Lyapunov stability analysis of Caputo fractional-order nonlinear systems

Imad **B**asdouri[a], Souad **K**asmi[b], Fehmi **M**abrouk[c] , Rim **M**essaoud[d]

[a]Gafsa University, Faculty of Sciences of Gafsa
Department of Mathematics, Zarroug Gafsa 2112 Tunisia
e-mail: basdourimed@yahoo.fr
[b]Sfax University, Faculty of Sciences of Sfax
Department of Mathematics, BP 1171 Sfax 3000 Tunisia
e-mail: souadkasmi90@gmail.com
[c]Sfax University, Faculty of Sciences of Sfax
Department of Mathematics, BP 1171 Sfax 3000 Tunisia
e-mail: fehmi.mabrouki@gmail.com
[d]Gafsa University, Faculty of Sciences of Gafsa
Department of Mathematics, Zarroug Gafsa 2112 Tunisia
e-mail: rimessaoud@yahoo.fr

Abstract. This paper deals with the stability analysis of the fractional nonlinear systems. It treats the asymptotic stability of the fractional nonlinear systems with Hurwitz state matrix, using the Lyapunov direct method. We give algebraic conditions under which the fractional nonlinear systems are asymptotical stable. Two numerical examples are provided to illustrate the proposed theoretical results.

§1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order that have recently proved to be valuable tools for the modeling of many physical phenomena, and have been the focus of many studies due to their frequent appearances in various applications, such as physics, biology, finance and fractional dynamics, engineering, signal processing and control theory [14], [9], [7], [16], [10] and [17]. Finding solutions to fractional differential systems is rather complicated, consequently, the stability results of the

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fractional differential systems have been the main goal of the previous studies. For example, in [18], the authors studied the Stability of fractional-order nonlinear dynamic systems using the Lyapunov direct method and generalized Mittag-Leffler stability. In [6], the authors provided a method for the asymptotic stabilization of fractional-order linear systems with saturation nonlinearity. In [11], Shahri and al. proposed a new stability condition for estimating the domain of attraction via ellipsoid approach based on saturation functions. In [12], Esmat and al. study the stability and the stabilization for a class of uncertain fractional order (FO) systems subject to input saturation. The authors investigate the problem of the robust stability of saturation control. In [13], the authors used the Lyapunov approach for the study of uncertain FO system stability analysis. To the best of our knowledge, the researches on the stability and stabilization of the fractional-order systems using the Lyapunov approach are not abundant enough.
In this paper, we studied the stability of Caputo fractional-order systems using the Lyapunov function. We suppose that the nonlinear part of the system satisfies the Lipschitz condition and the one-sides Lipschitz with the quadratic inner-bounded condition, we give some sufficient conditions which imply the asymptotical stability of the system.
The result of this paper is organized as follows. In Section 2, we introduce some Definitions and the necessary Lemmas. In Section 3, we given our main result. two examples are given to show the validity of the proposed method in Section 4. Finally, some conclusions are presented in Section 5.

§2 Notations and preliminaries

We start by introducing some notations that will be useful throughout the paper.

Notation:

$\mathbb{R}^n$: the real n-dimensional vector space.
$\mathbb{R}^{n\times n}$: the set of all $n \times n$ real matrices.
$\langle .,. \rangle$: the usual inner product on $\mathbb{R}^n$.
$\|x\|$: the norm of the vector x that belongs to $\mathbb{R}^n$, i.e. $\sqrt{\langle x, x\rangle} = \|x\|$.
Let $A \in \mathbb{R}^{n\times n}$:
the matrix A is positive semi-definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$;
A is negative semi definite if $-A$ is positive semi definite;
A^T denotes the transpose of the matrix A;
A is symmetric if $A^T = A$;
$\lambda(A)$ denotes the set of all the eigenvalues of A;
$\lambda_{max}(A) = max\{Re(\lambda) : \lambda \in \lambda(A)\}$, $\lambda_{min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}$.

In the following, we recall some classical definitions and results which will play important roles in our study.

