_0) \in I\!R \times I\!R : v = h(u, \epsilon_0), |u| < \delta_1, |\epsilon_0| < \delta_2, h(0,0) = 0, \\ Dh(0,0) = 0_2, \text{ for sufficiently small } \delta_1 \text{ and } \delta_2 \end{array} \right\}. \tag{2.13}$$

Consequently, according to Theorem 5.1 in [5], the dynamical behavior of system (2.12) at the origin, reduces to the study of the dynamics of the difference equation:

$$u_{n+1} = \lambda_1 u_n + \bar{f}(u_n, h(u_n, \epsilon_0), \epsilon_0) \tag{2.14}$$

where $|\lambda_1| = 1$. We suppose that $h(u_n, \epsilon_0)$ has the form:

$$v = h(u, \epsilon_0) = C_1 \epsilon_0 u + C_2 u^2 + C_3 \epsilon_0 u^2 + C_4 u^3 + O(\epsilon_0^2) + O((u + \epsilon_0)^4). \tag{2.15}$$

We can determine the coefficients of $h(u, \epsilon_0)$ applying the Taylor expansion to the center manifold equation (see Theorem 5.1 in [5], Theorem 6 in [48]), that derives from (2.12):

$$h(\lambda_1 u + \bar{f}(u, h(u, \epsilon_0), \epsilon_0), \epsilon_0) = \lambda_2 h(u, \epsilon_0) + \bar{g}(u, h(u, \epsilon_0), \epsilon_0). \tag{2.16}$$

Keeping the terms up to the third order in (2.16), we obtain the coefficients C_1, C_2, C_3 and C_4. Eventually, from center manifold theory (see e.g. [5], [48]), the dynamical behavior of the initial system is equivalent to the dynamics of the smooth map $G : I\!R^2 \to I\!R$ of difference equation in (2.14):

$$G(u, \epsilon_0) = \lambda_1 u + \bar{f}(u, h(u, \epsilon_0), \epsilon_0). \tag{2.17}$$

3 Neimark-Sacker Bifurcation of System (1.1)

In this section, we discuss the conditions under which the Neimark-Sacker bifurcation occurs in system (1.1) for small variation of the parameter a_1, as well as we study the direction and the stability of this bifurcation. We define:

$$A_1 = \frac{p_1 + p_2}{2}, \quad A_2 = \sqrt{1 - \left(\frac{p_1 + p_2}{2}\right)^2} \tag{3.1}$$

$$\begin{aligned}
G_{201} &= \frac{\sqrt{4-(p_1+p_2)^2}(p_1-p_2)}{4a_0} + \frac{(p_1-p_2)(p_1^2+p_2^2-2)}{4a_0\sqrt{4-(p_1+p_2)^2}} + \frac{d_1p_1(c_1-p_1)(p_1-p_2)}{2c_1\sqrt{4-(p_1+p_2)^2}} + \\
&\quad \frac{p_1p_2-1}{b_2\sqrt{4-(p_1+p_2)^2}} + \frac{a_2d_2p_2(c_2-p_2)(p_1^2+p_2^2-2)}{2b_2c_2(p_1p_2-1)\sqrt{4-(p_1+p_2)^2}} \\
G_{202} &= \frac{p_1p_2-1}{2a_0} + \frac{d_1p_1(c_1-p_1)}{2c_1} - \frac{a_2d_2p_2(c_2-p_2)(p_1-p_2)}{2b_2c_2(p_1p_2-1)} \\
G_{111} &= \frac{(p_1-p_2)(p_1p_2-1)}{2a_0\sqrt{4-(p_1+p_2)^2}} - \frac{d_1p_1(c_1-p_1)(p_1-p_2)}{2c_1\sqrt{4-(p_1+p_2)^2}} - \frac{p_1p_2-1}{b_2\sqrt{4-(p_1+p_2)^2}} + \\
&\quad \frac{a_2d_2p_2(c_2-p_2)}{b_2c_2\sqrt{4-(p_1+p_2)^2}} \\
G_{112} &= \frac{p_1p_2-1}{2a_0} - \frac{d_1p_1(c_1-p_1)}{2c_1} \\
G_{211} &= -\frac{3d_1^2p_1(c_1-p_1)(2p_1-c_1)}{8c_1^2} + \frac{3a_2^2d_2^2p_2(c_2-p_2)(2p_2-c_2)}{8b_2^2c_2^2(p_1p_2-1)} \\
G_{212} &= -\frac{3(p_1p_2-1)\sqrt{4-(p_1+p_2)^2}}{8a_0^2} - \frac{3(p_1-p_2)^2(p_1p_2-1)}{8a_0^2\sqrt{4-(p_1+p_2)^2}} + \frac{3d_1^2p_1(c_1-p_1)(2p_1-c_1)(p_1-p_2)}{8c_1^2\sqrt{4-(p_1+p_2)^2}} - \\
&\quad \frac{3(p_1p_2-1)}{2b_2^2\sqrt{4-(p_1+p_2)^2}} + \frac{3a_2^2d_2^2p_2(c_2-p_2)(2p_2-c_2)(p_1-p_2)}{8b_2^2c_2^2(p_1p_2-1)\sqrt{4-(p_1+p_2)^2}} \\
G_{021} &= -\frac{(p_1-p_2)\sqrt{4-(p_1+p_2)^2}}{4a_0} + \frac{(p_1-p_2)(p_1^2+p_2^2-2)}{4a_0\sqrt{4-(p_1+p_2)^2}} + \frac{d_1p_1(c_1-p_1)(p_1-p_2)}{2c_1\sqrt{4-(p_1+p_2)^2}} + \\
&\quad \frac{p_1p_2-1}{b_2\sqrt{4-(p_1+p_2)^2}} + \frac{a_2d_2p_2(c_2-p_2)(p_1^2+p_2^2-2)}{2b_2c_2(p_1p_2-1)\sqrt{4-(p_1+p_2)^2}} \\
G_{022} &= \frac{p_1^2+p_2^2-p_1p_2-1}{2a_0} + \frac{d_1p_1(c_1-p_1)}{2c_1} + \frac{a_2d_2p_2(c_2-p_2)(p_1-p_2)}{2b_2c_2(p_1p_2-1)}
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
a(0) = {} & \tfrac{A_1 G_{211}}{2} + \tfrac{A_2 G_{212}}{2} - \tfrac{1}{2((1-A_1)^2 + A_2^2)} \Bigg[\Bigg(A_1^2 (1-A_1)(1-2A_1) - \\
& A_2^2 (1-A_1)(1-2A_1) - 4A_1 A_2^2 (1-A_1) + 2A_1 A_2^2 (1-2A_1) + 2A_1^2 A_2^2 - 2A_2^4 \Bigg) \\
& (G_{201} G_{111} - G_{202} G_{112}) + \Bigg(2A_1 A_2 (1-A_1)(1-2A_1) + 2A_1^2 A_2 (1-A_1) - \\
& 2A_2^3 (1-A_1) - A_1^2 A_2 (1-2A_1) + A_2^3 (1-2A_1) + \\
& 4A_1 A_2^3 \Bigg) (G_{201} G_{112} - G_{202} G_{111}) \Bigg] - \tfrac{G_{111}^2 + G_{112}^2}{2} - \tfrac{G_{021}^2 + G_{022}^2}{4}
\end{aligned}
\tag{3.3}
$$

