Stability and flip bifurcation of a three dimensional exponential system of difference equations

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Abstract

In this paper, we study the stability of the zero equilibrium and the occurrence of flip bifurcation on the following system of difference equations:

$$x_{n+1} = a_1 \frac{y_n}{b_1 + y_n} + c_1 \frac{x_n e^{k_1 - d_1 x_n}}{1 + e^{k_1 - d_1 x_n}},$$

$$y_{n+1} = a_2 \frac{z_n}{b_2 + z_n} + c_2 \frac{y_n e^{k_2 - d_2 y_n}}{1 + e^{k_2 - d_2 y_n}},$$

$$z_{n+1} = a_3 \frac{x_n}{b_3 + x_n} + c_3 \frac{z_n e^{k_3 - d_3 z_n}}{1 + e^{k_3 - d_3 z_n}}$$

where a_i, b_i, c_i, d_i, k_i, for $i = 1, 2, 3$, are real constants and the initial values x_0, y_0 and z_0 are real numbers. We study the stability of this system in the special case when one of the eigenvalues is equal to -1 and the remaining eigenvalues have absolute value less than 1, using center manifold theory.

Keywords: Difference equations, stability, flip bifurcation, center manifold, dynamical systems, discrete dynamics.

1 Introduction

Difference equations and dynamical systems appear in a plethora of sciences and applied fields, as for example, in biological, economic and social sciences, celestial mechanics, fluid dynamics, nonlinear oscillations and more. In recent years, an attractive and far-reaching theory over systems of difference equations has emerged, as a result of the efforts of mathematicians and scientists from many disciplines.

Studying cyclic systems of difference equations, which are natural generalizations of symmetric ones (see, e.g., [1, 2]), seems to have been initiated by Iričanin and Stević in [3]. Natural generalizations of cyclic systems are the, so called, close–to–cyclic systems of difference equations (see, e.g., [4, 5, 6, 7, 8]). Close–to–cyclic systems are also natural generalizations of close–to–symmetric ones, which have attracted some considerable attention recently (see, e.g., [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]).

An interesting dynamical behavior is featured in difference equations systems whose the characteristic polynomial of their linearization has zeros belonging on the unit circle (e.g. [4, 5, 20, 35, 36, 37, 38]), a case for which Carr [39] and Kuznetsov [40] gave some classical results. A basic biological model of this category can be found in Berg and Stević [10] and in Stević [38], where the method of the last paper can be modified and applied to some other equations and systems, as for example, was conducted in [24, 29, 30, 31, 32, 33, 34, 41, 42, 43, 44]. In addition, Stević in [37] studied the asymptotic behavior of the solutions of a non-linear difference equation of a biological model with unity as a characteristic zero, in the very interesting case when the sum of the coefficients is equal to one. Moreover, some results and methods for investigating the existence of specific types of solutions are derived, for example, in [45, 46, 47, 48, 49].

Numerous applications in biology have the difference equations and the systems of difference equations with exponential terms, so a large number of papers dealing with related equations and systems have been published (see, for example, [12, 20, 36, 37, 50, 51, 52, 53]). Tilman and Wedin in [54] discuss an ecological model of grassland ecosystem incorporating plant inhibition by litter, given as:

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \quad L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}$$

where B is the living biomass, L the litter mass, N the total soil nitrogen, t the time in years and constants $a, b, c, d > 0$ and $0 < k < 1$. In this model, litter decay is determined by d and litter production is k times the living biomass. Motivated be this model, Papaschinopoulos *et al* in [16] studied the boundedness and the persistence of the positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium, as well as the existance of periodic solutions of the equation:

$$x_{n+1} = a\frac{x_n^2}{b+x_n} + c\frac{e^{k-dx_n}}{1+e^{k-dx_n}}$$

where $a \in (0,1)$, a, b, c, d, k are positive constants and x_0 is a positive real number. Moreover, in [20] the authors study the stability of zero equilibrium of the system:

$$x_{n+1} = a_1\frac{y_n}{b_1+y_n} + c_1\frac{x_n e^{k_1-d_1x_n}}{1+e^{k_1-d_1x_n}}, \quad y_{n+1} = a_2\frac{x_n}{b_2+x_n} + c_2\frac{y_n e^{k_2-d_2y_n}}{1+e^{k_2-d_2y_n}}$$

where a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, k_1, k_2, are real constants and the initial values x_0 and y_0 are real numbers.

Now, motivated by the above discrete time model, along with the recent studies of close–to–cyclic systems of difference equations and the potentials of difference equations systems with exponential terms, we study in this paper the stability of the non hyperbolic zero equilibrium and the conditions under which bifurcation and periodic-cycles occur, of the three dimensional system:

$$\begin{aligned} x_{n+1} &= a_1\frac{y_n}{b_1+y_n} + c_1\frac{x_n e^{k_1-d_1x_n}}{1+e^{k_1-d_1x_n}}, \\ y_{n+1} &= a_2\frac{z_n}{b_2+z_n} + c_2\frac{y_n e^{k_2-d_2y_n}}{1+e^{k_2-d_2y_n}}, \\ z_{n+1} &= a_3\frac{x_n}{b_3+x_n} + c_3\frac{z_n e^{k_3-d_3z_n}}{1+e^{k_3-d_3z_n}} \end{aligned} \tag{1.1}$$

where a_i, b_i, c_i, d_i, k_i, for $i = 1, 2, 3$, are real constants and the initial values x_0, y_0 and z_0 are real numbers. The zero equilibrium corresponds to the physical situation where quantities x, y, z vanish.

The analysis is conducted using center manifold reduction theorem, a method that is especially used in the case where the zero equilibrium is non hyperbolic. According to center manifold theory, we consider the system:

$$x_{n+1} = F(x_n), \quad n = 0, 1, ... \tag{1.2}$$

where $F : I\!R^k \to I\!R^k$ is a continuous function. Let $F(x^*) = x^*$ be a fixed point of F. Without loss of generality, we suppose that $x^* = 0_k$, where 0_k is the k-dimensional zero point. Let J_0 be the coefficient matrix of the linearized equation of (1.2) at the zero fixed point. If J_0 has q eigenvalues with modulus one and m eigenvalues with modulus less than one, with $q + m = k$, then system (1.2) can be transformed in the form:

$$H(u, u_1) = (Au + f(u, u_1), Bu + g(u, u_1)) \tag{1.3}$$

where $H : I\!R^{q+m} \to I\!R^{q+m}$, $u \in I\!R^q$, $u_1 \in I\!R^m$, A is an $q \times q$ matrix with eigenvalues lying on the unit circle, B is an $m \times m$ matrix with eigenvalues lying inside the unit circle, $f : I\!R^k \to I\!R^q$, $g : I\!R^k \to I\!R^m$ are C^2 functions with f, g and their first order derivatives are zero at the origin, and

$$J = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is the Jacobian matrix of system (1.3). Then, according to Theorem 6 in [39] there is a center manifold $h : I\!R^q \to I\!R^m$ for H, with $h(0_q) = 0_m$, $Dh(0_q) = 0_{m\times q}$, where $0_{m\times q}$ the $m \times q$ zero matrix, and $u_1 = h(u)$. The map h can be determined by the equation:

$$h(Au + f(u, h(u))) = Bh(u) + g(u, h(u)) \tag{1.4}$$

and the asymptotic behavior of the zero solution of (1.2) corresponds to the asymptotic behavior of the zero solution of:

$$u_{n+1} = Au_n + f(u_n, h(u_n)) \tag{1.5}$$

as is proved in Theorem 8 in [39].

