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Fractional weighted problems with a general nonlinearity or with concave-convex nonlinearities

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Summary

We consider nonlocal problems in which the leading operator contains a sign-changing weight which can be unbounded. We begin studying the existence and the properties of the first eigenvalue. Then we study a nonlinear problem in which the nonlinearity does not satisfy the usual Ambrosetti-Rabinowitz condition. Finally, we study a problem with general concave-convex nonlinearities.

KEYWORDS:
Fractional Laplacian, indefinite weight, first eigenvalue, superlinear problems, convex and concave nonlinearities

1 | INTRODUCTION

We are concerned with a class of nonlinear nonlocal problems in presence of a weight β, possibly unbounded, which is allowed to change sign. The prototype equations are

$$\begin{cases} (-\Delta)^s u + \beta(x)u = h(\lambda, x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

but, actually, we shall consider problems where the leading operator $(-\Delta)^s$ is replaced by more general nonlocal ones denoted by $\mathfrak{L}_K$, see Section 2 for the precise setting. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\lambda \in \mathbb{R}$ and h satisfies suitable structure conditions.

We shall start analyzing the eigenvalue problem

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1}$$

showing the existence of a principle eigenvalue $\hat{\lambda}_1$ enjoying the usual properties of the first eigenvalue in the classical locale case. This fact is far from being trivial, due to the fact that, at this step, β is assumed to be unbounded and sign–changing. Once the existence of $\hat{\lambda}_1$ is proved, it is standard to show the existence of a diverging sequence of eigenvalues solving (1), see Theorem 2 below.

After this preliminary result, we will look for solutions to problems of the form

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2}$$

with different assumptions on f. In particular, we produce two constant sign solutions in Theorem 3 by using the Mountain Pass Theorem.

Finally, we consider a problem of the form

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda g(x,u) + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3}$$

where $g(x,\cdot)$ has sublinear growth at infinity, while $f(x,\cdot)$ exhibits a superlinear growth. In this case we find two constant sign solutions, and we produce a third nontrivial one by using the Weiestrass Theorem, provided that λ is positive and small. Of course, this result has the flavour of the celebrated one in [2] for the local case. However, we shall treat a nonlinear source f which does not satisfy the usual Ambrosetti-Rabinowitz condition (AR-condition for short), as done in [10] for the local Neumann case. Indeed, we employ a more general condition introduced in [15], which covers the case of superlinear reactions with slower growth near $\pm\infty$ and which fail to satisfy the AR-condition; of course, the lack of the AR-condition makes the situation more complicated, since it is not clear if Palais-smale sequences are bounded. Thus, our result improves those in [3], where the existence of two solutions when $\beta = 0$ is proved in presence of pure powers. For related results, see also [5] for the spectral fractional Laplacian, recalling that such an operator is quite different from the one considered here, see [1, Section 2.3] for a detailed discussion on this fact. We also mention [7], where a probem like (3) with pure powers and with f having critical growth has been studied in presence of continuous and sign changing coefficients, showing the existence of two positive solutions for λ small enough. We conclude recalling that many other concave–convex problems have been studied in different situations, for instance in [4], [6], [9] and [22].

2 | MATHEMATICAL BACKGROUND

The underlying operator $\mathfrak{L}_K$ is defined as follows:

$$\mathfrak{L}_K u(x) = -\int_{\mathbb{R}^N} (u(x) - u(y))\, K(x-y)\, dy,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ is a function satisfying the following
κ-assumption:

1. $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\left\{1, |x|^2\right\}$;
2. there exist $\varkappa > 0$ such that $K(x) \geq \varkappa |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$;
3. $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

Notice that, up to some positive multiplicative constant, $\mathfrak{L}_K = -(-\Delta)^s$ when $K(x) = |x|^{/(N+2s)}$.

In order to work with the operator $\mathfrak{L}_K$, it is necessary to introduce a suitable functional setting.

From now on, we fix $s \in (0,1)$, $N > 2s$, and $\Omega \subset \mathbb{R}^N$ an open bounded set with Lipschitz Boundary. The space X is

$$X = \left\{ v : \mathbb{R}^N \to \mathbb{R} : v|_\Omega \in L^2(\Omega),\ (v(x) - v(y))\sqrt{K(x-y)} \in L^2(Q) \right\},$$

where $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$ and $\mathcal{O} = \Omega^c \times \Omega^c$. The space X is endowed with the norm

$$\|v\|_X = \|v\|_{L^2(\Omega)} + \left(\int_Q |v(x) - v(y)|^2 K(x-y)\, dx dy \right)^{\frac{1}{2}}.$$

Moreover, we set

$$X_0 = \left\{ v \in X : \ v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Like in the case of Sobolev spaces with integer s, it is possible to define a critical exponent that plays the same role in the embedding theorems. Precisely we define

$$2^\star = \frac{2N}{N-2s},$$

and we have the following

Lemma 1 ([20], Lemma 6)**.** Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfy the κ**-assumption**. Then

1. there exists a positive constant $c = c(N, s)$, such that for any $v \in X_0$
$$\|v\|^2_{L^{2^\star}(\Omega)} = \|v\|^2_{L^{2^\star}(\mathbb{R}^N)} \le c \int\limits_Q \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\, dxdy;$$
2. there exist a constant $C = (N, s, \lambda, \Omega) > 1$ such that for any $v \in X_0$
$$\int\limits_Q |v(x) - v(y)|^2 K(x - y)\, dxdy \le \|v\|^2_X \le C \int\limits_Q |v(x) - v(y)|^2 K(x - y)\, dxdy,$$
that is
$$\|v\|_{X_0} = \left(\int\limits_Q |v(x) - v(y)|^2 K(x - y)\, dxdy \right)^{\frac{1}{2}}$$
is a norm in X_0 equivalent to the usual one defined in X.

Lemma 2 ([20], Lemma 7)**.** $(X_0, \|\cdot\|_{X_0})$ endowed with the scalar product
$$\langle u, v\rangle_{X_0} = \int\limits_Q (u(x) - u(y))\,(v(x) - v(y))\,K(x - y)\, dxdy$$
is a Hilbert space.

Recalling that Ω has a Lipschitz boundary, we have:

Lemma 3 ([19], Lemma 9)**.** Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfies the κ**-assumption**. Then the following assertions hold true:

1. the embedding $X_0 \hookrightarrow L^p(\mathbb{R}^N)$ is compact for every $p \in [1, 2^\star)$;
2. the embedding $X_0 \hookrightarrow L^{2^\star}(\mathbb{R}^N)$ is continuous.

A fundamental compactness tool is the following

Definition 1. Let X be a Banach Space, and let X^* be its topological dual. Let $\varphi \in C^1(X)$; we say that φ satisfies the Cerami condition - (C) for short - if the following holds: every sequence $(u_n)_n \subset X$ such that
$$\big(\varphi(u_n)\big)_n \subset \mathbb{R} \text{ is bounded and } (1 + \|u_n\|_X)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$
admits a strongly convergent subsequence.

We shall use the following variant of the Mountain Pass Theorem, where the original Palais-Smale condition is replaced by (C), see [13] for a proof.

Theorem 1 (Mountain Pass Theorem)**.** If X is a Banach space, $\varphi \in C^1(X)$ satisfies (C), $u_0,\ u_1$ satisfy $\|u_1 - u_0\|_X > \rho > 0$
$$\max\left\{\varphi(u_0), \varphi(u_1)\right\} \le \inf\left\{\varphi(u)\ :\ \|u - u_0\|_X = \rho\right\} = \eta_\rho,$$
set $\Gamma := \left\{\gamma \in C\left([0,1], X\right)\ \ \gamma(0) = u_0,\ \gamma(1) = u_1\right\}$ and
$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)),$$
then $c \ge \eta_\rho$ and c is a critical value for φ.