Proposition 3.1 *Consider system (1.1), where a_1, b_1, b_2, c_2, k_1 are real positive constants, a_2, c_1, k_2 are real negative constants, d_1, d_2 are real constants and $a_1 = a_0 + \epsilon_0$, where ϵ_0 is a small number, is the bifurcation parameter. If*

$$
b_1 = \frac{a_0 a_2}{b_2 (p_1 p_2 - 1)}, \tag{3.4}
$$

where p_1, p_2 are given in (2.2), with

$$
0 < p_1 + p_2 < 2 \tag{3.5}
$$

and $\sigma = \pm 1$, the sign of $a(0)$ given in (3.3) with $a(0) \neq 0$, then, for $\sigma = 1$, the system (1.1) undergoes a subcritical Neimark-Sacker bifurcation near zero fixed point, while for $\sigma = -1$ the system (1.1) undergoes a supercritical Neimark-Sacker bifurcation. For small $\epsilon_0 > 0$ the origin is unstable and for small $|\epsilon_0|$ with $\epsilon_0 < 0$ the origin is asymptotically stable. In case the of the critical value $\epsilon_0 = 0$, if $\sigma = 1$ then the origin is unstable and if $\sigma = -1$ the origin is stable.

Proof. The Jacobian matrix of system (1.1) evaluated at the origin $(x_n, y_n, \epsilon_0) = (0, 0, 0)$ is given in (2.1). As (3.4) and (3.5) hold, the Jacobian matrix J_0 has the two complex conjugate eigenvalues

$$
\lambda_{1,2} = \frac{p_1 + p_2}{2} \pm i \frac{1}{2} \sqrt{-(p_1 - p_2)^2 - 4 \frac{a_0 a_2}{b_1 b_2}}.
$$

As relation (3.4) holds, the modulus of $\lambda_{1,2}$, $|\lambda_{1,2}| = \sqrt{p_1 p_2 - \frac{a_0 a_2}{b_1 b_2}}$ is equal to one, so the zero fixed point is non-hyperbolic. We observe that under the perturbation of the parameter $a_1 = a_0 + \epsilon_0$, for small $\epsilon_0 \neq 0$ the modulus and the argument of the eigenvalues change. So, for small ϵ_0 the eigenvalues can be written in the form

$$\lambda(\epsilon_0) = r(\epsilon_0)e^{i\theta(\epsilon_0)}, \quad \lambda_1 = \lambda(\epsilon_0), \quad \lambda_2 = \overline{\lambda}(\epsilon_0) \tag{3.6}$$

where $r(\epsilon_0) = 1 + \mu(\epsilon_0)$, $\mu(\epsilon_0) = \frac{1-p_1p_2}{a_0}\epsilon_0$ and $\theta(0) = \theta_0$ with

$$\cos\theta_0 = \frac{p_1+p_2}{2}, \quad \sin\theta_0 = \sqrt{1-\left(\frac{p_1+p_2}{2}\right)^2}. \tag{3.7}$$

From (3.5) we obtain $0 < \cos\theta_0 < 1$ and $0 < \sin\theta_0 < 1$, thus, $\theta_0 \in \left(0, \frac{\pi}{2}\right)$. Moreover, since $0 < \theta_0 < \frac{\pi}{2}$, we obtain $e^{ik\theta_0} \neq 1$, for $k = 1, 2, 3, 4$. In contrast, it holds that $e^{ik\theta_0} = 1$ if and only if $\theta_0 \in \left\{0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\right\}$ that is invalid in our case.

In addition, for the modulus $r(\epsilon_0)$ it holds that $r(0) = 1$ and $r'(0) \neq 0$ and from hypothesis it holds that $a(0) \neq 0$. Consequently, the transversality and nodegeneracy conditions (C.1), (C.2) and (C.3) of Theorems 4.5 and 4.6 in [1] (see also [2]):

(C.1) $r'(0) \neq 0$

(C.2) $e^{ik\theta_0} \neq 1$, for $k = 1, 2, 3, 4$

(C.2) $a(0) \neq 0$

are satisfied, hence, there are smooth invertible coordinate and parameter transformations forming the initial system into the complex normal form of Neimark-Sacker bifurcation:

$$\eta \to \lambda(\epsilon_0)\eta + c(\epsilon_0)\eta^2\bar{\eta} + O(|\eta|^4), \quad \eta \in \mathbb{C}, \quad \epsilon_0 \in I\!R \tag{3.8}$$

In this map and by extension in system (1.1), the origin is asymptotically stable if $\mu(\epsilon_0) < 0$ and unstable if $\mu(\epsilon_0) > 0$. Since $\frac{1-p_1p_2}{a_0} > 0$ under the former assumptions of the constant values of the initial system, we obtain that the origin is asymptotically stable if $\epsilon_0 < 0$ and becomes unstable if $\epsilon_0 > 0$. On the critical value $\epsilon_0 = 0$ the map undergoes the Neimark-Sacker bifurcation. For either $\epsilon_0 > 0$ or $\epsilon_0 < 0$, there exists an invariant circle with radius equal to $\sqrt{-\frac{\mu(\epsilon_0)}{a(\epsilon_0)}}$ (see [2]). The stability of the invariant circle is determined by $\sigma = \pm 1$, the sign of $a(0) = Re(e^{-i\theta_0}c(0))$.

