Spatial transmission and risk assessment of West Nile virus on a growing domain *

Liqiong Pu[a,b] and Zhigui Lin[a†]

[a] School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

[b] School of Mathematics and Statistics, Hexi University, Zhangye 734000, China.

Abstract. This paper is concerned with a West Nile virus (WNv) model on a growing domain, which accounts for habitat expansion of mosquitoes because of climate warming. We aim to understand the relationship of the growing rate and the transmission risk of WNv. The basic reproduction number, which is related to the growing rate and diffusion rate, is introduced through spectral theory. The conditions to determine whether the virus vanishes or spreads are deduced. The obtained results reveal that domain growth leads to increased risk of infection, and is detrimental to the control and prevention of WNv. To verify the feasibility of our analytical results on the long time behavior of WNv, some numerical simulations are given.

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Keywords: West Nile virus; Growing domain; Spatial transmission; Basic reproduction number

1 Introduction

Infectious diseases, which caused by various pathogens, can spread from person to person, animal to animal, or person to animal. Since ancient times, the prevalence of infectious diseases has brought enormous damage to people, therefore, many countries around the world pay more attention to the investigation of the infectious diseases. Various epidemic models have been proposed and analyzed for prevention and control strategies, especially for vector borne diseases [10, 15, 30].

West Nile virus (WNv) is an emerging mosquito-borne virus that causes a severe, life-threatening neurological disease in humans and horses, widely distributed throughout the world and with considerable impact both on public health and on animal health [9].

Many species of birds act as primary hosts [22] and main source of transmission for WNv [29]. The virus exists in the form of a mosquito-bird cycle, that is, when mosquitoes carrying the virus bite susceptible birds, they produce toxemia in the birds, and susceptible mosquitoes bite host birds and then participate in the spread of the virus, thus circulating [1]. More than 300 avian species have been identified as being associated with transmission of the virus. In general, the corvid and non-corvid families of birds have different responses to the virus, with corvids

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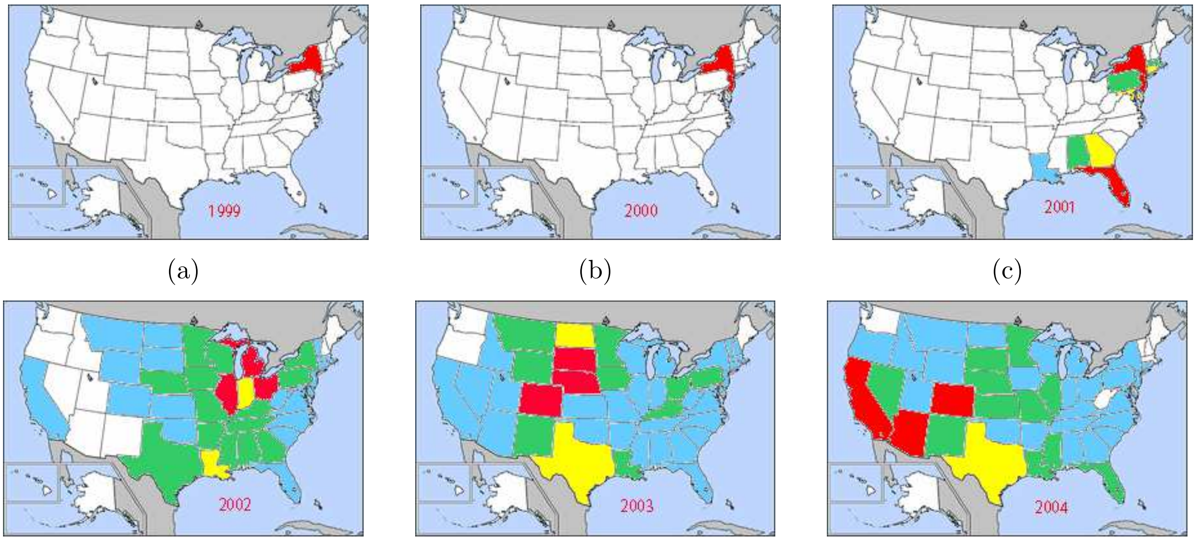
†Corresponding author. Email: zglin@yzu.edu.cn (Z. Lin).

suffering a higher disease-induced mortality rate, such as crows, large ravens, magpies, blue and grey birds [4, 16].

WNv was first identified in 1937 from the blood of a febrile woman in the west Nile region of Uganda [6], hence its name. An outbreak in Israel in 1957, in which the virus was first noted to be associated with central nervous system disease, was thought to be the cause of severe meningitis in the elderly [25]. Since 1950, the virus has been circulating in Africa, the Middle East, Asia and southern Europe, the first infected case was detected in 1999 during an outbreak of encephalitis in New York city [6, 19, 23, 30], and then propagated rapidly across the US [12, 19]. Fig. 1 shows the progress of human cases of WNv throughout the US, and indicate the growth process of infected areas from 1999 to 2004. The states were colored according to the percentage of all west Nile cases they represented in the US that year.



(a) (b) (c)

(d) (e) (f)

Fig. 1: Figures show the progress of human cases of WNv throughout the US. 0%(no cases) is white; Less than 1% is blue; Between 1% and 5% is green; Between 5% and 10% is yellow, and more than 10% is red. ([2], minor modification). All the data is based on the US Centers for Disease Control and Prevention [3].

For nearly two decades, many mathematical models for WNv have been proposed and analyzed, however most of the models are focused on the non-spatial transmission dynamics [8, 30, 31]. In order to analyze and evaluate two main anti-WNv preventive strategies, that is, mosquito reduction strategies and personal protection, Bowman et al. [8] proposed a single-season ordinary differential equation model for WNv in a mosquito-bird-human community, and showed that if the mosquito reduction strategy is carried out, which guarantee a certain threshold quantity less than 1, WNv can be eradicated from the mosquito-bird cycle, on the contrary, WNv persists in the mosquito-bird population.

But what we should actually do is considering the spatial spreading, which is an important factor to affect the persistence and eradication of WNv. For the sake of depicting the movement of birds and mosquitoes, Lewis et al. [19] researched the spatial spread of WNv, which is described by the reaction-diffusion model. The model was inspired and extended from Wonham et al. in [30], whose paper proposed and developed the non-spatial model for cross-infection between mosquitoes and birds.

To utilize the cooperative characteristic of cross-infection dynamics and estimate the spatial spread rate of infection, Lewis et al. in [19] proposed the modified spatial-independent WNv model

$$\begin{cases} \frac{\partial I_b}{\partial t} = \alpha_b \beta_b \frac{(N_b - I_b)}{N_b} I_m - \gamma_b I_b, & t > 0, \\ \frac{\partial I_m}{\partial t} = \alpha_m \beta_b \frac{(A_m - I_m)}{N_b} I_b - d_m I_m, & t > 0, \end{cases} \tag{1.1}$$

and spatial-dependent WNv model

$$\begin{cases} \frac{\partial I_b}{\partial t} = D_1 \Delta I_b + \alpha_b \beta_b \frac{(N_b - I_b)}{N_b} I_m - \gamma_b I_b, & x \in \Omega, t > 0, \\ \frac{\partial I_m}{\partial t} = D_2 \Delta I_m + \alpha_m \beta_b \frac{(A_m - I_m)}{N_b} I_b - d_m I_m, & x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

where N_b and A_m are the positive constants; $I_b(x,t)$ and $I_m(x,t)$ represent the populations of infected birds and mosquitos at the location x in the habitat $\Omega \subset \mathbb{R}^N$ and at time $t \geq 0$, respectively, and $I_b(x,0) + I_m(x,0) > 0$. The parameters are described in Table 1.

Table 1: Parameters Epidemiological Interpretations.

Parameter	Description
N_b	The total population of birds
A_m	The total population of mosquitoes
α_b	WNv transmission probability per bite to birds
α_m	WNv transmission probability per bite to mosquitoes
β_b	Biting rate of mosquitoes on birds
d_m	Adult mosquitos death rate
γ_b	Bird recovery rate from WNv
D_1	Diffusion coefficients for birds
D_2	Diffusion coefficients for mosquitoes

For model (1.1), the authors derived the basic reproduction number

$$\mathcal{R}_0 = \sqrt{\frac{\alpha_b \alpha_m \beta_b^2 A_m}{\gamma_b d_m N_b}} \tag{1.3}$$

by utilizing the next generation matrix method [28]. They pointed out that the virus vanishes for $\mathcal{R}_0 < 1$, while for $\mathcal{R}_0 > 1$, the disease-endemic equilibrium stabilizes. Moreover, in terms of (1.2), they considered the existence of traveling waves, and deduced that the spread rate, which is determined by linearized system, is equivalent to the minimal wave speed of the non-linear one.

In ecology, animals migrate over a certain distance for a variety of reasons. ones search for food to keep themselves alive, for instance, the king penguins look for fish and shrimp in nonfreezing seas; ones move for reproduction, such as Atlantic salmon and giant arapaima fish, they have to spawn in the upstream of freshwater river every year; ones (migrant birds) travel long distances in response to local climate change. In consideration of the mosquitos, which are extremely temperature and moisture sensitive. Moreover, up to a point, higher temperatures cause the mosquitoes to mature faster. Hence its habitat is incessantly expansive due to climate warming. For instance, according to the report of future Ontario climate change projections [33], the average temperature in Ontario is generally rising. It is estimated that by 2050, the average summer temperature (7-21°C) in southern and Northern Ontario will increase by 1.5-4°C and 4-7°C respectively in the 1980s; by 2050s, the average temperature in Ontario is likely to increase by about 3.6°C, and the number of warmer days will increase by 10-126 days. Therefore the range of mosquitoes to change and move northward beyond the current boundary.