3 | THE EIGENVALUE PROBLEM

In this section we give some results about the following nonlocal eigenvalue problem:
$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{P_λ}$$
where $\mathfrak{L}_K$ and Ω are as above. More precisely, we prove

Theorem 2. Let K satisfy the κ-**assumption** and let $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star-2}$. Then there exists a diverging sequence $(\hat{\lambda}_n)_n$ and associated eigenfunctions $(\hat{u}_n)_n \subset X_0 \setminus \{0\}$ such that $(\hat{\lambda}_n, \hat{u}_n)$ solve $(P_{\hat{\lambda}_n})$ for any $n \in \mathbb{N}$. Moreover, $\hat{\lambda}_1$ is simple with associated eigenfunction $\hat{u}_1 \geq 0$ a.e. in Ω.

The proof of Theorem 2 essentially consists in proving that the candidate first eigenvalue is finite, and this is the hardest part, because β is unbounded and sign-changing. Once the finiteness of $\hat{\lambda}_1$ is proved, the existence of a diverging sequence of eigenvalues follows in a standard way by applying the classical genus theory to a perturbed functional. Hence, we start from

Proposition 1. Let K satisfy the κ-**assumption** and let $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star-2}$. Then problem (P_λ) has a smallest eigenvalue $\hat{\lambda}_1 \in \mathbb{R}$ which is simple and has an eigenfunction $\hat{u}_1 \in X_0$ such that $\hat{u}_1 \geq 0$ a.e. in Ω.

Remark 1. If $\beta^+ \in L^\infty_{\text{loc}}(\Omega)$, or $K(x) = \frac{1}{|x|}$, we can conclude that $\hat{u}_1 > 0$ in Ω, for instance see [12] or [8, Remark 1.3].

Proof of Proposition 1. Let $\Psi : X_0 \to \mathbb{R}$ be the functional defined by

$$\Psi(u) = \|u\|^2_{X_0} + \int\limits_\Omega \beta u^2\, dx$$

and consider the set

$$M = \left\{ u \in X_0 : \int\limits_\Omega u^2 dx = 1 \right\}.$$

Set

$$\hat{\lambda}_1 = \inf_{u \in M} \Psi(u). \tag{4}$$

Claim 1: $\hat{\lambda}_1 > -\infty$.

Note that $q > \frac{2^\star}{2^\star-2}$, hence $2q' < 2^\star$. Then, if $u \in X_0$, by Theorem 3 we have that $u^2 \in L^{q'}(\Omega)$. Hence, by Hölder's inequality, we have that

$$\left| \int\limits_\Omega \beta u^2\, dz \right| \leq \|\beta\|_q \|u\|^2_{2q'}. \tag{5}$$

We know that $X_0 \hookrightarrow L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$ and the first embedding is compact. So, by Ehrling's inequality (for instance, see [17, Lemma 10.1.28], given $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\|u\|^2_{2q'} \leq \epsilon \|u\|^2_{X_0} + c(\epsilon)\|u\|^2_2 \quad \forall\, u \in X_0. \tag{6}$$

From (5) and (6) we get

$$\|u\|^2_{X_0} - \int\limits_\Omega \beta u^2\, dz \leq \|u\|^2_{X_0} + \epsilon \|\beta\|_q \|u\|^2_{X_0} + c(\epsilon)\|\beta\|_q \|u\|^2_2. \tag{7}$$

Now, we choose $\epsilon \in (0, 1/\|\beta\|_q)$. Reordering the terms from (7), we have

$$0 \leq \|u\|^2_{X_0}(1 - \epsilon\|\beta\|_q) \leq \Psi(u) + c(\epsilon)\|\beta\|_q \|u\|^2_2, \tag{8}$$

hence

$$-c(\epsilon)\|\beta\|_q \|u\|^2_2 \leq \Psi(u),$$

which implies $\hat{\lambda}_1 > -\infty$.

Claim 2: The infimum is obtained at a function $\hat{u}_1 \in M$ with $\hat{u}_1 \geq 0$ in Ω.

Let $(u_n)_n \subset M$ be a minimizing sequence for (4), i.e. $\Psi(u_n) \to \hat{\lambda}_1$ as $n \to \infty$. Now, from (8) we observe that $(u_n)_n$ is bounded, so we may assume that

$$u_n \rightharpoonup \hat{u}_1 \text{ in } X_0 \quad \text{and} \quad u_n \to \hat{u}_1 \text{ in } L^{2q'}(\Omega) \quad \text{as } n \to \infty.$$

By the weak sequential lower semicontinuity and Lemma 3, we have that

$$\|\hat{u}_1\|^2_{X_0} \leq \liminf_{n\to\infty} \|u_n\|^2_{X_0} \quad \text{and} \quad \lim_{n\to\infty} \int\limits_\Omega \beta u_n^2\, dx = \int\limits_\Omega \beta \hat{u}_1^2\, dx,$$

and thus $\Psi(\hat{u}_1) \leq \hat{\lambda}_1$. Since $\hat{u}_1 \in M$, we get that $\Psi(\hat{u}_1) = \hat{\lambda}_1$.

By the Lagrange multiplier rule, we have that $(\hat{\lambda}_1, \hat{u}_1)$ solve problem $(P_{\hat{\lambda}_1})$, and so $\hat{u}_1 \in X_0$ is an associated eigenfunction to $\hat{\lambda}_1$.

We observe that if u is a normalized eigenfunction for $(P_{\hat{\lambda}_1})$, by the triangle inequality we have

$$\begin{aligned}\hat{\lambda}_1 \leq \Psi(|u|) &= \iint_D (|u(x)| - |u(y)|)^2 K(x-y)\,dx dy + \int_\Omega \beta u^2\,dx \\ &\leq \iint_D (u(x)-u(y))^2 K(x-y)\,dxdy + \int_\Omega \beta u^2\,dx = \Psi(u) = \hat{\lambda}_1,\end{aligned}$$

hence we may assume $u \geq 0$.

Claim 3: $\hat{\lambda}_1$ is simple.

We start noticing that

$$\left\langle u^+, u^- \right\rangle_{X_0} = -\int_Q \left[u^+(y)u^-(x) + u^+(x)u^-(y)\right] K(x-y)\,dxdy \leq 0 \tag{9}$$

for every $u \in X_0$.

Now we improve Claim 2, showing that any weak solution $u \in X_0$ of $(P_{\hat{\lambda}_1})$, $u \neq 0$, is such that either

$$u \geq 0 \quad \text{in } \Omega$$

or

$$u \leq 0 \quad \text{in } \Omega.$$

Without loss of generality we assume that $\|u\|_2 = 1$ and by (9) we have

$$\begin{aligned}\hat{\lambda}_1 &= \Psi(u) = \Psi(u^+) + \Psi(u^-) - 2\left\langle u^+, u^-\right\rangle_{X_0} \\ &\geq \hat{\lambda}_1 \|u^+\|_2^2 + \hat{\lambda}_1 \|u^-\|_2^2 = \hat{\lambda}_1.\end{aligned}$$

Hence, in the previous inequality we find all equalities, and so

$$\Psi(u^+) = \hat{\lambda}_1 \|u^+\|_2^2 \quad \text{and} \quad \Psi(u^-) = \hat{\lambda}_1 \|u^-\|_2^2,$$

that is u^+ and u^- are weak solution of $(P_{\hat{\lambda}_1})$, as well. Moreover, we also get that $\langle u^+, u^-\rangle_{X_0} = 0$, that is

$$0 = \int_Q \left[u^+(y)u^-(x) + u^+(x)u^-(y)\right] K(x-y)\,dxdy.$$

Since $K > 0$, we get that

$$u^+(y)u^-(x) + u^+(x)u^-(y) = 0 \text{ a.e. in } Q \text{ and so in } \Omega.$$

As a consequence, $u^- = 0$, or $u^+ = 0$, as claimed.