In what follows, we will investigate the direction and stability of Neimark-Sacker bifurcation by computing the expression of $a(0)$ given in [1].

System (1.1) can be linearized and written in the form (2.7), where J_0 is the Jacodian matrix of the system evaluated at the origin. As mentioned before, under relations (3.4) and (3.5), the Jacodian matrix J_0 has two complex conjugate eigenvalues given in (3.6) and the corresponding eigenvectors are the conjugate arrays

$$q,\overline{q} = \begin{bmatrix} 1 \\ -\frac{b_1}{a_0}\left(\frac{p_1-p_2}{2}\right) \end{bmatrix} \pm i \begin{bmatrix} 0 \\ \frac{b_1}{a_0}\sqrt{1-\left(\frac{p_1+p_2}{2}\right)^2} \end{bmatrix}.$$

We let the coordinate transformation

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T_1 \begin{bmatrix} u_n \\ v_n \end{bmatrix} \tag{3.9}$$

where

$$T_1 = \begin{bmatrix} 0 & 1 \\ A & B \end{bmatrix},$$

with

$$A = \frac{b_1}{a_0}\sqrt{1-\left(\frac{p_1+p_2}{2}\right)^2}, \quad B = -\frac{b_1}{a_0}\left(\frac{p_1-p_2}{2}\right) \tag{3.10}$$

and the determinant of T_1 is nonzero. Thus, system (1.1) is transformed in the form

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + T_1^{-1} \begin{bmatrix} f(x_n(u_n,v_n), y_n(u_n,v_n), \epsilon_0) \\ g(x_n(u_n,v_n), y_n(u_n,v_n), \epsilon_0) \end{bmatrix},$$

where

$$f(x_n(u_n,v_n), y_n(u_n,v_n), \epsilon_0) = (a_0+\epsilon_0)\frac{Au_n+Bv_n}{b_1+Au_n+Bv_n} - \frac{a_0}{b_1}(Au_n+Bv_n) + c_1\frac{v_n e^{k_1-d_1 v_n}}{1+e^{k_1-d_1 v_n}} - \frac{c_1 e^{k_1}}{1+e^{k_1}}v_n,$$

$$g(x_n(u_n,v_n), y_n(u_n,v_n), \epsilon_0) = a_2\frac{v_n}{b_2+v_n} - \frac{a_2}{b_2}v_n + c_2\frac{(Au_n+Bv_n)e^{k_2-d_2(Au_n+Bv_n)}}{1+e^{k_2-d_2(Au_n+Bv_n)}} - \frac{c_2 e^{k_2}}{1+e^{k_2}}(Au_n+Bv_n)$$

and A_1, A_2 are the real and the imaginary part of the eigenvalue $\lambda(0) = e^{i\theta_0}$ respectively, given in (3.1). Equivalently, the system can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n,v_n,\epsilon_0) \\ \hat{g}(u_n,v_n,\epsilon_0) \end{bmatrix}, \tag{3.11}$$

where

$$\hat{f}(u_n, v_n, \epsilon_0) = -\tfrac{B}{A} f(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0) + \tfrac{1}{A} g(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0),$$
$$\hat{g}(u_n, v_n, \epsilon_0) = g(x_n(u_n, v_n), y_n(u_n, v_n), \epsilon_0).$$

Now, we apply the following linear complex transformation setting $z_n = u_n + iv_n$, or equivalently

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T_2 \begin{bmatrix} z_n \\ \bar{z}_n \end{bmatrix},$$

where

$$T_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

Consequently, the system (3.11) obtains the form

$$\begin{bmatrix} z_{n+1} \\ \bar{z}_{n+1} \end{bmatrix} = \begin{bmatrix} A_1 + iA_2 & 0 \\ 0 & A_1 - iA_2 \end{bmatrix} \begin{bmatrix} z_n \\ \bar{z}_n \end{bmatrix} + T_2^{-1} \begin{bmatrix} \hat{f}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0) \\ \hat{g}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0) \end{bmatrix},$$

which can be written for small ϵ_0 as follows

$$\begin{bmatrix} z_{n+1} \\ \bar{z}_{n+1} \end{bmatrix} = \begin{bmatrix} \lambda(\epsilon_0) & 0 \\ 0 & \bar{\lambda}(\epsilon_0) \end{bmatrix} \begin{bmatrix} z_n \\ \bar{z}_n \end{bmatrix} + \begin{bmatrix} F(z_n, \bar{z}_n, \epsilon_0) \\ G(z_n, \bar{z}_n, \epsilon_0) \end{bmatrix}, \tag{3.12}$$

where

$$F(z_n, \bar{z}_n, \epsilon_0) = \hat{f}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0) + i\hat{g}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0),$$
$$G(z_n, \bar{z}_n, \epsilon_0) = \hat{f}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0) - i\hat{g}(u_n(z_n, \bar{z}_n), v_n(z_n, \bar{z}_n), \epsilon_0).$$

Since the second component of (3.12) is simply the complex conjugate of the first component, all we really need to study is the difference equation

$$z_{n+1} = \lambda(\epsilon_0) z_n + F(z_n, \bar{z}_n, \epsilon_0)$$

where $F(z, \bar{z}, \epsilon_0)$ is a complex-valued smooth function of z, $\bar{z}$ and ϵ_0, whose Taylor expansion with respect to $(z, \bar{z})$ contains quadratic and higher order terms

$$F(z, \bar{z}, \epsilon_0) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\epsilon_0) z^k \bar{z}^l$$

with $k, l = 0, 1, \ldots$

According to Kuznetsov in [1] the coefficient $a(0)$, which determines the direction and the stability of the invariant circle in a generic system exhibiting the Neimark-Sacker bifurcation, can be computed by the following relation

$$a(0) = Re\left[\frac{e^{-i\theta_0}}{2} g_{21}\right] - Re\left[\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20}g_{11}\right] - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

where

$$g_{21} = \frac{\partial^3 F(0,0,\epsilon_0)}{\partial z^2 \partial \bar{z}}\bigg|_{\epsilon_0=0} = G_{211} + iG_{212}$$

$$g_{20} = \frac{\partial^2 F(0,0,\epsilon_0)}{\partial z^2}\bigg|_{\epsilon_0=0} = G_{201} + iG_{202}$$

$$g_{11} = \frac{\partial^2 F(0,0,\epsilon_0)}{\partial z \partial \bar{z}}\bigg|_{\epsilon_0=0} = G_{111} + iG_{112}$$

$$g_{02} = \frac{\partial^2 F(0,0,\epsilon_0)}{\partial \bar{z}^2}\bigg|_{\epsilon_0=0} = G_{021} + iG_{022}.$$

