AN EXPLICIT EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEM WITH APPLICATION TO A MODEL IN ELECTRICITY PRODUCTION

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Abstract. In this paper, we introduce a new explicit extragradient algorithm for solving Variational Inequality Problem (VIP) in Banach spaces. The proposed algorithm uses a linesearch method whose inner iterations is independent of any projection onto feasible sets. Under standard and mild assumption of pseudomonotonicity and uniform continuity of the VIP associated operator, we establish the strong convergence of the scheme. Further, we apply our algorithm to find an equilibrium point with minimal environmental cost for a model in electricity production. Finally, a numerical result is presented to illustrate the given model. Our result extends, improves and unifies other related results in the literature.

1. introduction

Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^*. Let $T : C \to C$ be a nonlinear mapping, a point $x \in C$ is called a fixed point of T if $x = Tx$. We denote the set of fixed points of T by $F(T)$. Let $A : C \to E^*$ be a continuous mapping. The Variational Inequality Problem (for short, VIP) is defined as: find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

We denote by $VI(C, A)$, the solution set of Problem (1.1). It is well known that x solves (1.1) if and only if x is the fixed point of the mapping T, where $T = P_C(I - \lambda A)$ or equivalently, x solves the residual equation $r_\lambda(x) = 0$, where

$$r_\lambda(x) := x - P_C(x - \lambda Ax), \tag{1.2}$$

for an arbitrary positive λ, see [26], for details. Therefore, the knowledge of fixed point algorithm is handful in obtaining the solutions of (1.1).

Variational inequality plays an important role in studying a wide class of unilateral, obstacle and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see [2, 8]) and the references therein. There have been extensive studies of this problem by several authors. Several iterative algorithms have been developed for solving variational inequalities and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, see ([5, 6, 7, 15, 21, 20, 29, 35, 46, 48]).

In 1976, Korpelevich [30] introduced the extragradient method which is given by

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n),\ n \geq 1, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, for approximating solutions of VIP in a finite-dimensional space, where $C \subset \mathbb{R}^n$ is nonempty, closed and convex and $A : C \to \mathbb{R}^n$ is monotone L-Lipschitz continuous. Several modifications and extensions of the extragradient method have been proposed in infinite-dimensional spaces (see [5, 23, 34]). However, there are some setbacks that come with the use of the extragradient method, this include having prior knowledge of the Lipschitz constant or some estimate of it at the least, also the projection onto the nonempty, closed and convex subset C. It is well known that the projection onto C is computationally expensive if the feasible set C is not simple. These reasons affect the effective usage and the efficiency of the extragradient method.

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In an attempt to overcome the difficulty resulting from the projection onto the set C, Bello and Iusem [9], replaced the feasible set C by a finite sequence of projections onto suitable halfspaces with explicit formula. The projection onto a suitably constructed halfspace can be calculated using an explicit formula given in [17].

Furthermore, overcoming the difficulty of having prior knowledge of the Lipschitz constant A or at least its estimation, is found in some prediction of a stepsize with its further correction (see [2, 22, 25, 39]) or in a usage of an Armijo linesearch rule along a feasible direction. Usually the Armijo linesearch rule has been found more effective since the former approach preserves the disadvantage brought about by projecting onto feasible set per iteration. Using the Armijo linesearch procedure and projected reflected gradient method (a modification of extragradient method), weak convergences results have been recently obtained in infinite dimensional real Hilbert spaces.

In some of those results, the monotonicity of A is required by the Lipschitz constant L of A may not necessarily be needed for input parameters see (Theroem 3.1 [31] and Theorem 4.4 [32]).

Very recently, Kanzow and Shehu [28] prove strong convergence of a double projection method for monotone variational inequality problem in a real Hilbert space. The method employed in [28] involves a stepsize rule which might need some evaluations of A in the inner iteration without additional projections. To be precise, they prove the strong convergence of the following porojection method:

Algorithm 1.1. *Projection-type method*

Step I: Choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ and take $\gamma, \sigma \in (0,1)$, $s > 0$. Let x_1 be a given starting point. Set $n := 1$

Step II: Set
$$w_n := (1-\alpha_n)x_n + \alpha_n x_1.$$
If $r(w_n) := 0$. STOP.

Step III: Let $y_n(\eta) := (1-\eta)w_n + \eta P_C(w_n - Aw_n)$ for $\eta \in \mathbb{R}$. Compute η_n as the maximum of the numbers $s, s\gamma, s\gamma^2 \dots$ such that
$$\langle Ay_n(\eta_n), r(w_n)\rangle \geq \frac{\eta}{2}||r(w_n)||^2$$
and define $y_n := y_n(\eta_n)$.

Step IV: Compute
$$\lambda_n := \frac{\langle Ay_n, w_n - y_n\rangle}{||Ay_n||^2},$$
$$x_{n+1} = ((1-\beta_n)w_n + \beta_n P_C(w_n - \lambda_n Ay_n)).$$

Step V: Set $n := n+1$ and go to step II.

The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and$\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$.