Now, we consider the WNv diffusive model on a growing domain through the following discussion. As discussed in [11,27], we assume that $\Omega(t) \subset \mathbb{R}^N$ be a simply connected bounded shifting domain at time $t \geq 0$ and $\partial\Omega(t)$ be the surface boundary of changing domain $\Omega(t)$. Let $I = (I_b(x(t),t), I_m(x(t),t))^T$ be a vector of two infected species (birds and mosquitos) at position $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \Omega(t)$. From the principle of mass conservation and

Reynolds transport theorem [26], we can derive the evolution equations for reaction-diffusion on growing domain. The continuous variation of the domain and its boundary produces a flow velocity $\mathbf{a}(x,t)$. And the advection terms and dilution terms are introduced in the growing domain $\Omega(t)$, the former is the amount of elements that shift with the flow due to local changes of domain, while the latter corresponds to changes of local volume. Consequently, the reaction-diffusion system of two infected species I_b and I_m with (1.2) on a continuously deforming domain becomes

$$\begin{cases} \underbrace{\frac{\partial I_b}{\partial t}}_{\text{The rate of change of } I_b} + \underbrace{\nabla I_b \cdot \mathbf{a}}_{\text{advection term}} + \underbrace{I_b(\nabla \cdot \mathbf{a})}_{\text{dilution term}} = \underbrace{D_1 \Delta I_b}_{\text{diffusion term}} + \underbrace{\alpha_b \beta_b \frac{(N_b - I_b)}{N_b} I_m - \gamma_b I_b}_{\text{reaction term}}, \\ \underbrace{\frac{\partial I_m}{\partial t}}_{\text{The rate of change of } I_m} + \underbrace{\nabla I_m \cdot \mathbf{a}}_{\text{advection term}} + \underbrace{I_m(\nabla \cdot \mathbf{a})}_{\text{dilution term}} = \underbrace{D_2 \Delta I_m}_{\text{diffusion term}} + \underbrace{\alpha_m \beta_b \frac{(A_m - I_m)}{N_b} I_b - d_m I_m}_{\text{reaction term}} \end{cases} \tag{1.4}$$

for $x \in \Omega(t),\ t \geq 0$, with the null Dirichlet boundary condition

$$I_b(x(t),t) = I_m(x(t),t) = 0, \quad x \in \partial\Omega(t),\ t \geq 0, \tag{1.5}$$

and the initial condition

$$I_b = I_b(x) \leq N_b,\ I_m = I_m(x) \leq A_m, \quad x \in \Omega(0),\ t = 0. \tag{1.6}$$

In order to circumvent the difficulty induced by the evolving domain, we want to transform problem (1.4)-(1.6) from a continuously changing domain to a fixed domain. To do this, we use Lagrangian transformations, see [21] for details. Assume $y_1, y_2, \ldots, y_n$ be fixed Cartesian coordinates in $\Omega(0)$ such that $x_1(t) = \hat{x}_1(y_1, y_2, \ldots, y_n, t)$, $x_2(t) = \hat{x}_2(y_1, y_2, \ldots, y_n, t)$, $\ldots$ $x_n(t) = \hat{x}_n(y_1, y_2, \ldots, y_n, t)$. Then (I_b, I_m) is mapped into the new function (u, v) defined as

$$\begin{aligned} I_b(x_1(t), x_2(t), \ldots, x_n(t), t) &= u(y_1, y_2, \ldots, y_n, t), \\ I_m(x_1(t), x_2(t), \ldots, x_n(t), t) &= v(y_1, y_2, \ldots, y_n, t). \end{aligned} \tag{1.7}$$

Although equations (1.4) can be translated to another form which are defined on the fixed domain $\Omega(0)$ with respect to $y = (y_1, y_2, \ldots, y_n)$, it is still difficult to deal with the new equations. To further simplify the model equations (1.4), we assume that domain evolution is uniform and isotropic. That is to say, the boundary of the domain deforms continuously at the same rate in all directions as time goes on. Mathematically, we let

$$(x_1(t), x_2(t), \ldots, x_n(t)) = \rho(t)(y_1, y_2, \ldots, y_n), \quad y \in \Omega(0), \tag{1.8}$$

where $\rho(t)$ is called growing ratio, which satisfies

$$\rho(t) \in C^1[0,\infty),\ \rho(0) = 1,\ \dot{\rho}(t) \geq 0,\ \lim_{t\to\infty} \rho(t) = \rho_\infty \geq 1 \text{ and } \lim_{t\to\infty} \dot{\rho}(t) = 0. \tag{1.9}$$

If $\rho(t) \equiv 1$ for any t, then $\Omega(t)$ is just the fixed domain $\Omega(0)$. Furthermore, assume that the flow velocity is identical to the evolving rate of domain and using (1.8) yields

$$\mathbf{a} = \dot{x}(t) = \dot{\rho}(t) y = \frac{\dot{\rho}(t)}{\rho(t)} x(t),$$

then it follows from (1.7)-(1.8) that

$$\begin{aligned} &u_t = \tfrac{\partial I_b}{\partial t} + \mathbf{a} \cdot \nabla I_b, \quad v_t = \tfrac{\partial I_m}{\partial t} + \mathbf{a} \cdot \nabla I_m, \quad \nabla \cdot \mathbf{a} = \tfrac{n\dot{\rho}}{\rho}, \\ &\Delta I_b = \tfrac{1}{\rho^2(t)} \Delta u, \quad \Delta I_m = \tfrac{1}{\rho^2(t)} \Delta v. \end{aligned}$$

Therefore, problem (1.4)-(1.6) is translated into

$$\begin{cases} u_t - \frac{D_1}{\rho^2(t)}\Delta u = \alpha_b\beta_b\frac{(N_b-u)}{N_b}v - (\gamma_b + \frac{n\dot{\rho}(t)}{\rho(t)})u, & y \in \Omega(0), t > 0, \\ v_t - \frac{D_2}{\rho^2(t)}\Delta v = \alpha_m\beta_b\frac{(A_m-v)}{N_b}u - (d_m + \frac{n\dot{\rho}(t)}{\rho(t)})v, & y \in \Omega(0), t > 0, \\ u(y,t) = v(y,t) = 0, & y \in \partial\Omega(0), t > 0, \\ u(y,0) = I_{b,0}(y) \le N_b, v(y,0) = I_{m,0}(y) \le A_m, & y \in \Omega(0). \end{cases} \tag{1.10}$$

As far as growth ratio $\rho(t)$ is concerned, see for example, the exponential growth rate $\rho(t) = e^{rt}$ with $r > 0$ and the logistic growth rate $\rho(t) = \frac{e^{Krt}}{1+\frac{1}{K}(e^{Krt}-1)}$ in [20], where K accounts for the carrying capacity (final domain size).

The rest of our paper is organised as follows: in Section 2, we introduce the basic reproduction numbers and study related properties. The corresponding preliminaries and the derivation of stability conditions for disease free equilibrium and the endemic equilibrium on a growing domain are presented in Section 3, while three examples are provided to illustrate our theoretical findings in the last section.

2 Basic reproduction number

It is well-known [13] that the basic reproduction number is a threshold of epidemiology models to describe the average number of secondary infections produced when a typical infected individual is introduced into a large susceptible population. In this section, the basic reproduction number is introduced and its related properties are given. For general ODE model, basic reproduction number can be calculated by the next generation matrix method [28], and subsequently developed by [32, 35], they presented by spectral radius of next infection operator for space-dependent models, it is also calculated by principal eigenvalue of a corresponding eigenvalue problem. In the following, the basic reproduction number for problem (1.10) is given by the eigenvalue method.

According to assumption $\lim\limits_{t\to\infty} \rho(t) = \rho_\infty$, as a consequence, consider the auxiliary problem of (1.10)

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta U = \alpha_b\beta_b\frac{(N_b-U)}{N_b}V - \gamma_b U, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta V = \alpha_m\beta_b\frac{(A_m-V)}{N_b}U - d_m V, & y \in \Omega(0), \\ U(y) = V(y) = 0, & y \in \partial\Omega(0), \end{cases} \tag{2.1}$$

and its related eigenvalue problem

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta \phi = \frac{1}{R}\alpha_b\beta_b\psi - \gamma_b\phi + \lambda(R)\phi, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta \psi = \frac{1}{R}\alpha_m\beta_b\frac{A_m}{N_b}\phi - d_m\psi + \lambda(R)\psi, & y \in \Omega(0), \\ \phi(y) = \psi(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{2.2}$$

Note that problem (2.2) is strongly cooperative if $R > 0$, by virtue of Theorem 2.3 [7], it exists a unique principal eigenvalue $\lambda(R)(R > 0)$, and then the positive eigenfunction pair (ϕ, ψ) corresponding to the principal eigenvalue $\lambda(R)$ is unique (subject to a constant multiple) for $y \in \Omega(0)$. Meanwhile, continuous function $\lambda(R)$ is strictly increasing to R, and satisfies $\lim\limits_{R\to 0^+} \lambda(R) < 0$, $\lim\limits_{R\to\infty} \lambda(R) > 0$. Based on the above argument, we derive that there is a unique value R_0^ρ such that $\lambda(R_0^\rho) = 0$, which satisfies the following problem

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta \Phi = \frac{1}{R_0^\rho}\alpha_b\beta_b\Psi - \gamma_b\Phi, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta \Psi = \frac{1}{R_0^\rho}\alpha_m\beta_b\frac{A_m}{N_b}\Phi - d_m\Psi, & y \in \Omega(0), \\ \Phi(y) = \Psi(y) = 0, & y \in \partial\Omega(0), \end{cases} \tag{2.3}$$

where (Φ, Ψ) is the positive eigenfunction pair corresponding to the principal eigenvalue R_0^ρ.

Clearly, the principal eigenvalue R_0^ρ of problem (2.3) can be explicitly written as

$$R_0^\rho = \sqrt{\frac{\alpha_b\beta_b \cdot \alpha_m\beta_b\frac{A_m}{N_b}}{(\frac{D_1\lambda^*}{\rho_\infty^2} + \gamma_b)(\frac{D_2\lambda^*}{\rho_\infty^2} + d_m)}} \tag{2.4}$$

and the positive eigenfunction pair corresponding to R_0^ρ

$$(\Phi, \Psi) = \left(\sqrt{\frac{\alpha_b\beta_b(\frac{D_2\lambda^*}{\rho_\infty^2} + d_m)}{\alpha_m\beta_b\frac{A_m}{N_b}(\frac{D_1\lambda^*}{\rho_\infty^2} + \gamma_b)}}\varphi^*, \quad \varphi^*\right), \tag{2.5}$$

where (λ^*, φ^*) is the principal eigen-pair of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & y \in \Omega(0), \\ \varphi(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{2.6}$$

Remark 2.1 *$R_0^\rho > \mathcal{R}_0$, where $\mathcal{R}_0$ has been given in* (1.3)*, which means the risk increases with respect to the growth of habitat.*

Remark 2.2 *It follows from problem* (2.3)*, equation* (2.5) *and problem* (2.6) *that*

$$-\Delta\Phi(y) \geq 0, \quad -\Delta\Psi(y) \geq 0, \quad y \in \Omega(0), \tag{2.7}$$

which will be used in the sequel.

