Constraint Minimizers of Inhomogeneous Mass Subcritical Minimization Problems

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Abstract

This paper considers minimizers of the following inhomogeneous L^2-subcritical energy functional

$$E(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u|^{p+1} dx,$$

under the mass constraint $\|u\|_2^2 = M$. Here $N \geq 1$, $p \in (1, 1 + \frac{4}{N})$, $M > 0$ and the inhomogeneous term $m(x)$ satisfies $0 < m(x) \leq 1$. Applying the concentration-compactness principle, we prove that this minimization problem admits minimizers for any $M \in (0, \infty)$. Further more, we also present a detail analysis on the influence of $m(x)$ on the limit behavior of minimizers as $M \to \infty$.

Keywords: Inhomogeneous; Mass subcritical; Minimizers; Mass concentration.

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1 Introduction and main results

In physical research, many physical phenomena can be described by L^2-constraint variational problems, such as the mass concentration of Bose-Einstein condensates (BEC), the collapse of pseudo-relativistic Boson stars, etc., (cf. [1, 7, 8, 23]). In addition, from the point of view of variation, the well-posed results of standing waves for nonlinear Schrödinger equations can be given by using the constraint variational method (cf. [2, 11]). Therefore, in recent years, the research of L^2-constraint variational problems has been widely concerned by many scholars at home and abroad, see, e.g., [2, 10, 11, 12, 13, 17, 18, 20, 21] and the references therein.

In this paper, we study constraint minimizers of the following L^2-subcritical constraint variational problem

$$I(M) := \inf_{\{u \in H^1(\mathbb{R}^N), \|u\|_2^2 = M\}} E(u), \tag{1.1}$$

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where the energy functional $E(u)$ is defined by

$$E(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u|^{p+1} dx. \tag{1.2}$$

Here $N \geq 1$, $p \in (1, 1+\frac{4}{N})$, $M > 0$ and the inhomogeneous term $m(x) \not\equiv 1$ satisfies

(M_1). $m(x) \in L^\infty_{loc}(\mathbb{R}^N) \bigcap C^\alpha(\mathbb{R}^N)(0 < \alpha < 1)$, $0 < m(x) \leq m(0) = \max_{x\in\mathbb{R}^N} m(x) = 1$ and $0 \leq \inf_{x\in\mathbb{R}^N} m(x) = \lim_{|x|\to\infty} m(x) = m_\infty \leq 1$,

(M_2). $0 \in \mathbb{R}^N$ is the unique global maximum point of $m(x)$ and $1 - m(x) = |x|^{s+2}\big(1 + o(1)\big)$ as $|x| \to 0$, where $s > 0$.

In particular, when $m(x) \equiv 1$, the research methods and results of this paper can be naturally generalized. Hence, we do not elaborate here.

Problem (1.1) is put forward by P. L. Lions in [21] where he considered orbital stability waves in nonlinear Schrödinger equations. In recent years, Guo and his collaborators (cf. [3, 4, 9, 10, 11, 13]) have studied a more general L^2-constraint variational problem:

$$I_{V(x)}(M) := \inf_{u\in H^1_{V(x)}} E_{V(x)}(u), \tag{1.3}$$

where the energy functional $E_{V(x)}(u)$ is defined by

$$E_{V(x)}(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u|^{p+1} dx,$$

$V(x) : \mathbb{R}^N \to \mathbb{R}$ is a suitable potential function, and $H^1_{V(x)}$ is defined by

$$H^1_{V(x)} := \big\{u \in H^1(\mathbb{R}^N) : \quad V(x)|u|^2 \in L^1(\mathbb{R}^N),\ \|u\|_2^2 = M\big\}.$$

These works (cf. [9, 10, 11, 13]) are mainly focused on the existence, uniqueness and mass concentration behavior for minimizers of (1.3) under different potential functions as $m(x) \equiv 1$ and $p = 1 + \frac{4}{N}$. As for the inhomogeneous case $m(x) \not\equiv 1$ and $p = 1 + \frac{4}{N}$, they (cf. [3, 4]) also presented a detailed analysis about the specific influence of the inhomogeneous term on the existence and limit behavior for minimizers of (1.3). Roughly speaking, when $p = 1 + \frac{4}{N}$ (the so-called L^2-critical case), there exists a threshold M^* such that (1.3) admits minimizers if $M < M^*$, and however, the influence of $m(x)$ on the existence results as $M = M^*$ will become uncertain and depends on the shape and some local profiles of $m(x)$.

When $1 < p < 1 + \frac{4}{N}$, (1.3) is an L^2-subcritical constraint variational problem. As for the homogeneous case $m(x) \equiv 1$, there are many results on studying (1.3), including the existence, uniqueness, symmetry and the concentration behavior of minimizers for (1.3), see [2, 12, 18, 19, 20, 21, 22, 24, 29] and the references therein. However, all these work mentioned require the trapping potential $V(x)$ to meet some certain conditions, such as $\lim_{|x|\to\infty} V(x) = \infty$. If $V(x)$ is bounded or even $V(x) \equiv 0$, as far as we know, there is no result on this case. On the other hand, when $m(x) \not\equiv 1$, there is also little work investigating the effect of $m(x)$ on this L^2-subcritical constraint variational problem. Motivated by these previous works, in this paper, we shall focus on analyzing the effect of the inhomogeneous term $m(x)$ on the existence results and limit behavior of minimizers for problem (1.3) with $V(x) \equiv 0$ and $1 < p < 1 + \frac{4}{N}$.

Setting $V(x) \equiv 0$, $m(x) \not\equiv 1$ and $1 < p < 1 + \frac{4}{N}$, problem (1.3) can be rewritten as (1.1), which is what we study in the present paper. Just as mentioned above, this paper is devoted to analyzing the existence results of minimizers for (1.1) and the limit behavior of minimizers as $M \to \infty$. Before stating our main results, we now introduce the following scalar field equation

$$-\Delta u + u = u^p, \quad u \in H^1(\mathbb{R}^N), \quad N \geq 1, \quad 1 < p < 1 + \frac{4}{N}. \tag{1.4}$$

It is well-known from [5, 15, 16, 26] that (1.4) admits a unique (up to translations) radially symmetric positive solution, which can be denoted as $Q = Q(|x|)$. Note also from [5, Proposition 4.1] that $Q(|x|)$ satisfies

$$Q(|x|), |\nabla Q(|x|)| = O(|x|^{-\frac{N-1}{2}} e^{-|x|}) \text{ as } |x| \to \infty. \tag{1.5}$$