The residual function $r_\lambda(x) = r_1(x) := x - P_C(x - Ax)$ from (1.2) with $\lambda = 1$.

In the infinite-dimensional case, many of the extragradient-like method schemes work for the larger class of monotone mappings A. However, this methods may fail to work for pseudomonotone mappings when the underlining space is infinite dimensional.

In this paper, motivated by the explicit method of Bello and Iusem [9] and the projection-type method of [28], we propose an explicit extragradient method for obtaining a solution of a VIP. Using this proposed method, we prove a strong convergence theorem for approximating a solution of VIP for a pseudomonotone operator A in the framework of 2-uniformly convex and smooth Banach space. The following are the advantages the current work have over some other works in this direction in the literature.

(i) Our method like the one in [28] employs the Armijo linesearch rule which only requires inner iteration without employing additional projections.
(ii) Our strong convergence algorithm solves the VIP where the underlining operator is pseudomonotone.
(iii) Our result is obtained in a real Banach space which is more general than the real Hilbert space that posses simple geometry.

The rest of the paper will be organized as follows: In Section 2, we recall some basic definitions and give some important results. We give some important discussions on the explicit extragradient method used in this paper in Section 3. The convergence analysis of our proposed method is given in Section 4. Some applications of our result, useful remarks and comments are given in Section 5.

2. PRELIMINARIES

We denote the weak and the strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$ respectively.

Let E be a real Banach space, given a function $g : E \to \mathbb{R}$,

- g is called Gâteaux differentiable at $x \in E$, if there exists an element of E, denoted by $g'(x)$ or $\nabla g(x)$ such that
$$\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t} = \langle y, g'(x)\rangle, \quad y \in E,$$
where $g'(x)$ or $\nabla g(x)$ is called Gâteaux differential or gradient of g at x. We say g is Gâteaux on E if for each $x \in E$, g is Gâteaux differentiable at x;
- g is called weakly lower semicontinuous at $x \in E$, if $x_n \rightharpoonup x$ implies $g(x) \le \liminf_{n\to\infty} g(x_n)$. We say that g is weakly lower semicontinuous on E, if for each $x \in E$, g is weakly lower semicontinuous at x;
- if g is a convex function, then it is said to be differentiable at a point $x \in E$ if the following set
$$\partial g(x) = \{f \in E : g(y) - g(x) \ge \langle f, y - x\rangle,\ y \in E\} \tag{2.1}$$
is nonempty. Each element $\partial g(x)$ is called a subgradient of g at x or the subdifferential of g and the inequality (2.1) is said to be the subdifferential inequality of g at x.
The function g is subdifferentiable at x, if g is subdifferntiable at every $x \in E$. It is well known that if g is Gâteaux differentiable at x, then g is subdifferentiable at x and $\partial g(x) = \{g'(x)\}$, that is, $\partial g(x)$ is just a singleton set. For more details on Gâteaux differentiable functions and other geometric properties of Banach space see [3, 38, 45, 47].

Let C be a nonempty, closed and convex subset of a real Banach space E with norm $||\cdot||$ and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by
$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2,\ \forall x \in E\},$$
where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ the duality pairing between the elements of E and E^*. Alber [4], introduced a generalized projection operator Π_C which is an analogue of the metric projection $P_C : H \to C$ in the Hilbert space H. The generalized projection $\Pi_C : E \to C$ is defined by
$$\Pi_C(x) = \inf_{y\in C}\{\phi(y,x),\ \forall x \in E\},$$
where ϕ is the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ defined by
$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2,\ \forall x, y \in E.$$
In the real Hilbert space, $P_C(x) \equiv \Pi_C(x)$ and $\phi(x,y) = ||x-y||^2$. It is obvious from the definition of the functional ϕ that
$$(||x|| - ||y||)^2 \le \phi(x,y) \le (||x|| + ||y||)^2.$$
The functional ϕ also satisfy the following important properties:
$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$
and
$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.3}$$
Note: If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$, see [13, 51].

We also define the functional $V : E \times E^* \to \mathbb{R}$ by
$$V(x,x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. It is well known that if E is a reflexive, strictly convex and smooth Banach space, then

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle \tag{2.5}$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [41].

Let C be a closed and convex subset of E and $T : C \to C$ be a mapping. A point $p \in C$ is called an asymptotic fixed point of T (see [40]) if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $||x_n - Tx_n|| \to 0$ as $n \to \infty$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$ (see [10, 11, 12]). T is said to be ϕ-nonexpansive if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$ and quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

It is known that the class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mapping which requires the strict condition $F(T) = \hat{F}(T)$, see ([10, 11, 12]).

Let E be a real Banach space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\}. \tag{2.6}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{||x + y||}{2} < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(||x + ty|| - ||x - ty||) - 1 : ||x|| = ||y|| = 1\}. \tag{2.7}$$

E is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $(p^{-1} + q^{-1} = 1)$. It is also known that every q-uniformly smooth Banach space is uniformly smooth.