3 Asymptotic profiles of solutions

In this section, to derive the property of disease free equilibrium and the endemic equilibrium of problem (1.10) on a growing domain, we will construct the Lyapunov function and use a super-subsolution technique. Before proceeding further, we here give the definition and relevant lemma of the supersolution and subsolution, which will be used to study the asymptotic profile behaviour of solutions.

3.1 Preliminaries

Throughout this paper, we denote the sector

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \{(u, v) : u_1 \leq u(y, t) \leq u_2, v_1 \leq v(y, t) \leq v_2, (y, t) \in \overline{\Omega}(0) \times [0, \infty)\}.$$

Definition 3.1 *A pair of functions $(\tilde{u}, \tilde{v})(y, t)$ and $(\hat{u}, \hat{v})(y, t)$ in $\mathcal{C}^{2,1}(\Omega(0) \times (0, \infty)) \bigcap \mathcal{C}(\overline{\Omega}(0) \times [0, \infty))$ is called ordered upper and lower solutions to problem* (1.10) *if $(0, 0) \leq (\hat{u}, \hat{v}) \leq (\tilde{u}, \tilde{v}) \leq (N_b, A_m)$ and*

$$\begin{cases} \tilde{u}_t - \frac{D_1}{\rho^2(t)}\Delta\tilde{u} \geq \alpha_b\beta_b\frac{(N_b - \tilde{u})}{N_b}\tilde{v} - (\gamma_b + \frac{n\dot{\rho}(t)}{\rho(t)})\tilde{u}, & y \in \Omega(0), t > 0, \\ \tilde{v}_t - \frac{D_2}{\rho^2(t)}\Delta\tilde{v} \geq \alpha_m\beta_b\frac{(A_m - \tilde{v})}{N_b}\tilde{u} - (d_m + \frac{n\dot{\rho}(t)}{\rho(t)})\tilde{v}, & y \in \Omega(0), t > 0, \\ \hat{u}_t - \frac{D_1}{\rho^2(t)}\Delta\hat{u} \geq \alpha_b\beta_b\frac{(N_b - \hat{u})}{N_b}\hat{v} - (\gamma_b + \frac{n\dot{\rho}(t)}{\rho(t)})\hat{u}, & y \in \Omega(0), t > 0, \\ \hat{v}_t - \frac{D_2}{\rho^2(t)}\Delta\hat{v} \geq \alpha_m\beta_b\frac{(A_m - \hat{v})}{N_b}\hat{u} - (d_m + \frac{n\dot{\rho}(t)}{\rho(t)})\hat{v}, & y \in \Omega(0), t > 0, \\ \tilde{u}(y, t) \geq 0 = \hat{u}(y, t), \ \tilde{v}(y, t) \geq 0 = \hat{v}(y, t), & y \in \partial\Omega(0), t > 0, \\ \tilde{u}(y, 0) \geq u(y, 0) \geq \hat{u}(y, 0), \ \tilde{v}(y, 0) \geq v(y, 0) \geq \hat{v}(y, 0), & y \in \Omega(0). \end{cases} \tag{3.1}$$

Lemma 3.1 *(Comparison principle) Let $(u,v)(y,t)$, $(\tilde{u},\tilde{v})(y,t)$ and $(\hat{u},\hat{v})(y,t)$ be a solution, ordered upper and lower solutions to problem* (1.10) *respectively, then*

$$(\hat{u},\hat{v})(y,t) \le (u,v)(y,t) \le (\tilde{u},\tilde{v})(y,t), \quad (y,t) \in \overline{\Omega}(0) \times [0,\infty).$$

Neglecting time variable t in Definition 3.1, we can get a straightforward definition of upper and lower solutions to problem (2.1).

Definition 3.2 *A pair of functions $(\tilde{U},\tilde{V})(y)$ and $(\hat{U},\hat{V})(y)$ in $\mathcal{C}^2(\Omega(0))\bigcap\mathcal{C}(\overline{\Omega}(0))$ is called ordered upper and lower solutions of problem* (2.1) *if it satisfies* $(0,0) \le (\hat{U},\hat{V}) \le (\tilde{U},\tilde{V}) \le (N_b, A_m)$ *and*

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta\tilde{U} \ge \alpha_b\beta_b\frac{(N_b-\tilde{U})}{N_b}\tilde{V} - \gamma_b\tilde{U}, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta\tilde{V} \ge \alpha_m\beta_b\frac{(A_m-\tilde{V})}{N_b}\tilde{U} - d_m\tilde{V}, & y \in \Omega(0), \\ -\frac{D_1}{\rho_\infty^2}\Delta\hat{U} \ge \alpha_b\beta_b\frac{(N_b-\hat{U})}{N_b}\hat{V} - \gamma_b\hat{U}, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta\hat{V} \ge \alpha_m\beta_b\frac{(A_m-\hat{V})}{N_b}\hat{U} - d_m\hat{V}, & y \in \Omega(0), \\ \tilde{U}(y) \ge 0 = \hat{U}(y),\ \tilde{V}(y) \ge 0 = \hat{V}(y), & y \in \partial\Omega(0). \end{cases} \tag{3.2}$$

Let

$$K_1 = \alpha_m\beta_b\frac{A_m}{N_b} + \gamma_b, \quad K_2 = \alpha_b\beta_b + d_m.$$

Using $(\overline{U}^{(0)},\overline{V}^{(0)}) = (\tilde{U}(y),\tilde{V}(y))$, $(\underline{U}^{(0)},\underline{V}^{(0)}) = (\hat{U}(y),\hat{V}(y))$ as the initial values, iteration sequences $(\overline{U}^{(k)},\overline{V}^{(k)})$ and $(\underline{U}^{(k)},\underline{V}^{(k)})$ are obtained through the following iterative process

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta\overline{U}^{(k)} + K_1\overline{U}^{(k)} = K_1\overline{U}^{(k-1)} + \alpha_b\beta_b(\frac{N_b-\overline{U}^{(k-1)}}{N_b})\overline{V}^{(k-1)} - \gamma_b\overline{U}^{(k-1)}, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta\overline{V}^{(k)} + K_2\overline{U}^{(k)} = K_2\overline{V}^{(k-1)} + \alpha_m\beta_b(\frac{A_m-\overline{V}^{(k-1)}}{N_b})\overline{U}^{(k-1)} - d_m\overline{V}^{(k-1)}, & y \in \Omega(0), \\ -\frac{D_1}{\rho_\infty^2}\Delta\underline{U}^{(k)} + K_1\underline{U}^{(k)} = K_1\underline{U}^{(k-1)} + \alpha_b\beta_b(\frac{N_b-\underline{U}^{(k-1)}}{N_b})\underline{V}^{(k-1)} - \gamma_b\underline{U}^{(k-1)}, & y \in \Omega(0), \\ -\frac{D_2}{\rho_\infty^2}\Delta\underline{V}^{(k)} + K_2\underline{U}^{(k)} = K_2\underline{V}^{(k-1)} + \alpha_m\beta_b(\frac{A_m-\underline{V}^{(k-1)}}{N_b})\underline{U}^{(k-1)} - d_m\underline{V}^{(k-1)}, & y \in \Omega(0), \\ \overline{U}^{(k)}(y) = \overline{V}^{(k)}(y) = \underline{U}^{(k)}(y) = \underline{V}^{(k)}(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{3.3}$$

where $k = 1,2,\cdots$, $(\overline{U}^{(k)},\overline{V}^{(k)})$ and $(\underline{U}^{(k)},\underline{V}^{(k)})$ are called the maximal and minimal sequences. From the comparison principle, it easily follows that the above sequences $(\overline{U}^{(k)},\overline{V}^{(k)})$ and $(\underline{U}^{(k)},\underline{V}^{(k)})$ admit the monotone property

$$(\hat{U},\hat{V}) \le (\underline{U}^{(k-1)},\underline{V}^{(k-1)}) \le (\underline{U}^{(k)},\underline{V}^{(k)}) \le (\overline{U}^{(k)},\overline{V}^{(k)}) \le (\overline{U}^{(k-1)},\overline{V}^{(k-1)}) \le (\tilde{U},\tilde{V}), \tag{3.4}$$

for $y \in \Omega(0)$ and therefore the limits exist, denoted by

$$\lim_{k\to\infty}(\overline{U}^{(k)},\overline{V}^{(k)}) = (\overline{U},\overline{V}), \quad \lim_{k\to\infty}(\underline{U}^{(k)},\underline{V}^{(k)}) = (\underline{U},\underline{V}),$$

which yields

$$\begin{aligned} (\hat{U},\hat{V}) &\le (\underline{U}^{(k-1)},\underline{V}^{(k-1)}) \le (\underline{U}^{(k)},\underline{V}^{(k)}) \le (\underline{U},\underline{V}) \\ &\le (\overline{U},\overline{V}) \le (\overline{U}^{(k)},\overline{V}^{(k)}) \le (\overline{U}^{(k-1)},\overline{V}^{(k-1)}) \le (\tilde{U},\tilde{V}), \end{aligned}$$

Furthermore, uniform estimate of the elliptic problem and Sobolev imbedding theorem assert that $(\overline{U},\overline{V})$ and $(\underline{U},\underline{V})$ meet with