Definition 1 (Caputo fractional derivative [4]). *Let $k \in \mathbb{N}^*$ and $k-1 \leq \alpha < k$, the Caputo fractional derivative of a function x of order $\alpha > 0$ is defined as*

$$
{}^{C}D_{t_0}^{\alpha}x(t) = \frac{1}{\Gamma(k-\alpha)} \int_{t_0}^{t} (t-s)^{k-\alpha-1} x^{(k)}(s) ds. \tag{1}
$$

Let the system described by

$$
{}^{C}D_{t_0}^{\alpha}x(t) = f(t,x), \tag{2}
$$

where the map $f : \mathbb{R} \times U \to \mathbb{R}^n$ is continuous locally Lipschitz, $f(t,0) = 0$, $\forall\, t \geq 0$ and U is an open set of $\mathbb{R}^n$. Denote $x(t,t_0)$ the solution of (9) starting at x_0 at time t_0.

Definition 2. *The equilibrium point $x = 0$ of the system (9) is said to be:*

i) *stable if*

$$
\forall \varepsilon > 0, \forall t_0 \geq 0, \exists \delta = \delta(t_0, \varepsilon) > 0, \textit{such that} \parallel x_0 \parallel < \delta \Longrightarrow \parallel x(t,t_0) \parallel < \varepsilon,\ \forall t \geq t_0
$$

ii) *attractive if there exists a neighborhood $\mathcal{V}$ of 0 such that for any initial condition x_0 belonging to $\mathcal{V}$, the corresponding solution $x(t,t_0)$ is defined for all $t \geq t_0$ and $\lim\limits_{t \to +\infty} x(t,t_0) = 0$.*
If $\mathcal{V} = \mathbb{R}^n$, $x = 0$ is globally attractive.

iii) *asymptotically stable if it is stable and attractive.*

iv) *globally asymptotically stable (GAS) if it is stable and globally attractive.*

Definition 3. *Let us consider the following control system :*

$$
\left\{ \begin{array}{l} {}^{C}D_{t_0}^{\alpha}x = X(x,u) \\ x \in U, u \in \mathcal{U} \end{array} \right. \tag{3}
$$

where U is an open set of $\mathbb{R}^n$,$\mathcal{U} \subset \mathbb{R}^m$, x is called the state of (3), u is called the control and $X : U \times \mathcal{U} \longrightarrow \mathbb{R}^n$ is a smooth function satisfying $X(0,0) = 0$. We say that the system (3) *is stabilizable (respectively globally stabilizable), if there exists a feedback function $u = u(x)$ such that the vector field $X(x,u(x))$ is at least continuous and the closed-loop system:*

$$
{}^{C}D_{t_0}^{\alpha}x = X(x,u(x))
$$

admits the origin as an asymptotically stable equilibrium point (respectively globally asymptotically stable).

Definition 4. *[3] A continuous function $\gamma : [0,t) \longrightarrow [0,+\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$.*

Lemma 1. *[15] Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_d = \{x \in \mathbb{R}^n : \|x\| < d\} \subset D$ for some $d > 0$. Then there exist class $\mathcal{K}$ functions λ_1 and λ_2 defined on $[0,d)$, such that*

$$
\lambda_1(\|x\|) \leq V(x) \leq \lambda_2(\|x\|), \tag{4}
$$

for all $x \in B_d$. If $D = \mathbb{R}^n$, the functions λ_1 and λ_2 are defined on $[0,\infty)$.

Theorem 1 (Fractional-order extension of Lyapunov direct method [5]).
Let $x = 0$ be the equilibrium point of the fractional-order system (9). Assume that there exists a fractional Lyapunov function $V(t, x(t)) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and class $\mathcal{K}$ functions $\lambda_i,\ i = 1, 2, 3$ satisfying:

(i) $\lambda_1(\|x\|) \leq V(t, x(t)) \leq \lambda_2(\|x\|)$,

(ii) ${}^C D^{\alpha}_{t_0} V(t, x(t)) \leq -\lambda_3(\|x\|)$.

Then the fractional-order system (9) is asymptotically stable.
Moreover, if $U = \mathbb{R}^n$, then the fractional-order system (9) is globally asymptotically stable.