It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. The following are some important and useful properties of J, for further details, see [1, 51]:

Let $C \subseteq E$ be a nonempty set. Then a mapping $A : C \to E$ is called

(a) monotone on E, if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in E$;
(b) pseudomonotone on E, if for all $x, y \in E$, $\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0$;
(c) Lipschitz continuous on E, if there exists a constant $L > 0$ such that $||Ax - Ay|| \leq L||x - y||$ for all $x, y \in E$.

Every monotone operator is pseudomonotone but the converse is not true (see for example [43]).

We now give the following useful and important lemmas that are needed in establishing our main results:

Lemma 2.1. [16] *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. For any $x \in E$ and $\lambda > 0$, we denote*

$$r_\lambda(x) = x - \Pi_C J^{-1}(Jx - \lambda Ax)$$

then

$$\min\{1, \lambda\}||r_1(x)|| \leq ||r_\lambda(x)|| \leq \max\{1, \lambda\}||r_1(x)||.$$

Lemma 2.2. [52] *Given a number $s > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that*

$$||tx + (1 - t)y||^2 \leq t||x||^2 + (1 - t)||y||^2 - t(1 - t)h(||x - y||),$$

for all $x, y \in X$, $t \in [0, 1]$, with $||x|| < s$ and $||y|| < s$.

Lemma 2.3. [27] *Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $||x_n - y_n|| \to 0$ as $n \to \infty$.*

Lemma 2.4. [4] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq \rangle \leq 0, \ \forall y \in C \tag{2.8}$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \ \forall y \in C, \ x \in X. \tag{2.9}$$

Lemma 2.5. [36] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \sigma_n + \gamma_n, \ n \geq 0$$

where

(a) $\{\alpha_n\} \subset [0, 1]$, $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup\limits_{n\to\infty} \sigma \leq 0$;
(c) $\gamma_n \geq 0$, $(n \geq 1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.6. [49, 50] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$: $a_{m_n} < a_{m_n+1}$ and $a_n < a_{m_n+1}$. In fact, $m_n = \max\{i \leq k : a_i < a_{i+1}\}$.*

The following is a special case of ([18], Lemma 1).

Lemma 2.7. *For all $v \neq 0 \in E$, $\bar{y} \in E$, $x \in d^+$ and $\bar{x} \in d^-$. We have that $\phi(\bar{x}, x) \geq \phi(\bar{x}, z) + \phi(z, x)$, where $z := \Pi_d x$ with $d := \{y \in E : \langle v, y - \bar{y} \rangle = 0\}$, whereas d^+ and d^- are defined by $d^+ := \{y \in E : \langle v, y - \bar{y} \rangle \geq 0\}$ and $d^- := \{y \in E : \langle v, y - \bar{y} \rangle \leq 0\}$, respectively.*

The following was stated and proved in ([19], Prop 2.11), see also ([18], Prop 4).

Lemma 2.8. *Let E_1 and E_2 be two real Banach spaces. Suppose $A : E_1 \to E_2$ is uniformly continuous on bounded subsets of E_1 and M is a bounded subset of E_1. Then, $A(M)$ is bounded.*

The following result was stated in real Hilbert space, see ([33]). The result can be applied on a real Banach space.

Lemma 2.9. *Consider VIP (1.1). If the mapping $f : [0, 1] \to E$ is defined as $f(t) = A(tx+(1-t)y)$ is continuous for all $x, y \in C$ (i.e f is hemicontinuous), then $M(A, C) \subset VI(C, A)$. Moreover, if A is pseudomonotone, then $VI(C, A)$ is closed and convex and $M(A, C) = VI(C, A)$, where $M(A, C) := \{x \in C : \langle Ay, y - x \rangle \geq 0, \forall y \in C\}$.*

3. Explicit Extragradient Method

In this section, we give a concise and precise statement of our algorithm, discuss some of its elementary properties and its convergence analysis. The convergence analysis is given in the next section.

Statement 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. For $i = 1, 2 \cdots m$, let $g_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinous and Gâteaux differentiable functions such that $g_i'(\cdot)$ is L_i-Lipschitz continuous with $L = \max_{1 \leq i \leq m} L_i$. Let $A : C \to E^*$ be a pseudomonotone operator which is uniformly continuous on bounded subsets of C.*

Assumption 3.2. *We require the following assumption for our operator and the solution set:*

A1. *$A : C \to E^*$ is a pseudomonotone and uniformly continuous on bounded subsets of C.*
A2. *The solution set $VI(C, A)$ is nonempty.*

Assumption 3.3. *For the convergence of the Algorithm 3.4, we make the following assumptions:*

B1. *The feasible set C is defined by*

$$C := \cap_{i=1}^m C^i$$

where

$$C^i := \{z \in E : g_i(z) \leq 0\};$$

B2. $\{\alpha_n\} \subset (0,1)$ *with* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;

B3. $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$.

Algorithm 3.4. *Explicit extragradient algorithm*

Step I: Choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying Assumption 3.3, take $\eta, \mu \in (0,1)$ and $\lambda \in (0,1)$. Let $x_0 \in C$ be a given starting point. Set $n = 1$.