Moreover, we recall from [26] the following Gagliardo-Nirenberg inequality

$$\|u\|_{p+1}^{p+1} \leq C_{GN}^{-1} \|\nabla u\|_2^{\frac{N}{2}(p-1)} \|u\|_2^{p+1-\frac{N}{2}(p-1)}, \quad u \in H^1(\mathbb{R}^N), \quad N \geq 1, \quad 1 < p < 1 + \frac{4}{N}, \tag{1.6}$$

where $C_{GN} > 0$ satisfies

$$C_{GN} := \|Q\|_2^{p-1} \Big(1 - \frac{p-1}{p+1}\frac{N}{2}\Big) \Big[\frac{N(p-1)}{2(p+1) - N(p-1)}\Big]^{\frac{N}{4}(p-1)}, \tag{1.7}$$

and the equality in (1.6) is achieved at $u = Q$. Moreover, one can derive from (1.4) and (1.6) that Q satisfies

$$\int_{\mathbb{R}^N} |\nabla Q|^2 dx = \frac{N}{2}\frac{p-1}{p+1} \int_{\mathbb{R}^N} Q^{p+1} dx = \frac{N(p-1)}{2(p+1) - N(p-1)} \int_{\mathbb{R}^N} Q^2 dx. \tag{1.8}$$

By using the above results, we shall establish the following existence theorem of minimizers for (1.1).

Theorem 1.1. *Assume $m(x)$ satisfies (M_1), and then there exists at least one minimizer of $I(M)$ for any $M \in (0, \infty)$.*

The key to complete the proof of of Theorem 1.1 is to overcome the possible loss of compactness of minimizing sequences. In order to overcome this difficulty, we adopt the concentration-compactness principle [20, 21]. Roughly speaking, we shall first prove the strict sub-additivity inequality of $I(M)$. Applying the GN inequality, one can further obtain the uniform boundedness of the minimizing sequences. Moreover, by using the concentration-compactness principle, the compactness of minimizing sequences can be obtained via a series of detailed analysis on ruling out the dichotomy case and the vanishing case.

Assume u_M is a minimizer of (1.1). One can note the fact that $E(u) \geq E(|u|)$ holds for any $u \in H^1(\mathbb{R}^N)$ due to the fact that $|\nabla u| \geq |\nabla |u||$ *a.e.* in $\mathbb{R}^N$, which indicates that u_M does not change the sign. Therefore, without loss of generality, one can suppose u_M is non-negative, i.e., $u_M \geq 0$. Moreover, one can derive that u_M satisfies the following Euler-Lagrange equation

$$-\Delta u_M - m(x)u_M^p = \mu_M u_M \text{ in } \mathbb{R}^N, \tag{1.9}$$

where $\mu_M \in \mathbb{R}$ is a suitable Lagrange multiplier associated to u_M.

Stimulated by [10, 19, 22, 25], we are next concerned with the limit behavior of minimizers u_M as $M \to \infty$. Our main result can be stated as the following theorem.

Theorem 1.2. *Suppose $m(x)$ satisfies (M_1) and (M_2). Let u_k be a nonnegative minimizer of $I(M_k)$. Then for any sequence $\{M_k\}$ with $M_k \to \infty$ as $k \to \infty$, there exists a subsequence of u_k, still denoted by u_k, such that u_k has a unique maximum point $\bar{z}_k$ and satisfies*

$$\lim_{k\to\infty} \epsilon_k^{\frac{2}{p-1}} u_k(\epsilon_k x + \bar{z}_k) = Q(x) \text{ in } H^1(\mathbb{R}^N), \tag{1.10}$$

where $\lim\limits_{k\to\infty} \bar{z}_k = 0$, $\epsilon_k := \left(\frac{M_k}{a^*}\right)^{-\frac{p-1}{4-N(p-1)}}$, $a^* := \|Q(x)\|_2^2$ and $Q(x)$ is the unique radially symmetric positive solution of (1.4).

Theorem 1.2 presents a refined description of the limit behavior of minimizers: each minimizer u_k must concentrate at a global maximum of the inhomogeneous term $m(x)$ as $k \to \infty$. One hard part of proving Theorem 1.2 is to obtain a suitable limit equation of (1.9), due to the unboundedness of u_k in $L^p(\mathbb{R}^N)(p \in (2, 2 + \frac{4}{N}))$. Motivated by [22], we rewrite the constraint variational problem (1.1) into the following equivalent form:

$$I_M := \inf_{\{v\in H^1(\mathbb{R}^N), \|v\|_2^2=1\}} E_M(v), \tag{1.11}$$

where $E_M(v)$ is defined by

$$E_M(v) := \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} m(x)|v|^{p+1} dx, \; 1 < p < 1 + \frac{4}{N}. \tag{1.12}$$

One can verify that, if u_M be a nonnegative minimizer of $I(M)$, then $v_M := M^{-\frac{1}{2}} u_M$ is a nonnegative minimizer of I_M and $I_M = M^{-1} \cdot I(M)$. In view of the relation between u_M and v_M, the proof of Theorem 1.2 can be equivalently convert into analyzing the limit behavior of minimizers for (1.11) as $M \to \infty$. Up to some necessary scaling of the minimizers for (1.11), one can obtain the boundedness of minimizers as $M \to \infty$.

Another challenge in studying the limit behavior of v_M is to locate the peak of v_M as $M \to \infty$. Inspired by [18, 19, 22], we introduce the following new constraint variational problem

$$\tilde{I}_M := \inf_{\{v\in H^1(\mathbb{R}^N), \|v\|_2^2=1\}} \tilde{E}_M(v), \tag{1.13}$$

where $\tilde{E}_M(v)$ is defined by

$$\tilde{E}_M(v) := \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |v|^{p+1} dx, \; 1 < p < 1 + \frac{4}{N}. \tag{1.14}$$

By establishing that $I_M - \tilde{I}_M \to 0$ as $M \to \infty$, one can derive that $\frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \big(1-m(x)\big)|v_M|^{p+1}dx \to 0$ as $M \to \infty$, which is a great aid to locate the peak of minimizers.

This paper is organized as follows. Section 2 is devoted to proving Theorem 1.1 on the existence of minimizers for (1.1). In Section 3, we firstly prove Theorem 3.1 on the limit behavior of minimizers for I_M as $M \to \infty$, upon which one can then complete the proof of Theorem 1.2.

2 Existence of minimizers for $I(M)$

This section is concerned with the proof of Theorem 1.1 on the existence of minimizers for (1.1). We shall firstly establish the strict sub-additivity inequality of $I(M)$, and then prove Theorem 1.1 by applying the concentration-compactness principle.