Lemma 2. *[8] Let $x(t) \in \mathbb{R}$ be a real continuous and differentiable function. Then, for any time $t \geq t_0$,*

$$\frac{1}{2}{}^C D^{\alpha}_{t_0} x^2(t) \leq x(t)^C D^{\alpha}_{t_0} x(t),\ for\ all\ 0 < \alpha < 1. \tag{5}$$

Remark 1. *[1] In the case when $x(t) \in \mathbb{R}^n$, lemma (2) is still valid. That is, $\alpha \in (0, 1)$ and $t \geq t_0$,*

$$\frac{1}{2}{}^C D^{\alpha}_{t_0} x^T(t) x(t) \leq x^T(t)^C D^{\alpha}_{t_0} x(t).$$

In addition, let $x(t) \in \mathbb{R}$ be a real continuous and differentiable function. Then, for any $p = 2n$, $n \in \mathbb{N}$, we have

$${}^C D^{\alpha}_{t_0} x^p \leq p\, x^{p-1}\ {}^C D^{\alpha}_{t_0} x(t),$$

where $0 < \alpha < 1$.

Lemma 3. *[2] Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable function. Then, for any time instant $t \geq t_0$, the following relationship holds*

$$\frac{1}{2}{}^C D^{\alpha}_{t_0} x^T(t) P x(t) \leq x^T(t) P^C D^{\alpha}_{t_0} x(t).$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, square, symmetric and positive definite matrix.

Remark 2. *Lemma 3 is also correct if P is a symmetric and positive semi-definite matrix*

Definition 5. *If there exists a nonnegative constant L satisfying the following inequality for any $x_1(t), x_2(t) \in \mathbb{R}^n$,*

$$\|f(t, x_1) - f(t, x_2(t))\| \leq L\|x_1(t) - x_2(t)\|, \tag{6}$$

then the function is said to be Lipschitz continuous.

Definition 6. *The nonlinear function $f(t, x)$ is said to be one-sided Lipschitz if there exist $\rho \in \mathbb{R}$ such that*

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq \rho\|x_1 - x_2\|^2, \tag{7}$$

where $\rho \in \mathbb{R}$ is called the one-sided Lipschitz constant.

Definition 7. *The nonlinear function $f(t, x)$ is called quadratically inner-bounded if there exist $\beta,\ \gamma \in \mathbb{R}$ such that*

$$[f(t, x_1) - f(t, x_2)]^T [f(t, x_1) - f(t, x_2)] \leq \beta\|x_1 - x_2\|^2 + \gamma\langle x_1 - x_2, f(t, x_1) - f(t, x_2)\rangle \tag{8}$$

Remark 3. *if any function $f(t,x)$ satisfies the Lipschitz condition, then it also satisfies the one-sides Lipschitz and quadratic innerbounded condition, but the converse is not true. Usually, the one-sided Lipschitz constant can be much smaller than the Lipschitz constant. In addition, the constants $\rho, \beta, \gamma \in \mathbb{R}$ can be arbitrary, while the Lipschitz constant must be positive. So the nonlinear part being considered in this paper is fairly general,*

§3 Main Result

In this section, we will pay attention to the following Caputo fractional differential system. The main purpose of this section is to analysis the asymptotical stability of the system.

$$ {}^{C}D^{\alpha}_{t_0}x(t) = f(t,x(t)) = Ax(t) + \mathfrak{a}(t,x(t)),\ x(t_0) = x_0 \tag{9}$$

where $\alpha \in (0,1)$, $x \in \mathbb{R}^n$ represents the state vector of the system, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and $\mathfrak{a} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function satisfying $\mathfrak{a}(t,0) = 0$, for every $t \geq 0$.
Let $\mathfrak{a}(t,x(t)) = 0$, then we obtain the following particular fractional nonlinear systems expressed by

$$ {}^{C}D^{\alpha}_{t_0}x(t) = Ax(t),\ x(t_0) = x_0 \tag{10}$$

The fractional systems define by (10) are called the Caputo fractional linear systems. We have the following results.