Step II: For $i = 1, 2 \cdots, m$ and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : g_i(w_n) + \langle g_i'(w_n), z - w_n\rangle \leq 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jx_n)$ and compute

$$z_n = \Pi_{C_n} J^{-1}(Jw_n - \lambda A w_n).$$

If $r_\lambda(w_n) = 0$: STOP.

Step III: Compute $y_n = w_n - \theta_n r_\lambda(w_n)$, $\quad n \geq 1$, where $\theta_n = \eta^{m_n}$ and m_n is the smallest positive whole number m such that

$$\langle Ay_n, r_\lambda(w_n)\rangle \geq \frac{\mu}{2}\phi(w_n, z_n).$$

Step IV: Define

$$x_{n+1} = J^{-1}((1-\beta_n)Jw_n + \beta_n J\Pi_{C_n}(w_n - \gamma_n A y_n)),$$

where

$$\gamma_n = \frac{\langle Ay_n, w_n - y_n\rangle}{||Ay_n||^2}.$$

Step V: Set $n := n+1$ and go to step I.

Remark 3.5. *From the definition of C and C_n, it is easy to see that $C \subset C_n$. Indeed, for each $i \in I$ and $x \in C^i$, we have by the subdifferential inequality that*

$$g_i(x_n) + \langle g_i'(x_n), x - x_n\rangle \leq g_i(x) \leq 0.$$

By the definition of C_n^i, we have that $x \in C_n^i$. Hence, $C^i \subset C_n^i$ for all $i \in I$ and therefore $C \subset C_n$ for all $n \geq 1$.

Note that if $r_\lambda(w_n) = w_n - z_n = 0$, we have arrived at the solution of the variational inequality. We will assume implicitly in our convergence analysis that this does not occur after finitely many iterations, so that Algorithm 3.4 generates an infinite sequence satisfying in particular $r_\lambda(w_n) \neq 0$ for all $n \in \mathbb{N}$. We will show that this property implies that Algorithm 3.4 is well defined.

Next we show that Algorithm 3.4 is well defined. This implies that the algorithm terminates after finitely many inner loops.

Proposition 3.6. *There exists a nonnegative integer η_{m_n} satisfying Step III.*

Proof. Let $n \in \mathbb{N}$ be an arbitrary number. Following our assumption $r_\lambda(w_n) \neq 0$. Assume the contrary, that is, the step size rule does not terminate after finitely many iterations. We have

$$\langle A((1-\eta_{m_n})w_n + \eta_{m_n} z_n), r_\lambda(w_n)\rangle < \frac{\mu}{2}\phi(w_n, z_n), \ \forall \ m_n = m \geq 0. \tag{3.1}$$

Letting $m \to \infty$, we obtain by the continuity of A, that

$$\langle Aw_n, r_\lambda(w_n)\rangle \leq \frac{\mu}{2}\phi(w_n, z_n).$$

Let $d_n = J^{-1}(Jw_n - \lambda Aw_n)$, then $Jd_n = Jw_n - \lambda Aw_n$ and

$$\frac{1}{\lambda}\langle Jw_n - Jd_n, r_\lambda(w_n)\rangle \leq \frac{\mu}{2}\phi(w_n, z_n)$$

that is

$$2\langle Jw_n - Jd_n, r_\lambda(w_n)\rangle \leq \lambda\mu\phi(w_n, z_n).$$

Using (2.2), we obtain

$$2\langle Jw_n - Jd_n, r_\lambda(w_n)\rangle = \phi(w_n, z_n) + \phi(w_n, d_n) - \phi(z_n, d_n),$$

that is

$$\phi(w_n, z_n) + \phi(w_n, d_n) - \phi(z_n, d_n) \leq \lambda\mu\phi(w_n, z_n).$$

Since $\lambda, \mu \in (0, 1)$ and $\phi(w_n, z_n)$, we obtain

$$\phi(w_n, z_n) + \phi(w_n, d_n) - \phi(z_n, d_n) \leq \phi(w_n, z_n).$$

Hence,

$$\phi(w_n, d_n) < \phi(z_n, d_n). \tag{3.2}$$

Since $d_n = J^{-1}(Jw_n - \lambda Aw_n)$ by definition and $w_n \in C$, inequality (3.2) contradicts the definition of the generalized projection operator Π_{C_n}. The result follows. □

We obtain the following as a consequence of Lemma 3.6. In the following result we show that γ_n defined in Step IV is well defined.

Corollary 3.7. $\langle Ay_n, x_n - y_n\rangle > 0$. *In particluar,* $Ay_n \neq 0$ *and therefore* γ_n *is well defined and positive.*

Proof. Consider again a fixed $n \in \mathbb{N}$. Recall by Lemma 2.1, that $||r_\lambda(w_n)|| > 0$ holds due to our implicit assumption regarding the termination of the algorithm. Since the step size rule in Step III is well defined by Lemma 3.6, the definition of y_n yields

$$\langle Ay_n, x_n - y_n\rangle = \theta_n\langle Ay_n, r_\lambda(w_n)\rangle \geq \frac{\mu\theta_n}{2}\phi(w_n, z_n) > 0. \tag{3.3}$$

Hence proved. □

4. Convergence Analyis

In this section, we show that Algorithm 3.4 generates a sequence $\{x_n\}$ which converges strongly to the solution of the variational inequality. Firstly, we prove a result which guarantees the existence of weak cluster points of the sequence. That is, we show that $\{x_n\}$ is bounded.

