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RESEARCH ARTICLE

Fractional Nonuniform Multiresolution Analysis in $L^2(\mathbb{R})$

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In order to provide a significantly richer representation of non-stationary signals appearing in various disciplines of science and engineering, we introduce here a novel fractional nonuniform multiresolution analysis (FrNUMRA) on the spectrum Λ given by $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, where $N \geqq 1$ is an integer and r is an odd integer with $1 \leqq r \leqq 2N-1$, such that r and N are relatively prime. The necessary and sufficient condition for the existence of nonuniform wavelets of fractional order is derived and an algorithm is also presented for the construction of fractional NUMRA starting from a fractional low-pass filter h_0^α with appropriate conditions. Moreover, we provide a complete characterization for the biorthogonality of the translates of the scaling functions of two fractional nonuniform multiresolution analyses and the associated fractional biorthogonal wavelet families.

KEYWORDS:
Fractional nonuniform multiresolution analysis, Fractional wavelet, Biorthogonal wavelet, Scaling function, Fractional Fourier transform.

1 | INTRODUCTION

A generalization of Mallat's celebrated theory of multiresolution analysis (MRA) [13] was presented by Gabardo and Nashed [9] for the dilation $2N$ and the translation set Λ given by

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z},$$

where $N \geqq 1$ is an integer and r is an odd integer with $1 \leqq r \leqq 2N-1$, such that r and N are relatively prime, acting on the scaling function ϕ, is no longer a group, but a union of two lattices, which is associated with a famous Fuglede conjecture on spectral pairs. Such constructions are called nonuniform MRA (NUMRA). An NUMRA is a non-decreasing family of closed subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R})$ satisfying the following axioms:
(i) $V_j \subset V_{j+1} \quad (j \in \mathbb{Z};$
(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iii) $f(\cdot) \in V_j$ if and only if $f(2N\cdot) \in V_{j+1}$;
(iv) There exists a function $\phi(\cdot) \in V_0$ such that $\{\phi(\cdot - \lambda) : \lambda \in \Lambda\}$ is an orthonormal basis for V_0.

The function ϕ whose existence is asserted in (iv) above is called a scaling function or father wavelet of the given NUMRA. It is worth mentioning that, when $N = 1$ and $\Lambda = \mathbb{Z}$, the nested family $\left\{V_j : j \in \mathbb{Z}\right\}$ reduces to the classical MRA. These studies were continued by Gabardo and his colleagues in [10], [11] and [22], wherein they derived an extension of Cohen's theorem which gives the necessary and sufficient condition for the orthonormality of the collection $\{\phi(\cdot - \lambda) : \lambda \in \Lambda\}$ and provided a complete characterization of associated wavelets by means of its dimension function. The theory of nonuniform wavelets was further studied and investigated by several researchers in such different directions as, for instance, nonuniform wavelet packets[3], nonuniform wavelet frames (see[16] and[17]), nonuniform wavelets and wavelet packets on local fields of positive characteristic (see[18],[19] and[20]) and vector-valued nonuniform wavelets and wavelet packets (see[1] and[14]).

On the other hand, fractional calculus is the outcome of multi-disciplinary endeavor that brought together mathematicians, physicists and engineers. During the past several decades, fractional calculus has been recognized as one of the valuable tools in order to describe many phenomena in engineering, mathematical biology, physical sciences, electrochemistry, acoustics, control theory, psychology and other areas of science that can be elegantly modelled by means of fractional-order derivatives (see, for example,[7],[12],[15] and[23]). Since an MRA is considered as the heart of the wavelet theory because it provides a natural framework for understanding and constructing discrete orthonormal wavelet bases. Keeping in view the exciting developments of the nonuniform MRA along with the profound applicability of the fractional calculus, we are deeply motivated to introduce a novel fractional nonuniform multiresolution analysis (FrNUMRA) in $L^2(\mathbb{R})$ and construct a new class of orthonormal nonuniform wavelets of fractional order. In this setup, the associated core subspace V_0 of $L^2(\mathbb{R})$ has an orthonormal basis, a collection of translates of a function ϕ of the form:

$$\left\{\phi(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda\right\},$$

where the translation set Λ is not necessarily a group, but it is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. In order to facilitate the motive, we establish a necessary and sufficient condition for the existence of the associated wavelets of fractional order and develop an algorithm for the construction of fractional NUMRA in $L^2(\mathbb{R})$ starting from a fractional low-pass filter $h_0^\alpha(u)$ with appropriate conditions. Moreover, we show that, if the translates of the scaling functions of two fractional nonuniform multiresolution analyses are biorthogonal, then the associated fractional wavelet families are also biorthogonal. Finally, it is hoped that the nonuniform wavelets of fractional order might provide significantly richer representations of the signals appearing in various disciplines of science and engineering, particularly in signal processing, multiplicative filtering, sampling theory, optics, biomedical imaging, oceanology, bioinformatics and operator theory.

The layout of the article is as follows. We start Section 2 with a brief overview of the fractional Fourier transform and then introduce the notion of fractional NUMRA on the spectrum Λ. A necessary and sufficient condition for the existence of fractional wavelets is also presented in the same section. Section 3 is devoted to the construction of a fractional NUMRA starting from a fractional low-pass filter h_0^α with appropriate conditions. In Section 4, we study some biorthogonal properties of the fractional nonuniform wavelets. Finally, in Section 5, we present the concluding remarks and observation.

2 | FRACTIONAL NONUNIFORM MULTIRESOLUTION ANALYSIS IN $L^2(\mathbb{R})$

We shall start this section with a brief overview of the fractional Fourier transform and then introduce the notion of the fractional NUMRA in $L^2(\mathbb{R})$.

Definition 2.1. (see[1] and[24]) For any function $f \in L^2(\mathbb{R})$, the α-order fractional Fourier transform (FrFT) is denoted by $\mathscr{F}^\alpha$ and defined by

$$\mathscr{F}^\alpha\big[f\big](u) = \widehat{f}(u) := \int_{\mathbb{R}} f(x)\,\mathcal{K}_\alpha(x,u)\,dx, \tag{2.1}$$

where $\mathcal{K}_\alpha(x,u)$ given by

$$\mathcal{K}_\alpha(x,u) = \begin{cases} \mathcal{A}_\alpha\, e^{\pi i(u^2+x^2)\cot\alpha - 2\pi i u x \csc\alpha} & (\alpha \neq k\pi) \\ \delta(x-u) & (\alpha = 2k\pi) \\ \delta(x+u) & \big(\alpha = (2k-1)\pi\big) \end{cases} \tag{2.2}$$

is the transform kernel with

$$\mathcal{A}_\alpha = \frac{e^{i\alpha/2}}{(i\sin\alpha)^{1/2}} \qquad (k \in \mathbb{Z}).$$

In case α is an integral multiple of π, the FrFT corresponds to a chirp multiplication. This case will be tacitly omitted throughout this paper. Moreover, for $\alpha = \pi/2$, the FrFT reduces to the classical Fourier transform. The inverse FrFT corresponding to (2.1) is given by

$$f(x)=\mathscr{F}^{-\alpha}\left\{\mathscr{F}^{\alpha}\big[f\big](u)\right\}(x)=\int_{\mathbb{R}}\mathscr{F}^{\alpha}\big[f\big](u)\,\overline{\mathcal{K}_{\alpha}(x,u)}\,du. \tag{2.3}$$

It is worth noticing that the new argument u in Definition 2.1 represents a new physical quantity extended from the frequency concept and is termed as the fractional Fourier domain-frequency. For any $f,g\in L^2(\mathbb{R})$, the Parseval formula for the FrFT states that

$$\Big\langle \mathscr{F}^{\alpha}\big[f\big],\mathscr{F}^{\alpha}\big[g\big]\Big\rangle=\big\langle f,g\big\rangle. \tag{2.4}$$

In particular, for $f=g$, we have the following energy preserving relation:

$$\Big\|\mathscr{F}^{\alpha}\big[f\big]\Big\|^2=\Big\|f\Big\|_2^2. \tag{2.5}$$

After the inception of the fractional Fourier transform[2], the fractional convolution and the associated convolution and product theorems have received considerable attention mainly due to the fact that the fractional Fourier transform has outlasted the classical Fourier transform in terms of applications to various fields of signal and image processing (see[15] and[8]). Among several available definitions of the fractional convolution, we shall follow the one mentioned in[21] mainly because of the elegant structure of the corresponding convolution theorem.

Definition 2.2. (see[21]) Given any two functions $f,g\in L^2(\mathbb{R})$, the α-order fractional convolution is denoted by $\circledast_\alpha$ and is defined as follows:

$$\big(f\circledast_\alpha g\big)(y):=\int_{\mathbb{R}} f(x)\,g(y-x)\,e^{\pi i(x^2-y^2)\cot\alpha}\,dx. \tag{2.6}$$

The convolution theorem corresponding to (2.6) states that, for any $f,g\in L^2(\mathbb{R})$,

$$\mathscr{F}^{\alpha}\big[f\circledast_\alpha g\big](u)=\mathscr{F}^{\alpha}\big[f\big](u)\mathscr{F}^{\pi/2}\big[g\big](u\csc\alpha). \tag{2.7}$$

Next, for an integer $N\geqq 1$ and an odd integer r with $1\leqq r\leqq 2N-1$ such that r and N are relatively prime, we define

$$\Lambda=\left\{0,\frac{r}{N}\right\}+2\mathbb{Z}=\left\{\frac{rk}{N}+2n\,:\,n\in\mathbb{Z}\ \text{ and }\ k=0,1\right\}. \tag{2.8}$$

It is easy to verify that Λ is neither a group nor a uniform discrete set, but it is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Indeed, Λ is the spectrum for the spectral set given by

$$\Gamma=\left[0,\frac{1}{2}\right)\cup\left[\frac{N}{2},\frac{N+1}{2}\right)$$

and the pair (Λ,Γ) is called a spectral pair[9].

We are now in a position to introduce a novel fractional nonuniform multiresolution analysis (FrNUMRA) as Definition 2.3 below.

Definition 2.3. For an integer $N\geqq 1$ and an odd integer r with $1\leqq r\leqq 2N-1$ such that r and N are relatively prime, a fractional NUMRA is a sequence of closed subspaces $\big\{V_j^\alpha\,:\,j\in\mathbb{Z}\big\}$ of $L^2(\mathbb{R})$ such that the following properties hold true:

(a) $V_j^\alpha\subset V_{j+1}^\alpha$ for all $j\in\mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}}V_j^\alpha$ is dense in $L^2(\mathbb{R})$;
(c) $\bigcap_{j\in\mathbb{Z}}V_j^\alpha=\{0\}$;
(d) $f(x)\in V_j^\alpha$ if and only if $f(2Nx)\,e^{\pi i\left((2Nx)^2-\lambda^2\right)\cot\alpha}\in V_{j+1}^\alpha$ for all $j\in\mathbb{Z}$;
(e) There exists a function ϕ in V_0^α such that

$$\left\{\phi(x-\lambda)\,e^{-\pi i(x^2-\lambda^2)\cot\alpha}\,:\,\lambda\in\Lambda\right\}$$

is a complete orthonormal basis for V_0^α.