Theorem 2. *Let $x = 0$ be an equilibrium point of the system (10). If the state matrix A is Hurwitz then the trivial solution of the fractional linear system (10) is fractional asymptotically stable.*

Proof We choose a Lyapunov candidate function $V(t,x(t)) = x(t)^T Px(t)$ where $A^T P + PA = -Q$ and Q is positive definite. The α Caputo derivative of V along the trajectories of (10) is given by

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}V(t) \leq x^T(t)P\,{}^{C}D^{\alpha}_{t_0}x(t) &= x^T(t)PAx(t) + x^T(t)A^T Px(t) \\ &= x^T(t)(PA + A^T P)x(t) \\ &\leq -\lambda_{min}(Q)\|x(t)\|^2 \end{aligned}$$

Using Theorem 1, we conclude that the trivial solution of the fractional system (10) is fractional asymptotically stable.

□

Now, we consider the perturbation term $\mathfrak{a}(t,x(t)) \neq 0$ for all $x(t) \neq 0$.

Theorem 3. *Let $x = 0$ be an equilibrium point of the system (10). Let that the state matrix A is Hurwitz and the condition $\|\mathfrak{a}(t,x(t))\| < \varrho\|x(t)\|$ holds. If* *there exist a positive definite matrix P such that the following inequality holds*

$$\varrho < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$$

where $A^TP + PA = -Q$, then the trivial solution of the fractional linear system (10) is fractional asymptotically stable.

Proof Let $V(t; x(t)) = x(t)^T Px(t)$ where $A^TP + PA = -Q$ and Q is positive definite. The α Caputo derivative of V along the trajectories of (10) is given by

$$\begin{aligned}
{}^C D^{\alpha}_{t_0} V(t) \leq x^T(t) P {}^C D^{\alpha}_{t_0} x(t) &= x^T(t)P[Ax(t) + \mathfrak{a}(t, x(t))] + [Ax(t) + \mathfrak{a}(t, x(t))]^T Px(t) \\
&= x^T(t)(PA + A^TP)x(t) + x^T(t)P\mathfrak{a}(t, x(t)) + \mathfrak{a}(t, x(t))^T Px(t) \\
&\leq -\lambda_{min}(Q)\|x(t)\|^2 + 2\lambda_{max}(P)\|\mathfrak{a}(t, x(t))\|\|x(t)\| \\
&\leq -\lambda_{min}(Q)\|x(t)\|^2 + 2\varrho\lambda_{max}(P)\|x(t)\|^2 \\
&\leq (-\lambda_{min}(Q) + 2\varrho\lambda_{max}(P))\|x(t)\|^2
\end{aligned}$$

Using Theorem 1, we conclude that the trivial solution of the fractional system (10) is fractional asymptotically stable. □

Theorem 4. *If the function $\mathfrak{a}(t, x(t))$ is Lipschitz continuous, L is Lipschitz constant. Assume that the following assumption is satisfied: There exists a positive symmetric matrix P and positive constant ϵ such that the following inequalities hold $A^TP + PA + \epsilon I < 0$ and $L < \frac{\epsilon}{2\lambda_{max}(P)}$, then the trivial solution of the fractional linear system (10) is fractional asymptotically stable.*

Proof Choose a Lyapunov function $V(t; x(t)) = x(t)^T Px(t)$. The α Caputo derivative of V along the trajectories of (10) is given by

$$\begin{aligned}
{}^C D^{\alpha}_{t_0} V(t) \leq x^T(t) P {}^C D^{\alpha}_{t_0} x(t) &= x^T(t)P[Ax(t) + \mathfrak{a}(t, x(t))] + [Ax(t) + \mathfrak{a}(t, x(t))]^T Px(t) \\
&= x^T(t)(PA + A^TP)x(t) + x^T(t)P\mathfrak{a}(t, x(t)) + \mathfrak{a}(t, x(t))^T Px(t) \\
&\leq -\epsilon\|x(t)\|^2 + 2\lambda_{max}(P)\|\mathfrak{a}(t, x(t))\|\|x(t)\| \\
&\leq -\epsilon\|x(t)\|^2 + 2L\lambda_{max}(P)\|x(t)\|^2 \\
&\leq (-\epsilon + 2L\lambda_{max}(P))\|x(t)\|^2
\end{aligned}$$