Now, let $u,\ v$ be non trivial solutions of $(P_{\hat{\lambda}_1})$. We have shown that we can suppose $u,\ v \geq 0$ with $\int_\Omega u > 0$ and $\int_\Omega v > 0$. Hence it is possible to solve the equation in α

$$0 = \int_\Omega (u - \alpha v)\ dx = \int_\Omega u\,dx - \alpha \int_\Omega v\,dx.$$

Recalling that $u - \alpha v$ is a solution of $(P_{\hat{\lambda}_1})$ as well, we have just seen that there are two available options: $u - \alpha v \geq 0$ with $u - \alpha v \neq 0$ or $u - \alpha v \equiv 0$; in the first case we would have $\int_\Omega (u - \alpha v)\ dz > 0$, and so we deduce that $u = \alpha v$, which proves the simplicity of $\hat{\lambda}_1$. □

Remark 2. If $\beta \in L^\infty(\Omega)$, then $\hat{u}_1 \in L^\infty(\Omega)$ by [11], and so by [18, Prop. 1.1] we get that $u \in C^s(\bar{\Omega})$.

Remark 3. From now on we will denote by $\hat{u}_1$ the first eigenfunction with $\|\hat{u}_1\|_2 = 1$ and $\hat{u}_1 \geq 0$ in Ω.

Proposition 2. Let $V = \left\{u \in X_0 : \int_\Omega \hat{u}_1 u\,dx = 0\right\}$ and set

$$\hat{\lambda}_V = \inf\left\{\Psi(u) : u \in M \cap V\right\}.$$

Then $\hat{\lambda}_1 < \hat{\lambda}_V$.

Proof. First of all, it is clear from the definition above that $\hat{\lambda}_1 \leq \hat{\lambda}_V$.

Suppose by contradiction that $\hat\lambda_1 = \hat\lambda_V$. Then we can find a sequence $(u_n)_n \subset M \cap V$ such that $\Psi(u_n) \to \hat\lambda_V = \hat\lambda_1$. By (8) we have

$$\|u_n\|^2_{X_0}(1-\epsilon\|\beta\|_q) \le \Psi(u_n) + c(\epsilon)\|\beta\|_q\|u_n\|_2^2 \to \hat\lambda_1 + c(\epsilon)\|\beta\|_q,$$

hence $(u_n)_n \subset X_0$ is bounded, and so we may assume

$$u_n \rightharpoonup u \text{ in } X_0 \quad \text{and} \quad u_n \to u \text{ in } L^{2q'}(\Omega). \tag{10}$$

Exploiting the sequential weak lower semicontinuity of Ψ, by (10) and since $u \in M \cap V$, we have

$$\hat\lambda_1 \le \Psi(u) \le \liminf_{n\to\infty} \Psi(u_n) = \hat\lambda_1 = \hat\lambda_V,$$

and hence

$$\Psi(u) = \hat\lambda_1.$$

By Proposition 1 this implies that $u = \pm\sigma\hat u_1$ for some $\sigma > 0$, a contradiction to the fact that $u \in M \cap V$. Thus $\hat\lambda_1 < \hat\lambda_V$. □

Proof of Theorem 2. The first part is contained in Proposition 1. Then, solving (P_λ) is equivalent to solving the eigenvalue problem

$$\begin{cases} \mathfrak{L}_K u + \tilde\beta(x)u = \Lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\tilde P_\Lambda$}$$

where $\tilde\beta = \beta - \hat\lambda_1 + 1$ and $\Lambda = \lambda - \hat\lambda_1 + 1$. Thus, in order to show that $(\tilde P_\Lambda)$ has a diverging sequence of eigenvalues, we apply the classical genus theorem in the form of[13, Theorem 9.26]. Hence, set

$$\phi(u) = \int_\Omega u^2 dx, \quad \psi(u) = \Psi(u) - (\hat\lambda_1 - 1)\int_\Omega u^2 dx$$

and

$$\mathcal{M} := \Big\{ u \in X_0 \ : \ \psi(u) = 1 \Big\}.$$

By definition of $\hat\lambda_1$, it is readily seen that $\int u^2 \le 1$ if $u \in \mathcal{M}$.As a consequence, by (8) we get that, if $u \in \mathcal{M}$, then

$$\|u\|^2_{X_0}(1-\epsilon\|\beta\|_q) \le 1 + |\hat\lambda_1 - 1| + c(\epsilon)\|\beta\|_q.$$

Hence, $\mathcal{M}$ is bounded. The other assumptions of[13, Theorem 9.26] are easily verified, and so there exists a sequence $\{(\Lambda_n, u_n)\}_n$ of solutions to $(\tilde P_\Lambda)$ with $\Lambda_n \ne 0$ and $1/\Lambda_n \to 0$, $\int u_n \to 0$ as $n \to \infty$. In particular,

$$1 = \psi(u_n) = (\Lambda_n - \hat\lambda_1 + 1)\int_\Omega u_n^2 dx \text{ for all } n \in \mathbb{N},$$

which implies that $\Lambda_n \to +\infty$ as $n \to \infty$, and so

$$\hat\lambda_n \to +\infty \text{ as } n \to \infty,$$

as claimed. □

4 | MOUNTAIN PASS SOLUTIONS BELOW THE FIRST EIGENVALUE

In this section, we study the following nonlinear fractional problem

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

where, as before, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and β may be sign changing. As for f, we shall assume

Hypothesis 1. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.e. $x \in \Omega$ and

(1) $|f(x,t)| \le a(x)(1 + |t|^{p-1})$ for a.e $x \in \Omega$, all $t \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+ = \{a \in L^\infty(\Omega) :: a \ge 0 \text{ a.e. in } \Omega\}$, $p \in (2, 2^\star)$;

(2) if $F(x,t) = \int_0^t f(x,w)\,dw$, then

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{t^2} = \infty \quad \text{uniformly for a.e } x \in \Omega;$$

(3) if $\xi(x,t) = f(x,t)t - 2F(x,t)$, then there exists $\beta^* \in L^1(\Omega)_+$ such that

$$\xi(x,t) \le \xi(x,y) + \beta^*(x) \quad \text{for a.e. } x \in \Omega \text{ and all } 0 \le t \le y, \text{ or } y \le t \le 0;$$

(4) there exist $\vartheta_0 \in L^\infty(\Omega)$ and $\eta_0 > 0$ such that

$$-\eta_0 \le \liminf_{t\to 0} \frac{f(x,t)}{t} \le \limsup_{t\to 0} \frac{f(x,t)}{t} \le \vartheta_0(x)$$

uniformly for a.e $x \in \Omega$, where ϑ_0 is such that one of the following conditions holds:

(i) $\beta^+ \in L^\infty_{\mathrm{loc}}(\Omega)$ or $K(x) = \dfrac{1}{|x|}$ and $\vartheta_0 \le \hat{\lambda}_1,\ \vartheta_0 \ne \hat{\lambda}_1$;

(ii) $\vartheta_0 < \hat{\lambda}_1$ a.e. in Ω.

Of course, the requirement in Hypothesis 1(4)**(i)** ensures that the first eigenfunction is strictly positive in Ω, see Remark 1.

Remark 4. Hypothesis 4.1(3) was introduced in[15] to replace the stronger Ambrosetti–Rabinowitz condition.