We find it to be worthwhile to note that the Definition 2.3 reduces to the ordinary NUMRA for $\alpha=\pi/2$. On the other hand, for $N=1,\alpha=\pi/2$ and $\Lambda=\mathbb{Z}$, one recovers the standard definition of a one-dimensional MRA with dyadic dilation. Moreover, when $N>1$, the dilation induced by $2N$ ensures that $2N\Lambda\subset 2\mathbb{Z}\subset\Lambda$.

For every $j \in \mathbb{Z}$, we define W_j^α to be the orthogonal compliment of V_j^α in V_{j+1}^α. Then we have

$$V_{j+1}^\alpha = V_j^\alpha \oplus W_j^\alpha \quad \text{and} \quad W_\ell^\alpha \perp W_k^\alpha \quad \text{if} \quad \ell \neq k. \tag{2.9}$$

It follows for $j > J$ that

$$V_j^\alpha = V_J^\alpha \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}^\alpha, \tag{2.10}$$

where all these subspaces are orthogonal. Condition (b) of Definition 2.3 implies that

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^\alpha, \tag{2.11}$$

is a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces.

Conditions (d) and (e) of the Definition 2.3 implies that

$$\phi_{1,\lambda}^\alpha(x) = (2N)^{1/2}\big((2N)x - \lambda\big) e^{-\pi i\left(x^2-(\lambda/2N)^2\right)\cot\alpha}, \tag{2.12}$$

constitute an orthonormal basis in V_1^α. Since $\phi \in V_0^\alpha \subset V_1^\alpha$ and the collection $\left\{\phi_{1,\lambda}^\alpha : \lambda \in \Lambda\right\}$ is an orthonormal basis in V_1^α, so we have

$$\phi(x) = (2N)^{1/2} \sum_{\lambda\in\Lambda} a_\lambda\, \phi\big(2Nx - \lambda\big)\, e^{-\pi i\left(x^2-(\lambda/2N)^2\right)\cot\alpha}, \tag{2.13}$$

where

$$a_\lambda = \int_{\mathbb{R}} \phi(x)\, e^{-\pi i x^2\cot\alpha}\, \overline{\phi_{1,\lambda}^\alpha(x)}\, dx \quad \text{with} \quad \sum_{\lambda\in\Lambda} |a_\lambda|^2 < \infty. \tag{2.14}$$

By taking the fractional Fourier transform on both sides of (2.13), we obtain

$$\hat{\phi}(2Nu\csc\alpha) = h_0^\alpha(u\csc\alpha)\,\hat{\phi}(u\csc\alpha), \tag{2.15}$$

where

$$h_0^\alpha(u\csc\alpha) = \frac{1}{\sqrt{2N}} \sum_{\lambda\in\Lambda} a_\lambda^\alpha\, e^{-2\pi i\lambda u\csc\alpha}. \tag{2.16}$$

Since

$$\Lambda = \{0, r/N\} + 2\mathbb{Z},$$

so the fractional low-filter h_0^α can be written as follows:

$$h_0^\alpha(u\csc\alpha) = h_0^{\alpha,1}(u\csc\alpha) + e^{-2\pi i(u\csc\alpha)r/N}\, h_0^{\alpha,2}(u\csc\alpha), \tag{2.17}$$

where $h_0^{\alpha,1}$ and $h_0^{\alpha,2}$ are locally L^2, $\frac{\sin\alpha}{2}$ periodic functions of fractional order α.

We note that the dilation factor in the fractional NUMRA is $2N$, so one expects the existence of $2N-1$ functions so that their translations by elements of Λ and dilations by the integral powers of $2N$ form an orthonormal basis for $L^2(\mathbb{R})$.

Definition 2.4. A set of functions $\left\{\psi_1^\alpha, \psi_2^\alpha, \cdots, \psi_{2N-1}^\alpha\right\}$ in $L^2(\mathbb{R})$ will be called a set of basic fractional wavelets associated with a given fractional NUMRA if the following family of functions:

$$\left\{\psi_\ell(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} : 1 \leqq \ell \leqq 2N-1,\ \lambda \in \Lambda\right\}$$

constitutes an orthonormal basis for W_0^α.

We thus need to look for a set of fractional wavelets $\left\{\psi_1^\alpha, \psi_2^\alpha, \cdots, \psi_{2N-1}^\alpha\right\}$ in W_0^α such that

$$\psi_{\ell,j,\lambda}^\alpha(x) = (2N)^{j/2}\psi_\ell\big((2N)^j x - \lambda\big)\, e^{-\pi i\left(x^2-\left(\lambda/(2N)^j\right)^2\right)\cot\alpha} \quad (1 \leqq \ell \leqq 2N-1;\ \lambda \in \Lambda) \tag{2.18}$$

forms an orthonormal basis for W_j^α. By the nested structure of the fractional NUMRA, this task can be reduced to find $\psi_\ell^\alpha \in W_0^\alpha$ such that

$$\psi_{\ell,0,\lambda}^\alpha(x) = \psi_\ell\big(x-\lambda\big)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \quad (1 \leqq \ell \leqq 2N-1;\ \lambda \in \Lambda) \tag{2.19}$$

constitute an orthonormal basis for W_0^α.

We set $\psi_0^\alpha = \phi_{0,0}^\alpha$, the scaling function and consider $2N-1$ functions $\psi_\ell^\alpha \;\; (1 \leqq \ell \leqq 2N-1)$ in W_0^α as possible candidates for fractional wavelets. Since

$$(2N)^{-1/2}\psi_\ell\big(x/2N\big)\, e^{-\pi i x^2 \cot\alpha} \in V_{-1}^\alpha \subset V_0^\alpha,$$

it follows from the property (d) of Definition 2.3 that, for each $\ell \;\; (0 \leqq \ell \leqq 2N-1)$, there exists a sequence $\left\{ b_{\ell,\lambda}^\alpha \right\}_{\lambda\in\Lambda}$ with $\sum_{\lambda\in\Lambda} \left| b_{\ell,\lambda}^\alpha \right|^2 < \infty$ such that

$$\psi_\ell\left(\frac{x}{2N}\right) e^{-\pi i x^2 \cot\alpha} = (2N)^{1/2} \sum_{\lambda\in\Lambda} b_{\ell,\lambda}^\alpha\, \phi_{0,\lambda}^\alpha(x). \tag{2.20}$$

By taking the fractional Fourier transform on both sides of (2.20), we obtain

$$\widehat{\psi_\ell}(2Nu\csc\alpha) = h_\ell^\alpha(u\csc\alpha)\,\widehat{\phi}(u\csc\alpha), \tag{2.21}$$

where

$$h_\ell^\alpha(u\csc\alpha) = \frac{1}{\sqrt{2N}} \sum_{\lambda\in\Lambda} b_{\ell,\lambda}^\alpha\, e^{-2\pi i\lambda u\csc\alpha}. \tag{2.22}$$

In view of the specific form of

$$\Lambda = \{0, r/N\} + 2\mathbb{Z},$$

we observe that

$$h_\ell^\alpha(u\csc\alpha) = h_\ell^{\alpha,1}(u\csc\alpha) + e^{-2\pi i(u\csc\alpha)r/N}\, h_\ell^{\alpha,2}(u\csc\alpha), \tag{2.23}$$

where $h_\ell^{\alpha,1}$ and $h_\ell^{\alpha,2}$ are locally L^2, $\frac{\sin\alpha}{2}$ periodic functions of fractional order α.

We are now in a position to establish Theorem 2.5 below on the completeness of the system given by

$$\left\{\psi_\ell(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, 1 \leqq \ell \leqq 2N-1 \quad \text{and} \quad \lambda\in\Lambda\right\}$$

in V_1^α. In fact, we will find two orthonormality conditions of the system by means of periodic functions h_ℓ^α as defined in the equation (2.23).

Theorem 2.5. Consider a fractional NUMRA with the associated parameters N and r as in Definition 2.3. Suppose that there exist $2N-1$ functions $\psi_\ell^\alpha \;\; (1 \leqq \ell \leqq 2N-1)$ in V_1^α. Then the following collection:

$$\psi_{\ell,0,\lambda}^\alpha(x) = \psi_\ell(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, 1 \leqq \ell \leqq 2N-1 \qquad (\lambda\in\Lambda) \tag{2.24}$$

forms an orthonormal system in V_1^α if and only if

$$\sum_{p=0}^{2N-1}\left[h_k^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N}\right)\overline{h_\ell^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N}\right)} + h_k^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N}\right)\overline{h_\ell^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N}\right)}\right] = \delta_{k,\ell} \tag{2.25}$$

and

$$\sum_{p=0}^{2N-1}\beta^p\left[h_k^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N}\right)\overline{h_\ell^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N}\right)} + h_k^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N}\right)\overline{h_\ell^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N}\right)}\right] = 0, \tag{2.26}$$

where $\beta = e^{-\pi i r/N}$.

Proof. First of all, we will prove the assertion is necessary. Indeed, by the orthonormality of the system (2.24), we have

$$\begin{aligned}
\left\langle \psi_{k,0,\lambda}^\alpha, \psi_{\ell,0,\sigma}^\alpha \right\rangle &= \left\langle \psi_k(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha}, \psi_\ell(x-\sigma)\, e^{-\pi i(x^2-\sigma^2)\cot\alpha} \right\rangle \\
&= \int_{\mathbb{R}} \psi_k(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha}\, \overline{\psi_\ell(x-\lambda)}\, e^{\pi i(x^2-\lambda^2)\cot\alpha}\, dx \\
&= e^{\pi i(\lambda^2-\sigma^2)\cot\alpha} \int_{\mathbb{R}} \psi_k(x-\lambda)\, \overline{\psi_\ell(x-\lambda)}\, dx \\
&= e^{\pi i(\lambda^2-\sigma^2)\cot\alpha}\, \delta_{k,\ell}\, \delta_{\lambda,\sigma},
\end{aligned}$$

where δ denotes the Kronecker delta function, $\lambda, \sigma \in \Lambda$ and $0 \leqq k, \ell \leqq 2N-1$. Equivalently, in the fractional frequency domain, we have

$$\delta_{k,\ell}\, \delta_{\lambda,\sigma} = \frac{1}{\sin\alpha} \int_{\mathbb{R}} \widehat{\psi_k}(u\csc\alpha)\, \overline{\widehat{\psi_\ell}(u\csc\alpha)}\, e^{-2\pi i(\lambda-\sigma)u\csc\alpha}\, du.$$

Upon setting $\lambda = 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}
\delta_{k,\ell}\, \delta_{m,n} &= \frac{1}{\sin\alpha} \int_{\mathbb{R}} \widehat{\psi_k}(u\csc\alpha)\, \overline{\widehat{\psi_\ell}(u\csc\alpha)}\, e^{-4\pi i(m-n)u\csc\alpha}\, du \\
&= \frac{1}{\sin\alpha} \int_{[0, N\sin\alpha)} e^{-4\pi i(m-n)u\csc\alpha} \sum_{j\in\mathbb{Z}} \widehat{\psi_k}(u\csc\alpha + Nj)\, \overline{\widehat{\psi_\ell}(u\csc\alpha + Nj)}\, du.
\end{aligned}$$