Lemma 2.1. *Assume $m(x)$ satisfies (M_1), and then for any $M \in (0,\infty)$, we have the following strict sub-additivity inequality*

$$I(M) < I(\alpha) + I(M-\alpha), \quad \forall\ \alpha \in (0, M). \tag{2.1}$$

Proof. Firstly, we claim that there exists a constant $C > 0$, independent of M, such that

$$I(M) \le -C < 0 \text{ for any } M \in (0,\infty). \tag{2.2}$$

In fact, set $u_\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$, where $\lambda > 0$ is a constant and $u \in H^1(\mathbb{R}^N)$ satisfies $\|u\|_2^2 = M$. One can deduce that, for any $M \in (0,\infty)$,

$$\begin{aligned} I(M) \le E(u_\lambda) &= \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 dx - \frac{2}{p+1}\int_{\mathbb{R}^N} m(x)|u_\lambda|^{p+1}dx \\ &= 2\Big(\frac{\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda^{\frac{N}{2}(p-1)}}{p+1}\int_{\mathbb{R}^N} m\big(\frac{x}{\lambda}\big)|u|^{p+1}dx\Big). \end{aligned}$$

It then follows that $E(u_\lambda) < 0$ for $\lambda > 0$ sufficiently small, due to the fact that $\frac{N}{2}(p-1) < 2$ and $m(x)$ satisfies (M_1), i.e., (2.2) holds.

Using (2.2), for any $\alpha \in (0, M\big)$ and $\theta \in (1, \frac{M}{\alpha}]$, one can now derive that

$$\begin{aligned} I(\theta\alpha) = &\inf_{\{u \in H^1(\mathbb{R}^N), \|u\|_2^2 = \theta\alpha\}} E(u) = \inf_{\{v \in H^1(\mathbb{R}^N), \|v\|_2^2 = \alpha\}} E(\theta^{\frac{1}{2}} v) \\ = &\inf_{\{v \in H^1(\mathbb{R}^N), \|v\|_2^2 = \alpha\}} \Big\{\theta \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{2\theta^{\frac{p+1}{2}}}{p+1}\int_{\mathbb{R}^N} m(x)|v|^{p+1}dx\Big\} \\ = &\inf_{\{v \in H^1(\mathbb{R}^N), \|v\|_2^2 = \alpha\}} \Big\{\theta\Big[\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{2}{p+1}\int_{\mathbb{R}^N} m(x)|v|^{p+1}dx\Big] \\ &\qquad + \frac{2(\theta - \theta^{\frac{p+1}{2}})}{p+1}\int_{\mathbb{R}^N} m(x)|v|^{p+1}dx\Big\} \\ < &\ \theta I(\alpha) < 0, \end{aligned}$$

where the penultimate "$<$" holds due to $\theta > 1$ and $p > 1$. This implies that for any $M \in (0, \infty)$,

$$I(\theta\alpha) < \theta I(\alpha), \quad \forall\ \alpha \in (0, M), \quad \forall\ \theta \in (1, \frac{M}{\alpha}]. \tag{2.3}$$

Further more, it follows from (2.3) that

$$I(M) = \frac{M-\alpha}{M} I\Big(\frac{M}{M-\alpha}(M-\alpha)\Big) + \frac{\alpha}{M} I\Big(\frac{M}{\alpha}\cdot\alpha\Big) < I(M-\alpha) + I(\alpha), \qquad \forall \alpha \in (0, M).$$

Hence, the proof of Lemma 2.1 is completed. $\square$

Proof of Theorem 1.1: For any given $M \in (0, \infty)$, assume $\|u\|_2^2 = M$. Applying (M_1) and GN inequality (1.6) to $E(u)$ then yields that

$$\begin{aligned} E(u) &= \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u(x)|^{p+1} dx \\ &\geq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} |u(x)|^{p+1} dx \\ &\geq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \frac{2}{p+1} C_{GN}^{-1} \|\nabla u\|_2^{\frac{N}{2}(p-1)} \|u\|_2^{p+1-\frac{N}{2}(p-1)} \\ &= \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - C\big(\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx\big)^{\frac{N(p-1)}{4}}, \end{aligned} \tag{2.4}$$

which implies that $E(u)$ is bounded from below for any $M \in (0, \infty)$. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence and satisfies $\|u_n\|_2^2 = M$ and $\lim\limits_{n\to\infty} E(u_n) = I(M)$. In view of (2.4), one can derive that $\{u_n\}$ is bounded uniformly in $H^1(\mathbb{R}^N)$. Further more, from the concentration-compactness principle [20, Lemma III. 1], one can conclude that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ satisfies compactness or dichotomy or vanishing. In what follows, we shall rule out the possibility of dichotomy and vanishing.

We first prove that vanishing does not occur. By contradiction, assume that vanishing occurs, i.e., for any $R < \infty$, there holds that

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} u_{n_k}^2(x) dx = 0.$$

According to [27, Lemma 1.21], one can further conclude that

$$u_{n_k} \xrightarrow{k} 0 \text{ in } L^q(\mathbb{R}^N) \text{ as } k \to \infty, \text{ where } 2 < q < 2^*.$$

It then directly follows that

$$\int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx \leq \int_{\mathbb{R}^N} |u_{n_k}|^{p+1} dx \xrightarrow{k} 0, \text{ as } k \to \infty.$$

Furthermore, one has

$$\begin{aligned} I(M) = \lim_{k\to\infty} E(u_{n_k}) &= \lim_{k\to\infty} \Big\{ \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx \Big\} \\ &\geq \lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx \geq 0, \end{aligned}$$

which contradicts to (2.2). Hence, vanishing does not occur.

Next, we shall prove that dichotomy does not occur. By contradiction again, suppose dichotomy occurs, and it then follows from [20, Lemma III.1] that there exist two subsequence of $\{u_{n_k}\}$, denoted as $\{u_{n_{k,1}}\}$ and $\{u_{n_{k,2}}\}$, such that

$$\begin{cases} \|u_{n_k} - (u_{n_{k,1}} + u_{n_{k,2}})\|_{L^q} \xrightarrow{k} 0, \text{ for } 2 \le q < 2^*, \\ \left| \int_{\mathbb{R}^N} u_{n_{k,1}}^2 dx - \alpha \right| \xrightarrow{k} 0, \quad \left| \int_{\mathbb{R}^N} u_{n_{k,2}}^2 dx - (M-\alpha) \right| \xrightarrow{k} 0, \ \forall\, \alpha \in (0, M), \\ \operatorname{dist}\Big(\operatorname{Supp} u_{n_{k,1}}, \operatorname{Supp} u_{n_{k,2}}\Big) \xrightarrow{k} \infty, \\ \lim\limits_{k\to\infty} \inf \int_{\mathbb{R}^N} \Big\{ |\nabla u_{n_k}|^2 - |\nabla u_{n_{k,1}}|^2 - |\nabla u_{n_{k,2}}|^2 \Big\} dx \ge 0. \end{cases} \tag{2.5}$$