Proposition 4.1. *Let Assumption 3.2, B2 and B3 of Assumption 3.3 hold. Then the sequence* $\{x_n\}$ *defined by Algorithm 3.4 is bounded.*

Proof. Define for each n the sets

$$\begin{aligned} h_n^- : &= \{z \in E : \langle Ay_n, z - y_n\rangle \leq 0\},\\ h_n : &= \{z \in E : \langle Ay_n, z - y_n\rangle = 0\},\\ h_n^+ : &= \{z \in E : \langle Ay_n, z - y_n\rangle \geq 0\}, \end{aligned} \tag{4.1}$$

where $\{y_n\}$ is defined as in Algorithm 3.4. Recall from Corollary 3.7, that $Ay_n \neq 0$.

Let $p \in VI(C, A)$. Since A is pseudomonotone, we have $\langle Ax, x - p\rangle \geq 0, \forall x \in C$. This implies $p \in h_n^-$ for all $n \in \mathbb{N}$ since $y_n \in C$. We again assume that Algorithm 3.4 does not terminate after finitely many iterations. We have $\langle Ay_n, w_n - y_n\rangle > 0$ by Corollary 3.7. Therefore, $w_n \in h_n^+$ and $w_n \notin h_n^-$ for all $n \in \mathbb{N}$. Using the definition of γ_n, we obtain

$$\begin{aligned} u_n &= w_n - \gamma_n Ay_n\\ &= w_n - \frac{\langle Ay_n, w_n - y_n\rangle}{||Ay_n||^2}Ay_n\\ &= \Pi_{h_n} w_n. \end{aligned}$$

Hence, u_n is the generalized projection of w_n onto the set h_n. In particular, we have that $u_n \in h_n$.

Using Lemma 2.7, we obtain

$$\phi(p, w_n) \geq \phi(p, u_n) + \phi(w_n, u_n). \tag{4.2}$$

Define $v_n := \Pi_{C_n}(w_n - \gamma_n A y_n) = \Pi_{C_n}(u_n)$. Using (2.2) and (2.8), we obtain

$$\phi(p, v_n) + \phi(u_n, v_n) - \phi(p, u_n) = 2\langle Ju_n - Jv_n, p - v_n\rangle \leq 0.$$

This implies

$$\phi(p, u_n) \geq \phi(p, v_n) + \phi(u_n, v_n). \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\phi(p, w_n) \geq \phi(p, v_n) + \phi(u_n, v_n) + \phi(w_n, u_n).$$

Therefore,

$$\phi(p, v_n) \leq \phi(p, w_n) - \phi(u_n, v_n) - \phi(w_n, u_n). \tag{4.4}$$

From the positivity of $\phi(\cdot,\cdot)$, we obtain

$$\phi(p, v_n) \leq \phi(p, w_n).$$

Further, from Algorithm 3.4, we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J^{-1}((1-\beta_n)Jw_n + \beta_n Jv_n)) \\
&= ||p||^2 - 2\langle p, (1-\beta_n)Jw_n + \beta_n Jv_n\rangle + ||(1-\beta_n)Jw_n + \beta_n Jw_n||^2 \\
&= ||p||^2 - 2\langle p, (1-\beta_n)Jw_n + \beta_n Jv_n\rangle + (1-\beta_n)||w_n||^2 + \beta_n||v_n||^2 - \beta_n(1-\beta_n)g(||Jw_n - Jv_n||) \\
&= (1-\beta_n)\phi(p, w_n) + \beta_n\phi(p, v_n) - \beta_n(1-\beta_n)g(||Jw_n - Jv_n||) \\
&= \phi(p, w_n) - \beta_n(1-\beta_n)g(||Jw_n - Jv_n||).
\end{aligned}$$

But,

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jx_n)) \\
&= ||p||^2 - 2\langle p, \alpha_n Jx_0 + (1-\alpha_n)Jx_n\rangle + ||\alpha_n Jx_0 + (1-\alpha_n)Jx_n||^2 \\
&= ||p||^2 - 2\alpha_n\langle p, Jx_0\rangle - 2(1-\alpha_n)\langle p, Jx_n\rangle + \alpha_n||x_0||^2 + (1-\alpha_n)||x_n||^2 - \alpha_n(1-\alpha_n)g(||Jx_0 - Jx_n||) \\
&\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, x_n). \qquad (4.5)
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, x_n) \\
&\leq \max\{\phi(p, x_0), \phi(p, x_n)\} \\
&\leq \vdots \\
&\leq \max\{\phi(p, x_0), \phi(p, x_0)\} = \phi(p, x_0).
\end{aligned}$$

This shows that the sequence $\{x_n\}$ is bounded. □

Note that in the proof of Proposition 4.1, we have not used the uniform continuity assumption on A on bounded subsets of C. Furthermore, it does not require condition B2 of Assumption 3.3. As a direct consequence, we have that $\{Ax_n\}$ is bounded. Also, the sequences $\{z_n\}$, $\{y_n\}$ and $\{Ay_n\}$ are bounded.