The corresponding investigation of the stability of zero equilibrium of system (1.1) is conducted in the special case when one eigenvalue of the characteristic equation of the linearized system is equal to -1 and the absolute value of the other eigenvalues is less than 1. In a discrete dynamical system the existence of an eigenvalue with value equal to -1 corresponds to the existence of flip bifurcation under certain conditions. In Kuznetsov [40] it is entirely analyzed the normal form of flip bifurcation. Moreover, in [35] it is studied the existence of flip bifurcation in a two dimensional discrete system using normal form analysis.

According to bifurcation theory, the one dimensional dynamical system depending on one parameter β:

$$\eta \to -(1+\beta)\eta + \sigma\eta^3 \tag{1.6}$$

where $\sigma = \pm 1$, undergoes flip bifurcation.

For $\sigma = 1$, the zero fixed point is linearly stable for small $|\beta|$ in a neighborhood of the origin with $\beta < 0$ and linearly unstable for $\beta > 0$. At $\beta = 0$ the equilibrium is non hyperbolic but is, nevertheless, non-linearly stable. Furthermore, for $\beta > 0$ there is a stable period-two cycle, which disappears as β approaches zero from above, and thus, a supercritical flip bifurcation occurs.

For $\sigma = -1$, the zero fixed point is linearly stable for small $|\beta|$ with $\beta < 0$ and linearly unstable for $\beta > 0$, but at $\beta = 0$ the equilibrium is unstable. Moreover, an unstable period-two cycle reveals for $\beta < 0$, which disappears at $\beta = 0$ and a subcritical flip bifurcation takes place.

In addition, on Theorem 4.3 in [40] are given the nondegeneracy conditions under which an one dimensional system can be transformed into the form of (1.6).

At last, in our study we apply center manifold theory to reduce the three dimensional system (1.1) to a corresponding one dimensional difference equation and afterwards we apply normal form analysis to investigate flip bifurcation on one parameter.

2 Stability of zero equilibrium of System (1.1)

In what follows, we prove the stability of the zero equilibrium of the system (1.1), using center manifold theory.

Proposition 2.1 *Let p_1 be a real negative constant, p_2, p_3 be real positive constants such that*

$$0 < p_3 < \frac{1}{3}(-1+\sqrt{10}), \tag{2.1}$$

$$\max\left\{0, \frac{1}{2}(-1-p_3) + \frac{1}{2}\sqrt{-3p_3^2 - 2p_3 + 3}\right\} < p_2 < \frac{1}{4}(-1-2p_3) + \frac{1}{4}\sqrt{-12p_3^2 - 4p_3 + 17}, \tag{2.2}$$

$$\max\left\{-1-p_2-p_3, \frac{-1-p_2p_3}{p_2+p_3}\right\} < p_1 < \frac{-\frac{3}{2}-p_2-p_3-p_2p_3}{1+p_2+p_3} \tag{2.3}$$

then, the equation

$$\lambda^2-(1+p_1+p_2+p_3)\lambda+1+p_1+p_2+p_3+p_1p_2+p_1p_3+p_2p_3=0 \tag{2.4}$$

has two real roots λ_2, λ_3 *such that* $|\lambda_2|<1$ *and* $|\lambda_3|<1$.

Proof. Firstly, we prove that

$$\max\left\{0, \frac{1}{2}(-1-p_3)+\frac{1}{2}\sqrt{-3p_3^2-2p_3+3}\right\} < \frac{1}{4}(-1-2p_3)+\frac{1}{4}\sqrt{-12p_3^2-4p_3+17} \tag{2.5}$$

and

$$\max\left\{-1-p_2-p_3, \frac{-1-p_2p_3}{p_2+p_3}\right\} < \frac{-\frac{3}{2}-p_2-p_3-p_2p_3}{1+p_2+p_3}. \tag{2.6}$$

Relation (2.1) implies that $-3p_3^2-2p_3+3>0$. Moreover since,

$$\frac{1}{3}(-1+\sqrt{10}) < \frac{1}{6}(-1+2\sqrt{13}),$$

from (2.1) we have that $-12p_3^2-4p_3+17>0$.

Using (2.1) and since

$$\frac{1}{3}(-1+\sqrt{10}) < \frac{1}{4}(-1+\sqrt{17}),$$

we can easily prove that $0<\frac{1}{4}(-1-2p_3)+\frac{1}{4}\sqrt{-12p_3^2-4p_3+17}$. Furthermore, since (2.1) holds, the inequality

$$\frac{1}{2}(-1-p_3)+\frac{1}{2}\sqrt{-3p_3^2-2p_3+3} < \frac{1}{4}(-1-2p_3)+\frac{1}{4}\sqrt{-12p_3^2-4p_3+17}$$

it true. Therefore, (2.5) are satisfied.

We prove now (2.6). We have that for p_2, p_3 positive numbers, inequality

$$-1-p_2-p_3 < \frac{-\frac{3}{2}-p_2-p_3-p_2p_3}{1+p_2+p_3}$$

is equivalent to

$$p_2^2+(1+p_3)p_2+p_3^2+p_3-\frac{1}{2}>0$$

which is true since (2.2) holds. Moreover, inequality

$$\frac{-1 - p_2p_3}{p_2 + p_3} < \frac{-\frac{3}{2} - p_2 - p_3 - p_2p_3}{1 + p_2 + p_3}$$

is equivalent to

$$p_2^2 + (\frac{1}{2} + p_3)p_2 + p_3^2 + \frac{1}{2}p_3 - 1 < 0$$

which holds since relation (2.2) is satisfied. Hence, (2.6) are true.

Followingly, we prove that equation (2.4) has two real roots with absolute value less than 1. Firstly, we prove that $\Delta > 0$, where

$$\Delta = (1 + p_1 + p_2 + p_3)^2 - 4(1 + p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3)$$

the discriminant of (2.4). It holds that $\Delta > 0$ when $3 + 2p_1 + 2p_2 + 2p_3 + 2p_1p_2 + 2p_1p_3 + 2p_2p_3 < 0$ or equivalently

$$p_1 < \frac{-\frac{3}{2} - p_2 - p_3 - p_2p_3}{1 + p_2 + p_3} \tag{2.7}$$

for p_2, p_3 positive numbers, which is true since (2.3) is satisfied. Further, the roots of (2.4) have absolute value less than 1, if and only if

$$|1 + p_1 + p_2 + p_3| < 2 + p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3 < 2. \tag{2.8}$$

Inequality (2.8) is equivalent with the following three inequalities:

$$1 + p_1 + p_2 + p_3 < 2 + p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3$$

or

$$\frac{-1 - p_2p_3}{p_2 + p_3} < p_1$$

which is true from (2.3),

$$-2 - p_1 - p_2 - p_3 - p_1p_2 - p_1p_3 - p_2p_3 < 1 + p_1 + p_2 + p_3$$

or

$$0 < 2(1 + p_1 + p_2 + p_3) + (1 + p_1p_2 + p_1p_3 + p_2p_3)$$

which holds when

$$\max\left\{-1 - p_2 - p_3, \frac{-1 - p_2p_3}{p_2 + p_3}\right\} < p_1$$

which is true from (2.3), and lastly,

$$2 + p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3 < 2$$

or

$$p_1 < \frac{-p_2 - p_3 - p_2p_3}{1 + p_2 + p_3}$$

which is true from (2.3). Therefore, the equation (2.4) has two real roots with absolute value less than 1.