Set $\bar{u}_{n_k} := m^{\frac{1}{p+1}}(x) u_{n_k}$, $\bar{u}_{n_{k,1}} := m^{\frac{1}{p+1}}(x) u_{n_{k,1}}$, $\bar{u}_{n_{k,2}} := m^{\frac{1}{p+1}}(x) u_{n_{k,2}}$. Applying (2.5), one can deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx - \int_{\mathbb{R}^N} m(x)|u_{n_{k,1}}|^{p+1} dx - \int_{\mathbb{R}^N} m(x)|u_{n_{k,2}}|^{p+1} dx \\ &\quad = \|\bar{u}_{n_k}\|_{p+1}^{p+1} - \|\bar{u}_{n_{k,1}}\|_{p+1}^{p+1} - \|\bar{u}_{n_{k,2}}\|_{p+1}^{p+1} = \|\bar{u}_{n_k}\|_{p+1}^{p+1} - \|\bar{u}_{n_{k,1}} + \bar{u}_{n_{k,2}}\|_{p+1}^{p+1} + o(1) \\ &\quad \le C\big(\|\bar{u}_{n_k}\|_{p+1} - \|\bar{u}_{n_{k,1}} + \bar{u}_{n_{k,2}}\|_{p+1}\big) + o(1) \\ &\quad \le C\big(\|\bar{u}_{n_k} - (\bar{u}_{n_{k,1}} + \bar{u}_{n_{k,2}})\|_{p+1}\big) + o(1) \\ &\quad \le C\big(\|u_{n_k} - (u_{n_{k,1}} + u_{n_{k,2}})\|_{p+1}\big) + o(1) \\ &\quad \le o(1), \quad \text{as } k \to \infty, \end{aligned}$$

where the second to last inequality holds due to $0 < m(x) \le 1$. This indicates that

$$\int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx = \int_{\mathbb{R}^N} m(x)|u_{n_{k,1}}|^{p+1} dx + \int_{\mathbb{R}^N} m(x)|u_{n_{k,2}}|^{p+1} dx + o(1), \ \text{ as } k \to \infty. \tag{2.6}$$

Applying (2.5) and (2.6) yields that

$$\begin{aligned} I(M) &= \lim_{k\to\infty} E(u_{n_k}) = \lim_{k\to\infty} \Big\{ \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx \Big\} \\ &\ge \lim_{k\to\infty} \Big\{ \int_{\mathbb{R}^N} \big(|\nabla u_{n_{k,1}}|^2 + |\nabla u_{n_{k,2}}|^2\big) dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)\big(|u_{n_{k,1}}|^{p+1} + |u_{n_{k,2}}|^{p+1}\big) dx \Big\} \\ &= \lim_{k\to\infty} E(u_{n_{k,1}}) + \lim_{k\to\infty} E(u_{n_{k,2}}) \ge I(\alpha) + I(M-\alpha), \end{aligned}$$

which contradicts to (2.1). Hence, dichotomy does not occur.

In view of the above conclusions, using the concentration-compactness lemma ([20, 21]), one can conclude that there exists a subsequence of $\{u_{n_k}\}$ (still denoted by $\{u_{n_k}\}$) and some $\{y_k\} \subset \mathbb{R}^N$ such that $\hat{u}_{n_k}(\cdot) := u_{n_k}(\cdot + y_k)$ satisfies

$$\begin{aligned} &\hat{u}_{n_k} \rightharpoonup u_0 \text{ in } H^1(\mathbb{R}^N) \text{ for some } u_0 \in H^1(\mathbb{R}^N), \\ &\text{and } \ \hat{u}_{n_k} \xrightarrow{k} u_0 \text{ in } L^q(\mathbb{R}^N) \text{ with } 2 \le q < 2^*. \end{aligned} \tag{2.7}$$

This further indicates that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} m(x)|\hat{u}_{n_k}|^{p+1} dx = \int_{\mathbb{R}^N} m(x)|u_0|^{p+1} dx.$$

Consequently, it then follows that $E(\cdot)$ is weak lower semicontinuous, *i.e.*,

$$E(u_0) \leq \lim_{k\to\infty} E(\hat{u}_{n_k}). \tag{2.8}$$

On the other hand, one can directly observe that $I(M) \leq \lim_{k\to\infty} E(\hat{u}_{n_k})$. If $I(M) = \lim_{k\to\infty} E(\hat{u}_{n_k})$, using (2.8), one can derive that

$$I(M) \leq E(u_0) \leq \lim_{k\to\infty} E(\hat{u}_{n_k}) = I(M), \tag{2.9}$$

which indicates that u_0 is a minimizer of $I(M)$. If $I(M) < \lim_{k\to\infty} E(\hat{u}_{n_k})$, we claim that $\{y_k\}$ is bounded uniformly in $\mathbb{R}^N$. Otherwise, assume $|y_k| \xrightarrow{k} \infty$, and then one has

$$\begin{aligned}
I(M) = \lim_{k\to\infty} E(u_{n_k}) &= \lim_{k\to\infty} \Big\{ \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx \Big\} \\
&= \lim_{k\to\infty} \Big\{ \int_{\mathbb{R}^N} |\nabla \hat{u}_{n_k}|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x+y_k)|\hat{u}_{n_k}|^{p+1} dx \Big\} \\
&= \lim_{k\to\infty} E(\hat{u}_{n_k}) + \lim_{k\to\infty} \frac{2}{p+1} \int_{\mathbb{R}^N} \Big(m(x) - m(x+y_k) \Big) |\hat{u}_{n_k}|^{p+1} dx \\
&\geq \lim_{k\to\infty} E(\hat{u}_{n_k}),
\end{aligned}$$

where the last inequality holds due to $\inf_{x\in\mathbb{R}^N} m(x) = \lim_{|x|\to\infty} m(x)$. This contradicts to the assumption $I(M) < \lim_{k\to\infty} E(\hat{u}_{n_k})$, and the claim then holds. With this claim, passing to a subsequence if necessary, one has $\lim_{k\to\infty} y_k = y_0$ for some $y_0 \in \mathbb{R}^N$. Then it follows from (2.7) that

$$u_{n_k} \xrightarrow{k} u_0(\cdot - y_0) \text{ in } L^q(\mathbb{R}^N) \text{ with } 2 \leq q < 2^*.$$

which further implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1} dx = \int_{\mathbb{R}^N} m(x)|u_0(x-y_0)|^{p+1} dx.$$

Similar to (2.8) and (2.9), one can conclude that

$$I(M) \leq E\big(u_0(\cdot - y_0)\big) \leq \lim_{k\to\infty} E(u_{n_k}) = I(M),$$

i.e., $u_0(\cdot - y_0)$ is a minimizer of $I(M)$. Hence, we complete the proof of Theorem 1.1. □

3 Mass concentration : proof of Theorem 1.2

In this section, we shall prove Theorem 1.2 on the concentration behavior of minimizers for $I(M)$ as $M \to \infty$. As mentioned in the introduction, we shall firstly establish the following Theorem 3.1 on the limit behavior of minimizers for I_M as $M \to \infty$, where I_M is defined in (1.11). Based on Theorem 3.1, one can then complete the proof of Theorem 1.2.