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta\overline{U} = \alpha_b\beta_b(\frac{N_b-\overline{U}}{N_b})\overline{V} - \gamma_b\overline{U}, & y\in\Omega(0),\\ -\frac{D_2}{\rho_\infty^2}\Delta\overline{V} = \alpha_m\beta_b(\frac{A_m-\overline{V}}{N_b})\overline{U} - d_m\overline{V}, & y\in\Omega(0),\\ -\frac{D_1}{\rho_\infty^2}\Delta\underline{U} = \alpha_b\beta_b(\frac{N_b-\underline{U}}{N_b})\underline{V} - \gamma_b\underline{U}, & y\in\Omega(0),\\ -\frac{D_2}{\rho_\infty^2}\Delta\underline{V} = \alpha_m\beta_b(\frac{A_m-\underline{V}}{N_b})\underline{U} - d_m\underline{V}, & y\in\Omega(0),\\ \overline{U}(y)=\overline{V}(y)=\underline{U}(y)=\underline{V}(y)=0, & y\in\partial\Omega(0). \end{cases}$$

Thus $(\overline{U},\overline{V})$ and $(\underline{U},\underline{V})$ are the solutions of problem (2.1) in the sector $\langle(\hat{U},\hat{V}),\,(\tilde{U},\tilde{V})\rangle$. For any solution (U_1,V_1) to problem (2.1) in the sector $\langle(\hat{U},\hat{V}),\,(\tilde{U},\tilde{V})\rangle$, if (U_1,V_1) and $(\underline{U},\underline{V})$ are a pair of super- and subsolutions of problem (2.1), we have $(U_1,V_1)\geq(\underline{U},\underline{V})$, in a similar way, we also have $(U_1,V_1)\leq(\overline{U},\overline{V})$. Therefore,

$$(\underline{U},\underline{V})\leq(U_1,V_1)\leq(\overline{U},\overline{V}) \ \text{ uniformly for } y\in\overline{\Omega}(0).$$

$(\overline{U},\overline{V})$ and $(\underline{U},\underline{V})$ are called the maximal and minimal solutions of problem (2.1) in the sector $\langle(\hat{U},\hat{V}),\,(\tilde{U},\tilde{V})\rangle$, respectively.

The following result is obvious, and is presented here for later application.

Lemma 3.2 *$(\overline{U}^{(k)},\overline{V}^{(k)})$ and $(\underline{U}^{(k)},\underline{V}^{(k)})$ generated by* (3.3) *converge monotonically to $(\overline{U},\overline{V})$ and $(\underline{U},\underline{V})$, respectively.*

3.2 The stability of the disease free equilibrium

Theorem 3.3 *If $\mathcal{R}_0<1$, then the disease free equilibrium* $(0,0)$ *of problem* (1.10) *is globally asymptotically stable.*

Proof: It follows from the assumption $\mathcal{R}_0<1$ that

$$\alpha_b\beta_b\cdot\alpha_m\beta_b\frac{A_m}{N_b}<\gamma_b d_m, \tag{3.5}$$

then there exits a small enough integer $\varepsilon_0>0$ such that

$$\alpha_b\beta_b\cdot\alpha_m\beta_b\frac{A_m}{N_b}<(1-\varepsilon_0)^2\gamma_b d_m. \tag{3.6}$$

Moreover, the above inequality implies that for any $t>0$

$$\int_{\Omega(0)}\alpha_b\beta_b uv dy+\theta\int_{\Omega(0)}\alpha_m\beta_b\frac{A_m}{N_b}uv dy\leq(1-\varepsilon_0)\int_{\Omega(0)}\gamma_b u^2dy+(1-\varepsilon_0)\theta\int_{\Omega(0)}d_m v^2dy, \tag{3.7}$$

where $\theta=\frac{2(1-\varepsilon_0)^2\gamma_b d_m-\alpha_b\beta_b\cdot\alpha_m\beta_b\frac{A_m}{N_b}}{(\alpha_m\beta_b\frac{A_m}{N_b})^2}>0$, and $(u,v)(y,t)$ is a solution of problem (1.10).

Construct the proper Lyapunov function

$$V(t)=\frac{1}{2}\int_{\Omega(0)}u^2dy+\frac{\theta}{2}\int_{\Omega(0)}v^2dy.$$

Calculating the derivative of $V(t)$ with respect to t yields

$$\begin{aligned}
V'(t) &= \int_{\Omega(0)} u\cdot u_t dy + \theta\int_{\Omega(0)} v\cdot v_t dy \\
&= \int_{\Omega(0)} u[\tfrac{D_1}{\rho^2(t)}\Delta u + \alpha_b\beta_b(\tfrac{N_b-u}{N_b})v - (\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)})u]dy \\
&\quad +\theta\int_{\Omega(0)} v[\tfrac{D_2}{\rho^2(t)}\Delta v + \alpha_m\beta_b(\tfrac{A_m-v}{N_b})u - (d_m + \tfrac{n\dot\rho(t)}{\rho(t)})v]dy \\
&= -\int_{\Omega(0)} \tfrac{D_1}{\rho^2(t)}|\nabla u|^2 dy - \theta\int_{\Omega(0)} \tfrac{D_2}{\rho^2(t)}|\nabla v|^2 dy + \int_{\Omega(0)} \alpha_b\beta_b uvdy \\
&\quad +\theta\int_{\Omega(0)} \alpha_m\beta_b\tfrac{A_m}{N_b}uvdy - \int_{\Omega(0)} \tfrac{\alpha_b\beta_b}{N_b}u^2vdy - \int_{\Omega(0)}(\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)})u^2dy \\
&\quad -\theta\int_{\Omega(0)} \tfrac{\alpha_m\beta_b}{N_b}v^2udy - \theta\int_{\Omega(0)}(d_m + \tfrac{n\dot\rho(t)}{\rho(t)})v^2dy \\
&\le -\int_{\Omega(0)} \tfrac{D_1}{\rho^2(t)}|\nabla u|^2 dy - \theta\int_{\Omega(0)} \tfrac{D_2}{\rho^2(t)}|\nabla v|^2 dy - \int_{\Omega(0)} \tfrac{\alpha_b\beta_b}{N_b}u^2vdy \\
&\quad -\theta\int_{\Omega(0)} \tfrac{\alpha_m\beta_b}{N_b}v^2udy - \int_{\Omega(0)} \tfrac{n\dot\rho(t)}{\rho(t)}u^2dy - \theta\int_{\Omega(0)} \tfrac{n\dot\rho(t)}{\rho(t)}v^2dy \\
&\quad -\varepsilon_0\int_{\Omega(0)}\gamma_b u^2dy - \varepsilon_0\theta\int_{\Omega(0)} d_m v^2dy \\
&\le -\varepsilon_0\int_{\Omega(0)}\gamma_b u^2dy - \varepsilon_0\theta\int_{\Omega(0)} d_m v^2dy.
\end{aligned} \tag{3.8}$$

Thanks to the Lyapunov stability theory, we have the convergence of L^1-norm. Further, using the technique of Theorem 4.2 in [18], we deduce that the disease free equilibrium $(0,0)$ is globally asymptotically stable when $\mathcal{R}_0 < 1$. □

Next, when $R_0^\rho < 1$, we exhibit another conclusion about the disease free equilibrium of problem (1.10).

To simplify the statement, we let

$$M = \min\{\frac{N_b}{\max_{y\in\overline{\Omega}(0)}\Phi(y)}, \frac{A_m}{\max_{y\in\overline{\Omega}(0)}\Psi(y)}\}, \tag{3.9}$$

where (Φ,Ψ) is the eigenfunction pair in problem (2.3).

Theorem 3.4 *Assume that $R_0^\rho < 1$, then any solution $(u,v)(y,t)$ to problem (1.10) satisfies*

$$\lim_{t\to\infty} u(y,t) = 0 \quad and \quad \lim_{t\to\infty} v(y,t) = 0 \quad uniformly\ for\ y\in\overline{\Omega}(0) \tag{3.10}$$

when the initial function meets with $(u(y,0), v(y,0)) \le (M\Phi(y), M\Psi(y))$ for $y\in\overline{\Omega}(0)$. That is to say, the sector $\langle(0,0),(M\Phi, M\Psi)\rangle$ is the stability region (or domain of attraction) of the disease free equilibrium $(0,0)$ to problem (1.10), where M is given by (3.9).

Proof: Let

$$\overline{u}(y,t) = Me^{-\delta t}\Phi(y), \quad \overline{v}(y,t) = Me^{-\delta t}\Psi(y),$$

where $\delta > 0$ is sufficiently small. Recalling Remark 2.2 and using (1.9) give

$$\begin{aligned}
&\overline{u}_t - \tfrac{D_1}{\rho^2(t)}\Delta\overline{u} - \alpha_b\beta_b\overline{v} + \big(\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)}\big)\overline{u} \\
=\ & Me^{-\delta t}\big[-\delta\Phi - \tfrac{D_1}{\rho^2(t)}\Delta\Phi - \alpha_b\beta_b\Psi + \big(\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)}\big)\Phi\big] \\
\ge\ & Me^{-\delta t}\big[-\delta\Phi - \tfrac{D_1}{\rho_\infty^2}\Delta\Phi - \alpha_b\beta_b\Psi + \gamma_b\Phi\big] \\
=\ & Me^{-\delta t}\big[-\delta\Phi + \big(\tfrac{1}{R_0^\rho} - 1\big)\alpha_b\beta_b\Psi\big] \\
\ge\ & 0
\end{aligned}$$

and

$$
\begin{aligned}
& \overline{v}_t - \tfrac{D_2}{\rho^2(t)}\Delta\overline{v} - \alpha_m\beta_b\tfrac{A_m}{N_b}\overline{u} + \big(d_m + \tfrac{n\dot\rho(t)}{\rho(t)}\big)\overline{v} \\
= \;& Me^{-\delta t}\big[-\delta\Psi - \tfrac{D_2}{\rho^2(t)}\Delta\Psi - \alpha_m\beta_b\tfrac{A_m}{N_b}\Phi + \big(d_m + \tfrac{n\dot\rho(t)}{\rho(t)}\big)\Psi\big] \\
\geq \;& Me^{-\delta t}\big[-\delta\Psi - \tfrac{D_2}{\rho^2_\infty}\Delta\Psi - \alpha_m\beta_b\tfrac{A_m}{N_b}\Phi + d_m\Psi\big] \\
= \;& Me^{-\delta t}\big[-\delta\Psi + \big(\tfrac{1}{R^\rho_0} - 1\big)\alpha_m\beta_b\tfrac{A_m}{N_b}\Phi\big] \\
\geq \;& 0
\end{aligned}
$$

because of $\frac{1}{R^\rho_0} - 1 > 0$. Hence, $(\overline{u}, \overline{v})(y,t)$ is the supersolution of problem (1.10) if

$$(u(y,0), v(y,0)) \leq (M\Phi(y), M\Psi(y)), \quad y \in \overline{\Omega}(0).$$

Moreover, from Lemma 3.1, we deduces

$$u(y,t) \leq \overline{u}(y,t), \quad v(y,t) \leq \overline{v}(y,t), \quad (y,t) \in \overline{\Omega}(0) \times (0,\infty),$$

which easily assert that

$$\lim_{t\to\infty} \overline{u}(y,t) = 0 \;\text{ and }\; \lim_{t\to\infty} \overline{v}(y,t) = 0 \;\text{ uniformly for } y \in \overline{\Omega}(0).$$

Therefore, $\langle(0,0), (M\Phi, M\Psi)\rangle$ is the stability region (or domain of attraction) of the disease free equilibrium $(0,0)$.