Let us now consider

$$\Gamma_{k,\ell}(u\csc\alpha) = \sum_{j\in\mathbb{Z}} \widehat{\psi_k}(u\csc\alpha + Nj)\, \overline{\widehat{\psi_\ell}(u\csc\alpha + Nj)}. \tag{2.27}$$

Then we have

$$\begin{aligned}
\delta_{k,\ell}\, \delta_{\lambda,\sigma} &= \frac{1}{\sin\alpha} \int_{[0, N\sin\alpha)} e^{-4\pi i(m-n)u\csc\alpha}\, \Gamma_{k,\ell}(u\csc\alpha)\, du \\
&= \frac{1}{\sin\alpha} \int_{[0, \frac{\sin\alpha}{2})} e^{-4\pi i(m-n)u\csc\alpha} \left[\sum_{p=0}^{2N-1} \Gamma_{k,\ell}\left(u\csc\alpha + \frac{p}{2}\right) \right] du
\end{aligned}$$

and

$$\sum_{p=0}^{2N-1} \Gamma_{k,\ell}\left(u\csc\alpha + \frac{p}{2}\right) = 2\,\delta_{k,\ell}. \tag{2.28}$$

Thus, by taking $\lambda = r/N + 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we find that

$$\begin{aligned}
0 &= \frac{1}{\sin\alpha} \int_{\mathbb{R}} e^{-2\pi i(r/N+2m-2n)u\csc\alpha}\, \widehat{\psi_k}(u\csc\alpha)\, \widehat{\psi_\ell}(u\csc\alpha)\, du \\
&= \frac{1}{\sin\alpha} \int_{\mathbb{R}} e^{-4\pi i(m-n)u\csc\alpha}\, e^{-2\pi i(u\csc\alpha)\, r/N}\, \widehat{\psi_k}(u\csc\alpha)\, \widehat{\psi_\ell}(u\csc\alpha)\, du \\
&= \frac{1}{\sin\alpha} \int_{[0, N\sin\alpha)} e^{-4\pi i(m-n)u\csc\alpha}\, e^{-2\pi i(u\csc\alpha)\, r/N} \sum_{j\in\mathbb{Z}} \widehat{\psi_k}\,(u\csc\alpha + Nj)\, \overline{\widehat{\psi_\ell}\,(u\csc\alpha + Nj)}\, du \\
&= \frac{1}{\sin\alpha} \int_{[0, N\sin\alpha)} e^{-4\pi i(m-n)u\csc\alpha}\, e^{-2\pi i(u\csc\alpha)\, r/N}\, \Gamma_{k,\ell}\,(u\csc\alpha)\, du \\
&= \frac{1}{\sin\alpha} \int_{[0, \frac{\sin\alpha}{2})} e^{-4\pi i(m-n)u\csc\alpha}\, e^{-2\pi i(u\csc\alpha)\, r/N} \left[\sum_{p=0}^{2N-1} e^{-\pi i p r/N}\, \Gamma_{k,\ell}\left(u\csc\alpha + \frac{p}{2}\right) \right] du.
\end{aligned}$$

Therefore, we conclude that

$$\sum_{p=0}^{2N-1} \beta^p\, \Gamma_{k,\ell}\left(u\csc\alpha + \frac{p}{2}\right) = 0, \tag{2.29}$$

where $\beta = e^{-\pi i r/N}$. Thus, clearly, the equations (2.28) and (2.29) are equivalent to the orthonormality of the system given by (2.24).

Next, we will represent the conditions (2.28) and (2.29) in terms of h_ℓ^α as follows:

$$\begin{aligned}
&\Gamma_{k,\ell}(2Nu\csc\alpha)\\
&=\sum_{j\in\mathbb{Z}}\widehat{\psi_k}\left(2N\left(u\csc\alpha+\frac{j}{2}\right)\right)\overline{\widehat{\psi_\ell}\left(2N\left(u\csc\alpha+\frac{j}{2}\right)\right)}\\
&=\sum_{j\in\mathbb{Z}}h_k^\alpha\left(u\csc\alpha+\frac{j}{2}\right)\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\overline{h_\ell^\alpha\left(u\csc\alpha+\frac{j}{2}\right)\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)}\\
&=\sum_{j\in\mathbb{Z}}h_k^\alpha\left(u\csc\alpha+\frac{j}{2}\right)\overline{h_\ell^\alpha\left(u\csc\alpha+\frac{j}{2}\right)}\left|\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\right|^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&\Gamma_{k,\ell}(2Nu\csc\alpha)\\
&=\left[h_k^{\alpha,1}(u\csc\alpha)\,\overline{h_\ell^{\alpha,1}(u\csc\alpha)}+h_k^{\alpha,2}(u\csc\alpha)\,\overline{h_\ell^{\alpha,2}(u\csc\alpha)}\right]\sum_{j\in\mathbb{Z}}\left|\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\right|^2\\
&\quad+\left[e^{2\pi i(u\csc\alpha)r/N}\,h_k^{\alpha,1}(u\csc\alpha)\,\overline{h_\ell^{\alpha,2}(u\csc\alpha)}\sum_{j\in\mathbb{Z}}\beta^{-j}\left|\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\right|^2\right]\\
&\quad+\left[e^{-2\pi i(u\csc\alpha)r/N}\,h_k^{\alpha,2}(u\csc\alpha)\,\overline{h_\ell^{\alpha,1}(u\csc\alpha)}\sum_{j\in\mathbb{Z}}\beta^{j}\left|\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\right|^2\right].
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\Gamma_{k,\ell}(2Nu\csc\alpha)\\
&=\left[h_k^{\alpha,1}(u\csc\alpha)\,\overline{h_\ell^{\alpha,1}(u\csc\alpha)}+h_k^{\alpha,2}(u\csc\alpha)\overline{h_\ell^{\alpha,2}(u\csc\alpha)}\right]\sum_{j=0}^{2N-1}\Gamma_{0,0}\left(u\csc\alpha+\frac{j}{2}\right)\\
&\quad+\left[h_k^{\alpha,1}(u\csc\alpha)\,\overline{h_\ell^{\alpha,2}(u\csc\alpha)}\,e^{2\pi i(u\csc\alpha)r/N}\sum_{j=0}^{2N-1}\beta^{-j}\,\Gamma_{0,0}\left(u\csc\alpha+\frac{j}{2}\right)\right]\\
&\quad+\left[h_k^{\alpha,2}(u\csc\alpha)\,\overline{h_\ell^{\alpha,1}(u\csc\alpha)}\,e^{-2\pi i(u\csc\alpha)\,r/N}\sum_{j=0}^{2N-1}\beta^{j}\,\Gamma_{0,0}\left(u\csc\alpha+\frac{j}{2}\right)\right]\\
&=2\left[h_k^{\alpha,1}(u\csc\alpha)\,\overline{h_\ell^{\alpha,1}(u\csc\alpha)}+h_k^{\alpha,2}(u\csc\alpha)\,\overline{h_\ell^{\alpha,2}(u\csc\alpha)}\right]. \qquad (2.30)
\end{aligned}$$

By combining the identities (2.28) to (2.30), we obtain the desired conditions (2.25) and (2.26).

Next, we shall prove the sufficiency part of the assertion. From (2.21) and (2.30), we observe that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\widehat{\psi_k}\left(2N\left(u\csc\alpha+\frac{j}{2}\right)\right)\overline{\widehat{\psi_\ell}\left(2N\left(u\csc\alpha+\frac{j}{2}\right)\right)}\\
&\quad=\sum_{j\in\mathbb{Z}}h_k^\alpha\left(u\csc\alpha+\frac{j}{2}\right)\overline{h_\ell^\alpha\left(u\csc\alpha+\frac{j}{2}\right)}\left|\hat{\phi}\left(u\csc\alpha+\frac{j}{2}\right)\right|^2\\
&\quad=2\left[h_k^{\alpha,1}\left(\frac{u\csc\alpha}{2N}\right)\overline{h_\ell^{\alpha,1}\left(\frac{u\csc\alpha}{2N}\right)}+h_k^{\alpha,2}\left(\frac{u\csc\alpha}{2N}\right)\overline{h_\ell^{\alpha,2}\left(\frac{u\csc\alpha}{2N}\right)}\right]\\
&\quad=2\sum_{p=0}^{2N-1}\left[h_k^{\alpha,1}\left(\frac{1}{2N}\left(u\csc\alpha+\frac{p}{2}\right)\right)\overline{h_\ell^{\alpha,1}\left(\frac{1}{2N}\left(u\csc\alpha+\frac{p}{2}\right)\right)}\right]\\
&\qquad\qquad+\left[h_k^{\alpha,2}\left(\frac{1}{2N}\left(u\csc\alpha+\frac{p}{2}\right)\right)\overline{h_\ell^{\alpha,2}\left(\frac{1}{2N}\left(u\csc\alpha+\frac{p}{2}\right)\right)}\right]\\
&\quad=2\,\delta_{k,\ell},
\end{aligned}$$

which proves the orthonormality of the system (2.24). This completes the proof of Theorem 2.5. □

The following result asserts the existence of a fractional nonuniform wavelet function.

Theorem 2.6. Let

$$\left\{\psi^{\alpha}_{\ell,0,\lambda} \,:\, 1 \leqq \ell \leqq 2N-1 \;\text{ and }\; \lambda \in \Lambda\right\}$$

be the system as defined in Theorem 2.5 and orthonormal in V_1^{α}. Then this system is complete in $W_0^{\alpha} \equiv V_1^{\alpha} \ominus V_0^{\alpha}$.

Proof. The completeness of the system (2.24) is equivalent to the completeness of the system $\left\{(2N)^{-1}\psi_{\ell}\big((2N)^{-1}x-\lambda\big) \,:\, 0 \leqq \ell \leqq 2N-1, \lambda \in \Lambda\right\}$ in V_0^{α}. Therefore, under given hypothesis, for every function $f \in V_0^{\alpha}$, there exist a unique function $h_0^{\alpha}(u\csc\alpha)$ of the form $\frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} a_{\lambda}\, e^{-i2\pi\lambda u\csc\alpha}$ with $\sum_{\lambda\in\Lambda}|a_{\lambda}|^2 < \infty$ such that

$$\widehat{f}(u\csc\alpha) = h^{\alpha}(u\csc\alpha)\,\widehat{\phi}(u\csc\alpha). \tag{2.31}$$

Therefore, it is enough to show that the family

$$\mathcal{P} = \left\{e^{-i2\pi(2N)\lambda u\csc\alpha}\, h_{\ell}^{\alpha}(u\csc\alpha)\,\chi_A(u\csc\alpha) \,:\, 0 \leqq \ell \leqq 2N-1 \;\text{ and }\; \lambda\in\Lambda\right\} \tag{2.32}$$

is complete in $L^2(A)$, where $A \subset \mathbb{R}$ with $0 < |A| < \infty$.