The following Lemma is required to prove our strong convergence.

Lemma 4.2. *The sequence $\{x_n\}$ generated by Algorithm 3.4 satisfies the following estimates:*

(i) $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$;
(ii) $-1 \leq \limsup_{n\to\infty} b_n < +\infty$.

where $a_n = \phi(p, x_n)$, $b_n = \langle x_{n+1} - p, Jx_0 - Jp\rangle$ and $p = \Pi_{VI(C,A)}x_0$.

Proof. From Algorithm 3.4 and (2.5), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J^{-1}((1-\beta_n)Jw_n + \beta_n Jv_n)) \\
&= ||p||^2 - 2\langle p, (1-\beta_n)Jw_n + \beta_n Jv_n\rangle + ||(1-\beta_n)Jw_n + \beta_n Jv_n||^2 \\
&= ||p||^2 - 2(1-\beta_n)\langle p, Jw_n\rangle - 2\beta_n\langle p, Jv_n\rangle + (1-\beta_n)||w_n||^2 + \beta_n||v_n||^2 - \beta_n(1-\beta_n)g(||Jw_n - Jv_n||) \\
&= (1-\beta_n)\phi(p, w_n) + \beta_n\phi(p, v_n) - \beta_n(1-\beta_n)g(||Jw_n - Jv_n||) \\
&\leq \phi(p, w_n), \qquad (4.6)
\end{aligned}$$

that is

$$
\begin{aligned}
\phi(p,x_{n+1}) &\leq \phi(p,J^{-1}(\alpha_n Jx_0+(1-\alpha_n)Jx_n))\\
&= V(p,\alpha_n Jx_0+(1-\alpha_n)Jx_n)\\
&\leq V(p,\alpha_n Jx_0+(1-\alpha_n)Jx_n-\alpha_n(Jx_0-Jp))+2\langle\alpha_n(Jx_0-Jp),J^{-1}(\alpha_n Jx_0+(1-\alpha_n)Jx_n)-p\rangle\\
&= V(p,\alpha_n Jp+(1-\alpha_n)Jx_n)+2\alpha_n\langle x_{n+1}-p,Jx_0-Jp\rangle\\
&\leq \alpha_n\phi(p,p)+(1-\alpha_n)\phi(p,x_n)+2\alpha_n\langle x_{n+1}-p,Jx_0-Jp\rangle\\
&\leq (1-\alpha_n)\phi(p,x_n)+2\alpha_n\langle x_{n+1}-p,Jx_0-Jp\rangle. \qquad (4.7)
\end{aligned}
$$

This establishes (i). Next we prove (ii). Since $\{x_n\}$ is bounded, we have

$$\sup_{n\geq 0} b_n \leq \sup 2||Jx_0-Jp||||x_{n+1}-p||<\infty.$$

This implies that $\limsup_{n\to\infty} b_n < \infty$. We now show that $\limsup_{n\to\infty} b_n \geq -1$. We assume the contrary, that is $\limsup_{n\to\infty} b_n < -1$. Then there exists $n_0\in\mathbb{N}$ such that $b_n < -1$, for all $n\geq n_0$. Then for all $n\geq n_0$, we get from that

$$
\begin{aligned}
a_{n+1} &\leq (1-\alpha_n)a_n+\alpha_n b_n\\
&< (1-\alpha_n)a_n-\alpha_n\\
&= a_n-\alpha_n(a_n+1)\\
&\leq a_n-\alpha_n
\end{aligned}
$$

By induction on the last inequality, we get

$$a_{n+1} \leq a_{n_0}-\sum_{i=n_0}^{n}\alpha_i. \qquad (4.8)$$

Taking superior limits of both sides in the inequality above, we obtain

$$\limsup_{n\to\infty} a_n \leq a_{n_0}-\lim_{n\to\infty}\sum_{i=n_0}^{n}\alpha_i=-\infty. \qquad (4.9)$$

This contradicts the fact that $\{a_n\}$ is nonnegative. Therefore (ii) holds. □

Recall by Proposition 4.1, that $\{x_n\}$ is bounded. This implies there exists at least one weak limit. The following result provides a condition under which each of such weak limit belongs to the solution of the variational inequality.