□

3.3 The stability of the endemic equilibrium

In this section, we analyze the stability of endemic equilibrium to problem (1.10) under the assumption $R^\rho_0 > 1$. We first construct the super- and subsolutions of problem (2.1).

Let

$$\hat{U}(y) = \delta_0\Phi(y), \quad \hat{V}(y) = \delta_0\Psi(y), \quad y \in \overline{\Omega}(0),$$

where $0 < \delta_0 \ll 1$ is arbitrarily small, and $(R^\rho_0; \Phi(y), \Psi(y))$ is the eigen-pair in problem (2.3). We have

$$
\begin{aligned}
& -\tfrac{D_1}{\rho^2_\infty}\Delta\hat{U} - \alpha_b\beta_b\tfrac{(N_b - \hat{U})}{N_b}\hat{V} + \gamma_b\hat{U} \\
= \;& \delta_0[-\tfrac{D_1}{\rho^2_\infty}\Delta\Phi - \alpha_b\beta_b\tfrac{(N_b - \delta_)\Phi)}{N_b}\Psi + \gamma_b\Phi] \\
= \;& \delta_0[\tfrac{1}{R^\rho_0}\alpha_b\beta_b\Psi - \alpha_b\beta_b\Psi + \delta_0\tfrac{\alpha_b\beta_b}{N_b}\Phi\Psi] \\
= \;& \delta_0\alpha_b\beta_b\Psi[(\tfrac{1}{R^\rho_0} - 1) + \tfrac{\delta_0}{N_b}\Phi]
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
& -\tfrac{D_2}{\rho^2_\infty}\Delta\hat{V} - \alpha_m\beta_b\tfrac{(A_m - \hat{V})}{N_b}\hat{U} + d_m\hat{V} \\
= \;& \delta_0[-\tfrac{D_2}{\rho^2_\infty}\Delta\Psi - \alpha_m\beta_b\tfrac{(A_m - \delta_0\Psi)}{N_b}\Phi + d_m\Psi] \\
= \;& \delta_0[\tfrac{1}{R^\rho_0}\alpha_m\beta_b\tfrac{A_m}{N_b}\Phi - \alpha_m\beta_b\tfrac{A_m}{N_b}\Phi + \delta_0\tfrac{\alpha_m\beta_b}{N_b}\Psi\Phi] \\
= \;& \delta_0\alpha_m\beta_b\tfrac{A_m}{N_b}\Phi[(\tfrac{1}{R^\rho_0} - 1) + \tfrac{\delta_0}{A_m}\Psi].
\end{aligned}
\tag{3.12}
$$

It is easily showed that (3.11) and (3.12) are both negative for sufficiently small δ_0 owing to the precondition $\frac{1}{R^\rho_0} < 1$. Hence, the pair $(\tilde{U}(y), \tilde{V}(y)) := \big(N_b, A_m\big)$ and the pair $(\hat{U}(y), \hat{V}(y)) = (\delta_0\Phi, \delta_0\Psi)$ are the super- and subsolutions of problem (2.1).

In what follows, we would like to derive the stability region of the endemic equilibrium of problem (1.10). We point out that, although we mainly follow the super- and subsolution method, some modifications are required. The major improvements of the method have two aspects, one is to circumvent the difficult that the super- and subsolution of the elliptic problem are not those of the associated parabolic problem, and the other is to derive the convergence of the sequence constructed by iteration for parabolic problem with specific initial value, which is evident for normal systems owing to the monotonicity, but the corresponding sequence for our system (1.10) is not monotone. In order to resolve these difficulties, we need the following two lemmas, see also Lemmas 4.5 and 4.6 [34].

Lemma 3.5 *If $R_0^\rho > 1$, there exist $T_0 > 0$ and δ^* such that (N_b, A_m) and $(\delta\Phi(y), \delta\Psi(y))$ are a pair of super- and subsolutions of problem* (1.10) *for $t \geq T_0$, provided $\delta < \delta^*$.*

Proof: On account of assumption (1.9), for arbitrary ε_0 $(0 < \varepsilon_0 < \rho_\infty)$, there exists a $T_0 > 0$ such that

$$\rho_\infty - \varepsilon_0 \leq \rho(t) \leq \rho_\infty, \quad 0 \leq \frac{\dot\rho(t)}{\rho(t)} \leq \varepsilon_0, \quad t \geq T_0. \tag{3.13}$$

Considering problem (2.3) and selecting $\sigma = \frac{1}{2}(R_0^\rho - 1)$ yields

$$\begin{cases} -\frac{D_1}{(\rho_\infty-\varepsilon)^2}\Delta\Phi \leq \frac{1}{R_0^\rho-\sigma}\alpha_b\beta_b\Psi - (\gamma_b+\varepsilon)\Phi, & y \in \Omega(0), \\ -\frac{D_2}{(\rho_\infty-\varepsilon)^2}\Delta\Psi \leq \frac{1}{R_0^\rho-\sigma}\alpha_m\beta_b\frac{A_m}{N_b}\Phi - (d_m+\varepsilon)\Psi, & y \in \Omega(0), \\ \Phi(y) = \Psi(y) = 0, & y \in \partial\Omega(0) \end{cases} \tag{3.14}$$

for the above ε_0.

Next, we pay attention to the auxiliary problem based on problem (1.10), what calls for special attention is that time begins at T_0, that is,

$$\begin{cases} u_t - \frac{D_1}{\rho^2(t)}\Delta u = \alpha_b\beta_b\frac{(N_b-u)}{N_b}v - (\gamma_b + \frac{n\dot\rho(t)}{\rho(t)})u, & y \in \Omega(0), t > T_0, \\ v_t - \frac{D_2}{\rho^2(t)}\Delta v = \alpha_m\beta_b\frac{(A_m-v)}{N_b}u - (d_m + \frac{n\dot\rho(t)}{\rho(t)})v, & y \in \Omega(0), t > T_0, \\ u(y,t) = v(y,t) = 0, & y \in \partial\Omega(0), t > T_0 \end{cases} \tag{3.15}$$

associated with the initial condition

$$u(y,T_0) = u_{T_0}(y) \leq N_b, \quad v(y,T_0) = v_{T_0}(y) \leq A_m, \quad y \in \Omega(0). \tag{3.16}$$

Apparently, for problem (3.15), (3.16), $(\tilde u(y,t), \tilde v(y,t)) := (N_b, A_m)$ is still a supersolution for $(y,t) \in \overline\Omega(0) \times [T_0, \infty)$. Let

$$\hat u(y,t) = \delta\Phi(y), \quad \hat v(y,t) = \delta\Psi(y), \quad y \in \overline\Omega(0), t \geq T_0,$$

where $\delta > 0$ is to be determined later and (Φ, Ψ) is the eigenfunction pair of problem (2.3). Apply Lemma 2.2, (3.13) and (3.14), we have

$$\begin{aligned} & \hat u_t - \tfrac{D_1}{\rho^2(t)}\Delta\hat u - \alpha_b\beta_b\tfrac{(N_b-\hat u)}{N_b}\hat v + (\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)})\hat u \\ = \;& -\delta\tfrac{D_1}{\rho^2(t)}\Delta\Phi - \alpha_b\beta_b\delta\Psi + \tfrac{\alpha_b\beta_b}{N_b}\delta^2\Phi\Psi + (\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)})\delta\Phi \\ \leq \;& -\delta\tfrac{D_1}{(\rho_\infty-\varepsilon)^2}\Delta\Phi - \alpha_b\beta_b\delta\Psi + \tfrac{\alpha_b\beta_b}{N_b}\delta^2\Phi\Psi + (\gamma_b + \tfrac{n\dot\rho(t)}{\rho(t)})\delta\Phi \\ \leq \;& \delta\big[\tfrac{1}{R_0^\rho-\sigma}\alpha_b\beta_b\Psi - \alpha_b\beta_b\Psi + \tfrac{\alpha_b\beta_b}{N_b}\delta\Phi\Psi\big] \\ = \;& \tfrac{\alpha_b\beta_b}{N_b}\delta\Psi\big[(\tfrac{1}{R_0^\rho-\sigma} - 1)N_b + \delta\Phi\big]. \end{aligned} \tag{3.17}$$

Similarly, we deduces

$$\hat{v}_t - \frac{D_2}{\rho^2(t)}\Delta\hat{v} - \alpha_m\beta_b\frac{(A_m-\hat{v})}{N_b}\hat{u} + (d_m + \frac{n\dot{\rho}(t)}{\rho(t)})\hat{v} \leq \frac{\alpha_m\beta_b}{N_b}\delta\Phi\big[(\frac{1}{R_0^\rho - \sigma} - 1)A_m + \delta\Psi\big]. \tag{3.18}$$

because of $\frac{1}{R_0^\rho - \sigma} - 1 < 0$, then there exists a sufficiently small $\delta_1 > 0$ such that inequalities (3.17) and (3.18) are both negative when $\delta \leq \delta_1$.