Now, we introduce the functional $\varphi : X_0 \to \mathbb{R}$ defined as

$$\varphi(u) = \frac{1}{2}\Psi(u) - \int_\Omega F(x,u(x))\,dx,$$

whose critical points are solutions of (P).

Proposition 3. If Hypotheses 1(1) − (3) hold and $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star - 2}$, then φ satisfies (C).

Proof. Let $(u_n)_n \subset X_0$ be a sequence such that

$$|\varphi(u_n)| \le M_1 \quad \text{for some } M_1 > 0, \quad \text{all } n \ge 1 \tag{11}$$

and

$$(1 + \|u_n\|_{X_0})\varphi'(u_n) \to 0 \text{ in } X_0^* \text{ as } n \to \infty. \tag{12}$$

We have

$$2\varphi(u_n)\varphi'(u_n)(u_n) = \int_\Omega \left[f(x,u_n)u_n - 2F(x,u_n) \right] dx.$$

By using (11) and (12), we immediately obtain that

$$\int_\Omega \xi(x,u_n)\,dx \le M_2 \quad \text{for all } n \ge 1. \tag{13}$$

Claim: $(u_n)_n \subset X_0$ is bounded. By contradiction we suppose that, up to a subsequence,

$$\|u_n\|_{X_0} \to \infty \text{ as } n \to \infty. \tag{14}$$

Let $y_n = \frac{u_n}{\|u_n\|_{X_0}}$, $n \ge 1$. Then $\|y_n\|_{X_0} = 1$ for all $n \ge 1$ and so we may assume that

$$y_n \rightharpoonup y \text{ in } X_0 \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{15}$$

First suppose that $y \ne 0$ and let $\Omega_0 = \{x \in \Omega : y(x) = 0\}$. Then

$$|u_n(x)| \to \infty \quad \text{for a.e } x \in \Omega_0^c := \{x \in \Omega \ : \ x \notin \Omega_0\}.$$

Then Hypothesis 1(2) and Fatou's Lemma imply that

$$\lim_{n\to\infty} \int_{\Omega_0^c} \frac{F(x,u_n(x))}{\|u_n\|^2_{X_0}}\,dx = \infty.$$

But

$$\int_{\Omega} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\,dx = \int_{\Omega_0} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\,dx + \int_{\Omega_0^c} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\,dx,$$

and so

$$\lim_{n\to\infty} \int_{\Omega_0^c} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\,dx = \infty. \tag{16}$$

On the other hand, from (11) we know that

$$\int_{\Omega} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\,dx \le M_3 \quad \text{for some } M_3 \text{ and all } n \ge 1,$$

which contradicts (16).

Now suppose that $y = 0$. We fix $\eta > 0$ and define

$$v_n = (2\eta)^{\frac{1}{2}} y_n \in X_0 \quad \text{for all } n \ge 1.$$

Since

$$v_n \to 0 \text{ in } L^p(\Omega),$$

we have

$$\int_{\Omega} F(x,v_n)\,dx \to 0 \quad \text{as } n \to \infty. \tag{17}$$

By (14), we can find $n_0 \ge 1$ such that

$$0 < (2\eta)^{\frac{1}{2}} \frac{1}{\|u_n\|_{X_0}} \le 1 \quad \text{for all } n \ge n_0. \tag{18}$$

Let $\zeta_n \in [0,1]$ be such that

$$\varphi(\zeta_n u_n) = \max_{0\le\zeta\le1} \varphi(\zeta u_n).$$

From (18) it follows that

$$\varphi(\zeta_n u_n) \ge \varphi(v_n) = \eta\Psi(y_n) - \int_{\Omega} F(x,v_n)\,dx \quad \text{for all } n \ge n_0. \tag{19}$$

As we have just seen,

$$\left|\int_{\Omega} \beta u^2\,dx\right| \le \|\beta\|_q \|u\|_{2q'}^2.$$

Again by Theorems 3, $X_0 \hookrightarrow L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$ and the first embedding is compact. By Ehrling's inequality, given $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\|u\|_{2q'}^2 \le \epsilon\|u\|_{X_0}^2 + c(\epsilon)\|u\|_2^2 \quad \text{for all } u \in X_0.$$

Like in (8), we get

$$(1 - \epsilon\|\beta\|_q)\|u\|_{X_0}^2 \le \Psi(u) + c(\epsilon)\|\beta\|_q\|u\|_2^2. \tag{20}$$

Now use (20) in (19) , so that

$$\varphi(\zeta_n u_n) \ge \eta\left[(1 - \epsilon\|\beta\|_q) - c(\epsilon)\|\beta\|_q\|y_n\|_2^2\right] - \int_{\Omega} F(x,v_n)\,dx \quad n \ge n_0. \tag{21}$$

Choose $\epsilon \in \left(0, 1/\|\beta\|_q\right)$ and note that

$$\|y_n\|_2^2 \to 0 \quad \text{as } n \to \infty, \tag{22}$$

see (15) and recall that $y = 0$. By (21), using (17) and (22), we get that

$$\liminf_{n\to\infty} \varphi(\zeta_n u_n) \ge \eta(1 - \epsilon\|\beta\|_q).$$

Since $\eta > 0$ is arbitrary, by letting $\eta \to \infty$ we conclude that

$$\varphi(\zeta_n u_n) \to \infty \quad \text{as } n \to \infty. \tag{23}$$

Notice that

$$\varphi(0) = 0 \text{ and } \varphi(u_n) \le M_1 \quad \text{for all } n \ge 1.$$

Therefore, (23) implies that exists $n_1 \geq n_0$ such that $\zeta_n \in (0,1)$ for all $n \geq n_1$, hence

$$\frac{d}{d\zeta}\varphi(\zeta u_n)|_{\zeta=\zeta_n} = 0 \quad \text{for all } n \geq n_1,$$

and so

$$\Psi(\zeta_n u_n) = \int_\Omega f(x,\zeta_n u_n)\zeta_n u_n \, dx \quad \text{for all } \; n \geq n_1. \tag{24}$$

Using Hypothesis 1(3) we have

$$\int_\Omega \xi(x,\zeta_n u_n)\, dx \leq \int_\Omega \xi(x,u_n)\, dx + \|\beta^*\|_1 \quad \text{for all } n \geq n_1.$$

Using the definition of ξ, (24) and (13) we obtain

$$2\varphi(\zeta_n u_n) = \Psi(\zeta_n u_n) - 2\int_\Omega F(x,\zeta_n u_n)\, dx = \int_\Omega \xi(x,\zeta_n u_n)\, dx \leq M_4 \tag{25}$$

fore some $M_4 > 0$ and all $n \geq n_1$. Comparing (23) and (25) we reach a contradiction. This proves the claim.

By the previous claim, now we may assume that

$$u_n \rightharpoonup u \text{ in } X_0 \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{26}$$

Choosing $u_n - u \in X_0$ as test function in (12), passing to the limit as $n \to \infty$ and using (26), we find

$$\lim_{n\to\infty} \langle \mathfrak{L}_K u_n, u_n - u\rangle = 0$$

which implies that $u_n \to u$ in X_0 as $n \to \infty$, and so φ satisfies (C). $\square$

Lemma 4. If Hypothesis 1(4) holds, then there exists $\alpha_0 > 0$ such that

$$\Sigma(u) = \Psi(u) - \int_\Omega \vartheta u^2 \, dx \geq \alpha_0 \|u\|^2_{X_0}.$$

Proof. The lines of the proof follow those of in the proof of[16, Lemma 18].