Proposition 2.2 *Consider system (1.1) where a_1, b_1, b_2, c_2, b_3, c_3 are real positive constants, c_1, a_2, a_3 are real negative constants and k_1, k_2, k_3, d_1, d_2, d_3 are real constants. Let*

$$p_1 = c_1\frac{e^{k_1}}{1 + e^{k_1}}, \quad p_2 = c_2\frac{e^{k_2}}{1 + e^{k_2}}, \quad p_3 = c_3\frac{e^{k_3}}{1 + e^{k_3}} \tag{2.9}$$

satisfy (2.1), (2.2), (2.3) and

$$b_3 = -\frac{a_1a_2a_3}{(1 + p_1)(1 + p_2)(1 + p_3)b_1b_2}. \tag{2.10}$$

If in addition b_1, $d_1 \in (-\epsilon, \epsilon)$, where ϵ is a sufficiently small positive number, then, the zero equilibrium of (1.1) is asymptotically stable.

Proof. At first, we can easily verify that the zero point $(x_n, y_n, z_n) = (0, 0, 0)$ is a fixed point of system (1.1). System (1.1) can be written as follows:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} + \begin{bmatrix} f_1(x_n, y_n, z_n) \\ f_2(x_n, y_n, z_n) \\ f_3(x_n, y_n, z_n) \end{bmatrix}, \tag{2.11}$$

where

$$J_0 = \begin{bmatrix} \frac{c_1e^{k_1}}{1+e^{k_1}} & \frac{a_1}{b_1} & 0 \\ 0 & \frac{c_2e^{k_2}}{1+e^{k_2}} & \frac{a_2}{b_2} \\ \frac{a_3}{b_3} & 0 & \frac{c_3e^{k_3}}{1+e^{k_3}} \end{bmatrix}$$

is the Jacobian matrix calculated at the zero equilibrium and

$$f_1(x, y, z) = \frac{a_1}{b_1}y\left(\frac{b_1}{b_1 + y} - 1\right) + c_1x\left(\frac{e^{k_1 - d_1x}}{1 + e^{k_1 - d_1x}} - \frac{e^{k_1}}{1 + e^{k_1}}\right),$$

$$f_2(x, y, z) = \frac{a_2}{b_2}z\left(\frac{b_2}{b_2 + z} - 1\right) + c_2y\left(\frac{e^{k_2 - d_2y}}{1 + e^{k_2 - d_2y}} - \frac{e^{k_2}}{1 + e^{k_2}}\right),$$

$$f_3(x,y,z) = \frac{a_3}{b_3}x\left(\frac{b_3}{b_3+x}-1\right) + c_3 z\left(\frac{e^{k_3-d_3 z}}{1+e^{k_3-d_3 z}} - \frac{e^{k_3}}{1+e^{k_3}}\right).$$

The characteristic equation of J_0 is

$$\lambda^3 - (p_1+p_2+p_3)\lambda^2 + (p_1p_2+p_1p_3+p_2p_3)\lambda - p_1p_2p_3 - \frac{a_1a_2a_3}{b_1b_2b_3} = 0. \quad (2.12)$$

If (2.10) holds, equation (2.12) has a solution $\lambda_1 = -1$. Then, we can write (2.12) as

$$(\lambda+1)(\lambda^2 - (1+p_1+p_2+p_3)\lambda + 1 + p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3) = 0.$$

Hence, from Proposition 2.1, the other two eigenvalues λ_2, λ_3 of J_0 satisfy $|\lambda_2| < 1$ and $|\lambda_3| < 1$. We assume that $\lambda_3 > \lambda_2$.
We let now

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix},$$

where T is the matrix that diagonalises J_0 defined by

$$T = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{bmatrix} \quad (2.13)$$

where

$$\begin{array}{l} r_1 = -\frac{b_1}{a_1}(p_1+1), \quad r_2 = -\frac{b_1}{a_1}(p_1-\lambda_2), \quad r_3 = -\frac{b_1}{a_1}(p_1-\lambda_3), \\ s_1 = \frac{b_1b_2}{a_1a_2}(p_1+1)(p_2+1), \quad s_2 = \frac{b_1b_2}{a_1a_2}(p_1-\lambda_2)(p_2-\lambda_2), \\ s_3 = \frac{b_1b_2}{a_1a_2}(p_1-\lambda_3)(p_2-\lambda_3). \end{array} \quad (2.14)$$

Thus,

$$T^{-1} = \frac{1}{R}\begin{bmatrix} r_2s_3 - r_3s_2 & s_2 - s_3 & r_3 - r_2 \\ r_3s_1 - r_1s_3 & s_3 - s_1 & r_1 - r_3 \\ r_1s_2 - r_2s_1 & s_1 - s_2 & r_2 - r_1 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

where

$$t_{11} = \frac{(p_1-\lambda_2)(p_1-\lambda_3)}{(1+\lambda_2)(1+\lambda_3)}, \ t_{12} = \frac{a_1(p_1+p_2-\lambda_2-\lambda_3)}{b_1(1+\lambda_2)(1+\lambda_3)}, \ t_{13} = \frac{a_1a_2}{b_1b_2(1+\lambda_2)(1+\lambda_3)},$$

$$t_{21} = \frac{(1+p_1)(p_1-\lambda_3)}{(1+\lambda_2)(\lambda_2-\lambda_3)}, \ t_{22} = \frac{a_1(p_1+p_2-\lambda_3+1)}{b_1(1+\lambda_2)(\lambda_2-\lambda_3)}, \ t_{23} = \frac{a_1a_2}{b_1b_2(1+\lambda_2)(\lambda_2-\lambda_3)},$$

$$t_{31}=\frac{(1+p_1)(p_1-\lambda_2)}{(1+\lambda_3)(\lambda_3-\lambda_2)},\ t_{32}=\frac{a_1(1+p_1+p_2-\lambda_2)}{b_1(\lambda_3-\lambda_2)(1+\lambda_3)},\ t_{33}=\frac{a_1a_2}{b_1b_2(\lambda_3-\lambda_2)(1+\lambda_3)}$$
(2.15)