The real and imaginary parts G_{klj} of g_{kl}, $k,l = 0,1,2,\ \ j = 1,2$ are given in (3.2) and we conclude that $a(0)$ is given from the expression in (3.3).

According to normal form Neimark-Sacker bifurcation we conclude that for $\sigma = 1$ and $\epsilon_0 < 0$, where $\sigma = \pm 1$ is the sign of $a(0)$, the system undergoes a subcritical Neimark-Sacker bifurcation, in which the stable zero fixed point is surrounded by a unique and unstable invariant circle that disappears as ϵ_0 crosses zero from negative to positive values, while for $\sigma = -1$ and $\epsilon_0 > 0$, the system undergoes a supercritical Neimark-Sacker bifurcation, in which the unstable zero fixed point is surrounded by a unique and stable invariant circle. In the case of $\epsilon_0 = 0$, the origin is stable if $\sigma = -1$ and unstable if $\sigma = 1$.

Example 1. Suppose that $a_0 = 3.6$, $b_2 = 4.1$, $d_1 = -2.7$, $d_2 = 3.9$, $c_1 = -1.6$, $k_1 = 2.3$, $k_2 = -2.6$, $c_2 = 23$, thus, from (2.2), we obtain $p_1 \cong -1.45$ and $p_2 \cong 1.59$. In addition, from (3.4) we take $b_1 \cong 1$ and from (2.6) $a_2 < -2.66$, let $a_2 = -3.8$.

From the above values and (3.3) we obtain $a(0) \cong -1.43$, consequently, the system (1.1) undergoes supercritical Neimark-Sacker bifurcation with an invariant stable circle arise for $\epsilon_0 > 0$, while the zero fixed point is asymptotically stable for $\epsilon_0 < 0$ and unstable for $\epsilon_0 > 0$.

4 Flip Bifurcation of System (1.1)

In this section we present some sufficient conditions for the existence of flip bifurcation in system (1.1), under variation of the parameter a_1, in a neighborhood of the zero fixed point.

We define:

$$
\begin{aligned}
&C_1 = \frac{(p_1+1)^2}{a_0(p_1+p_2+2)^2},\\
&C_2 = \frac{p_1+1}{(p_1+p_2)(p_1+p_2+2)}\left(\frac{(p_1+1)^2}{a_0} + \frac{p_2+1}{b_2} + \frac{d_1p_1(c_1-p_1)}{c_1} + \frac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2+1)}\right),\\
&C_3 = \frac{1}{(p_1+p_2+2)(p_1+p_2)}\Bigg((p_1+1)\left(\frac{(p_1+1)^2}{a_0^2} - \frac{2(p_1+1)(p_2+1)C_1}{a_0} - \frac{(p_2+1)C_2}{a_0} + \frac{2d_1p_1(c_1-p_1)C_1}{c_1}\right)+\\
&2\left(\frac{(p_1+1)(p_2+1)C_1}{b_2} + \frac{(p_1+1)(p_2+1)C_2}{a_0} - \frac{a_2d_2p_2(c_2-p_2)C_1}{b_2c_2}\right) + C_1\Bigg((p_1+1)\left(\frac{p_2+1}{b_2} + \frac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2+1)}\right)-\\
&(p_2+1)\left(\frac{(p_1+1)^2}{a_0} + \frac{d_1p_1(c_1-p_1)}{c_1}\right)\Bigg)\Bigg),\\
&C_4 = \frac{1}{(p_1+p_2+2)^2}\Bigg((p_1+1)\left(\frac{(p_1+1)^3}{a_0^2} - \frac{2(p_1+1)(p_2+1)C_2}{a_0} + \frac{2d_1p_1(c_1-p_1)C_2}{c_1} - \frac{d_1^2p_1(c_1-p_1)(c_1-2p_1)}{2c_1^2}\right)-\\
&2C_2\Bigg((p_1+1)\left(\frac{p_2+1}{b_2} + \frac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2+1)}\right) - (p_2+1)\left(\frac{(p_1+1)^2}{a_0} + \frac{d_1p_1(c_1-p_1)}{c_1}\right)\Bigg)-\\
&\frac{(p_1+1)(p_2+1)}{b_2^2} + \frac{2(p_1+1)(p_2+1)C_2}{b_2} - \frac{2a_2d_2p_2(c_2-p_2)C_2}{b_2c_2} + \frac{a_2^2d_2^2p_2(p_1+1)(c_2-p_2)(c_2-2p_2)}{2b_2^2c_2^2(p_2+1)^2}\Bigg),
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
&A_1 = -\frac{(p_1+1)(p_2+1)}{a_0(p_1+p_2+2)},\\
&A_2 = 2\left(\frac{p_1+1}{p_1+p_2+2}\left(\frac{p_2+1}{b_2} + \frac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2+1)}\right) - \frac{p_2+1}{p_1+p_2+2}\left(\frac{(p_1+1)^2}{a_0} + \frac{d_1p_1(c_1-p_1)}{c_1}\right)\right),\\
&A_3 = 6\Bigg(\frac{p_1+1}{p_1+p_2+2}\left(-\frac{p_2+1}{b_2^2} + \frac{2(p_2+1)C_2}{b_2} - \frac{2a_2d_2p_2(c_2-p_2)C_2}{b_2c_2(p_1+1)} + \frac{a_2^2d_2^2p_2(c_2-p_2)(c_2-2p_2)}{2b_2^2c_2^2(p_2+1)^2}\right)-\\
&\frac{p_2+1}{p_1+p_2+2}\left(\frac{(p_1+1)^3}{a_0^2} - \frac{2(p_1+1)(p_2+1)C_2}{a_0} + \frac{2d_1p_1(c_1-p_1)C_2}{c_1} - \frac{d_1^2p_1(c_1-p_1)(c_1-2p_1)}{2c_1^2}\right)\Bigg).
\end{aligned}
\tag{4.2}
$$