Of course $\Sigma(u) \geq 0$. By contradiction, we suppose the lemma is not true. Using the 2-homogenity of Σ, we can find $(u_n)_n \subset X_0$ such that

$$\|u_n\|_{X_0} = 1 \quad \text{for all } n \geq 1 \text{ and } \Sigma(u_n) \to 0^+ \text{ as } n \to \infty. \tag{27}$$

We may assume

$$u_n \rightharpoonup u \text{ in } X_0 \quad \text{and} \quad u_n \to u \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{28}$$

It follows from (28) and the lower weak semicontinuity of Ψ that $\Sigma(u) \leq 0$, and so

$$\Psi(u) \leq \int_\Omega \vartheta u^2 \, dx \leq \hat{\lambda}_1 \|u\|_2^2. \tag{29}$$

If $u = 0$ then from (8) applied to u_n and (28) we see that $u_n \to 0$ in X_0, a contradiction to the fact that $\|u_n\|_{X_0} = 1$ for all $n \geq 1$. Hence $u \neq 0$, but now from (29) and Proposition 1 we can deduce that $\Psi(u) = \hat{\lambda}_1\|u\|_2^2$, and so $u = \pm\sigma\hat{u}_1$ for some $\sigma > 0$. If Hypothesis 1.4**(i)** holds, then $\hat{u}_1(x) > 0$ for a.e. $x \in \Omega$, and so from the first inequality in (29) we have $\Psi(u) < \hat{\lambda}_1\|u\|_2^2$, again a contradiction; if 1.4**(ii)** holds, then the contradiction is reached using $\vartheta_0 < \hat{\lambda}_1$ and $u \neq 0$. The lemma is thus proved. $\square$

Now, we want to prove the existence of nontrivial solutions of constant sign. For this, we introduce the following truncations-perturbations of the reaction f:

$$\hat{f}_+(x,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ f(x,t) + \gamma t & \text{if } 0 < t \end{cases} \tag{30}$$

and

$$\hat{f}_-(x,t) = \begin{cases} f(x,t) + \gamma t & \text{if } \; t < 0 \\ 0 & \text{if } 0 \leq t, \end{cases} \tag{31}$$

where $\gamma > c(\epsilon)\|\beta\|_q$ once ϵ is chosen (see the Proof of proposition 3). We set

$$\hat{F}_{\pm}(x,t) = \int_0^t \hat{f}_{\pm}(x,w)\,dw.$$

Then, set $\hat{\beta}(x) = \beta + \gamma$ and define

$$\hat{\Psi}(u) = \|u\|^2_{X_0} + \int_\Omega \hat{\beta}(x)u^2\,dx \quad \text{for all } u \in X_0.$$

Finally, we consider the functionals $\hat{\varphi}_{\pm} : X_0 \to \mathbb{R}$ defined by

$$\hat{\varphi}_{\pm}(u) = \frac{1}{2}\hat{\Psi}(u) - \int_\Omega \hat{F}_{\pm}(x,u)\,dx \quad \text{for all } u \in X_0.$$

Remark 5. If we repeat the proof of Proposition 3 for the functionals $\hat{\varphi}_{\pm}$, we immediately have that they both satisfy (C).

Proposition 4. If Hypothesis 1 holds, $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star-2}$, then $u = 0$ is a strict local minimizer for φ and $\hat{\varphi}_{\pm}$.

Proof. We do the proof for the functional φ, for the others being similar. By Hypotheses 1(1) and 1(4), given $\epsilon > 0$ we can find c_ϵ such that

$$F(x,t) \le \frac{1}{2}(\vartheta(x)+\epsilon)t^2 + c_\epsilon|t|^p \quad \text{for a.e. } x \in \Omega,\ \text{all } t \in \mathbb{R},\ p > 2. \tag{32}$$

Then, for every $u \in X_0$ we have

$$\varphi(u) = \frac{1}{2}\Psi(u) - \int_\Omega F(x,u)\,dx \ge \frac{1}{2}\Psi(u) - \frac{1}{2}\int_\Omega \vartheta u^2\,dx - \frac{\epsilon}{2}\|u\|_2^2 - c_\epsilon\|u\|_p^p.$$

Recalling Lemma 3, we can find $C > 0$ such that $\|u\|_2^2 \le C\|u\|^2_{X_0}$ and $\|u\|_p^p \le C\|u\|^p_{X_0}$. Applying these inequalities to (4) toghther with Lemma 4, we obtain

$$\begin{aligned}\varphi(u) &\ge \frac{1}{2}\Psi(u) - \int_\Omega \vartheta u^2\,dz - \frac{\epsilon C}{2}\|u\|^2_{X_0} - C_\epsilon\|u\|^p_{X_0}\\ &\ge \frac{\alpha_0 - \epsilon C}{2}\|u\|^2_{X_0} - C_\epsilon\|u\|^p_{X_0}.\end{aligned}$$

Choosing $\epsilon \in (0, \alpha_0/C)$, we have

$$\varphi(u) \ge C_1\|u\|^2_{X_0} - C_2\|u\|^p_{X_0}. \tag{33}$$

for some $C_1,\ C_2 > 0$. Since $p > 2$, from (33) we get that $u = 0$ is a strict local minimizer of φ (and similarly for the functionals $\hat{\varphi}_{\pm}$). □

Proposition 5. If Hypothesis 1 holds and $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star-2}$, then for every $u \in X_0 \setminus \{0\}$, we have $\varphi(\zeta u) \to -\infty$ as $\zeta \to \pm\infty$.

Proof. By Hypothesis 1(1) and 1(2), given any $\mu > 0$ we can find $c_\mu > 0$ such that

$$F(x,t) \ge \frac{\mu}{2}t^2 - c_\mu \quad \text{for a.e } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Hence, for $u \in X_0$, $u \ne 0$ and $\zeta > 0$, choosing $\mu > 0$ big enough, we have

$$\varphi(\zeta u) \to -\infty \text{ as } \zeta \to \infty.$$

□

Theorem 3. If Hypothesis 1 holds, $\beta \in L^q(\Omega)$ with $q > \frac{2^\star}{2^\star-2}$, then problem (P) admits at least two nontrivial weak solutions $\hat{u},\ \hat{v} \in X_0$ such that

$$\hat{v}(x) \le 0 \le \hat{u}(x) \quad \text{a.e. in } \Omega.$$

Proof. By Proposition 4, we can find $\rho \in (0,1)$ so small that

$$\hat{\varphi}_+(0) = 0 < \inf\left\{\hat{\varphi}_+(u)\ \ \|u\|_{X_0} = \rho\right\} := \hat{m}_+. \tag{34}$$

Then (34), together with Proposition 5 and Remark 5, implies that we can use the Mountain Pass Theorem. So we can find $\hat{u} \in X_0$ such that

$$\hat{\varphi}_+(0) = 0 < \hat{m}_+ \le \hat{\varphi}_+(\hat{u}) \tag{35}$$

and

$$\hat{\varphi}_+'(\hat{u}) = 0. \tag{36}$$

From (35), we see that $\hat{u} \neq 0$, while from (36), we have

$$\mathfrak{L}_K \hat{u} + \hat{\beta}(x)\hat{u} = f_+(x, \hat{u}). \tag{37}$$

By (9)

$$\langle \hat{u}, \hat{u}^- \rangle_{X_0} = \left\langle \hat{u}^+ - \hat{u}^-, \hat{u}^- \right\rangle_{X_0} = \left\langle \hat{u}^+, \hat{u}^- \right\rangle_{X_0} - \|\hat{u}^-\|^2_{X_0} \le -\|\hat{u}^-\|^2_{X_0}. \tag{38}$$