Theorem 3.1. *Suppose $m(x)$ satisfies (M_1) and (M_2). Let v_k be a nonnegative minimizer of I_{M_k}, where $M_k \to \infty$ as $k \to \infty$. Then passing to a subsequence if necessary, v_k has a unique maximum point $\bar{z}_k$ as k is large enough and $\bar{z}_k$ satisfies $\lim_{k\to\infty} \bar{z}_k = 0$. Moreover, there also holds that*

$$\lim_{k\to\infty} \hat{\epsilon}_k^{\frac{N}{2}} v_k(\hat{\epsilon}_k x + \bar{z}_k) = (a^*)^{-\frac{2}{4-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4-N(p-1)}} x\Big) \text{ in } H^1(\mathbb{R}^N), \tag{3.1}$$

where $\hat{\epsilon}_k := M_k^{-\frac{p-1}{4-N(p-1)}}$, $a^* := \|Q(x)\|_2^2$ and $Q(x)$ is the unique positive solution of (1.4).

3.1 Energy estimates of I_M

This section is aimed at establishing the refined energy estimates of I_M by employing the analysis of $\tilde{I}_M$ defined in (1.13). Similar to the existence results for I_M, one can verify that $\tilde{I}_M$ admits minimizers for any $M \in (0, \infty)$ and all the minimizers don't change the sign. Therefore, without loss of generality, one can restrict minimizers for $\tilde{I}_M$ to nonnegative functions. As for the estimates of $\tilde{I}_M$, recall from [19][Lemma A.3] and one has the following lemma.

Lemma 3.1 (cf. [19][Lemma A.3])**.** *Suppose $\tilde{v}_M$ is a nonnegative minimizer of $\tilde{I}_M$. One then has*

$$\tilde{I}_M = -2\lambda_0 \Big(\frac{M}{a^*}\Big)^{\frac{2(p-1)}{4-N(p-1)}}, \tag{3.2}$$

and

$$\tilde{v}_M = \frac{1}{\sqrt{a^*}} \tilde{\alpha}_M^{\frac{N}{2}} Q(\tilde{\alpha}_M x), \tag{3.3}$$

where $\tilde{\alpha}_M := \left(\frac{M}{a^*}\right)^{\frac{p-1}{4-N(p-1)}}$, and λ_0 is defined by

$$\lambda_0 := \frac{1}{2} \frac{4-N(p-1)}{2(p+1)-N(p-1)}. \tag{3.4}$$

Based on Lemma 3.1, one can obtain the following energy estimates of I_M.

Lemma 3.2. *Suppose $m(x)$ satisfies (M_1), and then one has*

$$\lim_{M\to\infty} \frac{I_M}{\left(\frac{M}{a^*}\right)^{\frac{2(p-1)}{4-N(p-1)}}} = -2\lambda_0, \tag{3.5}$$

where $a^* := \|Q\|_2^2$, λ_0 is defined in (3.4), and $Q(x)$ is the unique positive solution of (1.4).

Proof. Let v_M be a nonmegative minimizer of I_M. As for the lower bound of I_M, one can deduce from (1.11), (1.12), (1.13), (1.14) and (3.2) that

$$\begin{aligned} I_M =& \tilde{E}_M(v_M) + \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \big(1-m(x)\big)|v_M|^{p+1} dx \\ \geq& \tilde{I}_M = -2\lambda_0 \Big(\frac{M}{a^*}\Big)^{\frac{2(p-1)}{4-N(p-1)}} \text{ as } M \to \infty. \end{aligned} \tag{3.6}$$

Next, we shall prove the upper bound of I_M by taking a suitable testing function. Set

$$v_\tau(x) := \frac{A_\tau}{\|Q\|_2}\tau^{\frac{N}{2}}Q(\tau x)\varphi(x), \tag{3.7}$$

where $\tau > 0$, A_τ is chosen so that $\|v_\tau\|_2^2 = 1$, and $\varphi(x) \in C^\infty(\mathbb{R}^N)$ is a cut-off function satisfying that $\varphi(x) = 1$ for $|x| \le 1$; $\varphi(x) = 0$ for $|x| \ge 2$; and $\varphi(x) \in (0,1)$ for $1 < |x| < 2$. From (1.5) and (1.8), one can deduce that

$$\begin{aligned} I_M \le & E_M(v_\tau) = \int_{\mathbb{R}^N} |\nabla v_\tau|^2 dx + \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N} \big(1-m(x)\big)|v_\tau|^{p+1}dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N} |v_\tau|^{p+1}dx \\ \le & \big(1+C_1e^{-2\tau}\big)\cdot \frac{N(p-1)}{2(p+1)-N(p-1)}\tau^2 - \big(1-C_2\tau^{-s-2}\big)\cdot\frac{4(a^*)^{\frac{1-p}{2}}M^{\frac{p-1}{2}}}{2(p+1)-N(p-1)}\tau^{\frac{N}{2}(p-1)}, \end{aligned}$$

as $\tau \to \infty$, where $s > 0$ is defined in (M_2). Setting $\tau = \left(\frac{M}{a^*}\right)^{\frac{p-1}{4-N(p-1)}}$ then yields that

$$\begin{aligned} I_M \le & \frac{N(p-1)-4}{2(p+1)-N(p-1)}\Big(\frac{M}{a^*}\Big)^{\frac{2(p-1)}{4-N(p-1)}} + C\Big(\frac{M}{a^*}\Big)^{\frac{-s(p-1)}{4-N(p-1)}} \\ \le & \frac{N(p-1)-4}{2(p+1)-N(p-1)}\Big(\frac{M}{a^*}\Big)^{\frac{2(p-1)}{4-N(p-1)}} + o(1) \\ = & -2\lambda_0\Big(\frac{M}{a^*}\Big)^{\frac{2(p-1)}{4-N(p-1)}} + o(1), \text{ as } M \to \infty. \end{aligned} \tag{3.8}$$

where λ_0 is defined in (3.4). Hence, (3.5) follows from (3.6) and (3.8) directly, and Lemma 3.2 is then proved. □

3.2 Blow-up analysis

In this section, we shall prove Theorem 3.1 and then complete the proof of Theorem 1.2. Motivated by [12, Lemma 2.2] and [22, Lemma 4.2], we firstly give the following lemma.