and

$$R=-r_2s_1+r_3s_1+r_1s_2-r_3s_2-r_1s_3+r_2s_3=\frac{b_1^2b_2(1+\lambda_2)(\lambda_3-\lambda_2)(1+\lambda_3)}{a_1^2a_2}\neq 0$$

the determinant of T. Then, (2.11) can be written as

$$\begin{bmatrix} u_{n+1}\\ v_{n+1}\\ w_{n+1}\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}u_n\\v_n\\w_n\end{bmatrix}+\begin{bmatrix}\bar{f}_1(u_n,v_n,w_n)\\\bar{f}_2(u_n,v_n,w_n)\\\bar{f}_3(u_n,v_n,w_n)\end{bmatrix}, \qquad (2.16)$$

where $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ derive from the product

$$T^{-1}\begin{bmatrix} f_1(x_n(u_n,v_n,w_n),y_n(u_n,v_n,w_n),z_n(u_n,v_n,w_n))\\ f_2(x_n(u_n,v_n,w_n),y_n(u_n,v_n,w_n),z_n(u_n,v_n,w_n))\\ f_3(x_n(u_n,v_n,w_n),y_n(u_n,v_n,w_n),z_n(u_n,v_n,w_n))\end{bmatrix}$$

and are equal to

$$\begin{aligned}&\bar{f}_i(u,v,w)=t_{i1}\Bigg(\tfrac{a_1}{b_1}\left(r_1u+r_2v+r_3w\right)\left(\tfrac{b_1}{b_1+r_1u+r_2v+r_3w}-1\right)+\\ &c_1\left(u+v+w\right)\left(\tfrac{e^{k_1-d_1(u+v+w)}}{1+e^{k_1-d_1(u+v+w)}}-\tfrac{e^{k_1}}{1+e^{k_1}}\right)\Bigg)+\\ &t_{i2}\Bigg(\tfrac{a_2}{b_2}\left(s_1u+s_2v+s_3w\right)\left(\tfrac{b_2}{b_2+s_1u+s_2v+s_3w}-1\right)+\\ &c_2\left(r_1u+r_2v+r_3w\right)\left(\tfrac{e^{k_2-d_2(r_1u+r_2v+r_3w)}}{1+e^{k_2-d_2(r_1u+r_2v+r_3w)}}-\tfrac{e^{k_2}}{1+e^{k_2}}\right)\Bigg)+\\ &t_{i3}\Bigg(\tfrac{a_3}{b_3}\left(u+v+w\right)\left(\tfrac{b_3}{b_3+u+v+w}-1\right)+\\ &c_3\left(s_1u+s_2v+s_3w\right)\left(\tfrac{e^{k_3-d_3(s_1u+s_2v+s_3w)}}{1+e^{k_3-d_3(s_1u+s_2v+s_3w)}}-\tfrac{e^{k_3}}{1+e^{k_3}}\right)\Bigg)\end{aligned}$$

for $i=1,2,3$.

For $A=-1$, $B=\begin{bmatrix}\lambda_2&0\\0&\lambda_3\end{bmatrix}$, $u_1=(v,w)$ and the C^2 functions

$$f(u,u_1)=\bar{f}_1(u,u_1),\ \ g(u,u_1)=\begin{bmatrix}\bar{f}_2(u,u_1)\\\bar{f}_3(u,u_1)\end{bmatrix},$$

with $f(0, 0_2) = 0$, $g(0, 0_2) = (0, 0)^T$ and $Df(0, 0_2) = 0_3$, $Dg(0, 0_2) = 0_{2\times 3}$, where 0_i is the i-dimensional zero vector and $0_{i\times j}$ is the $i \times j$ zero matrix, there exists a center manifold $h : I\!R \to I\!R^2$ with $h(0) = 0$, $h'(0) = (0, 0)^T$ and $u_1 = h(u)$. We let

$$h(u) = \left[\begin{array}{c} \psi_1(u) + O(u^4) \\ \psi_2(u) + O(u^4) \end{array} \right],$$

where $\psi_1(u) = C_1u^2+C_2u^3$, $\psi_2(u) = D_1u^2+D_2u^3$ and $\psi(u) = (\psi_1(u), \psi_2(u))$. The use of $\psi(u)$ as an approximation of $h(u)$ is justified by Theorem 7 in [39] . Consequently, according to Theorem 8 in [39] the asymptotic behavior of small solutions of system (1.1) corresponds to the asymptotic behavior of the zero equilibrium of the equation:

$$u_{n+1} = -u_n + \bar{f}_1(u_n, \psi(u_n)) = G(u_n) \tag{2.17}$$

where

$$\begin{aligned}
&\bar{f}_1(u_n, \psi(u_n)) = t_{11}\Bigg(\tfrac{a_1}{b_1}\left(r_1u_n + r_2\psi_1(u_n) + r_3\psi_2(u_n)\right)\left(\tfrac{b_1}{b_1+r_1u_n+r_2\psi_1(u_n)+r_3\psi_2(u_n)} - 1\right) + \\
&c_1\left(u_n + \psi_1(u_n) + \psi_2(u_n)\right)\left(\tfrac{e^{k_1-d_1(u_n+\psi_1(u_n)+\psi_2(u_n))}}{1+e^{k_1-d_1(u_n+\psi_1(u_n)+\psi_2(u_n))}} - \tfrac{e^{k_1}}{1+e^{k_1}}\right)\Bigg) + \\
&t_{12}\Bigg(\tfrac{a_2}{b_2}\left(s_1u_n + s_2\psi_1(u_n) + s_3\psi_2(u_n)\right)\left(\tfrac{b_2}{b_2+s_1u_n+s_2\psi_1(u_n)+s_3\psi_2(u_n)} - 1\right) + \\
&c_2\left(r_1u_n + r_2\psi_1(u_n) + r_3\psi_2(u_n)\right)\left(\tfrac{e^{k_2-d_2(r_1u_n+r_2\psi_1(u_n)+r_3\psi_2(u_n))}}{1+e^{k_2-d_2(r_1u_n+r_2\psi_1(u_n)+r_3\psi_2(u_n))}} - \tfrac{e^{k_2}}{1+e^{k_2}}\right)\Bigg) + \\
&t_{13}\Bigg(\tfrac{a_3}{b_3}\left(u_n + \psi_1(u_n) + \psi_2(u_n)\right)\left(\tfrac{b_3}{b_3+u_n+\psi_1(u_n)+\psi_2(u_n)} - 1\right) + \\
&c_3\left(s_1u_n + s_2\psi_1(u_n) + s_3\psi_2(u_n)\right)\left(\tfrac{e^{k_3-d_3(s_1u_n+s_2\psi_1(u_n)+s_3\psi_2(u_n))}}{1+e^{k_3-d_3(s_1u_n+s_2\psi_1(u_n)+s_3\psi_2(u_n))}} - \tfrac{e^{k_3}}{1+e^{k_3}}\right)\Bigg).
\end{aligned}$$