On (37), we act with $-\hat{u}^- \in X_0$. Then together with (38), we get

$$\Psi(\hat{u}^-) + \gamma\|\hat{u}^-\|_2^2 \le \langle \hat{u}, -\hat{u}^- \rangle_{X_0} + \int_\Omega \beta\,(\hat{u}^-)^2 \, dx + \gamma\|\hat{u}^-\|_2^2 = 0. \tag{39}$$

From (8) with $\epsilon > 0$ small enough, we have

$$(1 - \epsilon\|\beta\|_q)\|\hat{u}^-\|^2_{X_0} - c(\epsilon)\|\beta\|_q\|\hat{u}^-\|_2^2 \le \Psi(\hat{u}^-). \tag{40}$$

Since $\gamma > c(\epsilon)\|\beta\|_q$, from (39), (40) and recalling that $\|\hat{u}^-\|_2^2 \le C\|\hat{u}^-\|^2_{X_0}$ (see Lemma 3) it follows that

$$C\|\hat{u}^-\|^2_{X_0} \le 0 \tag{41}$$

for some $C > 0$, which implies that

$$\hat{u} \ge 0, \quad \hat{u} \neq 0.$$

So, (37) becomes

$$\mathfrak{L}_K \hat{u} + \beta(x)\hat{u} = f(x, \hat{u}).$$

i.e. $\hat{u}$ is a weak solution for (P).

In a similar fashion, working this time with $\hat{\varphi}_-$, we obtain another nontrivial solution $\hat{v} \in X_0$ having negative sign. □

5 | A PARAMETRIC PROBLEM WITH COMPETING NONLINEARITIES

In this section we study the following parametric nonlinear problem:

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda g(x, u(x)) + f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{E_λ}$$

$\lambda > 0$ being a parameter. Strengthening the previous assumption, here we will assume that $\beta \in L^\infty(\Omega)$. Moreover, f is a general superlinear function at ∞, while g is sublinear. In this case problem (E_λ) is an extension of the problem studied in[2] to the fractional setting and with more general nonlinearities and a sign changing weight in the operator.

Going into details, we impose the following conditions on g and f:

Hypothesis 2. $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x, 0) = 0$ for a.e. $x \in \Omega$ and

(1) there exist $b \in L^\infty(\Omega)$ and $\mathcal{P} \in (2, 2^\star)$ such that

$$|g(x,t)| \le b(x)(1 + |t|^{\mathcal{P}-1}) \quad \text{for a.e } x \in \Omega, \text{ all } t \in \mathbb{R};$$

(2)

$$\lim_{t\to\pm\infty} \frac{g(x,t)}{t} = 0 \quad \text{uniformly for a.e. } x \in \Omega;$$

(3) if $G(x,t) = \int_0^t g(x,z)\,dz$, then there exist $p,\ q \in (1,2)$, $\delta > 0$ and $\hat{\eta}_0,\ \eta_0 > 0$ such that

$$
\begin{aligned}
&0 < g(x,t) \le pG(x,t) \text{ for a.e. } x \in \Omega,\ 0 < |t| \le \delta,\\
&\operatorname*{ess\,inf}_{\Omega} G(\cdot, \pm\delta) > 0,\\
&\limsup_{t\to 0} \frac{g(x,t)}{|t|^{p-2}t} \le \hat{\eta}_0 \quad \text{uniformly for a.e } x \in \Omega \text{ and}\\
&\eta_0 |t|^q \le g(x,t)t \quad \text{for a.e. } x \in \Omega,\ \text{all } t \in \mathbb{R}.
\end{aligned}
$$

Hypothesis 3. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in \Omega$ and

(1) there exist $a \in L^\infty(\Omega)_+$ and $r \in (2, 2^\star)$ such that

$$|f(x,t)| \le a(x)(1 + |t|^{r-1}) \quad \text{for a.e. } x \in \Omega,\ \text{all } t \in \mathbb{R};$$

(2)

$$\lim_{t\to\pm\infty} \frac{f(x,t)}{t} = +\infty \quad \text{uniformly for a.e. } x \in \Omega;$$

(3)

$$\lim_{t\to 0} \frac{f(x,t)}{t} = 0 \quad \text{uniformly for a.e. } x \in \Omega.$$

Now, if $\lambda > 0$ set

$$\xi_\lambda(x,t) = \lambda g(x,t)t + f(x,t)t - \lambda p G(x,t) - pF(x,t),$$

where $F(x,t) = \int_0^t f(x,w)\,dw$.

Hypothesis 4. For every $\lambda > 0$, there exists $\beta^*_\lambda \in L^1(\Omega)$ such that

$$\xi_\lambda(x,t) \le \xi_\lambda(x,y) + \beta^*_\lambda(x)$$

for a.e $x \in \Omega$ and all $0 \le t \le y$ or $y \le t \le 0$.

Remark 6. The condition $\operatorname{ess\,inf}_\Omega G(\cdot, \pm\delta) > 0$ in Hypothesis 2(3) is automatically satisfied if stronger assumptions on g are required, see [14].

In what follows, for every $\lambda > 0$, by $\varphi_\lambda : X_0 \to \mathbb{R}$ we denote the energy functional associated to problem (E_λ) defined as

$$\varphi_\lambda(u) = \frac{1}{2}\Psi(u) - \lambda \int_\Omega G(x,u(x))\,dx - \int_\Omega F(x,u(x))\,dx$$

for all $u \in X_0$. Evidently, $\varphi_\lambda \in C^1(X_0)$.

As above, in order to generate nontrivial solutions of constant sign, we introduce suitable truncation–perturbations of the map $t \mapsto \lambda g(x,t) + f(x,t)$. To do that, from now on we assume that $\beta \in L^\infty(\Omega)$. So, by (6), fixed $\epsilon > 0$, if $\tau > \frac{2^\star}{2^\star - 2}$, we choose $\gamma > \|\beta\|_\infty$ and

$$\gamma > c(\epsilon)\|\beta\|_\infty |\Omega|^{\frac{1}{\tau}} \ge c(\epsilon)\|\beta\|_\tau \quad \text{with } c(\epsilon) \ge 1.$$

Now define

$$
\begin{aligned}
h^+_\lambda(x,t) &= \begin{cases} 0 & \text{if } t \le 0,\\ \lambda g(x,t) + f(x,t) + \gamma t & \text{if } 0 < t \end{cases}\\
h^-_\lambda(x,t) &= \begin{cases} \lambda g(x,t) + f(x,t) + \gamma t & \text{if } t < 0\\ 0 & \text{if } 0 \le t. \end{cases}
\end{aligned}
\tag{42}
$$

Both $h^\pm_\lambda$ are Carathéodory functions. We set

$$H^\pm_\lambda(x,t) = \int_0^t h^\pm_\lambda(x,w)\,dw$$

and consider the C^1-functionals $\varphi^\pm_\lambda : X_0 \to \mathbb{R}$ defined by

$$\varphi^\pm_\lambda(u) = \frac{1}{2}\Psi(u) + \frac{\gamma}{2}\|u\|_2^2 - \int_\Omega H^\pm_\lambda(x,u(x))\,dx \quad \text{for all } u \in X_0.$$

Note that using Hypotheses 2, 3, 4, the map $(x,t) \mapsto \lambda g(x,t) + f(x,t)$ satisfies Hypothesis 1, and so from Proposition 3, we have:

Proposition 6. If Hypotheses 2, 3 and 4 hold, $\lambda > 0$ and $\beta \in L^\infty(\Omega)$, then functionals φ_λ and $\varphi_\lambda^\pm$ satisfy (C).

The next two propositions show that for $\lambda > 0$ small, the functionals $\varphi_\lambda^\pm$ satisfy the Mountain Pass geometry.