The family $\left\{e^{-i2\pi\lambda u\csc\alpha}\,\chi_A(u\csc\alpha) \,:\, \lambda\in\Lambda\right\}$ constitutes an orthonormal basis for $L^2(A)$. Hence, clearly, every $g \in L^2(A)$ can be represented as

$$g(u\csc\alpha) = \left[g_1(u\csc\alpha) + e^{-i2\pi(u\csc\alpha)r/N} g_2(u\csc\alpha)\right]\chi_A(u\csc\alpha),$$

where g_1 and g_2 are locally square integrable functions. Suppose that g is orthogonal to all of the functions belonging to the collection (2.32). We thus observe that

$$\begin{aligned}
0 &= \int_A e^{-i2\pi(2N)\lambda u\csc\alpha}\, h_{\ell}^{\alpha}(u\csc\alpha)\,\overline{g(u\csc\alpha)}\,du \\
&= \int_{[0,\frac{\sin\alpha}{2})} e^{-i2\pi(2N)\lambda u\csc\alpha}\left[h_{\ell}^{\alpha}(u\csc\alpha)\,\overline{g(u\csc\alpha)} + h_{\ell}^{\alpha}\left(u\csc\alpha+\frac{N}{2}\right)\overline{g\left(u\csc\alpha+\frac{N}{2}\right)}\right]du \\
&= \int_{[0,\frac{\sin\alpha}{2})} e^{-i2\pi(2N)\lambda u\csc\alpha}\left[h_{\ell}^{\alpha,1}(u\csc\alpha)\,\overline{g_1(u\csc\alpha)} + h_{\ell}^{\alpha,2}(u\csc\alpha)\,\overline{g_2(u\csc\alpha)}\right]du.
\end{aligned}$$

For the choice $\lambda = 2m$, $m\in\mathbb{Z}$ and $\ell = 0,1,2,\cdots,2N-1$, we define

$$\Upsilon_{\ell}(u\csc\alpha) = h_{\ell}^{\alpha,1}(u\csc\alpha)\,\overline{g_1(u\csc\alpha)} + h_{\ell}^{\alpha,2}(u\csc\alpha)\,\overline{g_2(u\csc\alpha)}, \tag{2.33}$$

so that

$$\begin{aligned}
0 &= \int_{[0,\frac{\sin\alpha}{2})} e^{-i2\pi(2N)u\csc\alpha(4N)m}\,\Upsilon_{\ell}(u\csc\alpha)\,du \\
&= \int_{[0,\frac{\sin\alpha}{4N})} e^{-i2\pi u\csc\alpha(4N)m}\sum_{j=0}^{2N-1}\Upsilon_{\ell}\left(u\csc\alpha+\frac{j}{4N}\right)du.
\end{aligned}$$

Since this equality holds true for all $m\in\mathbb{Z}$, we have

$$\sum_{j=0}^{2N-1}\Upsilon_{\ell}\left(u\csc\alpha+\frac{j}{4N}\right) = 0 \quad \text{a.e.} \tag{2.34}$$

Similarly, by taking $\lambda = 2m + r/N$ $(m\in\mathbb{Z})$, we obtain

$$\begin{aligned}
0 &= \int_{[0,\frac{\sin\alpha}{2})} e^{-i2\pi u\csc\alpha\,(4N)m}e^{-i2\pi(2r)u\csc\alpha}\,\Upsilon_{\ell}(u\csc\alpha)\,du \\
&= \int_{[0,\frac{\sin\alpha}{4N})} e^{-i2\pi u\csc\alpha\,(4N)m}e^{-i4\pi ru\csc\alpha}\sum_{j=0}^{2N-1}\beta^j\,\Upsilon_{\ell}\left(u\csc\alpha+\frac{j}{4N}\right)du,
\end{aligned}$$

from which we deduce that

$$\sum_{j=0}^{2N-1} \beta^j \, \Upsilon_\ell \left(u \csc\alpha + \frac{j}{4N} \right) = 0 \quad \text{a.e,}$$

which proves our claim. This completes the proof of Theorem 2.6. □

If $\psi_0^\alpha, \psi_1^\alpha, \ldots, \psi_{2N-1}^\alpha \in V_1^\alpha$ are as in Theorem 2.5, one can get from them an orthonormal basis for $L^2(\mathbb{R})$ by following the standard methodology for construction of wavelets from a given NUMRA [9,14]. It is easy to verify that for every $j \in \mathbb{Z}$, the system

$$\left\{ \psi_{\ell,j,\lambda}^\alpha \,:\, 0 \leqq \ell \leqq 2N-1 \quad \text{and} \quad \lambda \in \Lambda \right\}$$

given by (2.18) constitutes a complete orthonormal system for V_{j+1}. Therefore, it follows immediately from (2.11) that the system (2.18) forms a complete orthonormal system for $L^2(\mathbb{R})$.

In the following theorem, we present a necessary and sufficient condition for the existence of fractional wavelets associated with fractional NUMRA.

Theorem 2.7. Let us consider a fractional NUMRA with associated parameters N and r as in Definition 2.3 such that the corresponding space V_0^α has an orthonormal system of the form $\left\{ \phi(x-\lambda)\, e^{-\pi i (x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda \right\}$ and $\widehat{\phi}$ satisfies the two scale relation (2.15). Define

$$H_0^\alpha(u \csc\alpha) = \left| h_0^{\alpha,1}(u\csc\alpha) \right|^2 + \left| h_0^{\alpha,2}(u\csc\alpha) \right|^2, \tag{2.35}$$

where $h_0^{\alpha,1}$ and $h_0^{\alpha,2}$ are locally L^2-functions of fractional order α. Then a necessary and sufficient condition for the existence of associated fractional wavelets $\psi_1^\alpha, \psi_2^\alpha, \cdots, \psi_{2N-1}^\alpha$ is that H_0^α satisfies the following identity:

$$H_0^\alpha\left(u\csc\alpha + \frac{1}{4} \right) = H_0^\alpha(u\csc\alpha). \tag{2.36}$$

Proof. The orthonormality of the system $\left\{ \phi(x-\lambda)\, e^{-\pi i (x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda \right\}$, which satisfies the condition (2.15), implies the following identities as shown in the proof of Theorem 2.5:

$$\sum_{p=0}^{2N-1} \left[\left| h_0^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right) \right|^2 + \left| h_0^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right) \right|^2 \right] = 1 \tag{2.37}$$

and

$$\sum_{p=0}^{2N-1} \beta^p \left[\left| h_0^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right) \right|^2 + \left| h_0^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right) \right|^2 \right] = 0. \tag{2.38}$$

Similarly, if $\psi_\ell^\alpha, \ell = 0, 1, \cdots, 2N-1$ are the basic fractional wavelets associated with the given fractional NUMRA, then it satisfies the identity (2.21) and the orthonormality of the system $\left\{ \psi_\ell^\alpha \,:\, 0 \leqq \ell \leqq 2N-1 \right\}$ in V_1^α is equivalent to the following identities:

$$\begin{aligned} \sum_{p=0}^{2N-1} \Bigg[& h_k^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right) \overline{h_\ell^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right)} \\ & + h_k^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right) \overline{h_\ell^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right)} \Bigg] = \delta_{k,\ell} \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} \sum_{p=0}^{2N-1} \beta^p \Bigg[& h_k^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right) \overline{h_\ell^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right)} \\ & + h_k^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right) \overline{h_\ell^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right)} \Bigg] = 0, \end{aligned} \tag{2.40}$$

where $0 \leqq k, \ell \leqq 2N-1$. Moreover, if

$$a_\ell(p) = h_\ell^{\alpha,1}\left(u\csc\alpha + \frac{p}{4N} \right) \quad \text{and} \quad b_\ell(p) = h_\ell^{\alpha,2}\left(u\csc\alpha + \frac{p}{4N} \right)$$

are vectors in $\mathbb{C}^{2N}$ for $p = 0, 1, \cdots, 2N-1$ and $0 \leqq \ell \leqq 2N-1$, where $u \in [0, 1/4]$ is fixed, then the solvability of the system of equations (2.39) and (2.40) is equivalent to

$$H_0^\alpha\left(u\csc\alpha + \frac{(p+N)}{4N}\right) = H_0^\alpha\left(u\csc\alpha + \frac{p}{4N}\right) \quad \left(u \in \left[0, \frac{1}{4N}\right]\right),$$

for $p = 0, 1, \cdots, 2N-1$, which (in turn) is equivalent to (2.36). The proof of this fact can be proved in similar lines as Lemma 3.5 in[9]. This completes the proof of Theorem 2.7. □

3 | CONSTRUCTION OF FRACTIONAL NUMRA

The basic idea behind this section is to construct a fraction NUMRA starting from a fractional polynomial of degree $2N-1$ and is of the form

$$h_0^\alpha(u\csc\alpha) = h_0^{\alpha,1}(u\csc\alpha) + e^{-i2\pi(u\csc\alpha)r/N}\, h_0^{\alpha,2}(u\csc\alpha), \tag{3.1}$$

where $N \geqq 1$ is an integer and r is an odd integer with $1 \leqq r \leqq 2N-1$ such that r and N are relatively prime and $h_0^{\alpha,1}$ and $h_0^{\alpha,2}$ are locally square integrable functions of fractional order α. In other words, we build up conditions under which the solutions of scaling equating (2.13) generates a fractional NUMRA in $L^2(\mathbb{R})$. The father wavelet ϕ associated with the given fractional NUMRA should satisfy the following scaling identity:

$$\hat{\phi}(u\csc\alpha) = h_0^\alpha\left(\frac{u\csc\alpha}{2N}\right)\hat{\phi}\left(\frac{u\csc\alpha}{2N}\right). \tag{3.2}$$

Thus, by replacing $u\csc\alpha$ by $u\csc\alpha/2N$ in relation (3.2), we obtain

$$\hat{\phi}\left(\frac{u\csc\alpha}{2N}\right) = h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^2}\right)\hat{\phi}\left(\frac{u\csc\alpha}{(2N)^2}\right)$$

and then

$$\hat{\phi}(u\csc\alpha) = h_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^2}\right)\hat{\phi}\left(\frac{u\csc\alpha}{(2N)^2}\right).$$

Continuing like this, we obtain

$$\hat{\phi}(u\csc\alpha) = \hat{\phi}\left(\frac{u\csc\alpha}{(2N)^n}\right)\prod_{k=1}^{n} h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right).$$

Taking $n \to \infty$ and noting that $1/(2N)^k \to 0$ as $k \to \infty$, the above relation reduces to

$$\hat{\phi}(u\csc\alpha) = \hat{\phi}(0)\prod_{k=1}^{\infty} h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right). \tag{3.3}$$

As in the standard case, we assume that $\hat{\phi}(u)$ is continuous at zero and that $\hat{\phi}(0) = 1$. Then the equation (3.3) becomes

$$\hat{\phi}(u\csc\alpha) = \prod_{k=1}^{\infty} h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right). \tag{3.4}$$

It follows immediately from (3.2) that $h_0^\alpha(0) = 1$, which is essential for convergence of the infinite product:

$$\prod_{k=1}^{\infty} h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right).$$

Define

$$H_0^\alpha(u\csc\alpha) = \left|h_0^{\alpha,1}(u\csc\alpha)\right|^2 + \left|h_0^{\alpha,2}(u\csc\alpha)\right|^2. \tag{3.5}$$

We also assume that the following condition holds true:

$$\sum_{p=0}^{2N-1} H_0^\alpha\left(u\csc\alpha + \frac{p}{4N}\right) = 1 \tag{3.6}$$

and

$$\sum_{p=0}^{2N-1} \beta^p H_0^\alpha \left(u \csc\alpha + \frac{p}{4N}\right) = 0 \qquad \left(\beta = e^{-\pi i r/N}\right). \tag{3.7}$$

Then, for any h_0^α of the form (3.1), the conditions (3.6) and (3.7) imply that $\left|h_0^\alpha\right| \leqq 1$ a.e. On the other hand, the case when $\left|h_0^\alpha\right| > 1$ will imply that

$$\left|h_0^{\alpha,1}(u\csc\alpha)\right|^2 + \left|h_0^{\alpha,2}(u\csc\alpha)\right|^2 > 1,$$

which is equivalent to $|H_0^\alpha(u\csc\alpha)| > 1$, and hence contradicts the identity (3.6).