To determine the center manifold $h(u)$, we conclude from (1.4) that the map h must satisfy the equation:

$$h(-u + \bar{f}_1(u, h(u))) = \left[\begin{array}{cc} \lambda_2 & 0 \\ 0 & \lambda_3 \end{array} \right] h(u) + \left[\begin{array}{c} \bar{f}_2(u, h(u)) \\ \bar{f}_3(u, h(u)) \end{array} \right]. \tag{2.18}$$

Keeping the terms up to u^3, from (2.18), we obtain:

$$C_1 = \frac{1}{1-\lambda_2}\left(\frac{-t_{21}a_1r_1^2}{b_1^2} - \frac{t_{21}d_1p_1(c_1-p_1)}{c_1} - \frac{t_{22}a_2s_1^2}{b_2^2} - \frac{t_{22}r_1^2d_2p_2(c_2-p_2)}{c_2}\right.$$
$$\left. - \frac{t_{23}a_3}{b_3^2} - \frac{t_{23}s_1^2d_3p_3(c_3-p_3)}{c_3}\right), \tag{2.19}$$

$$D_1 = \frac{1}{1-\lambda_3}\left(\frac{-t_{31}a_1r_1^2}{b_1^2} - \frac{t_{31}d_1p_1(c_1-p_1)}{c_1} - \frac{t_{32}a_2s_1^2}{b_2^2} - \frac{t_{32}r_1^2d_2p_2(c_2-p_2)}{c_2}\right.$$
$$\left. - \frac{t_{33}a_3}{b_3^2} - \frac{t_{33}s_1^2d_3p_3(c_3-p_3)}{c_3}\right). \tag{2.20}$$

Thus, from (2.17),(2.19) and (2.20) we conclude that $G'(0) = -1$ and

$$G'''(0) = t_{11}\left(6\frac{a_1r_1}{b_1^2}\left(-2M_1 + \frac{r_1^2}{b_1}\right) + \frac{3(c_1-p_1)p_1d_1}{c_1}\left(-\frac{d_1(2p_1-c_1)}{c_1} - 4M_2\right)\right) +$$
$$t_{12}\left(6\frac{a_2s_1}{b_2^2}\left(-2M_3 + \frac{s_1^2}{b_2}\right) + \frac{3(c_2-p_2)p_2d_2r_1}{c_2}\left(-\frac{d_2(2p_2-c_2)r_1^2}{c_2} - 4M_1\right)\right) +$$
$$t_{13}\left(6\frac{a_3}{b_3^2}\left(-2M_2 + \frac{1}{b_3}\right) + \frac{3(c_3-p_3)p_3d_3s_1}{c_3}\left(-\frac{d_3(2p_3-c_3)s_1^2}{c_3} - 4M_3\right)\right) \tag{2.21}$$

where

$$M_1 = C_1r_2 + D_1r_3, \quad M_2 = C_1 + D_1, \quad M_3 = C_1s_2 + D_1s_3. \tag{2.22}$$

Then, from (2.10), (2.14), (2.15), (2.21) and (2.22) we can write $G'''(0)$ as follows:

$$G'''(0) = \Phi_1 b_1^2 + \Phi_2 b_1 + \Phi_3 \tag{2.23}$$

and Φ_1, Φ_2, Φ_3 are continuous functions of a_1, a_2, a_3, b_2, c_1, c_2, c_3, d_1, d_2, d_3, k_1, k_2 and k_3.

Moreover, we can write Φ_3 as follows:

$$\Phi_3 = \phi_1 d_1^2 + \phi_2 d_1 + \phi_3 \tag{2.24}$$

where

$$\phi_1 = \frac{3p_1(c_1-p_1)(p_1-\lambda_2)(p_1-\lambda_3)}{c_1(1+\lambda_2)(1+\lambda_3)}\left(1-\frac{2p_1}{c_1}+\frac{4p_1(1+p_1)(c_1-p_1)(1+p_1\lambda_2-\lambda_2^2+p_1\lambda_3-\lambda_2\lambda_3-\lambda_3^2)}{c_1(-1+\lambda_2^2)(-1+\lambda_3^2)}\right),$$

$$\phi_2 = \frac{12p_1(c_1-p_1)(1+p_1)^2(p_1-\lambda_2)(p_1-\lambda_3)}{(1+\lambda_2)(1+\lambda_3)}\left(\frac{(p_1-\lambda_2)(p_1-\lambda_3)(\lambda_2+\lambda_3)}{a_1c_1(-1+\lambda_2^2)(-1+\lambda_3^2)}+\right.$$

$$\left.\frac{(1+p_1)(1+p_1\lambda_2-\lambda_2^2+p_1\lambda_3-\lambda_2\lambda_3-\lambda_3^2)}{a_1c_1(-1+\lambda_2^2)(-1+\lambda_3^2)}\right),$$

$$\phi_3 = \frac{(\lambda_2-p_1)(\lambda_3-p_1)}{(1+\lambda_2)(1+\lambda_3)}\left(-\frac{6(1+p_1)^3}{a_1^2}+\frac{12(1+p_1)^4(\lambda_2-p_1)(\lambda_3-p_1)(\lambda_2+\lambda_3)}{a_1^2(\lambda_2^2-1)(\lambda_3^2-1)}\right).$$

Consequently, fixing a_1, a_2, a_3, b_2, c_1, c_2, c_3, d_2, d_3, k_1, k_2, k_3 satisfying (2.1), (2.2), (2.3), (2.10) and by using (2.23), (2.24), $G'''(0)$ can be written as a continuous function of b_1 and d_1 as follows:

$$G'''(0) = K(b_1, d_1) = \Phi_1 b_1^2 + \Phi_2 b_1 + \phi_1 d_1^2 + \phi_2 d_1 + \phi_3. \tag{2.25}$$

Now, we will prove that $\phi_3 > 0$. From (2.1), (2.2), (2.3) we have that $|\lambda_2| < 1$ and $|\lambda_3| < 1$, where λ_2, λ_3 are the roots of equation (2.4). Thus, $(1+\lambda_2)(1+\lambda_3) > 0$ and $(\lambda_2^2-1)(\lambda_3^2-1) > 0$. Furthermore, from (2.3) we can easily prove that $p_1 < -1$, so, $(\lambda_2 - p_1)(\lambda_3 - p_1) > 0$ and $(1+p_1)^3 < 0$, where p_1 is given in (2.9). Moreover, as λ_2 and λ_3 are the solutions of (2.4), we have that $\lambda_2 + \lambda_3 = 1 + p_1 + p_2 + p_3 > 0$ because $p_1 > -1 - p_2 - p_3$ from (2.3), where p_2, p_3 are given in (2.9). So, we proved that $\phi_3 > 0$ under relations (2.1), (2.2), (2.3).

In conclusion, we have that $K(0,0) = \phi_3 > 0$. Since K is a continuous function, there exists a sufficiently small positive ϵ such that for $|b_1| < \epsilon$, $|d_1| < \epsilon$ we have $K(b_1, d_1) > 0$. This implies that for b_1, $d_1 \in (-\epsilon, \epsilon)$ we have $G'''(0) > 0$. Hence, $SG(0) < 0$, where

$$SG(0) = -G'''(0) - \frac{3}{2}(G''(0))^2$$

the Schwarzian derivative of G at $u = 0$.