Lemma 3.3. *Suppose $m(x)$ satisfies (M_1) and (M_2). Let v_M be a nonnegative minimizer of I_M, and one then has*

$$\frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}\big(1-m(x)\big)|v_M|^{p+1}dx \to 0, \ \text{as } M\to\infty. \tag{3.9}$$

Proof. The key to prove this lemma is to verify that

$$I_M - \tilde{I}_M \to 0, \text{ as } M \to \infty, \tag{3.10}$$

where $\tilde{I}_M$ is defined by (1.13). In fact, combing (3.2) and (3.8) yields that

$$I_M \le \tilde{I}_M + o(1), \text{ as } M\to\infty. \tag{3.11}$$

On the other hand, one can deduce that

$$I_M - \tilde{I}_M \ge E_M(v_M) - \tilde{E}_M(v_M) = \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}\big[1-m(x)\big]|v_M|^{p+1}dx \ge 0. \tag{3.12}$$

Hence, (3.10) follows from (3.11) and (3.12). Further more, one can derive from (3.10) that

$$\frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}\big(1-m(x)\big)|v_M|^{p+1}dx = I_M-\tilde{E}_M(v_M)\le I_M-\tilde{I}_M\to 0,\ \text{as } M\to\infty. \tag{3.13}$$

□

Spired by [11, 19, 25, 28, 29], we shall establish the following lemma.

Lemma 3.4. *Suppose $m(x)$ satisfies (M_1) and (M_2). Let v_k be a nonnegative minimizer of I_{M_k}, where $M_k\to\infty$ as $k\to\infty$. Then we have follows.*

(1). There exist a sequence $\{y_k\}\subset\mathbb{R}^N$ and positive constants η, $R_0>0$ such that

$$\omega_k(x):=\hat{\epsilon}_k^{\frac{N}{2}}v_k(\hat{\epsilon}_kx+\hat{\epsilon}_ky_k) \tag{3.14}$$

satisfies

$$\liminf_{k\to\infty}\int_{B_{R_0}(0)}|\omega_k|^{p+1}dx\ge\eta>0. \tag{3.15}$$

where $\hat{\epsilon}_k:=M_k^{-\frac{p-1}{4-N(p-1)}}$. Morever, for the above sequence $\{y_k\}$, passing to a subsequence if necessary, there holds that

$$\lim_{k\to\infty}\hat{\epsilon}_ky_k=0. \tag{3.16}$$

(2). There exists a subsequence of $\{\omega_k\}$, still denoted by $\{\omega_k\}$, such that

$$\omega_k\xrightarrow{k}\omega_0:=(a^*)^{-\frac{2}{4-N(p-1)}}Q\Big((a^*)^{-\frac{p-1}{4-N(p-1)}}x+\hat{x}_0\Big)\ \textit{strongly in } H^1(\mathbb{R}^N), \tag{3.17}$$

where $\hat{x}_0\in\mathbb{R}^N$, $a^*:=\|Q\|_2^2$ and Q is the unique positive solution of (1.4).

Proof. Since the proof of (1) is similar to that given in [13, Lemma 2.3], here we omit it and mainly give the proof of (2).

Firstly, we shall prove the boundedness of $\|\nabla\omega_k\|_2^2$ and $\|\omega_k\|_{p+1}^{p+1}$, *i.e.*, there exist positive constants C_1, C_2, $C_1^{'}$ and $C_2^{'}$, which are independent of k, such that ω_k satisfies

$$0<C_1\le\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx\le C_2 \text{ and } 0<C_1'\le\int_{\mathbb{R}^N}|\omega_k|^{p+1}dx\le C_2'. \tag{3.18}$$

In fact, one can deduce from (3.14) that

$$I_{M_k}=\hat{\epsilon}_k^{-2}\Big[\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx+\hat{\epsilon}_k^2\frac{2M_k^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}\big(1-m(x)\big)|v_k|^{p+1}dx-\frac{2}{p+1}\int_{\mathbb{R}^N}|w_k|^{p+1}dx\Big].$$

This implies from (3.5) and (3.9) that

$$(a^*)^{\frac{2(p-1)}{4-N(p-1)}}\Big[\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx-\frac{2}{p+1}\int_{\mathbb{R}^N}|w_k|^{p+1}dx\Big]\xrightarrow{k}-2\lambda_0<0. \tag{3.19}$$

If $\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx\to\infty$ as $k\to\infty$, setting $\gamma_k^2:=\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx$ and $\omega_k(x):=\gamma_k^{\frac{N}{2}}\nu_k(\gamma_kx)$, one can deduce that $\|\nabla\nu_k\|_2^2=\|\nu_k\|_2^2=1$. Applying the GN inequality (1.6) then yields $\|\nu_k\|_{p+1}^{p+1}\le C_{GN}^{-1}$. Moreover, one can deduce that

$$\frac{\int_{\mathbb{R}^N}|w_k|^{p+1}dx}{\int_{\mathbb{R}^N}|\nabla\omega_k|^2dx}=\frac{\|\nu_k\|_{p+1}^{p+1}}{\|\nabla\nu_k\|_2^2}\gamma_k^{\frac{N}{2}(p-1)-2}\le C_{GN}^{-1}\gamma_k^{\frac{N}{2}(p-1)-2}\to 0,\ \text{as } k\to\infty. \tag{3.20}$$

However, from (3.19), one has

$$\frac{\int_{\mathbb{R}^N} |w_k|^{p+1}dx}{\int_{\mathbb{R}^N} |\nabla \omega_k|^2 dx} \to 1, \text{ as } k \to \infty,$$

which contradicts (3.20), and we know that $\int_{\mathbb{R}^N} |\nabla \omega_k|^2 dx \le C$. Applying the GN inequality (1.6) further yields $\int_{\mathbb{R}^N} |\omega_k|^{p+1}dx \le C_2'$. On the other hand, from (3.19) one can observe that $\int_{\mathbb{R}^N} |\omega_k|^{p+1}dx \ge C_1' > 0$. Using the GN inequality again, one has $\int_{\mathbb{R}^N} |\nabla \omega_k|^2 dx \ge C_1$. Hence, one can thus conclude that (3.18) holds. Moreover, since ω_k is bounded uniformly in $H^1(\mathbb{R}^N)$, then passing to a subsequence if necessary, there exist some $\omega_0 \in H^1(\mathbb{R}^N)$ such that

$$\omega_k \rightharpoonup \omega_0 \ge 0 \text{ as } k \to \infty. \tag{3.21}$$

Since v_k is a nonnegative minimizer of I_{M_k}, one can derive that v_k satisfies the following Euler-Lagrange equation

$$-\Delta v_k = \mu_k v_k + M_k^{\frac{p-1}{2}} m(x) v_k^p \ \text{ in } \mathbb{R}^N, \tag{3.22}$$

where $\mu_k \in \mathbb{R}$ is a suitable Lagrange multiplier and satisfies

$$\mu_k = I_{M_k} - \frac{(p-1)M_k^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} m(x)|v_k|^{p+1}dx.$$