Moveover, for the initial function $(u(y,T_0), v(y,T_0))$, then there exists a $\delta_2 = \delta_2(u(y,T_0), v(y,T_0), T_0) > 0$ such that

$$\delta_2\Phi(y) \leq u(y,T_0), \quad \delta_2\Psi(y) \leq v(y,T_0).$$

Letting $\delta^* = \min(\delta_1, \delta_2)$, it follows that $(\hat{u}(y,t), \hat{v}(y,t))$ is a subsolution to problem (3.15), and then the proof is completed. $\square$

Next, we consider problem (3.15) associated with the different initial conditions

$$u(y,T_0) = \delta\Phi(y), \quad v(y,T_0) = \delta\Psi(y), \quad y \in \Omega(0), \tag{3.19}$$

and

$$u(y,T_0) = N_b, \quad v(y,T_0) = A_m, \quad y \in \Omega(0), \tag{3.20}$$

respectively, where T_0 is given in (3.13). According to Lemma 3.5, we know that (N_b, A_m) and $(\delta\Phi(y), \delta\Psi(y))$ are a pair of super- and subsolutions of problem (1.10) for $(y,t) \in \overline{\Omega}(0) \times [T_0, +\infty)$. We denote by $(\underline{u}(y,t), \underline{v}(y,t))$ and $(\overline{u}(y,t), \overline{v}(y,t))$ the solutions to problem (3.15), (3.19), and (3.15), (3.20), respectively. Therefore, it follows from Theorem 6.1 in [24, Chapter 9] that both $(\overline{u}(y,t), \overline{v}(y,t))$ and $(\underline{u}(y,t), \underline{v}(y,t))$ exist and are unique, and satisfy

$$(\delta\Phi(y), \delta\Psi(y)) \leq (\underline{u}(y,t), \underline{v}(y,t)) \leq (N_b, A_m), \tag{3.21}$$

and

$$(\delta\Phi(y), \delta\Psi(y)) \leq (\overline{u}(y,t), \overline{v}(y,t)) \leq (N_b, A_m). \tag{3.22}$$

Furthermore, we have the following conclusion.

Lemma 3.6 $\liminf\limits_{t\to\infty}(\underline{u}(y,t), \underline{v}(y,t)) \geq (\underline{u}(y), \underline{v}(y))$, *and* $\limsup\limits_{t\to\infty}(\overline{u}(y,t),\ \overline{v}(y,t)) \leq (\overline{u}(y), \overline{v}(y))$, *where both* $(\underline{u}(y), \underline{v}(y))$ *and* $(\overline{u}(y), \overline{v}(y))$ *are the solutions to problem* (2.1).

Proof: First of all, recalling that if the coefficients of the diffusion term is independent of t, then both $(\underline{u}(y), \underline{v}(y))$ and $(\overline{u}(y), \overline{v}(y))$ are monotonously convergent as $t \to +\infty$, whereas in our problem (3.15), the coefficients are $\frac{D_1}{\rho^2(t)}$ and $\frac{D_2}{\rho^2(t)}$, we here use a different technique.

Put

$$s = \int_{T_0}^{t} \frac{1}{\rho^2(\tau)} d\tau, \tag{3.23}$$

then $s'(t) = \frac{1}{\rho^2(t)} > 0$, which means that there exists an inverse transformation $t = h(s)$ and

$$\lim_{s\to\infty} t = \lim_{s\to\infty} h(s) = +\infty. \tag{3.24}$$

Furthermore, we let

$$\underline{u}(y,t) = w(y,s), \quad \underline{v}(y,t) = z(y,s), \quad y \in \overline{\Omega}(0),\ s > 0, \tag{3.25}$$

then obtain

$$\underline{u}_t = w_s \cdot \frac{1}{\rho^2(t)}, \quad \Delta \underline{u}(y,t) = \Delta w(y,s),$$
$$\underline{v}_t = z_s \cdot \frac{1}{\rho^2(t)}, \quad \Delta \underline{v}(y,t) = \Delta z(y,s),$$

and

$$(\delta\Phi(y), \delta\Psi(y)) \leq (w(y,0), z(y,0)) \leq (N_b, A_m) \tag{3.26}$$

by (3.21). Subsequently, problem (3.15), (3.19) is translated into

$$\begin{cases} w_s - D_1\Delta w = \rho^2(h(s))\alpha_b\beta_b\frac{N_b-w}{N_b}z - \rho^2(h(s))\big(\gamma_b + \frac{n\dot{\rho}(h(s))}{\rho(h(s))}\big)w, & y \in \Omega(0), s > 0, \\ z_s - D_2\Delta z = \rho^2(h(s))\alpha_m\beta_b\frac{A_m-z}{N_b}w - \rho^2(h(s))\big(d_m + \frac{n\dot{\rho}(h(s))}{\rho(h(s))}\big)z, & y \in \Omega(0), s > 0, \\ w(y,s) = z(y,s) = 0, & y \in \partial\Omega(0), s > 0, \\ w(y,0) = \delta\Phi(y), \quad z(y,0) = \delta\Psi(y), & y \in \Omega(0). \end{cases} \tag{3.27}$$

On account of (1.9), (3.23) and (3.24), for arbitrary $\varepsilon(0 < \varepsilon < \rho_\infty)$, there exists a $T_{1\varepsilon} > 0$ such that

$$\rho_\infty - \varepsilon < \rho(t) < \rho_\infty, \quad \text{for} \quad t > T_0 + T_{1\varepsilon},$$

take $s_1 = \int_{T_0}^{T_0+T_{1\varepsilon}} \frac{1}{\rho^2(\tau)}d\tau$, we then have

$$\rho_\infty - \varepsilon \leq \rho(h(s)) \leq \rho_\infty + \varepsilon, \quad s > s_1. \tag{3.28}$$

Assume that $(\underline{w}^\varepsilon(y,s), \underline{z}^\varepsilon(y,s))$ is solution of the auxiliary problem as follows

$$\begin{cases} W_s - D_1\Delta W = (\rho_\infty - \varepsilon)^2\alpha_b\beta_b\frac{N_b-W}{N_b}Z - (\rho_\infty + \varepsilon)^2(\gamma_b + \varepsilon)W, & y \in \Omega(0), s > s_1, \\ Z_s - D_2\Delta Z = (\rho_\infty - \varepsilon)^2\alpha_m\beta_b\frac{A_m-Z}{N_b}W - (\rho_\infty + \varepsilon)^2(d_m + \varepsilon)Z, & y \in \Omega(0), s > s_1, \\ W(y,s) = Z(y,s) = 0, & y \in \partial\Omega(0), s > s_1, \\ W(y,s_1) = \delta\Phi(y), \quad Z(y,s_1) = \delta\Psi(y), & y \in \Omega(0). \end{cases} \tag{3.29}$$

It is easy to check that $(\underline{w}^\varepsilon(y,s), \underline{z}^\varepsilon(y,s))$ is the subsolution of problem (3.27). Recalling that $(\delta\Phi(y), \delta\Psi(y))$ is subsolution of (3.29), utilizing the method of the proof to Lemma 11.2 in [24, Chapter 7], we claim that $(\underline{w}^\varepsilon(y,s), \underline{z}^\varepsilon(y,s))$ is monotonically increasing with respect to s for $y \in \overline{\Omega}(0)$. In fact, $\underline{w}^\varepsilon(y,s) \geq \delta\Phi(y)$ when $s > s_1$, then $\underline{w}^\varepsilon(y,\eta) \geq \delta\Phi(y)$ for any $\eta > s_1$. As a result, $\underline{w}^\varepsilon(y, s+\eta)$ is the supersolution of (3.29). Moreover, $\underline{w}^\varepsilon(y,s)$ for $(y,s) \in \overline{\Omega}(0) \times [s_1, +\infty)$ is the solution of (3.29), by comparison principle, this implies that $\underline{w}^\varepsilon(y, s+\eta) \geq \underline{w}^\varepsilon(y,s)$ for $(y,s) \in \overline{\Omega}(0) \times [s_1, +\infty)$. In the same way, we deduce $\underline{z}^\varepsilon(y, s+\eta) \geq \underline{z}^\varepsilon(y,s)$ for $(y,s) \in \overline{\Omega}(0) \times [s_1, +\infty)$. By virtue of (3.26), one can derive that $\lim_{s\to\infty}(\underline{w}^\varepsilon(y,s), \underline{z}^\varepsilon(y,s))$ exists for $y \in \overline{\Omega}(0)$. Denote

$$\lim_{s\to\infty}(\underline{w}^\varepsilon(y,s), \underline{z}^\varepsilon(y,s)) = (\underline{w}^\varepsilon(y), \underline{z}^\varepsilon(y)), \quad \text{for} \quad y \in \overline{\Omega}(0).$$

Next, we will show that $(\underline{w}^\varepsilon(y), \underline{z}^\varepsilon(y))$ is the solution of the following problem

$$\begin{cases} -D_1\Delta\underline{w}^\varepsilon(y) = (\rho_\infty - \varepsilon)^2\alpha_b\beta_b\frac{N_b-\underline{w}^\varepsilon(y)}{N_b}\underline{z}^\varepsilon(y) - (\rho_\infty + \varepsilon)^2(\gamma_b + \varepsilon)\underline{w}^\varepsilon(y), & y \in \Omega(0), \\ -D_2\Delta\underline{z}^\varepsilon(y) = (\rho_\infty - \varepsilon)^2\alpha_m\beta_b\frac{A_m-\underline{z}^\varepsilon(y)}{N_b}\underline{w}^\varepsilon(y) - (\rho_\infty + \varepsilon)^2(d_m + \varepsilon)\underline{z}^\varepsilon(y), & y \in \Omega(0), \\ \underline{w}^\varepsilon(y) = \underline{z}^\varepsilon(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{3.30}$$