Proposition 4.1 *Consider system (1.1), where a_1, b_1, b_2, c_2, k_1 are real positive constants, a_2, c_1, k_2 are real negative constants, d_1, d_2 are real constants and $a_1 = a_0 + \epsilon_0$, where ϵ_0 is a small number, is the bifurcation parameter. If*

$$b_1 = \frac{a_0a_2}{b_2(p_1+1)(p_2+1)}, \tag{4.3}$$

where p_1, p_2 are given in (2.2), with

$$-2 < p_1 < -1, \quad 0 < p_2 < 1, \tag{4.4}$$

$$A_1 \neq 0, \quad \frac{1}{2}A_2^2 + \frac{1}{3}A_3 \neq 0 \tag{4.5}$$

where A_1, A_2, A_3 are given in (4.2), and $\sigma = \pm 1$, the sign of $\frac{1}{2}A_2^2 + \frac{1}{3}A_3$, then, for $\sigma = 1$, the system (1.1) undergoes a supercritical flip bifurcation near zero fixed point and a stable period-two cycle exists for small $|\epsilon_0|$ with $\epsilon_0 < 0$ and disappears as ϵ_0 approaches zero, while for $\sigma = -1$ the system (1.1) undergoes a subcritical flip bifurcation near the origin and an unstable cycle of period two appears for $\epsilon_0 > 0$ and disappears at $\epsilon_0 = 0$.

Proof. The Jacobian matrix of system (1.1) evaluated at the origin $(x_n, y_n, \epsilon_0) = (0, 0, 0)$ is given in (2.1). As (4.3) and (4.4) hold, the Jacobian matrix J_0 has the two distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p_1 + p_2 + 1$, where p_1, p_2 are given in (2.2). Moreover, from (4.4) we obtain that $|\lambda_2| < 1$, otherwise the zero equilibrium would be unstable.

The system (1.1) can be written in the form of (2.7). We, now, apply the coordinate transformation given in (2.8), where T is the matrix that diagonalizes J_0, corresponding to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p_1 + p_2 + 1$ of J_0, defined in (2.9), where:

$$A = -\frac{b_1}{a_0}(p_1 + 1), \quad B = \frac{b_1}{a_0}(p_2 + 1).$$

with the determinant

$$D = \frac{b_1(p_1 + p_2 + 2)}{a_0} \neq 0.$$

Applying the coordinate transformation, the system (1.1) can be written in the form of (2.11). To apply the center manifold theorem depending on the parameter ϵ_0 we increase the number of equations by writing the system (2.11) in the form of (2.12).

As $|\lambda_1| = 1$, $|\lambda_2| \neq 1$ and $\bar{f}$, $\bar{g}$ are C^2 functions with $\bar{f}(0,0,0) = 0$, $\bar{g}(0,0,0) = 0$ and $D\bar{f}(0,0,0) = 0_3$, $D\bar{g}(0,0,0) = 0_3$, where 0_3 the 3-dimensional zero vector, according to Theorem 5.1 in [5], the system (2.12) can be represented locally by the center manifold M_c with the form of (2.13). Consequently, according to Theorem 5.1 in [5], the dynamical behavior of system (2.12) at the origin, reduces to the study of the dynamics of (2.14).

We suppose that $h(u_n, \epsilon_0)$ has the form of (2.15). We can determine the coefficients of $h(u, \epsilon_0)$ applying the Taylor expansion to the center manifold equation given in (2.16).

Keeping the terms up to the third order, we obtain the coefficients C_1, C_2, C_3 and C_4 given in (4.1). From center manifold theory (see e.g. [5], [48]), the dynamical behavior of the initial system is equivalent to the dynamics of the smooth map $G : I\!R^2 \to I\!R$ given in (2.17).

The map G can be written in a neighborhood of $(u, \epsilon_0) = (0, 0)$ as $F : I\!R^2 \to I\!R$:

$$\begin{aligned}
&F(u,\epsilon_0) = -u - \tfrac{(p_1+1)(p_2+1)}{a_0(p_1+p_2+2)}\epsilon_0 u + \\
&\Bigg(\tfrac{p_1+1}{p_1+p_2+2}\Big(\tfrac{p_2+1}{b_2} + \tfrac{a_2 d_2 p_2 (c_2-p_2)}{b_2 c_2 (p_2+1)}\Big) - \tfrac{p_2+1}{p_1+p_2+2}\Big(\tfrac{(p_1+1)^2}{a_0} + \tfrac{d_1 p_1 (c_1-p_1)}{c_1}\Big)\Bigg) u^2 + \\
&\Bigg(-\tfrac{p_2+1}{p_1+p_2+2}\Big(\tfrac{(p_1+1)^2}{a_0^2} - \tfrac{2(p_1+1)(p_2+1)C_1}{a_0} - \tfrac{(p_2+1)C_2}{a_0} + \tfrac{2 d_1 p_1 (c_1-p_1) C_1}{c_1}\Big) + \\
&\tfrac{2}{p_1+p_2+2}\Big(\tfrac{(p_1+1)(p_2+1)C_1}{b_2} - \tfrac{a_2 d_2 p_2 (c_2-p_2) C_1}{b_2 c_2}\Big)\Bigg)\epsilon_0 u^2 + \\
&\Bigg(\tfrac{p_1+1}{p_1+p_2+2}\Big(-\tfrac{p_2+1}{b_2^2} + \tfrac{2(p_2+1)C_2}{b_2} - \tfrac{2a_2 d_2 p_2 (c_2-p_2) C_2}{b_2 c_2 (p_1+1)} + \tfrac{a_2^2 d_2^2 p_2 (c_2-p_2)(c_2-2p_2)}{2 b_2^2 c_2^2 (p_2+1)^2}\Big) - \\
&\tfrac{p_2+1}{p_1+p_2+2}\Big(\tfrac{(p_1+1)^3}{a_0^2} - \tfrac{2(p_1+1)(p_2+1)C_2}{a_0} + \tfrac{2 d_1 p_1 (c_1-p_1) C_2}{c_1} - \tfrac{d_1^2 p_1 (c_1-p_1)(c_1-2p_1)}{2 c_1^2}\Big)\Bigg) u^3 + \\
&O(\epsilon_0^2) + O((u+\epsilon_0)^4).
\end{aligned}$$