Theorem 3.1. Let h_0^α be a fractional polynomial of the form (3.1) and H_0^α satisfying (3.6) and (3.7). Let $\phi(x)$ be defined by (3.4) and assume that the infinite product defining $\widehat{\phi}$ converges a.e on $\mathbb{R}$. Then the function ϕ belongs to $L^2(\mathbb{R})$.

Proof. Let

$$I_1 = \int_{|u|\leqq N/2} H_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) du, \quad \text{and}$$

$$I_M = \int_{|u|\leqq N(2N)^{M-1}} H_0^\alpha\left(\frac{u\csc\alpha}{(2N)^M}\right) \prod_{k=1}^{M-1}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du, \quad M \geqq 2.$$

Then we have

$$I_1 = \int_{[0,N)} H_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) du = \int_{[0,\frac{\sin\alpha}{2})} \sum_{p=0}^{2N-1} H_0^\alpha\left(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\right) du = \frac{1}{2}$$

$$\begin{aligned} I_M &= \int_{[0,N(2N)^M]} H_0^\alpha\left(\frac{u\csc\alpha}{(2N)^{M+1}}\right) \prod_{k=1}^{M}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du \\ &= \int_{[0,N(2N)^{M-1}]} \sum_{p=0}^{2N-1}\left|h_0^\alpha\left(u\csc\alpha + \frac{p}{2}\right)\right|^2 H_0^\alpha\left(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\right) \prod_{k=1}^{M-1}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du. \end{aligned}$$

Moreover, we observe that

$$\begin{aligned} &\sum_{p=0}^{2N-1}\left|h_0^\alpha\left(u\csc\alpha + \frac{p}{2}\right)\right|^2 H_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) \\ &= \left[\left|h_0^{\alpha,1}(u\csc\alpha)\right|^2 + \left|h_0^{\alpha,2}(u\csc\alpha)\right|^2\right] \sum_{p=0}^{2N-1} H_0^\alpha\left(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\right) \\ &+ \left[\overline{h_0^{\alpha,1}(u\csc\alpha)}\, h_0^{\alpha,2}(u\csc\alpha)\, e^{-i2\pi(u\csc\alpha)r/N} \sum_{p=0}^{2N-1} \beta^p H_0^\alpha\left(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\right)\right] \\ &+ \left[h_0^{\alpha,1}(u\csc\alpha)\, \overline{h_0^{\alpha,2}(u\csc\alpha)}\, e^{i2\pi(u\csc\alpha)r/N} \sum_{p=0}^{2N-1} \beta^{-p} H_0^\alpha\left(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\right)\right] \\ &= \left|h_0^{\alpha,1}(u\csc\alpha)\right|^2 + \left|h_0^{\alpha,2}(u\csc\alpha)\right|^2 \\ &= H_0^\alpha(u\csc\alpha). \end{aligned}$$

This shows that

$$I_M = \int_{[0,N(2N)^{M-1}]} H_0^\alpha\left(\frac{u\csc\alpha}{(2N)^M}\right) \prod_{k=1}^{M-1}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du = I_{M-1} \qquad (\forall\, M \geqq 1).$$

We thus conclude that

$$I_M = I_{M-1} = I_{M-2} = \cdots = I_2 = I_1 = \frac{1}{2}.$$

Hence we have

$$
\begin{aligned}
&\int\limits_{|u|\leqq (N(2N)^{M-1})/2} \left|\widehat{\phi}(u\csc\alpha)\right|^2 du \\
&\leqq \int\limits_{|u|\leqq (N(2N)^{M-1})/2} \left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^M}\right)\right|^2 \prod_{k=1}^{M-1}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du \\
&\leqq \int\limits_{|u|\leqq (N(2N)^{M-1})/2} 2H_0^\alpha\left(\frac{u\csc\alpha}{(2N)^M}\right) \prod_{k=1}^{M-1}\left|h_0^\alpha\left(\frac{u\csc\alpha}{(2N)^k}\right)\right|^2 du \\
&= 2I_M \\
&= 1.
\end{aligned}
$$

Since M is arbitrary, it follows that $\phi \in L^2(\mathbb{R})$. The proof of Theorem 3.1 is completed. □

Next, we shall construct the fractional NUMRA in $L^2(\mathbb{R})$ from a fractional polynomial h_0^α of the form (3.1) which satisfies (3.6) and (3.7) together with the condition $h_0^\alpha(0) = 1$. In order to facilitate this, it is necessary to determine the orthonormality of the system given by

$$\left\{\phi(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda\right\}$$

in $L^2(\mathbb{R})$. Therefore, if the orthonormality condition is satisfied, then we can define V_0^α and V_j^α as follows:

$$V_0^\alpha = \overline{\text{Span}}\left\{\phi(x-\lambda)\, e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda\right\} \tag{3.8}$$

and

$$f(x) \in V_j^\alpha \quad \text{if and only if} \quad f\left((2N)^{-j}x\right) e^{-\pi i x^2\cot\alpha} \in V_0^\alpha,\ j \in \mathbb{Z}, \tag{3.9}$$

respectively, so that the axioms (d) and (e) of Definition 2.3 hold true. The identity (2.15) implies that (a) also holds true. The rest of the conditions (b) and (c) of the Definition 2.3 shall follow from the following results (Theorems 3.2 and 3.3)which are analogies of the results in standard wavelet theory (see[9] and[8]).

For $j \in \mathbb{Z}$, $\lambda \in \Lambda$ and $\alpha \in \mathbb{R}$, we define

$$\phi_{j,\lambda}^\alpha(x) = (2N)^j \phi\Big((2N)^j x - \lambda\Big) e^{-\pi i\left(x^2-\left(\lambda/(2N)^j\right)^2\right)\cot\alpha}. \tag{3.10}$$

Also, for each $j \in \mathbb{Z}$, we define the orthogonal projection P_j^α of $L^2(\mathbb{R})$ onto V_j as follows:

$$P_j^\alpha f = \sum_{\lambda\in\Lambda}\left\langle f, \phi_{j,\lambda}^\alpha\right\rangle \phi_{j,\lambda}^\alpha \tag{3.11}$$

Theorem 3.2. Let $\left\{V_j^\alpha \,:\, j \in \mathbb{Z}\right\}$ be a collection of subspaces defined by (3.9) with given $\phi \in L^2(\mathbb{R})$. If

$$\left\{\phi(x-\lambda)e^{-\pi i(x^2-\lambda^2)\cot\alpha} \,:\, \lambda \in \Lambda\right\}$$

is an orthonormal basis in V_0^α,, then

$$\bigcap_{j\in\mathbb{Z}} V_j^\alpha = \{0\}\,.$$

Proof. Let g be a compactly supported continuous function in some interval I_ε and satisfies $\left\|f-g\right\|_2 < \varepsilon$ for all $f \in \bigcap_{j\in\mathbb{Z}} V_j^\alpha$. Then

$$\left\|f - P_j^\alpha g\right\|_2 = \left\|P_j^\alpha(f-g)\right\|_2 \leqq \left\|f-g\right\|_2 < \varepsilon,$$

so that

$$\left\|f\right\|_2 < \varepsilon + \left\|P_j^\alpha g\right\|_2.$$

Since

$$\left\{\phi_{j,\lambda}^\alpha \,:\, j \in \mathbb{Z}, \lambda \in \Lambda\right\}$$

is an orthonormal basis for V_j^α, therefore, for each $j \in \mathbb{Z}$, we have

$$\left\|P_j^\alpha g\right\|_2^2 = \sum_{\lambda\in\Lambda} \left|\left\langle P_j^\alpha g, \phi_{j,\lambda}^\alpha \right\rangle\right|^2 = \sum_{\lambda\in\Lambda} \left|\left\langle g, \phi_{j,\lambda}^\alpha \right\rangle\right|^2. \tag{3.12}$$

We now choose $I_\varepsilon = [-1/4, 1/4]$. Then, for small enough values of j, we find that

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \left|\left\langle g, \phi_{j,\lambda}^\alpha \right\rangle\right|^2 &= (2N)^j \sum_{\lambda\in\Lambda} \left| \int_{-1/4}^{1/4} g(x)\, \phi\left(\left(2N\right)^j x - \lambda\right) e^{\pi i\left(x^2 - \left(\lambda/(2N)^j\right)^2\right)\cot\alpha}\, dx \right|^2 \\
&\leqq \frac{(2N)^j K^2}{2} \sum_{\lambda\in\Lambda} \int_{-1/4}^{1/4} \left|\phi\left((2N)^j x - \lambda\right)\right|^2 dx \\
&= \frac{K^2}{2} \int_{\bigcup_{\lambda\in\Lambda}[-\lambda-(2N)^j/4, -\lambda+(2N)^j/4]} \chi_{\bigcup_{\lambda\in\Lambda}[-\lambda-(2N)^j/4, -\lambda+(2N)^j/4]} \left|\phi(y)\right|^2 dy,
\end{aligned}$$

where $K = \|g\|_\infty$ is the supremum norm of g. Thus, by applying Lebesgue's dominated convergence theorem, it follows that

$$\lim_{j\to-\infty} \left\|P_j^\alpha g\right\| = 0.$$

Therefore, we conclude that $\left\|f\right\|_2 < \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, $f = 0$, hence

$$\bigcap_{j\in\mathbb{Z}} V_j^\alpha = \{0\}.$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. Suppose $\phi \in L^2(\mathbb{R})$ is such that

$$\left\{\phi_{0,\lambda}^\alpha : \lambda \in \Lambda\right\}$$

is an orthonormal basis in V_0^α and let $\left\{V_j^\alpha : j \in \mathbb{Z}\right\}$ be the family of subspaces as defined in (3.9). Let us assume that $\widehat{\phi}(u\csc\alpha)$ is bounded and continuous near $u = 0$, with $\left|\widehat{\phi}(0)\right| \neq 0$. Then

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j^\alpha} = L^2(\mathbb{R}).$$