Lemma 4.3. *Let $\{w_{n_k}\}$ and $\{v_{n_k}\}$ be subsequences of $\{w_n\}$ and $\{v_n\}$ respectively, such that $||w_{n_k}-v_{n_k}||=0$ as $k\to\infty$, then $\lim_{k\to\infty}\langle Aw_{n_k},x-w_{n_k}\rangle\geq 0$ and $\lim_{k\to\infty}||w_{n_k}-z_{n_k}||=0$.*

Proof. Let u_n and v_n be defined as above. We divide the proof into three steps. First, we show that $\langle Ay_{n_k},w_{n_k}-y_{n_k}\rangle=0$ as $k\to\infty$ holds on the chosen subsequence. To this end, replace p by q in (4.4), then

$$
\begin{aligned}
\phi(u_{n_k},w_{n_k}) &\leq \phi(q,w_{n_k})-\phi(q,v_{n_k})\\
&= ||w_{n_k}||^2-||v_{n_k}||^2+2\langle q,Jv_{n_k}-Jw_{n_k}\rangle\\
&\leq ||w_{n_k}-v_{n_k}||(||v_{n_k}||+||w_{n_k}||)+2||q||||Jw_{n_k}-Jv_{n_k}|| \to 0 \quad \text{as} \quad k\to\infty. \qquad (4.10)
\end{aligned}
$$

By Lemma 2.3, we obtain

$$\lim_{k\to\infty}||u_{n_k}-w_{n_k}||=0. \qquad (4.11)$$

Now since $u_n\in d_n$, we have

$$0=\langle Ay_n,u_n-y_n\rangle=\langle Ay_n,u_n-w_n\rangle+\langle Ay_n,w_n-y_n\rangle.$$

Using the boundedness of $\{||Ay_n||\}$, we get

$$|\langle Ay_{n_k},w_{n_k}-y_{n_k}\rangle|\leq ||Ay_{n_k}||||w_{n_k}-u_{n_k}||\to 0, \quad k\to\infty. \qquad (4.12)$$

Therefore, $\lim_{k\to\infty}\langle Ay_{n_k},w_{n_k}-y_{n_k}\rangle=0$. Secondly, we show that there exists $\{w_{n_k}\}$ such that, for all $x\in C$ we have $\lim_{k\to\infty}\langle Aw_{n_k},x-w_{n_k}\rangle\geq 0$.

In so doing, choose the subsequence $\{z_{n_k}\}$ of $\{z_n\}$ with $z_{n_k} := \Pi_{C_{n_k}} J^{-1}(Jw_{n_k} - \lambda A w_{n_k})$ for all $k \in \mathbb{N}$. We now consider two cases depending on the behaviour of the bounded sequence of step sizes $\{\theta_{n_k}\}$.

Case 1: Suppose that $\lim_{k\to\infty} \theta_{n_k} = 0$. Subsequencing if necessary, we may assume without loss of generality that $\lim_{k\to\infty} \theta_{n_k} = 0$. We first show that $\lim_{k\to\infty} ||w_{n_k} - z_{n_k}|| = 0$. It suffices to show that $\limsup_{k\to\infty} ||w_{n_k} - z_{n_k}|| = 0$ holds. Assume the contrary that $\limsup_{k\to\infty} ||w_{n_k} - z_{n_k}|| = \delta > 0$. Observe that $\delta \neq +\infty$, $\{w_{n_k} - z_{n_k}\}$ is bounded. Hence, $\delta < +\infty$. Let $\bar{y_k} = \frac{1}{l}\theta_{n_k} z_{n_k} + (1 - \frac{1}{l}\theta_{n_k}) w_{n_k}$ or equivalently, $\bar{y_k} - w_{n_k} = \frac{1}{l}\theta_{n_k}(z_{n_k} - w_{n_k})$. Since $\{z_{n_k} - w_{n_k}\}$ is bounded and $\lim_{k\to\infty} \theta_{n_k} = 0$, it follows that

$$\lim_{k\to\infty} ||\bar{y_k} - w_{n_k}|| = 0. \tag{4.13}$$

From the step size rule and the definition of $\bar{y_k}$, we get

$$\langle A\bar{y_k}, w_{n_k} - z_{n_k}\rangle < \frac{\mu}{2}\phi(w_{n_k}, z_{n_k}), \quad k \in \mathbb{N}. \tag{4.14}$$

Since A is uniformly continuous on bounded subsets of C, $\mu \in (0,1)$ and the right hand side is bounded from below by a positive constant, we obtain from (4.13), that there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} 0 &= 2\langle Aw_{n_k}, w_{n_k} - z_{n_k}\rangle - 2\langle Aw_{n_k}, w_{n_k} - z_{n_k}\rangle \\ &> \mu\phi(w_{n_k}, z_{n_k}), \quad k \in \mathbb{N}, \quad K \geq N. \end{aligned} \tag{4.15}$$

This is a contradiction since $\mu > 0$ and $\phi(\cdot,\cdot) \geq 0$. Therefore, $\limsup_{k\to\infty} \phi(w_{n_k}, z_{n_k}) = 0$, by Lemma 2.3, we obtain

$$\lim_{k\to\infty} ||w_{n_k} - z_{n_k}|| = 0. \tag{4.16}$$

Case 2: Suppose $\liminf_{k\to\infty} \theta_{n_k} > 0$. Then there exists a constant θ such $\theta_{n_k} \geq \theta > 0$ holds for all $k \in \mathbb{N}$. It follows from the step size rule in Algorithm 3.4, that

$$\langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle \geq \frac{\mu}{2}\theta_{n_k}\phi(w_{n_k}, z_{n_k}). \tag{4.17}$$