It then follows that ω_k satisfies

$$-\Delta \omega_k = \hat{\epsilon}_k^2 \mu_k \omega_k + \omega_k^p + \big(m(\hat{\epsilon}_k x + \hat{\epsilon}_k y_k) - 1\big)\omega_k^p \ \text{ in } \mathbb{R}^N. \tag{3.23}$$

Next, we shall derive the limit equation of (3.23) as $k \to \infty$. As for $\hat{\epsilon}_k^2 \mu_k$, applying (3.5), (3.9), (3.14) and (3.18), one can deduce that

$$\begin{aligned} \hat{\epsilon}_k^2 \mu_k &= \hat{\epsilon}_k^2 \Big[I_{M_k} - \frac{(p-1)M_k^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} m(x)|v_k|^{p+1}dx \Big] \\ &= \hat{\epsilon}_k^2 I_{M_k} + \hat{\epsilon}_k^2 \frac{(p-1)M_k^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \big(1 - m(x)\big)|v_k|^{p+1}dx - \hat{\epsilon}_k^2 \frac{(p-1)M_k^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |v_k|^{p+1}dx \\ &= -2\lambda_0 (a^*)^{\frac{-2(p-1)}{4-N(p-1)}} - \frac{p-1}{p+1} \|\omega_k\|_{p+1}^{p+1} + o(1) < 0, \text{ as } k \to \infty. \end{aligned} \tag{3.24}$$

This implies from (3.18) that $\{\hat{\epsilon}_k^2 \mu_k\}$ is bounded uniformly as $k \to \infty$, *i.e.*, passing to a subsequence if necessary, there exist some $\beta \in \mathbb{R}^+$ such that

$$\hat{\epsilon}_k^2 \mu_k \to -\beta, \text{ as } k \to \infty. \tag{3.25}$$

As for the inhomogeneous term, by (3.9) and (3.14), one can deduce that

$$\int_{\mathbb{R}^N} \big(m(\hat{\epsilon}_k x + \hat{\epsilon}_k y_k) - 1\big)\omega_k^{p+1} = \hat{\epsilon}_k^2 M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} \big(m(x) - 1\big)|v_k|^{p+1}dx \to 0, \text{ as } k \to \infty. \tag{3.26}$$

Combing (3.21), (3.25) and (3.26) and letting $k \to \infty$, one can thus obtain the following limit equation of (3.23),

$$-\Delta\omega_0 = -\beta\omega_0 + \omega_0^p \quad \text{in } \mathbb{R}^N. \tag{3.27}$$

Applying the strong maximum principle, together with (3.15), one has $\omega_0 > 0$. Since the equation (1.4), up to translations, admits a unique positive solution Q, it then follows from (3.27) that, there exists some $\hat{x}_0 \in \mathbb{R}^N$ such that

$$\omega_0 = \beta^{\frac{1}{p-1}} Q(\beta^{\frac{1}{2}} x + \hat{x}_0). \tag{3.28}$$

Finally, we shall prove that

$$\|\omega_0\|_2^2 = 1, \quad \beta = \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}}. \tag{3.29}$$

It follows from (3.28) that

$$\begin{cases} \|Q\|_2^2 = \beta^{\frac{N(p-1)-4}{2(p-1)}} \|\omega_0\|_2^2, \\ \|\nabla Q\|_2^2 = \beta^{\frac{N(p-1)-4}{2(p-1)}-1} \|\nabla\omega_0\|_2^2, \\ \|Q\|_{p+1}^{p+1} = \beta^{\frac{N(p-1)-4}{2(p-1)}-1} \|\omega_0\|_{p+1}^{p+1}. \end{cases} \tag{3.30}$$

Applying the Fatou Lemma yields $\|\omega_0\|_2^2 \le \lim\limits_{k\to\infty} \|\omega_k\|_2^2 = 1$. Hence, it follows from (3.30) that $\beta \le \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}}$. On the other hand, substituting (3.30) into the identity (1.8) then yields that

$$\|\nabla\omega_0\|_2^2 = \frac{N}{2}\frac{p-1}{p+1}\|\omega_0\|_{p+1}^{p+1} = \frac{N(p-1)}{2(p+1)-N(p-1)}\beta\|\omega_0\|_2^2. \tag{3.31}$$

Apply the GN inequality (1.6) and one has

$$\begin{aligned} C_{GN} \le & \frac{\|\nabla\omega_0\|_2^{\frac{N}{2}(p-1)} \|\omega_0\|_2^{p+1-\frac{N}{2}(p-1)}}{\|\omega_0\|_{p+1}^{p+1}} \\ = & \frac{\Big(\frac{N(p-1)}{2(p+1)-N(p-1)}\beta\|\omega_0\|_2^2\Big)^{\frac{N}{4}(p-1)} \|\omega_0\|_2^{p+1-\frac{N}{2}(p-1)}}{\frac{2(p+1)}{2(p+1)-N(p-1)}\beta\|\omega_0\|_2^2} \\ = & \Big(\frac{N(p-1)}{2(p+1)-N(p-1)}\Big)^{\frac{N}{4}(p-1)}\Big(1-\frac{N(p-1)}{2(p+1)}\Big)\beta^{\frac{N}{4}(p-1)-1}\|\omega_0\|_2^{p-1} \\ \le & \frac{C_{GN}}{\|Q\|_2^{p-1}}\beta^{\frac{N(p-1)-4}{4}}, \end{aligned} \tag{3.32}$$

where the last"$\le$" holds due to $\|\omega_0\|_2 \le 1$. It then follows from (3.32) that $\beta \ge \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}}$. Thus, we get that $\beta = \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}}$. Further, substitute $\beta = \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}}$ into (3.30) and we have $\|\omega_0\|_2^2 = 1$. Hence, (3.29) is proved.