Thus, we proved that $G'(0) = -1$ and $SG(0) < 0$. This implies that the zero equilibrium of scalar equation (2.17) is asymptotically stable (see Theorem 1.6, [55]). Consequently, the zero equilibrium of system (1.1) is asymptotically stable.

3 Flip Bifurcation of System (1.1)

In this section we discuss the sufficient conditions for the existence of flip bifurcation of system (1.1) occurring at zero equilibrium for the bifurcation parameter a_1, using the center manifold reduction theorem and the normal form bifurcation analysis. At first, we define:

$$A_1 = 2\bigg(\frac{d_1p_1(p_1-c_1)t_{11}}{c_1} - \frac{a_0r_1^2t_{11}}{b_1^2} + \frac{d_2p_2r_1^2(p_2-c_2)t_{12}}{c_2} - \frac{a_2s_1^2t_{12}}{b_2^2} - \frac{b_1^2b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2t_{13}}{a_0^2a_2^2a_3} - \frac{d_3p_3s_1^2(c_3-p_3)t_{13}}{c_3}\bigg), \tag{3.1}$$

$$A_2 = 3\bigg(\frac{d_1p_1(c_1-p_1)\big(c_1(d_1-4M_5)-2d_1p_1\big)t_{11}}{c_1^2} + \frac{2a_0r_1(r_1^2-2b_1M_4)t_{11}}{b_1^3} + \frac{d_2p_2r_1(c_2-p_2)(c_2d_2r_1^2-2d_2p_2r_1^2-4M_4c_2)t_{12}}{c_2^2} + \frac{2a_2s_1(s_1^2-2M_6b_2)t_{12}}{b_2^3} - \frac{2b_1^2b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2\big(2a_0a_2a_3M_5+b_1b_2(1+p_1)(1+p_2)(1+p_3)\big)t_{13}}{a_0^3a_2^3a_3^2} + \frac{d_3p_3s_1(c_3-p_3)(c_3d_3s_1^2-2d_3p_3s_1^2-4M_6c_3)t_{13}}{c_3^2}\bigg) \tag{3.2}$$

where p_1, p_2, p_3 are given in (2.9),

$$M_4 = C_2r_2 + D_2r_3, \quad M_5 = C_2 + D_2, \quad M_6 = C_2s_2 + D_2s_3, \tag{3.3}$$

$$\begin{aligned}
&C_1 = -\tfrac{r_1t_{21}}{b_1(1+\lambda_2)}, \quad D_1 = -\tfrac{r_1t_{31}}{b_1(1+\lambda_3)},\\
&C_2 = \tfrac{1}{1-\lambda_2}\bigg(\Big(\tfrac{d_1p_1(p_1-c_1)}{c_1} - \tfrac{a_0r_1^2}{b_1^2}\Big)t_{21} - \Big(\tfrac{d_2p_2r_1^2(c_2-p_2)}{c_2} + \tfrac{a_2s_1^2}{b_2^2}\Big)t_{22} - \Big(\tfrac{b_1^2b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2}{a_0^2a_2^2a_3} + \tfrac{d_3p_3s_1^2(c_3-p_3)}{c_3}\Big)t_{23}\bigg),\\
&D_2 = \tfrac{1}{1-\lambda_3}\bigg(\Big(\tfrac{d_1p_1(p_1-c_1)}{c_1} - \tfrac{a_0r_1^2}{b_1^2}\Big)t_{31} - \Big(\tfrac{d_2p_2r_1^2(c_2-p_2)}{c_2} + \tfrac{a_2s_1^2}{b_2^2}\Big)t_{32} - \Big(\tfrac{b_1^2b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2}{a_0^2a_2^2a_3} + \tfrac{d_3p_3s_1^2(c_3-p_3)}{c_3}\Big)t_{33}\bigg),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&r_1 = -\tfrac{b_1}{a_0}(p_1+1), \quad r_2 = -\tfrac{b_1}{a_0}(p_1-\lambda_2), \quad r_3 = -\tfrac{b_1}{a_0}(p_1-\lambda_3),\\
&s_1 = \tfrac{b_1b_2}{a_0a_2}(p_1+1)(p_2+1), \quad s_2 = \tfrac{b_1b_2}{a_0a_2}(p_1-\lambda_2)(p_2-\lambda_2),\\
&s_3 = \tfrac{b_1b_2}{a_0a_2}(p_1-\lambda_3)(p_2-\lambda_3),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&t_{11} = \tfrac{(p_1-\lambda_2)(p_1-\lambda_3)}{(1+\lambda_2)(1+\lambda_3)}, \quad t_{12} = \tfrac{a_0(p_1+p_2-\lambda_2-\lambda_3)}{b_1(1+\lambda_2)(1+\lambda_3)}, \quad t_{13} = \tfrac{a_0a_2}{b_1b_2(1+\lambda_2)(1+\lambda_3)},\\
&t_{21} = \tfrac{(1+p_1)(p_1-\lambda_3)}{(1+\lambda_2)(\lambda_2-\lambda_3)}, \quad t_{22} = \tfrac{a_0(p_1+p_2-\lambda_3+1)}{b_1(1+\lambda_2)(\lambda_2-\lambda_3)}, \quad t_{23} = \tfrac{a_0a_2}{b_1b_2(1+\lambda_2)(\lambda_2-\lambda_3)},\\
&t_{31} = \tfrac{(1+p_1)(p_1-\lambda_2)}{(1+\lambda_3)(\lambda_3-\lambda_2)}, \quad t_{32} = \tfrac{a_0(1+p_1+p_2-\lambda_2)}{b_1(\lambda_3-\lambda_2)(1+\lambda_3)}, \quad t_{33} = \tfrac{a_0a_2}{b_1b_2(\lambda_3-\lambda_2)(1+\lambda_3)}.
\end{aligned} \tag{3.6}$$

Proposition 3.1 *Under conditions of Propositions 2.1 and 2.2, if*

$$\frac{1}{2}A_1^2 + \frac{1}{3}A_2 \neq 0 \tag{3.7}$$

and $a_1 = a_0 + \epsilon_0$, where ϵ_0 is a small parameter, the system (1.1) undergoes a supercritical flip bifurcation near zero fixed point, which is stable for $\epsilon_0 \geq 0$ and unstable for $\epsilon_0 < 0$. Moreover, a stable period-two cycle exists for small $|\epsilon_0|$ with $\epsilon_0 < 0$ and disappears as ϵ_0 approaches zero.