Proof. For $f \in \left(\bigcup_{j\in\mathbb{Z}} V_j^\alpha\right)^\perp$ and $\varepsilon > 0$, we choose a compactly supported continuous function f_ε such that $\left\|f - f_\varepsilon\right\| < \varepsilon$. Then, for each $j \in \mathbb{Z}$, we observe from (3.11) that

$$\left\|P_j^\alpha f\right\|_2^2 = \left\langle P_j^\alpha f, P_j^\alpha f\right\rangle_2 = \left\langle f, P_j^\alpha f\right\rangle_2 = 0, \quad \text{and} \tag{3.13}$$

$$\left\|P_j^\alpha f_\varepsilon\right\|_2^2 = \left\|P_j^\alpha (f - f_\varepsilon)\right\|_2 \leqq \left\|f - f_\varepsilon\right\|_2 < \varepsilon. \tag{3.14}$$

Since the family

$$\left\{\phi_{j,\lambda}^\alpha : j \in \mathbb{Z} \quad \text{and} \quad \lambda \in \Lambda\right\}$$

constitutes an orthonormal basis for V_j^α and f_ε is of compact support, we have

$$\left\|P_j^\alpha f_\varepsilon\right\|_2^2 = \sum_{\lambda\in\Lambda} \left|\left\langle f_\varepsilon, \phi_{j,\lambda}^\alpha \right\rangle\right|^2 = \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} \mathscr{F}^\alpha\left[f_\varepsilon\right](u)\, \overline{\mathcal{K}_\alpha\left(u, \frac{\lambda}{(2N)^j}\right) \widehat{\phi}\left(\frac{u\csc\alpha}{(2N)^j}\right)}\, du \right|^2. \tag{3.15}$$

We now choose j to be sufficiently large so that $\operatorname{supp} f_\varepsilon \subseteq [-1/4, 1/4]$ and, for this choice of j, we assume that

$$\Phi(u\csc\alpha) = \mathscr{F}^\alpha\left[f_\varepsilon\right](u)\, \overline{\widehat{\phi}\left(\frac{u\csc\alpha}{(2N)^j}\right)}, \tag{3.16}$$

for some function Φ of the form:

$$\Phi(u\csc\alpha) = h_1^\alpha(u\csc\alpha) + e^{2\pi i(u\csc\alpha)r/N} h_2^\alpha(u\csc\alpha), \tag{3.17}$$

where h_1^α and h_2^α are locally square integrable periodic functions. If $\Phi(u\csc\alpha)$ has the expansion of the form:

$$\sum_{\lambda\in\Lambda} b_\lambda^\alpha e^{-2\pi i\lambda u\csc\alpha}$$

on the set $A=[0,1/2)\cup[N/2,(N+1)/2)$, then

$$\begin{aligned} b_\lambda^\alpha &= \int_A \Phi(u\csc\alpha)\, e^{-2\pi i\lambda u\csc\alpha}\, du \\ &= \int_{\mathbb{R}} \mathscr{F}^\alpha\left[f_\varepsilon\right]\left((2N)^j u\right)\overline{\hat{\phi}(u\csc\alpha)}\, e^{2\pi i\lambda u\csc\alpha}\, du \quad (\lambda\in\Lambda). \end{aligned}$$

Taking $\lambda=2m$, where $m\in\mathbb{Z}$, we have

$$\begin{aligned} &2\int_{[0,\frac{\sin\alpha}{2})} h_1^\alpha(u\csc\alpha)\, e^{2\pi i(2k)u\csc\alpha}\, du \\ &= \int_{[0,\frac{\sin\alpha}{2})} \sum_{k\in\mathbb{Z}} \mathscr{F}^\alpha\left[f_\varepsilon\right]\left((2N)^j u+\frac{k(2N)^j\sin\alpha}{2}\right)\overline{\hat{\phi}\left(u\csc\alpha+\frac{k}{2}\right)}\, e^{2\pi i(2k)u\csc\alpha}\, du. \end{aligned}$$

Therefore, we find that

$$h_1^\alpha(u\csc\alpha)=\frac{1}{2}\sum_{j\in\mathbb{Z}} \mathscr{F}^\alpha\left[f_\varepsilon\right]\left((2N)^j u+\frac{k(2N)^j\sin\alpha}{2}\right)\overline{\hat{\phi}\left(u\csc\alpha+\frac{k}{2}\right)}.$$

Similarly, on taking $\lambda=2m+r/n$, where $m\in\mathbb{Z}$, we obtain

$$h_2^\alpha(u\csc\alpha)=\frac{1}{2}\sum_{k\in\mathbb{Z}} \mathscr{F}^\alpha\left[f_\varepsilon\right]\left((2N)^j u+\frac{k(2N)^j\sin\alpha}{2}\right)\overline{\hat{\phi}\left(u\csc\alpha+\frac{k}{2}\right)}\, e^{2\pi i(u\csc\alpha+k/2)\,r/N}.$$

Consequently, we have

$$\Phi(u\csc\alpha)=\frac{1}{2}\sum_{k\in\mathbb{Z}} \mathscr{F}^\alpha\left[f_\varepsilon\right]\left((2N)^j u+\frac{k(2N)^j\sin\alpha}{2}\right)\overline{\hat{\phi}\left(u\csc\alpha+\frac{k}{2}\right)}\left(1+\beta^k\right).$$

Since $\operatorname{supp} f_\varepsilon\subseteq[-1/4,1/4]$, therefore, for large values of j, (3.15) becomes

$$\left\|P_j^\alpha f_\varepsilon\right\|_2^2 \leqq \int_{\cup_{j\in\mathbb{Z}}[-1/4+Nj,1/4+Nj]} \left|\mathscr{F}^\alpha\left[f_\varepsilon\right](\eta\sin\alpha)\,\overline{\hat{\phi}(\eta/(2N)^j)}\right|^2 d\eta.$$

By invoking Lebesgue's dominated convergence theorem once again, we observe that the right-hand side of above inequality converges to $|\hat{\phi}(0)|^2\|f_\varepsilon^\alpha\|_2^2$, as $j\to\infty$. Therefore, we have

$$\varepsilon > \left\|P_j^\alpha f_\varepsilon\right\|_2^2 = \left\|\mathscr{F}^\alpha[f_\varepsilon]\right\|_2^2 = \left\|f_\varepsilon\right\|_2^2.$$

Consequently, we get

$$\left\|f\right\|_2 < \varepsilon + \left\|f_\varepsilon\right\|_2 < 2\varepsilon.$$

Since ε is arbitrary, therefore, $f=0$. This completes the proof of Theorem 3.3. □

4 | BIORTHOGONAL PROPERTIES OF FRACTIONAL NONUNIFORM WAVELETS

Orthogonality has long been assumed as a key property in virtually all standard approaches when analyzing or synthesizing signals. A higher-level signal processing technique involves the concept of biorthogonality in which two (cross-orthogonal) sets are used: one for the analysis and the other one synthesis. During the early 1990s, biorthogonal wavelets brought a major breakthrough into image compression, thanks to their natural feature of concentrating energy in a few transform coefficients (see[65] and[4]).

Let $\left\{V_j^\alpha : j \in \mathbb{Z}\right\}$ and $\left\{\tilde{V}_j^\alpha : j \in \mathbb{Z}\right\}$ be biorthogonal fractional NUMRA's with scaling functions ϕ and $\tilde{\phi}$, respectively. Then there exists the integral periodic functions $h_0\alpha$ and $\tilde{h}_0\alpha$ of fractional order α such that

$$\widehat{\phi}(2Nu\csc\alpha) = h_0^\alpha(u\csc\alpha)\,\widehat{\phi}(u\csc\alpha) \quad \text{and} \quad \widehat{\tilde{\phi}}(2Nu\csc\alpha) = \tilde{h}_0^\alpha(u\csc\alpha)\,\widehat{\tilde{\phi}}(u\csc\alpha).$$

Suppose that there exists the integral periodic functions h_ℓ^α and $\tilde{h}_\ell^\alpha$ $(1 \leqq \ell \leqq 2N-1)$ such that

$$H^\alpha(u\csc\alpha)\,\tilde{H}^\alpha(u\csc\alpha) = I, \tag{4.1}$$

where

$$H^\alpha(u\csc\alpha) = \begin{pmatrix} h_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) & h_0^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & h_0^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \\ h_1^\alpha\left(\frac{u\csc\alpha}{2N}\right) & h_1^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & h_1^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \\ \vdots & & & \\ h_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}\right) & h_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & h_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \end{pmatrix}$$

and

$$\tilde{H}^\alpha(u\csc\alpha) = \begin{pmatrix} \tilde{h}_0^\alpha\left(\frac{u\csc\alpha}{2N}\right) & \tilde{h}_0^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & \tilde{h}_0^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \\ \tilde{h}_1^\alpha\left(\frac{u\csc\alpha}{2N}\right) & \tilde{h}_1^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & \tilde{h}_1^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \\ \vdots & & & \\ \tilde{h}_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}\right) & \tilde{h}_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{1}{4N}\right) & \cdots & \tilde{h}_{2N-1}^\alpha\left(\frac{u\csc\alpha}{2N}+\frac{2N-1}{4N}\right) \end{pmatrix}.$$

For $1 \leqq \ell \leqq 2N-1$, we define the associated fractional biorthgonal nonuniform wavelets as ψ_ℓ^α and $\tilde{\psi}_\ell^\alpha$ by

$$\widehat{\psi_\ell}(2Nu\csc\alpha) = h_\ell^\alpha(u\csc\alpha)\,\widehat{\phi}(u\csc\alpha)$$

and

$$\widehat{\tilde{\psi}_\ell}(2Nu\csc\alpha) = \tilde{h}_\ell^\alpha(u\csc\alpha)\,\widehat{\tilde{\phi}}(u\csc\alpha).$$

Definition 4.1. A pair of fractional nonuniform multiresolution analyses $\{V_j^\alpha : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j^\alpha : j \in \mathbb{Z}\}$ with scaling functions ϕ and $\tilde{\phi}$, respectively, are said to be biorthogonal to each other if

$$\{\phi_{0,\lambda}^\alpha(x) = \phi(x-\lambda)\,e^{-\pi i(x^2-\lambda^2)\cot\alpha} : \lambda \in \Lambda\}$$

and

$$\left\{\tilde{\phi}_{0,\lambda}^\alpha(x) = \tilde{\phi}(x-\lambda)\,e^{-\pi i(x^2-\lambda^2)\cot\alpha} : \lambda \in \Lambda\right\}$$

are biorthogonal.