Using Theorem 1, we conclude that the trivial solution of the fractional system (10) is fractional asymptotically stable. □

Theorem 5. *If the function $\mathfrak{a}(t, x(t))$ satisfies the conditions (7) and (8) with constants ρ, β and γ. Assume that there exists two matrices P and Q which verifies:*

$$A^TP + PA = -Q, \tag{11}$$

$$\{(\beta + 1) + \rho(\gamma + 2)\} \lambda_{\max}(P) < \lambda_{\min}(P) + \lambda_{\min}(Q), \tag{12}$$

then the origin of system (10) is fractional asymptotically stable.

Proof Let us choose a Lyapunov functional candidate as follows

$$V(x) = x^T(t)Px(t) \tag{13}$$

From Lemma 3, we can conclude

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}V(t) &\leq x^T(t)(A^TP+PA)x(t)+2x^T(t)P\mathfrak{a}(t,x(t)) \\ &\leq -x^T(t)Qx(t)+2x^T(t)P\mathfrak{a}(t,x(t)) \end{aligned}$$

We denote $\mathfrak{a}(t,x(t)) := \mathfrak{a}$. Since $\mathfrak{a}(t,0)=0$, we have

$$2x^TP\mathfrak{a} = [x+\mathfrak{a}]^T P [x+\mathfrak{a}] - x^TPx - \mathfrak{a}^TP\mathfrak{a} \tag{14}$$

Since P satisfies the following inequality,

$$\lambda_{\min}(P)\|\mathfrak{a}\|^2 \leq \mathfrak{a}^TP\mathfrak{a} \leq \lambda_{\max}(P)\|\mathfrak{a}\|^2 \tag{15}$$

So,

$$[x+\mathfrak{a}]^T P [x+\mathfrak{a}] \leq \lambda_{\max}(P)[x+\mathfrak{a}]^T [x+\mathfrak{a}] \tag{16}$$

Using the quadratic inner-bounded, we get

$$\begin{aligned} [x+\mathfrak{a}]^T[x+\mathfrak{a}] &= x^Tx+2x^T\mathfrak{a}+\mathfrak{a}^T\mathfrak{a} \\ &\leq x^Tx+2x^T\mathfrak{a}+\beta x^Tx+\gamma\langle x,\mathfrak{a}\rangle \\ &\leq (\beta+1)x^Tx+(\gamma+2)x^T\mathfrak{a} \end{aligned}$$

From (16) and one-sided Lipschitz, we get

$$\begin{aligned} 2x^TP\mathfrak{a} &\leq \lambda_{\max}(P)\left[(\beta+1)x^Tx+(\gamma+2)x^T\mathfrak{a}\right] - x^TPx - \lambda_{\min}(P)\|\mathfrak{a}\|^2 \\ &\leq \{\lambda_{\max}(P)[(\beta+1)+\rho(\gamma+2)] - \lambda_{\min}(P)\} x^Tx \end{aligned}$$

So, we deduce that

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}V(t) &\leq -x^TQx + \{\lambda_{\max}(P)[(\beta+1)+\rho(\gamma+2)] - \lambda_{\min}(P)\} x^Tx \\ &\leq -\{\lambda_{\min}(Q)+\lambda_{\min}(P)-\lambda_{\max}(P)[(\beta+1)+\rho(\gamma+2)]\}\|x\|^2 \end{aligned}$$

By Theorem1, it is easy to verify that the origin of system (10) is fractional asymptotically stable, the proof is completed. □

Let consider the following fractional differential system

$${}^{C}D^{\alpha}_{t_0}x(t) = Ax(t)+Bu+\mathfrak{a}(t,x(t)),\ x(t_0)=x_0. \tag{17}$$

Where $B \in \mathbb{R}^{n\times p}$ is a constant matrix and $u \in \mathbb{R}^p$ is the control input to be defined.