Proof. Consider the system (1.1), where $a_1 = a_0 + \epsilon_0$ is the bifurcation parameter. The system (1.1) can be written as:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} + \begin{bmatrix} f_1(x_n, y_n, z_n, \epsilon_0) \\ f_2(x_n, y_n, z_n, \epsilon_0) \\ f_3(x_n, y_n, z_n, \epsilon_0) \end{bmatrix}, \tag{3.8}$$

where

$$J_0 = \begin{bmatrix} \frac{c_1 e^{k_1}}{1+e^{k_1}} & \frac{a_0}{b_1} & 0 \\ 0 & \frac{c_2 e^{k_2}}{1+e^{k_2}} & \frac{a_2}{b_2} \\ \frac{a_3}{b_3} & 0 & \frac{c_3 e^{k_3}}{1+e^{k_3}} \end{bmatrix}$$

is the Jacobian matrix of the system at $(x, y, z, \epsilon_0) = (0, 0, 0, 0)$ and

$$f_1(x, y, z, \epsilon_0) = \frac{a_0}{b_1} y \left(\frac{b_1}{b_1 + y} - 1 \right) + \epsilon_0 \frac{y}{b_1 + y} + c_1 x \left(\frac{e^{k_1 - d_1 x}}{1 + e^{k_1 - d_1 x}} - \frac{e^{k_1}}{1 + e^{k_1}} \right),$$

$$f_2(x, y, z, \epsilon_0) = \frac{a_2}{b_2} z \left(\frac{b_2}{b_2 + z} - 1 \right) + c_2 y \left(\frac{e^{k_2 - d_2 y}}{1 + e^{k_2 - d_2 y}} - \frac{e^{k_2}}{1 + e^{k_2}} \right),$$

$$f_3(x, y, z, \epsilon_0) = \frac{a_3}{b_3} x \left(\frac{b_3}{b_3 + x} - 1 \right) + c_3 z \left(\frac{e^{k_3 - d_3 z}}{1 + e^{k_3 - d_3 z}} - \frac{e^{k_3}}{1 + e^{k_3}} \right).$$

When (2.10) holds, for $a_1 = a_0$, the characteristic equation (2.12) of J_0 has a root $\lambda_1 = -1$. Moreover, if λ_2, λ_3 are the other two roots of (2.12) with $|\lambda_2| \neq 1$ and $|\lambda_3| \neq 1$, then a flip bifurcation occurs for the non-hyperbolic zero equilibrium point. In Proposition 2.1 we proved that when (2.1), (2.2), (2.3) hold, then $|\lambda_2| < 1$, $|\lambda_3| < 1$.

We diagonalize the matrix J_0, applying the coordinate transformation $(x, y, z)' = T(u, v, z)'$, where T is given in (2.13) and r_i, s_i, $i = 1, 2, 3$ in (3.5). Thus, the system (3.8) is written as:

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} + \begin{bmatrix} \bar{f}_1(u_n, v_n, w_n, \epsilon_0) \\ \bar{f}_2(u_n, v_n, w_n, \epsilon_0) \\ \bar{f}_3(u_n, v_n, w_n, \epsilon_0) \end{bmatrix}, \tag{3.9}$$

where

$$\bar{f}_1(u,v,w,\epsilon_0) = t_{11}\Bigg(\tfrac{a_0}{b_1}\left(r_1u+r_2v+r_3w\right)\left(\tfrac{b_1}{b_1+r_1u+r_2v+r_3w}-1\right)+$$

$$\epsilon_0\tfrac{r_1u+r_2v+r_3w}{b_1+r_1u+r_2v+r_3w}+c_1\left(u+v+w\right)\left(\tfrac{e^{k_1-d_1(u+v+w)}}{1+e^{k_1-d_1(u+v+w)}}-\tfrac{e^{k_1}}{1+e^{k_1}}\right)\Bigg)+$$

$$t_{12}\Bigg(\tfrac{a_2}{b_2}\left(s_1u+s_2v+s_3w\right)\left(\tfrac{b_2}{b_2+s_1u+s_2v+s_3w}-1\right)+$$

$$c_2\left(r_1u+r_2v+r_3w\right)\left(\tfrac{e^{k_2-d_2(r_1u+r_2v+r_3w)}}{1+e^{k_2-d_2(r_1u+r_2v+r_3w)}}-\tfrac{e^{k_2}}{1+e^{k_2}}\right)\Bigg)+$$

$$t_{13}\Bigg(\tfrac{a_3}{b_3}\left(u+v+w\right)\left(\tfrac{b_3}{b_3+u+v+w}-1\right)+$$

$$c_3\left(s_1u+s_2v+s_3w\right)\left(\tfrac{e^{k_3-d_3(s_1u+s_2v+s_3w)}}{1+\epsilon^{k_3-d_3(s_1u+s_2v+s_3w)}}-\tfrac{e^{k_3}}{1+e^{k_3}}\right)\Bigg)$$

and

$$\bar{f}_i(u,v,w,\epsilon_0) = t_{i1}\Bigg(\tfrac{a_0}{b_1}\left(r_1u+r_2v+r_3w\right)\left(\tfrac{b_1}{b_1+r_1u+r_2v+r_3w}-1\right)+$$

$$c_1\left(u+v+w\right)\left(\tfrac{e^{k_1-d_1(u+v+w)}}{1+e^{k_1-d_1(u+v+w)}}-\tfrac{e^{k_1}}{1+e^{k_1}}\right)\Bigg)+$$

$$t_{i2}\Bigg(\tfrac{a_2}{b_2}\left(s_1u+s_2v+s_3w\right)\left(\tfrac{b_2}{b_2+s_1u+s_2v+s_3w}-1\right)+$$

$$c_2\left(r_1u+r_2v+r_3w\right)\left(\tfrac{e^{k_2-d_2(r_1u+r_2v+r_3w)}}{1+e^{k_2-d_2(r_1u+r_2v+r_3w)}}-\tfrac{e^{k_2}}{1+e^{k_2}}\right)\Bigg)+$$

$$t_{i3}\Bigg(\tfrac{a_3}{b_3}\left(u+v+w\right)\left(\tfrac{b_3}{b_3+u+v+w}-1\right)+$$

$$c_3\left(s_1u+s_2v+s_3w\right)\left(\tfrac{e^{k_3-d_3(s_1u+s_2v+s_3w)}}{1+\epsilon^{k_3-d_3(s_1u+s_2v+s_3w)}}-\tfrac{e^{k_3}}{1+e^{k_3}}\right)\Bigg)$$

for $i=2,3$.