Proposition 7. Assume Hypotheses 2, 3 and 4 hold, $\lambda > 0$ and $\beta \in L^\infty(\Omega)$. Then:

1. There exists $\lambda_+^* > 0$ such that for all $\lambda \in (0, \lambda_+^*)$ there exists $\rho_\lambda^+ > 0$ such that
$$\inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+\right\} := m_\lambda^+ > 0.$$
2. There exists $\lambda_-^* > 0$ such that for all $\lambda \in \left(0, \lambda_-^*\right)$ there exists $\rho_\lambda^- > 0$ for which we have
$$\inf\left\{\varphi_\lambda^-(u) : \|u\|_{X_0} = \rho_\lambda^-\right\} := m_\lambda^- > 0.$$

Proof. Without loss of generality, we assume $\mathcal{P} \le r$ (otherwise r is replaced by $\mathcal{P}$ in the calculations below).

Hypotheses 2 and 3 imply that given $\vartheta > 0$ we can find $C = C(\vartheta) > 0$ such that
$$H_\lambda^+(x,t) \le \frac{\vartheta}{2}(t^+)^2 + \lambda C(t^+)^p + C(1+\lambda)(t^+)^r \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R} \tag{43}$$
since $|t|^2 \le |t|^p + |t|^r$ for every $t \in R$.

Then, for all $u \in X_0$, using (8) and Theorem 3 we have
$$\begin{aligned}\varphi_\lambda^+(u) &\ge \frac{(1-\epsilon\|\beta\|_\tau)}{2}\|u\|_{X_0}^2 - \frac{c(\epsilon)\|\beta\|_\tau}{2}\|u\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 - \frac{\vartheta}{2}\|u\|_2^2 \\ &- \lambda C\|u\|_p^p - (1+\lambda)C\|u\|_r^r \\ &\ge \frac{1}{2}(1-\epsilon\|\beta\|_\tau)\|u\|_{X_0}^2 + \frac{1}{2}(-c(\epsilon)\|\beta\|_\tau + \gamma - \vartheta)\|u\|_2^2 \\ &- \lambda B\|u\|_{X_0}^p - (1+\lambda)D\|u\|_{X_0}^r \\ &\ge \left(A - \lambda B\|u\|_{X_0}^{p-2} - D(1+\lambda)\|u\|_{X_0}^{r-2}\right)\|u\|_{X_0}^2,\end{aligned} \tag{44}$$
for some $A,\ B,\ D > 0$.

Now, we consider the function
$$k_\lambda(y) = \lambda B y^{p-2} + D(1+\lambda)y^{r-2} \quad \text{for all } y \in \mathbb{R}.$$
Evidently, $k_\lambda \in C^1(0,\infty)$ and since $p < 2 < r$ (see Hypotheses 2 and 3), we have
$$k_\lambda(y) \to \infty \text{ as } y \to 0^+ \text{ and as } y \to \infty.$$
So, we can find $y_0 \in (0,\infty)$ such that
$$k_\lambda(y_0) = \min\left\{k_\lambda(y) : y > 0\right\} \Rightarrow k_\lambda'(y_0) = 0,$$
that is $\lambda B(2-p) = D(1+\lambda)(r-2)y_0^{r-p}$, and so
$$y_0(\lambda) = \left[\frac{\lambda B(2-p)}{D(1+\lambda)(r-2)}\right]^{\frac{1}{r-p}}.$$
Then, observing that
$$1 + \frac{p-2}{r-p} = \frac{r-2}{r-p} > 0,$$
we can deduce that $k_\lambda(y_0) \to 0^+$ as $\lambda \to 0^+$ and so we can find $\lambda_+^* > 0$ such that for every $\lambda \in (0, \lambda_+^*)$ we have
$$k_\lambda(y_0) < A.$$
So, from (44) it follows that
$$\inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+ = y_0(\lambda)\right\} = m_\lambda^+ \ge A - k_\lambda(y_0) > 0$$
for all $\lambda \in (0, \lambda_+^*)$. In a similar fashion, we show the corresponding result for φ_λ^-. □

For the next result, we set
$$\lambda^* = \min\left\{\lambda_+^*, \lambda_-^*\right\}.$$

Proposition 8. If Hypotheses 2, 3, 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then for every $u \in X_0$, with $u \geq 0$ and $\|u\|_2 = 1$, we have $\varphi_\lambda^+(\zeta u) \to -\infty$ as $\zeta \to \infty$.

Proof. Hypotheses 2(1) and 2(2) imply that, given $\vartheta > 0$, there exists $C_1 = C_1(\vartheta) > 0$ such that

$$\lambda G(x,t) \geq -\frac{\vartheta}{2}t^2 - C_1 \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}, \text{ all, } \lambda \in (0, \lambda^*). \tag{45}$$

Similarly, Hypotheses 3(1) and 3(2) imply that, given $\mu > 0$, we can find $C_2 = C_2(\mu) > 0$ such that

$$F(x,t) \geq \frac{\mu}{2}t^2 - C_2 \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}. \tag{46}$$

Let $u \in X_0$, with $u \geq 0$ and $\|u\|_2 = 1$, and let $\zeta > 0$. Then, from (42), (45) and (46), we have

$$\varphi_\lambda^+(\zeta u) \leq \frac{\zeta^2}{2}\left[\|u\|_{X_0}^2 + (\|\beta\|_\infty + \vartheta - \mu + C)\right] \quad (\text{since } \|u\|_2 = 1) \tag{47}$$

for some $C > 0$.

Since $\vartheta > 0$ and $\mu > 0$ are arbitrary, we can choose $\vartheta > 0$ so small and $\mu > 0$ so large that $\mu - \vartheta > \|u\|_{X_0}^2 + \|\beta\|_\infty + C$. Then, from (47), we infer that

$$\varphi_\lambda^+(\zeta u) \to -\infty \quad \text{as } \zeta \to \infty.$$

□

Remark 7. In a similar fashion, we show that if $u \in X_0$, with $u \leq 0$ and $\|u\|_2 = 1$, then

$$\varphi_\lambda^-(\zeta u) \to -\infty \quad \text{as } \zeta \to \infty.$$

Next we will produce two nontrivial constant sign solution.

Proposition 9. If Hypotheses 2, 3 and 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (E_λ) admits at least two nontrivial weak solution such that

$$v_0(x) \leq 0 \leq u_0(x) \quad \text{for a.e } x \in \Omega.$$

Proof. We do the proof for the functional φ_λ^+. By Proposition 7, for every $\lambda \in (0, \lambda^*)$ it is possible to find $\rho_\lambda^+ > 0$ such that

$$m_\lambda^+ = \inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+\right\} > 0. \tag{48}$$

Thanks to (48) and Propositions 6 and 8, we can apply the Mountain Pass Theorem. So we can find $u_0 \in X_0$ such that

$$\varphi_\lambda^+(0) = 0 < m_\lambda^+ \leq \varphi_\lambda^+(u_0) \tag{49}$$

and

$$\varphi_\lambda^{+'}(u_0) = 0. \tag{50}$$

The inequalities in (49) tell us that $u_0 \neq 0$, while from (50) we have

$$\mathfrak{L}_K u_0 + (\beta(x) + \gamma)u_0 = h_\lambda^+(x, u_0). \tag{51}$$

Like we have already done in Proposition 3, thanks to (9) we have

$$\left\langle u_0, u_0^- \right\rangle_{X_0} = \left\langle u_0^+ - u_0^-, u_0^- \right\rangle_{X_0} = \left\langle u_0^+, u_0^- \right\rangle_{X_0} - \|u_0^-\|_{X_0} \leq -\|u_0^-\|_{X_0}. \tag{52}$$