In fact, we consider the following auxiliary problem

$$\begin{cases} -D_1\Delta\xi^\varepsilon = (\rho_\infty-\varepsilon)^2\alpha_b\beta_b\frac{N_b-\underline{w}^\varepsilon}{N_b}\underline{z}^\varepsilon-(\rho_\infty+\varepsilon)^2(\gamma_b+\varepsilon)\underline{w}^\varepsilon, & y\in\Omega(0),\\ -D_2\Delta\eta^\varepsilon = (\rho_\infty-\varepsilon)^2\alpha_m\beta_b\frac{A_m-\underline{z}^\varepsilon}{N_b}\underline{w}^\varepsilon-(\rho_\infty+\varepsilon)^2(d_m+\varepsilon)\underline{z}^\varepsilon, & y\in\Omega(0),\\ \xi^\varepsilon(y)=\eta^\varepsilon(y)=0, & y\in\partial\Omega(0), \end{cases} \tag{3.31}$$

which exists a unique generalized solution $(\xi^\varepsilon,\eta^\varepsilon)$, where $\xi^\varepsilon,\eta^\varepsilon\in W^{2,p}(\Omega(0))$ for any $p>1$. Taking

$$W^\varepsilon(y,s)=\underline{w}^\varepsilon(y,s)-\xi^\varepsilon(y),\quad V^\varepsilon(y,s)=\underline{z}^\varepsilon(y,s)-\eta^\varepsilon(y),$$

then $W^\varepsilon(y,s)$ meets with

$$\begin{cases} W^\varepsilon_s-D_1\Delta W^\varepsilon=(\rho_\infty-\varepsilon)^2\alpha_b\beta_b\frac{N_b-\underline{w}^\varepsilon(y,s)}{N_b}\underline{z}^\varepsilon(y,s)-(\rho_\infty+\varepsilon)^2(\gamma_b+\varepsilon)\underline{w}^\varepsilon(y,s) & \\ \qquad -(\rho_\infty-\varepsilon)^2\alpha_b\beta_b\frac{N_b-\underline{w}^\varepsilon(y)}{N_b}\underline{z}^\varepsilon(y)+(\rho_\infty+\varepsilon)^2(\gamma_b+\varepsilon)\underline{w}^\varepsilon(y) & \\ \qquad := F^\varepsilon(y,s), & y\in\Omega(0), s>s_1,\\ W^\varepsilon(y,s)=0, & y\in\partial\Omega(0), s>s_1,\\ W^\varepsilon(y,s_1)=\delta\Phi(y)-\xi^\varepsilon(y), & y\in\overline{\Omega}(0). \end{cases}$$

Noting that

$$(\underline{w}^\varepsilon(y,s),\underline{z}^\varepsilon(y,s))\to(\underline{w}^\varepsilon(y),\underline{z}^\varepsilon(y))\ \text{ as }\ s\to\infty,$$

which implies that $F^\varepsilon(y,s)\to 0$, which leads to $W^\varepsilon(y,s)\to 0$ in $L^2(\Omega(0))$. Consequently, $\xi^\varepsilon(y)=\underline{w}^\varepsilon(y)$ in $L^2(\Omega(0))$, according to the L^P-theory for elliptic equations, we deduce that $\underline{w}^\varepsilon(y)\in \mathrm{W}^{2,p}(\Omega(0))$ for $p>1$. Analogously, $\eta^\varepsilon(y)=\underline{z}^\varepsilon(y)$ in $L^2(\Omega(0))$ and $\underline{w}^\varepsilon(y)\in W^{2,p}(\Omega(0))$ for $p>1$. Therefore, $(\underline{w}^\varepsilon(y),\underline{z}^\varepsilon(y))$ is a generalized solution to problem (3.30). By choosing an appropriate $p>n$ and applying Sobolev's imbedding theorem, then there exists an $\alpha\in(0,1)$ such that $\underline{w}^\varepsilon(y)$, $\underline{z}^\varepsilon(y)\in C^{1+\alpha}(\overline{\Omega}(0))$, which deduces that the right-hand sides of the first two equations in problem (3.31) are in $C^\alpha(\overline{\Omega}(0))$ further. In terms of the Schauder theory, $(\underline{w}^\varepsilon(y),\underline{z}^\varepsilon(y))$ is the classical solution of problem (3.30).

Letting $\varepsilon\to 0$ in (3.30) yields

$$\begin{cases} -\frac{D_1}{\rho_\infty^2}\Delta\underline{w}=\alpha_b\beta_b\frac{N_b-\underline{w}}{N_b}\underline{z}-\gamma_b\underline{w}, & y\in\Omega(0),\\ -\frac{D_2}{\rho_\infty^2}\Delta\underline{z}=\alpha_m\beta_b\frac{A_m-\underline{z}}{N_b}\underline{w}-d_m\underline{z}, & y\in\Omega(0),\\ \underline{w}(y)=\underline{z}(y)=0, & y\in\partial\Omega(0), \end{cases} \tag{3.32}$$

which implies that $(\underline{w}(y),\underline{z}(y))$ is the solution of (2.1). Noting that $(\underline{w}^\varepsilon(y,s),\underline{z}^\varepsilon(y,s))$ is the subsolution of problem (3.27) gives that

$$\liminf_{s\to\infty}(w(y,s),z(y,s))\geq(\underline{w}(y),\underline{z}(y)),\quad y\in\overline{\Omega}(0).$$

Then

$$\liminf_{t\to\infty}(\underline{u}(y,t),\underline{v}(y,t))=\liminf_{s\to\infty}(w(y,s),z(y,s))\geq(\underline{w}(y),\underline{z}(y)):=(\underline{u}(y),\underline{v}(y)),\quad y\in\overline{\Omega}(0).$$

Similarly to the above we can prove that

$$\limsup_{t\to\infty}(\overline{u}(y,t),\overline{v}(y,t))=\limsup_{s\to\infty}(w(y,s),z(y,s))\leq(\overline{w}(y),\overline{z}(y)):=(\overline{u}(y),\overline{v}(y)),\quad y\in\overline{\Omega}(0),$$

where $(\overline{u}(y),\overline{v}(y))$ and $(\underline{u}(y),\underline{v}(y))$ are all the solutions of the elliptic problem (2.1). □

The aforementioned conclusions present a new treatment skill to study the property of the endemic equilibrium, which is different from the classical approach. Next, we give the persistence of the reaction diffusion system.

Theorem 3.7 *Suppose $R_0^\rho > 1$, for any solution $(u(y,t), v(y,t))$ to problem (1.10) satisfies*

$$(\underline{U}(y), \underline{V}(y)) \le \liminf_{t\to\infty}(u(y,t), v(y,t)) \le \limsup_{t\to\infty}(u(y,t), v(y,t)) \le (\overline{U}(y), \overline{V}(y)), \tag{3.33}$$

uniformly for $y \in \overline{\Omega}(0)$, where $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ are the maximal and minimal solutions of problem (2.1) in the sector $\langle(\hat{U}, \hat{V}),\ (\tilde{U}, \tilde{V})\rangle$ obtained by the iteration sequences in (3.10). Furthermore, the sector $\langle(\delta\Phi, \delta\Psi), (N_b, A_m)\rangle$ is the stability region of problem (1.10), where $\delta > 0$ is small enough.

Proof: First, combining Lemma 3.5 and comparison principle, we derive that

$$(\delta\Phi(y), \delta\Psi(y)) \le (u(y,t), v(y,t)) \le (N_b, A_m), \quad (y,t) \in \overline{\Omega}(0) \times [T_0, +\infty),$$

which shows that the sector $\langle(\delta\Phi, \delta\Psi), (N_b, A_m)\rangle$ is the stability region of problem (1.10).

According to comparison principle yields

$$(\underline{u}(y,t), \underline{v}(y,t)) \le (u(y,t), v(y,t)) \le (\overline{u}(y,t), \overline{v}(y,t)), \quad (y,t) \in \overline{\Omega}(0) \times [T_0, +\infty), \tag{3.34}$$

where $(\underline{u}(y,t), \underline{v}(y,t))$ and $(\overline{u}(y,t), \overline{v}(y,t))$ have been defined in Lemma 3.6. Meanwhile, it follows from Lemma 3.6 that

$$\liminf_{t\to\infty}(\underline{u}(y,t), \underline{v}(y,t)) \ge (\underline{u}(y), \underline{v}(y)), \quad \limsup_{t\to\infty}(\overline{u}(y,t), \overline{v}(y,t)) \le (\overline{u}(y), \overline{v}(y))$$

as well as both $(\underline{u}(y), \underline{v}(y))$ and $(\overline{u}(y), \overline{v}(y))$ are the solutions to problem (2.1). Therefore, letting $t \to \infty$ in (3.34) yields

$$\begin{aligned}(\underline{U}(y), \underline{V}(y)) &\le (\underline{u}(y), \underline{v}(y)) \le \liminf_{t\to\infty}(u(y,t), v(y,t)) \\ &\le \limsup_{t\to\infty}(u(y,t), v(y,t)) \le (\overline{u}(y), \overline{v}(y)) \le (\overline{U}(y), \overline{V}(y)).\end{aligned}$$

Thus, we admit (3.33), and the proof of Theorem 3.7 is finished. □

Specifically, if either $\overline{U}(y) = \underline{U}(y)$ or $\overline{V}(y) = \underline{V}(y)$, then $(\overline{U}, \overline{V})(y) = (\underline{U}, \underline{V})(y) := (U^*, V^*)(y)$ is the unique solution to problem (2.1), we derive the following conclusion.

Theorem 3.8 *Suppose $R_0^\rho > 1$, if either $\overline{U}(y) = \underline{U}(y)$ or $\overline{V}(y) = \underline{V}(y)$, for any initial value of problem (1.10), we have $\lim_{t\to\infty}(u(y,t), v(y,t)) = (U^*, V^*)(y)$, that is to say, the endemic equilibrium (U^*, V^*) is globally asymptotically stable.*

4 Numerical simulation and discussion

In this section, in order to illustrate the impact of growing domain on the transmission of WNv, let us carry out numerical simulations for problem (1.10) with coefficients and initial functions as follows

$$\begin{aligned}&D_1 = 0.46,\ \alpha_b = 3,\ \beta_b = 0.5,\ \gamma_b = 0.14,\ N_b = 20,\\ &D_2 = 0.2,\ \alpha_m = 2,\ d_m = 0.07,\ A_m = 30,\\ &u(y,0) = 0.5\sin\pi y, \quad v(y,0) = 0.75\sin\pi y, \quad \Omega(0) = (0,1),\end{aligned}$$

and subsequently $\lambda^* = \pi^2,\ \varphi^* = \sin\pi y$.