To apply the center manifold theorem depending on parameter ϵ_0 we increase the number of equations by writing the system (3.9) in the form:

$$\begin{bmatrix} u_{n+1} \\ \epsilon_{0_{n+1}} \\ v_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -u_n \\ \epsilon_{0_n} \\ \lambda_2 v_n \\ \lambda_3 w_n \end{bmatrix} + \begin{bmatrix} \bar{f}_1(u_n,v_n,w_n,\epsilon_0) \\ 0 \\ \bar{f}_2(u_n,v_n,w_n,\epsilon_0) \\ \bar{f}_3(u_n,v_n,w_n,\epsilon_0) \end{bmatrix}. \tag{3.10}$$

As $|\lambda_{2,3}| \neq 1$ and $\bar{f}_i$, $i = 1,2,3$ are C^2 functions with $\bar{f}_i(0,0,0,0) = 0$ and $D\bar{f}_i(0,0,0,0) = 0_4$, where 0_4 the 4-dimensional zero vector, there is a center manifold M_c with the form:

$$M_c = \left\{ \begin{array}{l} (u,\epsilon_0,v,w) : (v,w)^T = h(u,\epsilon_0), |u| < \delta_1, |\epsilon_0| < \delta_2, h(0,0) = (0,0)^T, Dh(0,0) = 0_{2\times 2}, \\ for \;\; sufficiently \;\; small \;\; \delta_1 \;\; and \;\; \delta_2 \end{array} \right\}$$

where

$$h(u,\epsilon_0) = \left[\begin{array}{c} \psi_1(u,\epsilon_0) + O(\epsilon_0^2) + O(u^4), \\ \psi_2(u,\epsilon_0) + O(\epsilon_0^2) + O(u^4) \end{array} \right],$$

and

$$\psi_1(u,\epsilon_0) = C_1\epsilon_0 u + C_2 u^2 + C_3\epsilon_0 u^2 + C_4 u^3$$

$$\psi_2(u,\epsilon_0) = D_1\epsilon_0 u + D_2 u^2 + D_3\epsilon_0 u^2 + D_4 u^3$$

for small ϵ_0. We can determine $h(u,\epsilon_0)$ applying (1.4):

$$h(-u + \bar{f}_1(u, h(u,\epsilon_0),\epsilon_0),\epsilon_0) = \left[\begin{array}{cc} \lambda_2 & 0 \\ 0 & \lambda_3 \end{array} \right] h(u,\epsilon_0) + \left[\begin{array}{c} \bar{f}_2(u, h(u,\epsilon_0),\epsilon_0) \\ \bar{f}_3(u, h(u,\epsilon_0),\epsilon_0) \end{array} \right]. \tag{3.11}$$

Keeping the terms up to the third order and comparing the coefficients of $\epsilon_0 u$, u^2, $\epsilon_0 u^2$, u^3, from (3.11), we obtain the constants C_i and D_i, $i = 1,2,3,4$.

According to Theorem 5.1 in [55] the dynamics restricted to M_c are given locally by the smooth map $G : I\!R^2 \to I\!R$

$$G(u_n,\epsilon_0) = -u_n + \bar{f}_1(u_n, h(u,\epsilon_0),\epsilon_0). \tag{3.12}$$

The map G can be written in a neighborhood of $(u_n,\epsilon_0) = (0,0)$ as F :

$\mathbb{R}^2 \to \mathbb{R}$ with

$$F(u,\epsilon_0) = -u + \frac{r_1 t_{11}}{b_1}\epsilon_0 u + \left(\frac{d_1 p_1(p_1-c_1)t_{11}}{c_1} - \frac{a_0 r_1^2 t_{11}}{b_1^2} + \frac{d_2 p_2 r_1^2(p_2-c_2)t_{12}}{c_2} - \frac{a_2 s_1^2 t_{12}}{b_2^2} - \frac{b_1^2 b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2 t_{13}}{a_0^2 a_2^2 a_3} - \frac{d_3 p_3 s_1^2(c_3-p_3)t_{13}}{c_3}\right)u^2 +$$
$$2\left(\frac{M_2 d_1 p_1(p_1-c_1)t_{11}}{c_1} - \frac{r_1^2 t_{11}}{2b_1^2} + \frac{M_4 t_{11}}{2b_1} - \frac{M_1 a_0 r_1 t_{11}}{b_1^2} + \frac{M_1 d_2 p_2 r_1(p_2-c_2)t_{12}}{c_2} - \frac{M_3 a_2 s_1 t_{12}}{b_2^2} - \frac{M_2 b_1^2 b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2 t_{13}}{a_0^2 a_2^2 a_3} - \frac{M_3 d_3 p_3 s_1(c_3-p_3)t_{13}}{c_3}\right)\epsilon_0 u^2 +$$
$$\frac{1}{2}\left(\frac{d_1 p_1(c_1-p_1)(c_1(d_1-4M_5)-2d_1p_1)t_{11}}{c_1^2} + \frac{2a_0 r_1(r_1^2-2b_1M_4)t_{11}}{b_1^3} + \frac{d_2p_2r_1(c_2-p_2)(c_2d_2r_1^2-2d_2p_2r_1^2-4M_4c_2)t_{12}}{c_2^2} + \frac{2a_2s_1(s_1^2-2M_6b_2)t_{12}}{b_2^3} - \frac{2b_1^2b_2^2(1+p_1)^2(1+p_2)^2(1+p_3)^2(2a_0a_2a_3M_5+b_1b_2(1+p_1)(1+p_2)(1+p_3))t_{13}}{a_0^3a_2^3a_3^2} + \frac{d_3p_3s_1(c_3-p_3)(c_3d_3s_1^2-2d_3p_3s_1^2-4M_6c_3)t_{13}}{c_3^2}\right)u^3 + O(\epsilon_0^2) + O(u^4) \tag{3.13}$$

where M_1, M_2, M_3 are given in (2.22), M_4, M_5, M_6 in (3.7), C_1, C_2, D_1, D_2 in (3.4) and r_i, s_i, t_{ij}, $i,j=1,2,3$ in (3.5) and (3.6) respectively.

We can easily verify that $F(0,0)=0$, $F_u(0,0)=-1$,

$$F_{u\epsilon_0}(0,0) = \frac{r_1t_{11}}{b_1} = -\frac{(1+p_1)(p_1-\lambda_2)(p_1-\lambda_3)}{a_0(1+\lambda_2)(1+\lambda_3)} \neq 0,$$

$$F_{uu}(0,0) = A_1, \;\; F_{uuu}(0,0) = A_2$$

where A_1 and A_2 are given in (3.1) and (3.2) respectively. Thus, as (3.7) holds, the non-degeneracy conditions (B.1) and (B.2) of Theorem 4.3 in [40]:

(B.1) $\frac{1}{2}(F_{uu}(0,0))^2 + \frac{1}{3}F_{uuu}(0,0) \neq 0$

(B.2) $F_{u\epsilon_0}(0,0) \neq 0$

are satisfied. Consequently, the dynamic behavior of system (1.1) is equivalent to the dynamic behavior of the one dimensional dymanical system (1.6), where

$$\beta = \frac{(1+p_1)(p_1-\lambda_2)(p_1-\lambda_3)}{a_0(1+\lambda_2)(1+\lambda_3)}\epsilon_0$$

with $\sigma = \pm 1$, the sign of $\frac{1}{2}(F_{uu}(0,0))^2 + \frac{1}{3}F_{uuu}(0,0)$.

Finally, under the conditions of Propositions 2.1 and 2.2, we have that $(1+p_1)(p_1-\lambda_2)(p_1-\lambda_3)/(a_0(1+\lambda_2)(1+\lambda_3)) < 0$, so the zero fixed point is stable for small $\epsilon_0 > 0$ and unstable for $\epsilon_0 < 0$. In Proposition 2.2 we proved that the zero fixed point is stable for $\epsilon_0 = 0$. Consequently, a supercritical flip bifurcation occurs, as the stable period-two cycle, that exists for small $\epsilon_0 < 0$, disappears as ϵ_0 approaches zero.

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