Theorem 6. *If the function $\mathfrak{a}(t,x(t))$ satisfies the conditions* (7) *and* (8) *with constants ρ, β and γ. Assume that there exists a positive symmetric matrix P, a constant matrix $K \in \mathbb{R}^{p\times n}$ and positive constant ϵ such that:*

$$(A+BK)^TP+P(A+BK) = -\epsilon I \tag{18}$$

$$\{(\beta+1)+\rho(\gamma+2)\}\lambda_{\max}(P) < \lambda_{\min}(P)+\epsilon, \tag{19}$$

then the control law $u(x)=Kx$ render the system (17) fractional asymptotically stable.

Proof Let us choose a Lyapunov functional candidate as follows $V(x) = x^T(t)Px(t)$ From Lemma 2, we can conclude

$$\begin{aligned} {}^C D^{\alpha}_{t_0} x(t) &\le x^T(t)((A+BK)^T P + P(A+BK))x(t) + 2x^T(t)P\mathfrak{a}(t,x(t)) \\ &\le -\epsilon x^T(t)x(t) + 2x^T(t)P\mathfrak{a}(t,x(t)) \\ &\le -\epsilon x^T(t)x(t) + \{\lambda_{\max}(P)\,[(\beta+1)+\rho(\gamma+2)] - \lambda_{\min}(P)\}\, x^T(t)x(t) \\ &\le -\{\epsilon + \lambda_{\min}(P) - \lambda_{\max}(P)\,[(\beta+1)+\rho(\gamma+2)]\}\, \|x(t)\|^2 \end{aligned}$$

By Theorem 1, it is easy to verify that the origin of the closed-loop system $T^{\alpha}x(t) = Ax(t) + BKx(t) + \mathfrak{a}(t,x(t))$ is fractional asymptotically stable. $\square$

§4 Numerical example

In this section, two examples will be provided to demonstrate the effectiveness of the proposed results.

Example 1. *Consider the following fractional system:*

$$ {}^C D^{\alpha}_{t_0} x(t) = Ax(t) + \mathfrak{a}(t,x(t)) \tag{20}$$

Where $x(t) = (x_1(t), x_2(t))$, $A = \begin{pmatrix} -3 & 1 \\ 1 & -5 \end{pmatrix}$ *and* $\mathfrak{a}(t,x(t)) = (\sin x_1(t), \sin x_2(t))$, *where* $P = I^2$. *The* α *Caputo derivative of* V *along the trajectories of (20) is given by*

$$\begin{aligned} {}^C D^{\alpha}_{t_0} V(t) \le x^T P\, {}^C D^{\alpha}_{t_0} x &= x^T P[Ax + \mathfrak{a}(t,x)] + [Ax + \mathfrak{a}(t,x)]^T Px \\ &= x^T(PA + A^T P)x + x^T P\mathfrak{a}(t,x) + \mathfrak{a}(t,x)^T Px \\ &\le -3x_1^2 + 2x_1x_2 - 5x_2^2 + 2x_1 \sin x_1 + 2x_2 \sin x_2 \\ &\le -(x_1 - x_2)^2 - 2x_2^2 \end{aligned}$$

Hence ${}^C D^{\alpha}_{t_0} V(t)$ *is negative definite which implies the trivial solution of the fractional nonlinear system (20) is fractional asymptotically stable. This conclusion can be obtained by applying the Theorem 4. To see that, we can remark the state matrix* A *is Hurwitz and the condition* $L < \frac{\epsilon}{2\lambda_{max}(P)}$ *is hold, with* $L = 1, \epsilon = 3$ *and* $\lambda_{max}(P) = 1$, *thus the trivial solution of (20) is fractional asymptotically stable.*