Acting on (51) with u_0^- and using (52), we obtain

$$\Psi(u_0^-) + \gamma\|u_0^-\|_2^2 \leq \left\langle u_0, -u_0^- \right\rangle_{X_0} + \int_\Omega \beta(u_0^-)^2\,dx + \gamma\|u_0^-\| = 0. \tag{53}$$

Recalling now (8) with $\epsilon > 0$ small enough, we have

$$(1 - \epsilon\|\beta\|_\tau)\|u_0^-\|_{X_0}^2 - c(\epsilon)\|\beta\|_\tau\|u_0^-\|_2^2 \leq \Psi(u_0{}^-), \tag{54}$$

with $\gamma > c(\epsilon)\|\beta\|_\tau$. Hence, by (53), (54) and recalling that $\|u_0{}^-\|_2^2 \leq C\|u_0{}^-\|_{X_0}^2$ (see 3) it follows that

$$\tilde{C}\|u_0^-\|_{X_0}^2 \leq 0$$

for some $\tilde{C} > 0$, which implies that

$$u_0 \geq 0, \quad u_0 \neq 0.$$

So, 51 becomes

$$\mathfrak{L}_K u_0 + \beta(x)u_0 = f(x, u_0) + \lambda g(x, u),$$

i.e. u_0 is weak solution for (E_λ).

Analogously, working with $\hat{\varphi}_\lambda^-$, we obtain the other nontrivial constant sign solution $v_0 \in X_0$. □

In the next proposition we produce a third nontrivial solution for (E_λ) when $\lambda \in (0, \lambda^*)$.

Proposition 10. If Hypotheses 2, 3, 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (E_λ) has a third nontrivial weak solution $y_0 \in [u_0, v_0]$.

Proof. Let u_0 and v_0 the two constant sign solutions found in Proposition 9. With γ as before, we consider the following truncation perturbation of the reaction in problem (E_λ):

$$d_\lambda(x,t) = \begin{cases} \lambda g(x, v_0(x)) + f(x, v_0(x)) + \gamma v_0(x), & \text{if } t < v_0(x), \\ \lambda g(x,t)) + f(x,t) + \gamma t, & \text{if } v_0(x) < t < u_0(x), \\ \lambda g(x, u_0(x)) + f(x, u_0(x)) + \gamma u_0(x), & \text{if } t > u_0(x). \end{cases} \tag{55}$$

This is a Carathéodory function. Set $D_\lambda = \int_0^x d_\lambda(x,w)\,dw$ and consider the C^1-functional $\Xi_\lambda : X_0 \to \mathbb{R}$ defined by

$$\Xi_\lambda(u) = \frac{1}{2}\Psi(u) + \frac{\gamma}{2}\|u\|_2^2 - \int_\Omega D_\lambda(x, u(x))\,dx \quad \text{for all } u \in X_0.$$

By (8), since $\gamma > \|\beta\|_\infty$, we get that

$$\Psi(u) + \gamma\|u\|_2^2 \geq C\|u\|_{X_0}^2 \tag{56}$$

for some $C > 0$.

From (56) and (55), it is clear that Ξ_λ is coercive. Moreover, it is sequentially weakly lower semicontinuous. So, by the Weierstrass Theorem (see[21, Theorem 1.2]), we can find $y_0 \in X_0$ such that

$$\Xi_\lambda(y_0) = \inf\left\{\Xi_\lambda(u) : u \in X_0\right\}. \tag{57}$$

By Hypothesis 3 (3), given $\epsilon > 0$ we can find $\delta = \delta(\epsilon) > 0$ such that

$$-F(x,t) \leq \frac{\epsilon}{2}t^2 \quad \text{for a.e. } x \in \Omega, \text{ all } |t| \leq \delta. \tag{58}$$

Recalling Remark 2, for $\zeta \in (0,1)$ small enough, we have that $\zeta\hat{u}_1 \in (0,\delta]$ for all $x \in \Omega$, see Remark 2. Then

$$\Xi_\lambda(\zeta\hat{u}_1) \leq \frac{\zeta^2}{2}\left[\hat{\lambda}_1 + \gamma + \epsilon\right] - \frac{\lambda\eta_0\zeta^q}{q}\|\hat{u}_1\|_q^q,$$

see (4), (42), (58) and Hypothesis 2 (3).

Since by Hypothesis 2 (3) $q < 2$, by choosing $\zeta \in (0,1)$ even smaller if necessary, we have

$$\Xi_\lambda(\zeta\hat{u}_1) < 0,$$

so that from (57)

$$\Xi_\lambda(y_0) < 0 = \Xi_\lambda(0), \tag{59}$$

and hence $y_0 \neq 0$.

From 57 we have $\Xi'_\lambda(y_0) = 0$, that is

$$\mathfrak{L}_K y_0 + (\beta(x) + \gamma)\, y_0 = d_\lambda(x, y_0). \tag{60}$$

On (60) we act with $(v_0 - y_0)^+ \in X_0$. Then, by (55) we get

$$\begin{aligned} &\left\langle y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \int_\Omega (\beta(x) + \gamma)\, y_0 (v_0 - y_0)^+\,dx \\ &\qquad = \int_\Omega d_\lambda(x, y_0)(v_0 - y_0)^+\,dx = \int_\Omega d_\lambda(x, v_0)(v_0 - y_0)^+\,dx \\ &\qquad\qquad = \left\langle v_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \int_\Omega (\beta(x) + \gamma)\, v_0(v_0 - y_0)^+\,dx, \end{aligned}$$

hence
$$\left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \int_\Omega (\beta(x) + \gamma)\,(v_0 - y_0)(v_0 - y_0)^+ \, dx = 0.$$
Then, recalling the choice of γ, there exists $\tilde{C} > 0$ such that
$$\left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \tilde{C} \int_\Omega \left(v_0 - y_0\right)(v_0 - y_0)^+ \, dx \leq 0,$$
and so, using (9) in the first inequality, we get
$$\begin{aligned}\|(v_0 - y_0)^+\|_{X_0}^2 &\leq \|(v_0 - y_0)^+\|_{X_0}^2 - \left\langle (v_0 - y_0)^-, (v_0 - y_0)^+ \right\rangle_{X_0} \\ &\quad + \tilde{C} \int_\Omega \left[(v_0 - y_0)^+\right]^2 \\ &= \left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \tilde{C} \int_\Omega (v_0 - y_0)(v_0 - y_0)^+ \leq 0.\end{aligned}$$
Thus we deduce that $v_0 \leq y_0$ in Ω.

Similarly, acting on on (60) with $(y_0 - u_0)^+ \in X_0$ and repeating analogous calculations, we obtain $y_0 \leq u_0$. Putting together the two inequalities, we have $y_0 \in \left[v_0, u_0\right] = \left\{u \in X_0 \,:\, v_0(x) \leq u(x) \leq u_0(x) \;\forall x \in \Omega\right\}$.

Then (60) becomes
$$\mathfrak{L}_K y_0 + \beta(x) y_0 = \lambda g(x, y_0) + f(x, y_0),$$
i.e. y_0 is a weak solution of problem (E_λ). □

So, summarizing the situation for problem (E_λ), we can state the following multiplicity theorem:

Theorem 4. If Hypotheses 2, 3, 4 hold and $\beta \in L^\infty(\Omega)$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (E_λ) has at least three nontrivial weak solutions $u_0,\ v_0,\ y_0 \in C^s(\bar{\Omega}) \setminus \{0\}$ such that
$$\begin{aligned}&u_0 \geq 0 \text{ for a.e. } x \in \Omega, \\ &v_0 \leq 0 \text{ for a.e. } x \in \Omega, \\ &y_0 \in \left[u_0, v_0\right].\end{aligned}$$

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$\square$