Since $\|\omega_k\|_2^2 = \|\omega_0\|_2^2 = 1$, one can derive that $\omega_k \to \omega_0$ in $L^2(\mathbb{R}^N)$ as $k \to \infty$. Then, using the interpolation inequality and the Sobolev inequality, one can deduce that

$$\omega_k \to \omega_0 \text{ in } L^q(\mathbb{R}^N)(2 \le q < 2^*), \text{ as } k \to \infty. \tag{3.33}$$

Further, one can derive from (3.23), (3.27) and (3.33) that $\omega_k \to \omega_0$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Moreover, substituting $\beta = \|Q\|_2^{-\frac{4(p-1)}{4-N(p-1)}} = (a^*)^{-\frac{2(p-1)}{4-N(p-1)}}$ into (3.28) then yields (3.17). Hence, we complete the proof of Lemma 3.4. $\square$

Proof of Theorem 3.1: We shall firstly prove that, passing to a subsequence of $\{\omega_k\}$, there holds that

$$\omega_k \to 0 \text{ as } |x| \to \infty \text{ uniformly for large } k. \tag{3.34}$$

Following from (3.33), one can deduce that

$$\int_{|x|\geq\gamma} |\omega_k|^2 dx \to 0 \text{ as } \gamma \to \infty \text{ uniformly for large } k. \tag{3.35}$$

Applying (3.25) and (3.26), one can derive from (3.23) that $-\Delta\omega_k - c(x)\omega_k \leq 0$, where $c(x) = \omega_k^{p-1}(x)$. Using De-Giorgi-Nash-Moser theory [14, Theorem 4.1] yields that

$$\max_{B_1(\xi)} \omega_k(x) \leq C\Big(\int_{B_2(\xi)} |\omega_k(x)|^2 dx\Big)^{\frac{1}{2}}, \tag{3.36}$$

where $\xi \in \mathbb{R}^N$. Hence, (3.34) follows from (3.35) and (3.36).

From (3.34), one knows that v_k admits at least one global maximum point. Let $\bar{z}_k$ be any global maximum point of v_k and denote $z_k := \hat{\epsilon}_k y_k \to 0$ as $k \to \infty$. Hence, one can derive from (3.14) that $\omega_k(x)$ attains its global maximum point at $x_k' := \frac{\bar{z}_k - z_k}{\hat{\epsilon}_k}$. One can verify that $\{\frac{\bar{z}_k - z_k}{\hat{\epsilon}_k}\}$ is bounded uniformly in $\mathbb{R}^N$. Otherwise, it follows from (3.34) that $\lim\limits_{k\to\infty} \|\omega_k\|_\infty = 0$ as $|x_k'| \xrightarrow{k} \infty$, which contradicts to (3.15). This further indicates that, passing to a subsequence if necessary,

$$\lim_{k\to\infty} \bar{z}_k = \lim_{k\to\infty} z_k = 0. \tag{3.37}$$

Set

$$\bar{\omega}_k := \hat{\epsilon}_k^{\frac{N}{2}} v_k(\hat{\epsilon}_k x + \bar{z}_k) = \hat{\epsilon}_k^{\frac{N}{2}} v_k\Big(\hat{\epsilon}_k\Big(x + \frac{\bar{z}_k - z_k}{\hat{\epsilon}_k}\Big) + \hat{\epsilon}_k y_k\Big). \tag{3.38}$$

By (3.14) and (3.17), one can derive that there exist a subsequence of $\{\bar{\omega}_k\}$, still denoted by $\bar{\omega}_k$, such that

$$\bar{\omega}_k(x) = \omega_k\Big(x + \frac{\bar{z}_k - z_k}{\hat{\epsilon}_k}\Big) \to \bar{\omega}_0(x) \text{ in } H^1(\mathbb{R}^N) \text{ as } k \to \infty, \tag{3.39}$$

where $\bar{\omega}_0(x) := (a^*)^{-\frac{2}{4-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4-N(p-1)}}\big(x + y_1\big) + \hat{x}_0\Big)$ and $y_1 \in \mathbb{R}^N$ is some fixed point. Since $m(x) \in C^\alpha(\mathbb{R}^N)$, using the discussion similar to that given in [13, Lemma 3.1], one can deduce that

$$\bar{\omega}_k(x) \to \bar{\omega}_0(x) \text{ in } C^2_{loc}(\mathbb{R}^N) \text{ as } k \to \infty. \tag{3.40}$$

From (3.38) we know that the origin is a local maximum point of $\bar{\omega}_k$ for all $k > 0$. Hence, from (3.40), it is also a local maximum point of $\bar{\omega}_0$. On the other hand, since Q is radially symmetric about origin and strictly decreasing about $|x|$ ([6, 16, 26]), one can thus conclude that the origin is the unique local maximum point of $\bar\omega_0$, *i.e.*,

$$\bar\omega_0(x) = (a^*)^{-\frac{2}{4-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4-N(p-1)}} x\Big). \tag{3.41}$$

Therefore, (3.1) follows from (3.37) and (3.41).

Finally, we shall prove the uniqueness of maximum point for u_k as k is large enough. Applying (3.23) and (3.38), one can deduce that $\bar\omega_k$ satisfies the following equation

$$-\Delta\bar\omega_k = \hat\epsilon_k^2\mu_k\bar\omega_k + \bar\omega_k^p + \big(m(\hat\epsilon_k x + \bar z_k) - 1\big)\bar\omega_k^p \quad \text{in } \mathbb{R}^N. \tag{3.42}$$

Suppose x_k is any local maximum point of $\bar\omega_k$, and then $-\Delta\bar\omega_k(x_k) \geq 0$. From (3.25), (3.26) and (3.42), one can deduce that $\bar\omega_k(x_k) \geq C > 0$ for large k. It then follows from (3.34) that all local maximum points of $\bar\omega_k$ must stay in a finite ball $B_R(0)$ as k is large enough, where $R > 0$ is independent of k. Since the origin is the unique maximum point of $\bar\omega_0$ and $\bar\omega_k(x) \to \bar\omega_0(x)$ in $C^2_{loc}(B_R(0))$ as $k \to \infty$, one can thus derive that $x_k = 0$ is the unique maximum point of $\bar\omega_k$ as $k \to \infty$, *i.e.*, $\bar z_k$ is the unique maximum point of v_k when k is large enough. Hence, we complete the proof of Theorem 3.1. □

Proof of Theorem 1.2 Now, based on Theorem 3.1, we shall complete the proof of Theorem 1.2. Let u_k be a nonnegative minimizer of $I(M_k)$ and $\epsilon_k := \left(\frac{M_k}{a^*}\right)^{-\frac{p-1}{4-N(p-1)}}$. Note from Theorem 3.1 that $\epsilon_k = (a^*)^{\frac{p-1}{4-N(p-1)}}\hat\epsilon_k$ and $u_k = M_k^{\frac{1}{2}} v_k$, and some calculations then yield that

$$\epsilon_k^{\frac{2}{p-1}} u_k(\epsilon_k x + \bar z_k) = (a^*)^{\frac{2}{4-N(p-1)}} \hat\epsilon_k^{\frac{N}{2}} v_k\Big((a^*)^{\frac{p-1}{4-N(p-1)}}\hat\epsilon_k x + \bar z_k\Big) \to Q(x), \tag{3.43}$$

in $H^1(\mathbb{R}^N)$ as $k \to \infty$, *i.e.*, (1.10) holds. Moreover, as for the uniqueness and limit behavior of the local maximum point $\bar z_k$ of u_k, one can directly obtain the same conclusions from Theorem 3.1.

□

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