Lemma 4.2. Let $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ be given. Then the system

$$\{\phi_{0,\lambda}^\alpha : \lambda \in \Lambda\}$$

is biorthogonal to $\{\tilde{\phi}_{0,\lambda}^\alpha : \lambda \in \Lambda\}$ if and only if

$$\sum_{\lambda\in\Lambda} \widehat{\phi}(u\csc\alpha+\lambda)\,\overline{\widehat{\tilde{\phi}}(u\csc\alpha+\lambda)} = \frac{1}{\sin\alpha}. \tag{4.2}$$

Proof. For all $\lambda, \sigma \in \Lambda$, we observe that

$$\left\langle \phi_{0,\lambda}^\alpha, \tilde{\phi}_{0,\sigma}^\alpha \right\rangle = e^{\pi i(\lambda^2-\sigma^2)\cot\alpha}\,\delta_{\lambda,\sigma} \iff \left\langle \phi_{0,0}^\alpha, \tilde{\phi}_{0,\sigma}^\alpha \right\rangle = e^{-\pi i\sigma^2\cot\alpha}\,\delta_{0,\sigma}.$$

By Parseval's identity, we have

$$
\begin{aligned}
&\left\langle \phi^{\alpha}_{0,0}, \tilde{\phi}^{\alpha}_{0,\sigma} \right\rangle \\
&= \left\langle \mathscr{F}^{\alpha}\left[\phi^{\alpha}_{0,0}\right](u), \mathscr{F}^{\alpha}\left[\tilde{\phi}^{\alpha}_{0,\sigma}\right](u) \right\rangle \\
&= \frac{e^{-\pi i \sigma^2 \cot\alpha}}{\sin\alpha} \int\limits_{\mathbb{R}} \widehat{\phi}(u\csc\alpha)\, \overline{\widehat{\tilde{\phi}}(u\csc\alpha)}\, e^{2\pi i (u\csc\alpha)\sigma}\, du \\
&= \frac{e^{-\pi i \sigma^2 \cot\alpha}}{\sin\alpha} \int\limits_{[0,\frac{\sin\alpha}{2})} \left[\sum_{p\in\mathbb{Z}} \widehat{\phi}\Big(u\csc\alpha + \frac{p}{2}\Big)\, \overline{\widehat{\tilde{\phi}}\Big(u\csc\alpha + \frac{p}{2}\Big)}\, e^{\pi i \sigma p} \right] e^{-2\pi i (u\csc\alpha)\sigma}\, du.
\end{aligned}
\tag{4.3}
$$

Using the fact that $\{e^{-i2\pi(u\csc\alpha)\sigma} : \sigma \in \Lambda\}$ is an orthonormal basis of $L^2[0, \frac{\sin\alpha}{2})$, we get the desired result. This completes the proof of Lemma 4.2. □

Let ϕ and $\tilde{\phi}$ be scaling functions for the fractional biorthogonal nonuniform multiresolution analyses $\left\{V^{\alpha}_j : j \in \mathbb{Z}\right\}$ and $\left\{\tilde{V}^{\alpha}_j : j \in \mathbb{Z}\right\}$, respectively. For each $j \in \mathbb{Z}$, we define the fractional-order operators P^{α}_j and $\tilde{P}^{\alpha}_j$ on $L^2(\mathbb{R})$ by

$$
P^{\alpha}_j f = \sum_{\lambda\in\Lambda} \langle f, \tilde{\phi}^{\alpha}_{j,\lambda} \rangle \phi^{\alpha}_{j,\lambda} \quad \text{and} \quad \tilde{P}^{\alpha}_j f = \sum_{\lambda\in\Lambda} \langle f, \phi^{\alpha}_{j,\lambda} \rangle \tilde{\phi}^{\alpha}_{j,\lambda},
$$

respectively. It is easy to verify that both these fractional operators are uniformly bounded on $L^2(\mathbb{R})$ and both the series are convergent in $L^2(\mathbb{R})$.

Remark 4.3. The fractional-order operators P^{α}_j and $\tilde{P}^{\alpha}_j$ satisfy the following properties:

(a) $P^{\alpha}_j f = f$ if and only if $f \in V^{\alpha}_j$ and $\tilde{P}^{\alpha}_j f = f$ if and only if $f \in \tilde{V}^{\alpha}_j$.

(b) $\lim\limits_{j\to\infty} \left\| P^{\alpha}_j f - f \right\|_2 = 0$ and $\lim\limits_{j\to-\infty} \left\| P^{\alpha}_j f \right\|_2 = 0$ for every $f \in L^2(\mathbb{R})$.

Theorem 4.4. Let ϕ and $\tilde{\phi}$ be the scaling functions for the fractional biorthogonal nonuniform multiresolution analyses $\left\{V^{\alpha}_j : j \in \mathbb{Z}\right\}$ and $\left\{\tilde{V}^{\alpha}_j : j \in \mathbb{Z}\right\}$, respectively. If ψ^{α}_{ℓ} and $\tilde{\psi}^{\alpha}_{\ell}$, $1 \leqq \ell \leqq 2N-1$ are the associated wavelets satisfying (4.1). Then
(i) $\left\{\psi^{\alpha}_{\ell,0,\lambda} : 1 \leqq \ell \leqq 2N-1 \text{ and } \lambda \in \Lambda\right\}$ is biorthogonal to $\left\{\tilde{\psi}^{\alpha}_{\ell,0,\lambda} : 1 \leqq \ell \leqq 2N-1, \lambda \in \Lambda\right\}$.

(ii) $\left\langle \psi^{\alpha}_{\ell,0,\lambda}, \phi^{\alpha}_{0,\sigma} \right\rangle = \left\langle \tilde{\psi}^{\alpha}_{\ell,0,\lambda}, \tilde{\phi}^{\alpha}_{0,\sigma} \right\rangle \quad (\forall\ \lambda, \sigma \in \Lambda)$.

Proof. To prove Part (i), we observe that

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \left[\widehat{\psi}_{\ell}\Big(u\csc\alpha + \frac{j}{2}\Big)\, \overline{\widehat{\tilde{\psi}}_{\ell}\Big(u\csc\alpha + \frac{j}{2}\Big)} \right] \\
&= \sum_{j\in\mathbb{Z}} \left[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{4N}\Big)\, \widehat{\phi}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{4N}\Big)\, \overline{\tilde{h}^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{4N}\Big)\, \widehat{\tilde{\phi}}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{4N}\Big)} \right] \\
&= \sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \left[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{2} + \frac{p}{4N}\Big)\, \widehat{\phi}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{2} + \frac{p}{4N}\Big) \right. \\
&\qquad\qquad \left. \times \overline{\tilde{h}^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{2} + \frac{p}{4N}\Big)\, \widehat{\tilde{\phi}}\Big(\frac{u\csc\alpha}{2N} + \frac{j}{2} + \frac{p}{4N}\Big)} \right] \\
&= \frac{1}{\sin\alpha} \sum_{p=0}^{2N-1} \left[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\Big)\, \overline{\tilde{h}^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N} + \frac{p}{4N}\Big)} \right] \\
&= \frac{1}{\sin\alpha}.
\end{aligned}
$$

By virtue of Lemma 4.2, we obtain the desired result in Part (i).

We now prove Part (ii). For fixed $\lambda, \sigma \in \Lambda$, an application of the Plancherel formula yields

$$\begin{aligned}
&\left\langle \psi^{\alpha}_{\ell,0,\lambda}, \phi^{\alpha}_{0,\sigma} \right\rangle \\
&= \left\langle \mathscr{F}^{\alpha}\left[\psi^{\alpha}_{\ell,0,\lambda}\right](u), \mathscr{F}^{\alpha}\left[\phi^{\alpha}_{0,\sigma}\right](u) \right\rangle \\
&= \frac{e^{\pi i(\lambda^2-\sigma^2)\cot\alpha}}{\sin\alpha} \int\limits_{\mathbb{R}} \widehat{\psi}_{\ell}(u\csc\alpha)\, \overline{\widehat{\tilde{\phi}}(u\csc\alpha)}\, e^{-i2\pi(\lambda-\sigma)u\csc\alpha}\, du \\
&= \frac{e^{-\pi i\sigma^2\cot\alpha}}{\sin\alpha} \int\limits_{\mathbb{R}} h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N}\Big)\, \widehat{\phi}\Big(\frac{u\csc\alpha}{2N}\Big)\, \overline{\tilde{h}^{\alpha}_{0}\Big(\frac{u\csc\alpha}{2N}\Big)\, \widehat{\tilde{\phi}}\Big(\frac{u\csc\alpha}{2N}\Big)} e^{-i2\pi(\lambda-\sigma)u\csc\alpha}\, du \\
&= \frac{e^{-\pi i\sigma^2\cot\alpha}}{\sin\alpha} \int\limits_{[0,\frac{\sin\alpha}{2})} \sum_{j\in\mathbb{Z}} \Bigg[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\Big)\, \widehat{\phi}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\Big) \\
&\qquad\qquad \times \overline{\tilde{h}^{\alpha}_{0}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\Big)\, \widehat{\tilde{\phi}}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\Big)} \Bigg]\, e^{-i2\pi(\lambda-\sigma)u\csc\alpha}\, du \\
&= \frac{e^{-\pi i\sigma^2\cot\alpha}}{\sin\alpha} \int\limits_{[0,\frac{\sin\alpha}{2})} \sum_{p=o}^{2N-1} \sum_{j\in\mathbb{Z}} \Bigg[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\Big)\, \widehat{\phi}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\Big) \\
&\qquad\qquad \times \overline{\tilde{h}^{\alpha}_{0}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\Big)\, \widehat{\tilde{\phi}}\Big(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\Big)} \Bigg]\, e^{-i2\pi(\lambda-\sigma)u\csc\alpha}\, du \\
&= \frac{e^{-\pi i\sigma^2\cot\alpha}}{\sin\alpha} \int\limits_{[0,\frac{\sin\alpha}{2})} \sum_{p=o}^{2N-1} \Bigg[h^{\alpha}_{\ell}\Big(\frac{u\csc\alpha}{2N}+\frac{p}{4N}\Big) \overline{\tilde{h}^{\alpha}_{0}\Big(\frac{u\csc\alpha}{2N}+\frac{p}{4N}\Big)} \Bigg]\, e^{-i2\pi(\lambda-\sigma)u\csc\alpha}\, du \\
&= 0.
\end{aligned}$$

Similarly, we can show that

$$\left\langle \tilde{\psi}^{\alpha}_{\ell,0,\lambda}, \tilde{\phi}^{\alpha}_{0,\sigma} \right\rangle = 0 \qquad (\forall\ \lambda, \sigma \in \Lambda).$$

This completes the proof of the Theorem 4.4. □

Theorem 4.5. Let $\phi, \tilde{\phi}, \psi^{\alpha}_{\ell}$, and $\tilde{\psi^{\alpha}_{\ell}}$ $(1 \leqq \ell \leqq 2N-1)$ be as in Theorem 4.4. Let us put $\psi^{\alpha}_{0} = \phi^{\alpha}_{0,0}$ and $\tilde{\psi}^{\alpha}_{0} = \tilde{\phi}^{\alpha}_{0,0}$. Then, for every $f \in L^2(\mathbb{R})$,
(i)

$$Q^{\alpha} f = P^{\alpha}_{0} f + \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}^{\alpha}_{\ell,0,\lambda} \right\rangle \psi^{\alpha}_{\ell,0,\lambda}, \quad \text{and} \tag{4.4}$$

$$\tilde{Q}^{\alpha} f = \tilde{P}^{\alpha}_{0} f + \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \psi^{\alpha}_{\ell,0,\lambda} \right\rangle \tilde{\psi}^{\alpha}_{\ell,0,\lambda}, \tag{4.5}$$

where the series (4.4) and (4.5) converges in $L^2(\mathbb{R})$.