We can easily verify that $F_u(0,0) = -1$, $F_{u\epsilon_0}(0,0) = A_1$, $F_{uu}(0,0) = A_2$ and $F_{uuu}(0,0) = A_3$, where A_1, A_2, A_3 are given in (4.2). Hence, the non-degeneracy conditions (B.1) and (B.2) of Theorem 4.3 in [1]:

(B.1) $\frac{1}{2}(F_{uu}(0,0))^2 + \frac{1}{3}F_{uuu}(0,0) \neq 0$

(B.2) $F_{u\epsilon_0}(0,0) \neq 0$

are satisfied from (4.5), thus, there are smooth invertible coordinate and parameter changes transforming the map F into:

$$\eta \to -(1 + \mu(\epsilon_0))\eta + \sigma\eta^3$$

where $\sigma = \pm 1$ the sign of (B.1) and

$$\mu(\epsilon_0) = \frac{(p_1+1)(p_2+1)}{a_0(p_1+p_2+2)}\epsilon_0.$$

According to normal form bifurcation analysis (see [1], [2]), the map η undergoes flip bifurcation depending on the sign of $\mu(\epsilon_0)$.

Since (4.4) holds, we obtain that $-2 < p_1 < -1$ and $0 < p_2 < 1$. Moreover since $a_1 > 0$, we have that $a_0 > 0$ for small $|\epsilon_0|$. Thus, $\frac{(p_1+1)(p_2+1)}{a_0(p_1+p_2+2)} < 0$ and the sign of $\mu(\epsilon_0)$ depends on the sign of ϵ_0.

Consequently, for $\sigma = 1$, the zero fixed point is unstable for small $|\epsilon_0|$ with $\epsilon_0 < 0$ and stable for $\epsilon_0 > 0$. At $\epsilon_0 = 0$, the equilibrium is non-hyperbolic, but is however stable. In addition, a period-two cycle arise which is stable for $\epsilon_0 < 0$ and disappears as ϵ_0 approaches zero. This is the case of supercritical flip bifurcation.

On the other hand, for $\sigma = -1$, the zero fixed point has the same stability as in the previous case, but at the critical value $\epsilon_0 = 0$ the equilibrium is unstable, thus, a subcritical flip bifurcation occurs and an unstable cycle of period two arise for $\epsilon_0 > 0$ which disappears at $\epsilon_0 = 0$.

Example 2. Suppose that $a_0 = 1.3$, $b_2 = 0.5$, $k_1 = 1.7$, $k_2 = -0.7$, $d_1 = 2.6$, $d_2 = 1.3$. From (4.4) $-2.37 < c_1 < -1.18$, let $c_1 = -1.9$ and $0 < c_2 < 3.01$, let $c_2 = 2.2$, thus, from (2.2), we obtain $p_1 \cong -1.6$ and $p_2 \cong 0.7$. In addition, from (4.3) we take $b_1 \cong 3.96$ and from (2.5) $-2.08 < a_2 < 0$, let $a_2 = -1.6$.

From the above values we obtain $D \neq 0$, $|\lambda_2| < 1$, from (4.1) we have $C_1 \cong 0.22$, $C_2 \cong 1.19$, $C_3 \cong 1.36$, $C_4 \cong 4.55$ and from (4.2) we have $A_1 \cong 0.72 \neq 0$, $A_2 \cong -1.35$, $A_3 \cong 23.02$. Consequently, $\sigma = 1$, the sign of $\frac{1}{2}A_2^2 + \frac{1}{3}A_3 > 0$, and system (1.1) undergoes supercritical flip bifurcation. The stability of the fixed point $\eta^* = 0^*$ and the periodic cycle are shown in Figure 1.

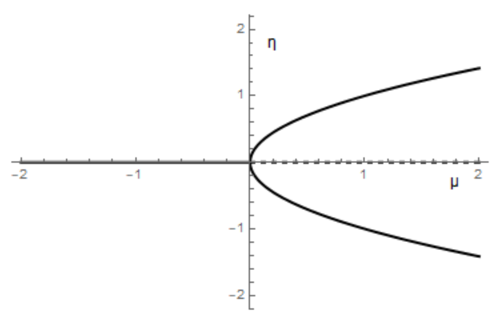

Figure 1: Bifurcation diagram in (μ, η)-plane. For the $\mu < 0$ plane we have $\epsilon_0 > 0$ and for $\mu > 0$ we have $\epsilon_0 < 0$. The horizontal axis corresponds to the zero fixed point while the parabola represents the stable cycle of period two. The solid line depicts stability, while the dashed line depicts instability.

5 Transcritical Bifurcation of System (1.1)

In this section we present some sufficient conditions for the existence of transcritical bifurcation in system (1.1), under variation of the parameter a_1, in a neighborhood of the zero fixed point.