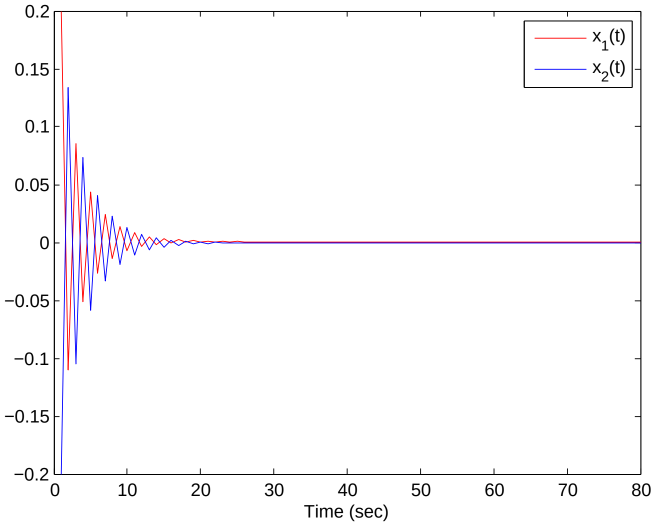

Fig 1: Evolution of the state $x_1(t)$ and $x_2(t)$ of Example 1 and initial conditions $x_1(0) = -0.2$ and $x_2(0) = 0.2$.

The numerical solution to the system (20) is shown in the Figure 1 for some suitable value of fractional order $\alpha = 0.4$. It indicates that the zero solution is asymptotically stable

Example 2. *Consider the following fractional system:*

$$
{}^{C}D_{t_0}^{\alpha}x(t) = Ax(t) + Bu + \mathfrak{a}(t, x(t)). \tag{21}
$$

Where $x(t) = (x_1(t), x_2(t))$, $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$,$B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *and* $\mathfrak{a}(t, x(t)) = (x_1^2(t) + x_2^2(t)) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.
We note the parameters of the one-sided Lipschitz condition and quadratic inner-boundedness inequality are $\rho = 0$, $\beta = -100$ *and* $\gamma = -99$. *Now select* $K = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ *then,* $A_K = A + BK = \begin{pmatrix} -3 & 11 \\ 1 & -4 \end{pmatrix}$ *is Hurwitz, we also choose* $\epsilon = 1$. *The matrix P is given by* $P = \begin{pmatrix} 9.8571 & 2.6429 \\ 2.6429 & 0.7857 \end{pmatrix}$, *the condition 18 and the condition 19 are holds, and*

$$
\begin{aligned}
T^{\alpha}V(t) &= x^T(t)((A + BK)^T P + P(A + BK))x(t) + 2x^T(t)Pf(t, x(t)) \\
&\leq -\{\epsilon + \lambda_{\min}(P) - \lambda_{\max}(P)[(\beta + 1) + \rho(\gamma + 2)]\} \|x(t)\|^2 \\
&\leq -1.4596||x(t)||^2
\end{aligned}
.
$$

Hence the system (21) is fractional asymptotically stable.

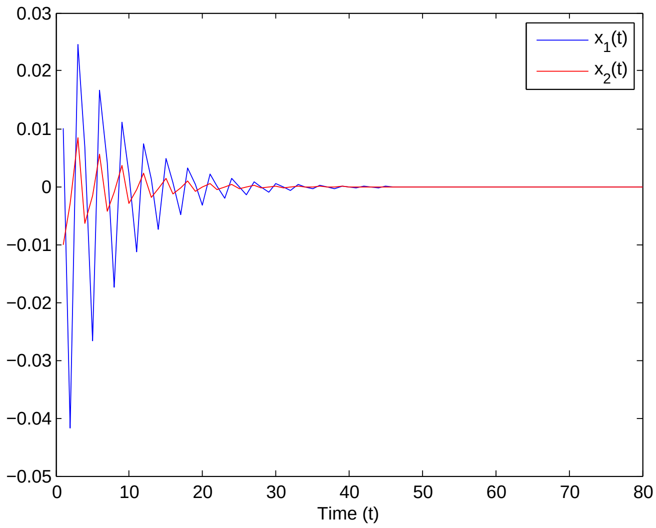

Fig 2: Evolution of the state $x_1(t)$ and $x_2(t)$ of Example 2 and initial conditions $x_1(0) = -0.1$ and $x_2(0) = 0.1$.

The numerical solution to the system (21) is shown in the Figure 2 for some suitable value of fractional order $\alpha = 0.24$. It indicates that the zero solution is asymptotically stable

§5 Conclusion

We have discussed in this paper the asymptotic stability and stabilization of the fractional nonlinear system with Hurwitz state matrix. It contributes to giving practical conditions under which the fractional nonlinear systems with perturbation terms are asymptotically stable.

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