(ii) The collection $\left\{\psi^{\alpha}_{\ell,j,\lambda} : 1 \leqq \ell \leqq 2N-1, j \in \mathbb{Z} \text{ and } \lambda \in \Lambda\right\}$ is biorthogonal to $\left\{\tilde{\psi}^{\alpha}_{\ell,j,\lambda} : 1 \leqq \ell \leqq 2N-1 \text{ and } j \in \mathbb{Z}, \lambda \in \Lambda\right\}$.

Proof. In order to prove Part (i), we shall only prove the identity (4.4), because the proof of (4.5) will follows along similar lines. Moreover, it is sufficient to prove (4.4) in the weak sense, that is, for all $f, g \in L^2(\mathbb{R})$,

$$\left\langle Q^{\alpha} f, g \right\rangle = \left\langle P^{\alpha}_{0} f, g \right\rangle + \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}^{\alpha}_{\ell,0,\lambda} \right\rangle \overline{\left\langle g, \psi^{\alpha}_{\ell,0,\lambda} \right\rangle} = \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}^{\alpha}_{\ell,0,\lambda} \right\rangle \overline{\left\langle g, \psi^{\alpha}_{\ell,0,\lambda} \right\rangle}.$$

Therefore, we have

$$
\begin{aligned}
&\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\left\langle f,\tilde{\psi}^{\alpha}_{\ell,0,\lambda}\right\rangle\overline{\left\langle g,\psi^{\alpha}_{\ell,0,\lambda}\right\rangle}\\
&=\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\left[\int_{\mathbb{R}}\mathscr{F}^{\alpha}[f](u)\,\overline{\mathcal{K}_{\alpha}(u,\lambda)\,\widehat{\tilde{\psi}}_{\ell}(u\csc\alpha)}\,du\right]\left[\int_{\mathbb{R}}\overline{\mathscr{F}^{\alpha}[g](u)}\,\mathcal{K}_{\alpha}(u,\lambda)\,\widehat{\psi}_{\ell}(u\csc\alpha)\,du\right]\\
&=\frac{1}{\sin\alpha}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\left[\int_{[0,\frac{\sin\alpha}{2})}\sum_{j\in\mathbb{Z}}\mathscr{F}^{\alpha}[f]\left(u+\frac{j\sin\alpha}{2}\right)\overline{\widehat{\tilde{\psi}}_{\ell}\left(u\csc\alpha+\frac{j}{2}\right)}\,e^{i2\pi\lambda u\csc\alpha}\,du\right]\\
&\qquad\times\left[\int_{[0,\frac{\sin\alpha}{2})}\sum_{k\in\mathbb{Z}}\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{k\sin\alpha}{2}\right)}\widehat{\psi}_{\ell}\left(u\csc\alpha+\frac{j}{2}\right)\,e^{-i2\pi\lambda u\csc\alpha}\,du\right]\\
&=\frac{1}{\sin\alpha}\sum_{\ell=0}^{2N-1}\int_{[0,\frac{\sin\alpha}{2})}\left[\sum_{j\in\mathbb{Z}}\mathscr{F}^{\alpha}[f]\left(u+\frac{j\sin\alpha}{2}\right)\overline{\widehat{\tilde{\psi}}_{\ell}\left(u\csc\alpha+\frac{j}{2}\right)}\right]\\
&\qquad\times\left[\sum_{k\in\mathbb{Z}}\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{k\sin\alpha}{2}\right)}\widehat{\psi}_{\ell}\left(u\csc\alpha+\frac{k}{2}\right)\right]du\\
&=\frac{1}{\sin\alpha}\int_{[0,\frac{\sin\alpha}{2})}\sum_{\ell=0}^{2N-1}\left\{\left[\sum_{j\in\mathbb{Z}}\mathscr{F}^{\alpha}[f]\left(u+\frac{j\sin\alpha}{2}\right)\overline{\tilde{h}^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\right)\widehat{\tilde{\phi}}\left(\frac{u\csc\alpha}{2N}+\frac{j}{4N}\right)}\right]\right.\\
&\times\left.\left[\sum_{k\in\mathbb{Z}}\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{k\sin\alpha}{2}\right)}h^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{k}{4N}\right)\widehat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{4N}\right)\right]\right\}du\\
&=\frac{1}{\sin\alpha}\int_{[0,\frac{\sin\alpha}{2})}\sum_{\ell=0}^{2N-1}\left\{\left[\sum_{p=0}^{2N-1}\sum_{j\in\mathbb{Z}}\mathscr{F}^{\alpha}[f]\left(u+\frac{Nj\sin\alpha}{2}+\frac{p\sin\alpha}{4N}\right)\overline{\tilde{h}^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\right)}\right.\right.\\
&\qquad\times\left.\overline{\widehat{\tilde{\phi}}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\right)}\right]\left[\sum_{q=0}^{2N-1}\sum_{k\in\mathbb{Z}}\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{kN\sin\alpha}{2}+\frac{q\sin\alpha}{4N}\right)}\right.\\
&\qquad\qquad\left.\left.\cdot h^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}+\frac{q}{4N}\right)\widehat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}+\frac{q}{4N}\right)\right]\right\}du\\
&=\frac{1}{\sin\alpha}\int_{[0,\frac{\sin\alpha}{2})}\sum_{p=0}^{2N-1}\sum_{q=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left[\sum_{\ell=0}^{2N-1}\overline{\tilde{h}^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{p}{4N}\right)}h^{\alpha}_{\ell}\left(\frac{u\csc\alpha}{2N}+\frac{q}{4N}\right)\right]\\
&\cdot\mathscr{F}^{\alpha}[f]\left(u+\frac{Nj\sin\alpha}{2}+\frac{p\sin\alpha}{2}\right)\overline{\widehat{\tilde{\phi}}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\right)\mathscr{F}^{\alpha}[g]\left(u+\frac{kN\sin\alpha}{2}+\frac{q\sin\alpha}{2}\right)}\\
&\qquad\cdot\widehat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}+\frac{q}{4N}\right)du\\
&=\frac{1}{\sin\alpha}\int_{[0,\frac{\sin\alpha}{2})}\sum_{p=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left[\mathscr{F}^{\alpha}[f]\left(u+\frac{Nj\sin\alpha}{2}+\frac{p\sin\alpha}{2}\right)\overline{\widehat{\tilde{\phi}}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}+\frac{p}{4N}\right)}\right.\\
&\qquad\left.\times\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{kN\sin\alpha}{2}+\frac{p\sin\alpha}{2}\right)}\widehat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}+\frac{p}{4N}\right)\right]du\\
&=\frac{1}{\sin\alpha}\sum_{p=0}^{2N-1}\int_{[0,(p+1/2)\sin\alpha)}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left[\mathscr{F}^{\alpha}[f]\left(u+\frac{Nj\sin\alpha}{2}\right)\overline{\widehat{\tilde{\phi}}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}\right)}\right.\\
&\qquad\left.\times\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{kN\sin\alpha}{2}\right)}\widehat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}\right)\right]du.
\end{aligned}\tag{4.6}
$$

By using similar lines, we can show that

$$\sum_{\lambda\in\Lambda}\left\langle f,\tilde{\phi}^{\alpha}_{1,\lambda}\right\rangle\overline{\left\langle g,\phi^{\alpha}_{1,\lambda}\right\rangle}=\frac{1}{\sin\alpha}\int_{[0,\frac{\sin\alpha}{2})}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left[\mathscr{F}^{\alpha}[f]\left(u+\frac{Nj\sin\alpha}{2}\right)\overline{\hat{\tilde{\phi}}\left(\frac{u\csc\alpha}{2N}+\frac{j}{2}\right)}\right.$$
$$\left.\cdot\,\overline{\mathscr{F}^{\alpha}[g]\left(u+\frac{kN\sin\alpha}{2}\right)}\,\hat{\phi}\left(\frac{u\csc\alpha}{2N}+\frac{k}{2}\right)\right]du. \tag{4.7}$$

From equations (4.6) and (4.7), we obtain the desired result (4.4).

For proving Part (ii), we show for each $j\in\mathbb{Z}$ and $1\leqq\ell\leqq 2N-1$ that

$$\left\langle\psi^{\alpha}_{\ell,j,\lambda},\tilde{\psi}^{\alpha}_{\ell,j,\lambda}\right\rangle=e^{\pi i(\lambda^2-\sigma^2)\cot\alpha}\,\delta_{\lambda,\sigma}.$$

For $j=0$, this claim follows immediately by applying Theorem 4.4. For $j\neq 0$, we have

$$\left\langle\psi^{\alpha}_{\ell,j,\lambda},\tilde{\psi}^{\alpha}_{\ell,j,\lambda}\right\rangle=\left\langle\delta_{-j}\,\psi^{\alpha}_{\ell,0,\lambda},\delta_{-j}\,\tilde{\psi}^{\alpha}_{\ell,0,\lambda}\right\rangle=\left\langle\psi^{\alpha}_{\ell,0,\lambda},\tilde{\psi}^{\alpha}_{\ell,0,\lambda}\right\rangle=e^{\pi i(\lambda^2-\sigma^2)\cot\alpha}\,\delta_{\lambda,\sigma}.$$

This completes the proof of the Theorem 4.5. □

Theorem 4.6. Let $\phi,\tilde{\phi},\psi^{\alpha}_{\ell}$, and $\tilde{\psi^{\alpha}_{\ell}}$ $(1\leqq\ell\leqq 2N-1)$ be given as in Theorem 4.5. Then, for every $f\in L^2(\mathbb{R})$,

$$f=\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left\langle f,\tilde{\psi}^{\alpha}_{\ell,j,\lambda}\right\rangle\psi^{\alpha}_{\ell,j,\lambda}=\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left\langle f,\psi^{\alpha}_{\ell,j,\lambda}\right\rangle\tilde{\psi}^{\alpha}_{\ell,j,\lambda}, \tag{4.7}$$

where the series converges in $L^2(\mathbb{R})$.

Proof. The result asserted by Theorem 4.4 follows immediately by using Remark 4.3 and Theorem 4.5. □

5 | CONCLUDING REMARKS AND OBSERVATIONS

Here, in our present investigation, we have provided significantly richer representation of non-stationary signals appearing in various disciplines of science and engineering. Our methodology is based essentially upon a novel fractional nonuniform multiresolution analysis (FrNUMRA) on the spectrum Λ given by $\Lambda=\left\{0,\frac{r}{N}\right\}+2\mathbb{Z}$, where $N\geqq 1$ is an integer and r is an odd integer with $1\leqq r\leqq 2N-1$ such that r and N are relatively prime. We have successfully derived the necessary and sufficient condition for the existence of nonuniform wavelets of fractional order. We have also presented an algorithm for the construction of fractional NUMRA starting from a fractional low-pass filter h^{α}_0 under appropriate conditions. Furthermore, we have obtained a complete characterization for the biorthogonality of the translates of the scaling functions of two fractional nonuniform multiresolution analyses and the associated fractional biorthogonal wavelet families.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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