In particular, $\Omega(t)$ is the fixed domain $\Omega(0)$ if $\rho(t) = \beta_\infty = 1$, then problem (1.10) turns to

$$\begin{cases} u_t^\Delta - D_1 \Delta u^\Delta = \alpha_b \beta_b \frac{(N_b - u^\Delta)}{N_b} v^\Delta - \gamma_b u^\Delta, & y \in \Omega(0), t > 0, \\ v_t^\Delta - D_2 \Delta v^\Delta = \alpha_m \beta_b \frac{(A_m - v^\Delta)}{N_b} u^\Delta - d_m v^\Delta, & y \in \Omega(0), t > 0, \\ u^\Delta(y,t) = v^\Delta(y,t) = 0, & y \in \partial\Omega(0), t > 0, \\ u^\Delta(y,0) = I_{b,0}(x(0)) \le N_b, v^\Delta(y,0) = I_{m,0}(x(0)) \le A_m, & y \in \Omega(0), \end{cases} \tag{4.1}$$

whose basic reproduction number can be represented as

$$R_0^{\Omega(0)} = \sqrt{\frac{\alpha_b \alpha_m \beta_b^2 A_m}{N_b (D_1 \lambda^* + \gamma_b)(D_2 \lambda^* + d_m)}}, \tag{4.2}$$

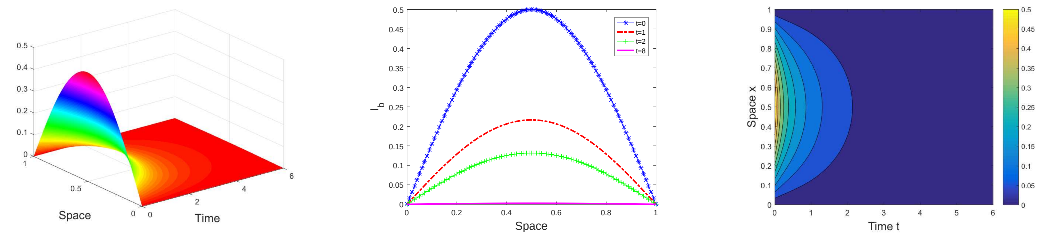
where λ^* is defined by (2.6). It is evidently that $R_0^{\Omega(0)} < R_0^\rho$. Moreover, it follows from Theorem 3.4 that if $R_0^{\Omega(0)} < 1$, for any solution $(u^\Delta, v^\Delta)(y,t)$ satisfies

$$\lim_{t\to\infty} u^\Delta(y,t) = 0, \quad \lim_{t\to\infty} v^\Delta(y,t) = 0 \;\text{ uniformly for } y \in \overline{\Omega}(0). \tag{4.3}$$

Example 4.1 According to the above parameter selection and (4.2), it follows that

$$R_0^{\Omega(0)} = \sqrt{\frac{3 \times 0.76 \times 2 \times 0.76 \times 30}{20 \times (0.46\pi^2 + 0.14)(0.2\pi^2 + 0.07)}} \approx \sqrt{\frac{5.198}{4.680 \times 2.044}} \approx 0.737 < 1,$$

as a result, when $t \to \infty$, the variable $u^\Delta(y,t)$ of problem (4.1) decays to zero quickly for $y \in [0,1]$, see Fig. 2, which implies that WNv is vanishing on the fixed domain.



(a) (b) (c)

Fig. 2: $\rho(t) = 1$. $R_0^{\Omega(0)} < 1$, which implies that WNv is vanishing in the fixed domain. Graph (a) shows that the variable I_b decays to zero. Graphs (b) and (c) are the cross-sectional view and contour one, respectively.

Next, we choosing logistic growth function $\rho(t) = \frac{e^{Krt}}{1+\frac{1}{K}(e^{Krt}-1)}$, which satisfies the assumption (1.9). In all our simulations we fix $r = 1$ and vary K to account for the long-time asymptotic behavior of solutions and the evolution of domain $\Omega(t)$.

Example 4.2 Set $K = 1.1$, then $\rho_\infty = 1.1$. One can easily compute from (2.4), (2.5) and (3.9) that

$$R_0^\rho = \sqrt{\frac{3 \times 0.76 \times 2 \times 0.76 \times \frac{30}{20}}{[\frac{0.46\pi^2}{1.1^2} + 0.14][\frac{0.2\pi^2}{1.1^2} + 0.07]}} \approx \sqrt{\frac{5.198}{3.892 \times 1.701}} \approx \sqrt{0.785} \approx 0.886 < 1,$$

$$\Phi(y) \approx 0.662 \sin \pi y, \;\; \Psi(y) = \sin \pi y,$$

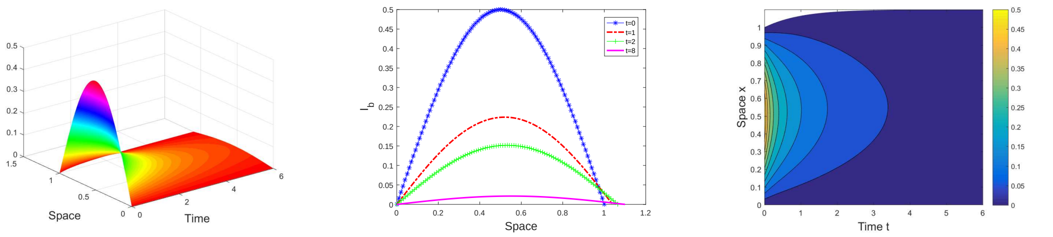
and

$$M = \min\left(\frac{N_b}{\max_{y\in\overline{\Omega}(0)} \Phi(y)}, \frac{A_m}{\max_{y\in\overline{\Omega}(0)} \Psi(y)}\right) = \min\left(\frac{20}{0.662}, \frac{30}{1}\right) = 30.$$

Obviously,

$$u(y,0) \le M\Phi(y), \quad v(y,0) \le M\Psi(y).$$

It follows from Theorem 3.4 that $u(y,t) \to 0$ uniformly for $y \in [0,1]$ when $t \to \infty$, which means that $I_b(x,t) \to 0$ uniformly on any compact subset of $[0,1.1)$ as $t \to \infty$. Fig. 3(a) shows that the infectious bird density I_b decays to zero as time elapses, which implies that the virus vanishes eventually. The corresponding cross-sectional view and contour one for I_b, see for Fig. 3(b) and (c), which illustrate the changing trends of the variable I_b, and that the domain $\Omega(t)$ is growing gradually from the interval $[0,1)$ to $[0,1.1)$. A comparison of the above-mentioned two examples reveal that the small growth rate makes the virus still vanishes.



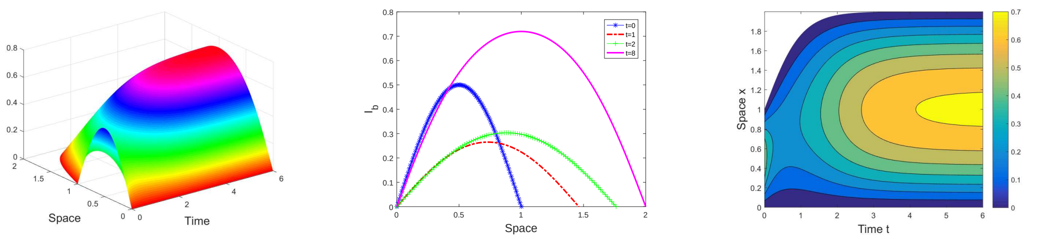
(a) (b) (c)

Fig. 3: The small growing ratio $\rho(t)$ guarantees $R_0^\rho < 1$ and WNv is still vanishing. Graph (a) shows that the variable I_b decays to zero. Graphs (b) and (c) are the cross-sectional view and contour one, respectively, from which we can clearly observe the change of domain.

Example 4.3 Select $K = 2$, then $\rho_\infty = 2$. It follows from (2.4) that

$$R_0^\rho = \sqrt{\frac{5.198}{1.275 \times 0.5635}} \approx 2.690 > 1.$$

According to the outcome of Theorem 3.7, we can obtain that the virus is spreading as time goes on. From Fig. 4, we can find that the variable $I_b(x,t)$ tends to a positive steady state and $\Omega(t)$ is growing from the interval $[0,1)$ to $[0,2)$. It's worth noting that the big growing ratio $\rho(t)$ results in the spread of the virus.



(a) (b) (c)

Fig. 4: For the big growing ratio $\rho(t)$, we acquire $R_0^\rho > 1$ and the virus is spreading. Graph (a) shows that the variable I_b stabilizes to a steady state, and graphs (b) and (c) exhibit the enlargement of domain.

Comparing Example 4.1 with 4.2 shows that if $R_0^\Delta < 1$ on the fixed domain, we still have $R_0^\rho < 1$ for the model on the growing domain with the small growth rate. However, the basic reproduction number could be great than 1 if the growth rate is big, which can be seen by a comparison of Examples 4.1 with 4.3. We can be fairly sure from the aforementioned examples that domain growth is closely related to the transmission risk of WNv, and that the bigger the growing ratio, the greater the risk.

It is well-known that WNv is carried by mosquitoes, and mosquitoes replicate vigorously in temperate climates (warm summer). Climate warming has led to the gradual expansion of mosquitoes into new areas, which were previously cold and unsuitable for survival of mosquitoes, but are now warming up, and leading to larger mosquito habitats. Similarly, birds, as hosts of WNv, have strong reproductive capacity. They are moving around in search of food, breeding, or responding to seasonal changes, which leads to the expansion of their habitat. In our paper, habitat expansion of birds and mosquitoes is described as domain growth in mathematical models. It follows from our theoretical result and numerical simulation that when WNv vanishes on a fixed domain, one still vanishes on the growing domain with the small growth rate, but WNv can be spread on the growing domain with the large growth rate. Therefore, climate warming, which results in habitat expansion of birds and mosquitoes, is harmful to the prevention and control of WNv.

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