We define:

$$
\begin{aligned}
&C_1 = \tfrac{(p_1-1)^2}{a_0(p_1+p_2-2)^2},\\
&C_2 = \tfrac{p_1-1}{(p_1+p_2-2)^2}\Big(\tfrac{(p_1-1)^2}{a_0} + \tfrac{p_2-1}{b_2} + \tfrac{d_1p_1(c_1-p_1)}{c_1} + \tfrac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2-1)}\Big),\\
&C_3 = \tfrac{1}{(p_1+p_2-2)^2}\Bigg((p_1-1)\Big(\tfrac{(p_1-1)^2}{a_0^2} - \tfrac{2(p_1-1)(p_2-1)C_1}{a_0} - \tfrac{(p_2-1)C_2}{a_0} + \tfrac{2d_1p_1(c_1-p_1)C_1}{c_1}\Big) +\\
&2\Big(\tfrac{(p_1-1)(p_2-1)C_1}{b_2} - \tfrac{(p_1-1)(p_2-1)C_2}{a_0} - \tfrac{a_2d_2p_2(c_2-p_2)C_1}{b_2c_2}\Big) + C_1\Big((p_1-1)\Big(\tfrac{p_2-1}{b_2} + \tfrac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2-1)}\Big) -\\
&(p_2-1)\Big(\tfrac{(p_1-1)^2}{a_0} + \tfrac{d_1p_1(c_1-p_1)}{c_1}\Big)\Big)\Bigg),\\
&C_4 = \tfrac{1}{(p_1+p_2-2)^2}\Bigg((p_1-1)\Big(\tfrac{(p_1-1)^3}{a_0^2} - \tfrac{2(p_1-1)(p_2-1)C_2}{a_0} + \tfrac{2d_1p_1(c_1-p_1)C_2}{c_1} - \tfrac{d_1^2p_1(c_1-p_1)(c_1-2p_1)}{2c_1^2}\Big) +\\
&2C_2\Big((p_1-1)\Big(\tfrac{p_2-1}{b_2} + \tfrac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2-1)}\Big) - (p_2-1)\Big(\tfrac{(p_1-1)^2}{a_0} + \tfrac{d_1p_1(c_1-p_1)}{c_1}\Big)\Big) -\\
&\tfrac{(p_1-1)(p_2-1)}{b_2^2} + \tfrac{2(p_1-1)(p_2-1)C_2}{b_2} - \tfrac{2a_2d_2p_2(c_2-p_2)C_2}{b_2c_2} + \tfrac{a_2^2d_2^2p_2(p_1-1)(c_2-p_2)(c_2-2p_2)}{2b_2^2c_2^2(p_2-1)^2}\Bigg),
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&A_1 = -\tfrac{(p_1-1)(p_2-1)}{a_0(p_1+p_2-2)},\\
&A_2 = 2\Bigg(\tfrac{p_1-1}{p_1+p_2-2}\Big(\tfrac{p_2-1}{b_2} + \tfrac{a_2d_2p_2(c_2-p_2)}{b_2c_2(p_2-1)}\Big) - \tfrac{p_2-1}{p_1+p_2-2}\Big(\tfrac{(p_1-1)^2}{a_0} + \tfrac{d_1p_1(c_1-p_1)}{c_1}\Big)\Bigg).
\end{aligned}
\tag{5.2}
$$

Proposition 5.1 *Consider system (1.1), where a_1, b_1, b_2, c_2, k_1 are real positive constants, a_2, c_1, k_2 are real negative constants, d_1, d_2 are real constants and $a_1 = a_0 + \epsilon_0$, where ϵ_0 is a small number, is the bifurcation parameter. If*

$$b_1 = \frac{a_0a_2}{b_2(p_1-1)(p_2-1)}, \tag{5.3}$$

where p_1, p_2 are given in (2.2), with

$$-1 < p_1 < 0, \quad 1 < p_2 < 2, \tag{5.4}$$

$$A_1 \neq 0, \quad A_2 \neq 0 \tag{5.5}$$

where A_1, A_2 are given in (5.2), and $\sigma = \pm 1$, the sign of $-\frac{A_2}{A_1}$, then, for $\sigma = 1$, the system (1.1) undergoes a subcritical transcritical bifurcation near zero fixed point while for $\sigma = -1$ the system (1.1) undergoes a supercritical transcritical bifurcation. Moreover, for small $\epsilon_0 > 0$ the origin is asymptotically stable and for small $|\epsilon_0|$ with $\epsilon_0 < 0$ the origin is unstable. In case the of $\epsilon_0 = 0$, the origin is non-hyperbolic, but nevertheless unstable.

Proof. The Jacobian matrix of system (1.1) evaluated at the origin $(x_n, y_n, \epsilon_0) = (0, 0, 0)$ is given in (2.1). As (5.3) and (5.4) hold, the Jacobian matrix J_0 has the two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = p_1 + p_2 - 1$, where p_1, p_2 are given in (2.2). Moreover, from (5.4) we obtain that $|\lambda_2| < 1$, otherwise the zero equilibrium would be unstable.

The system (1.1) can be written in the form of (2.7). We, now, apply the coordinate transformation given in (2.8), where T is the matrix that diagonalizes J_0, corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = p_1 + p_2 - 1$ of J_0, defined in (2.9), where:

$$A = -\frac{b_1}{a_0}(p_1 - 1), \quad B = \frac{b_1}{a_0}(p_2 - 1).$$

with the determinant

$$D = \frac{b_1(p_1 + p_2 - 2)}{a_0} \neq 0.$$

Applying center manifold reduction theory in an analogous way as in the case of flip bifurcation in Section 4, we obtain the coefficients C_1, C_2, C_3 and C_4 of $h(u, \epsilon_0)$ in (2.15) given in (5.1). Hence, from center manifold theory (see e.g. [5], [48]), the dynamical behavior of the initial system is equivalent to the dynamics of the smooth map $G : I\!R^2 \to I\!R$ given in (2.17).

The map G can be written in a neighborhood of $(u, \epsilon_0) = (0, 0)$ as

$F : I\!R^2 \to I\!R$:

$$
\begin{aligned}
&F(u,\epsilon_0) = u - \tfrac{(p_1-1)(p_2-1)}{a_0(p_1+p_2-2)}\epsilon_0 u + \\
&\left(\tfrac{p_1-1}{p_1+p_2-2}\left(\tfrac{p_2-1}{b_2} + \tfrac{a_2 d_2 p_2 (c_2-p_2)}{b_2 c_2 (p_2-1)}\right) - \tfrac{p_2-1}{p_1+p_2-2}\left(\tfrac{(p_1-1)^2}{a_0} + \tfrac{d_1 p_1 (c_1-p_1)}{c_1}\right)\right) u^2 + \\
&\left(-\tfrac{p_2-1}{p_1+p_2-2}\left(\tfrac{(p_1-1)^2}{a_0^2} - \tfrac{2(p_1-1)(p_2-1)C_1}{a_0} - \tfrac{(p_2-1)C_2}{a_0} + \tfrac{2d_1 p_1 (c_1-p_1) C_1}{c_1}\right) + \right. \\
&\left. \tfrac{2}{p_1+p_2-2}\left(\tfrac{(p_1-1)(p_2-1)C_1}{b_2} - \tfrac{a_2 d_2 p_2 (c_2-p_2) C_1}{b_2 c_2}\right)\right)\epsilon_0 u^2 + \\
&\left(\tfrac{p_1-1}{p_1+p_2-2}\left(-\tfrac{p_2-1}{b_2^2} + \tfrac{2(p_2-1)C_2}{b_2} - \tfrac{2a_2 d_2 p_2 (c_2-p_2) C_2}{b_2 c_2 (p_1-1)} + \tfrac{a_2^2 d_2^2 p_2 (c_2-p_2)(c_2-2p_2)}{2b_2^2 c_2^2 (p_2-1)^2}\right) - \right. \\
&\left. \tfrac{p_2-1}{p_1+p_2-2}\left(\tfrac{(p_1-1)^3}{a_0^2} - \tfrac{2(p_1-1)(p_2-1)C_2}{a_0} + \tfrac{2d_1 p_1 (c_1-p_1) C_2}{c_1} - \tfrac{d_1^2 p_1 (c_1-p_1)(c_1-2p_1)}{2c_1^2}\right)\right) u^3 + \\
&O(\epsilon_0^2) + O((u+\epsilon_0)^4).
\end{aligned}
$$

We can easily verify that $F_u(0,0) = 1$, $F_{\epsilon_0}(0,0) = 0$, $F_{u\epsilon_0}(0,0) = A_1$ and $F_{uu}(0,0) = A_2$, where A_1, A_2 are given in (5.2). Hence, the non-degeneracy and transversality conditions (A.1) and (A.2) (see e.g. [2]):

(A.1) $F_{\epsilon_0}(0,0) = 0$

(A.2) $F_{u\epsilon_0}(0,0) \neq 0$ and $F_{uu}(0,0) \neq 0$

are satisfied from (5.5), thus, there are smooth invertible coordinate and parameter changes transforming the map F into:

$$\eta \to (1+\mu(\epsilon_0))\eta + \sigma\eta^2$$

where $\sigma = \pm 1$ the sign of $-F_{uu}(0,0)/F_{u\epsilon_0}(0,0)$ and

$$\mu(\epsilon_0) = -\frac{(p_1-1)(p_2-1)}{a_0(p_1+p_2-2)}\epsilon_0.$$

According to normal form bifurcation analysis (see [1], [2]), the map η undergoes transcritical bifurcation depending on the sign of $\mu(\epsilon_0)$.

Since (5.4) holds, we obtain that $-1 < p_1 < 0$ and $1 < p_2 < 2$. Moreover since $a_1 > 0$, we have that $a_0 > 0$ for small $|\epsilon_0|$. Thus, $-\frac{(p_1-1)(p_2-1)}{a_0(p_1+p_2-2)} < 0$ and the sign of $\mu(\epsilon_0)$ depends on the sign of ϵ_0.

In transcritical bifurcation there are always two equilibria which collide and their stability is exchanged depending in the parameter ϵ_0. One of those two equilibria is always the origin. For $\epsilon_0 > 0$, the origin is asymptotically stable and the second equilibrium is unstable. For $\epsilon_0 < 0$, the origin becomes unstable by transferring its stability to the other equilibrium. At the bifurcation value $\epsilon_0 = 0$, the two fixed points coalesce at the origin, which is non-hyperbolic, but nevertheless unstable.

At last, the two equilibria have the same stability for $\sigma = 1$ and $\sigma = -1$. Although, for $\sigma = 1$, the system undergoes a subcritical transcritical bifurcation and the origin is semi-stable from the left and for $\sigma = -1$, the system undergoes a supercritical transcritical bifurcation and the origin is semi-stable from the right.

Example 3. Suppose that $a_0 = 1.3$, $b_2 = 1.8$, $k_1 = 1.9$, $k_2 = -0.6$, $d_1 = 1.1$, $d_2 = 2.9$. From (5.4) $-1.15 < c_1 < 0$, let $c_1 = -0.8$ and $2.82 < c_2 < 5.64$, let $c_2 = 3.2$, thus, from (2.2), we obtain $p_1 \cong -0.7$ and $p_2 \cong 1.13$. In addition, from (5.3) we take $b_1 \cong 4.77$ and from (2.5) $-5.53 < a_2 < 0$, let $a_2 = -1.5$.

From the above values we obtain $D \neq 0$, $|\lambda_2| < 1$, from (5.1) we have $C_1 \cong 0.91$, $C_2 \cong 7.66$, $C_3 \cong 9.77$, $C_4 \cong 125.65$ and from (5.2) we have $A_1 \cong -0.11 \neq 0$, $A_2 \cong -28.17 \neq 0$. Consequently, $\sigma = -1$, the sign of $-A_2/A_1$, and system (1.1) undergoes supercritical transcritical bifurcation. The stability of the two fixed points $\eta^* = 0^*$ and $\eta^* = \mu^*$ is shown in Figure 2.

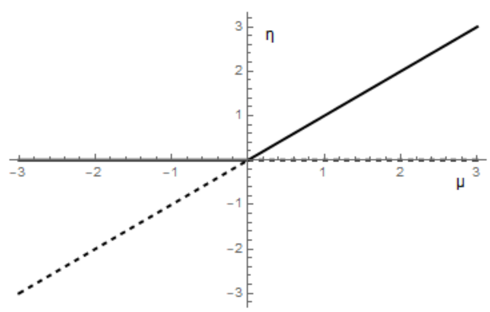

Figure 2: Bifurcation diagram in (μ, η)-plane. For the $\mu < 0$ plane we have $\epsilon_0 > 0$ and for $\mu > 0$ we have $\epsilon_0 < 0$. The solid line depicts that the fixed point is stable, while the dashed line depicts that the fixed point is unstable.

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