Therefore, by the first step and Lemma 2.3, we have $\lim_{k\to\infty} ||w_{n_k} - z_{n_k}|| = 0$. □

Lemma 4.4. *There exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that for all $x \in C$, $0 \leq \liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle$.*

Proof. By the definition of z_{n_k} together with (2.8), we obtain

$$\langle Jw_{n_k} - \lambda Aw_{n_k} - Jz_{n_k}, x - z_{n_k}\rangle \leq 0, \quad x \in C_{n_k}. \tag{4.18}$$

This implies that

$$\langle Jw_{n_k} - Jz_{n_k}, x - z_{n_k}\rangle \leq \lambda\langle Aw_{n_k}, x - z_{n_k}\rangle, \quad \forall x \in C_{n_k}.$$

Hence

$$\langle Jw_{n_k} - Jz_{n_k}, x - z_{n_k}\rangle + \lambda\langle Aw_{n_k}, z_{n_k} - w_{n_k}\rangle \leq \lambda\langle Aw_{n_k}, x - w_{n_k}\rangle, \quad \forall x \in C_{n_k}.$$

Fix $x \in C_{n_k}$ and let $k \to \infty$. Using the continuity of J on bounded subsets of E, $\lambda \in (0,1)$ and the fact that $\lim_{k\to\infty} ||w_{n_k} - z_{n_k}|| = 0$. We have that $0 \leq \liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle$ for all $x \in C_{n_k}$. Thus, conclusion follows from this, the fact that $w_{n_k} \in C$ and $C \subset C_{n_k}$. □

Next we show that $q \in VI(C, A)$

Lemma 4.5. *Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup q$, then $q \in VI(C, A)$.*

Proof. First, we show that $q \in C$. Indeed, it follows from $z_{n_k} \in C_{n_k}$ that

$$g_i(w_{n_k}) + \langle g_i'(w_{n_k}), z_{n_k} - w_{n_k}\rangle \leq 0.$$

By using Cauchy Schwartz inequality, we have

$$\begin{aligned} g_i(w_n) &\leq \langle g_i'(w_{n_k}), w_{n_k} - z_{n_k}\rangle \\ &\leq ||g_i'(w_{n_k})|| \cdot ||w_{n_k} - z_{n_k}||. \end{aligned}$$

Since g_i' is Lipschitz continuous and $\{w_{n_k}\}$ is bounded, we have that $\{g_i'(w_{n_k})\}$ is bounded. Thus, there exists $L_i > 0$ such that $||g_i'(w_{n_k})||$ for each i. Therefore, we obtain

$$g_i(w_{n_k}) \leq L \cdot ||w_{n_k} - z_{n_k}||,$$

where $L = \max_{1\leq i\leq m}\{L_i\}$. Hence, by the weakly continuity of g_i and Lemma 4.3, we have

$$g_i(q) \leq \liminf_{k\to\infty} g_i(w_{n_k}) \leq \lim_{k\to\infty} L \cdot ||w_{n_k} - z_{n_k}|| = 0.$$

Thus, $q \in C$.

Now, take an arbitrary $x \in C$ and fix a positive number ϵ. Using the previous result, we can obtain N large enough such that

$$\langle Aw_{n_k}, x - w_{n_k}\rangle + \epsilon \geq 0, \quad \forall\ k \geq N.$$

For some $b_{n_k} \in E$ satisfying $\langle Aw_{n_k}, b_{n_k}\rangle = 1$, since $Aw_{n_k} \geq 0$, we can rewrite the above inequality as

$$\langle Aw_{n_k}, x + \epsilon b_{n_k} - w_{n_k}\rangle \geq 0, \quad k \geq N. \tag{4.19}$$

Using the fact A is pseudomonotone in (4.19), we get

$$\langle A(x + \epsilon b_{n_k}), x + \epsilon b_{n_k} - w_{n_k}\rangle \geq 0, \quad \forall k \geq N.$$

Thus,

$$\langle Ax, x - w_{n_k}\rangle \geq \langle Ax - A(x + \epsilon b_{n_k}), x + \epsilon b_{n_k} - w_{n_k}\rangle - \epsilon\langle Ax, b_{n_k}\rangle, \quad k \geq N.$$

Let $\epsilon \to 0$, then by the continuity of A and boundedness of $\{w_n\}$ we have

$$\liminf_{k\to\infty}\langle Ax, x - w_{n_k}\rangle \geq 0.$$

Since $||Jw_{n_k} - Jx_{n_k}|| \leq \alpha_{n_k}||Jx_0 - Jx_{n_k}|| \to 0$, by the uniform continuity of J on bounded subsets of C, we obtain $||w_{n_k} - x_{n_k}|| \to 0$ as $k \to \infty$. Thus, $x_{n_k} \rightharpoonup q$ implies $w_{n_k} \rightharpoonup q$. We, therefore, have for all $x \in C$,

$$\langle Ax, x - q\rangle = \lim_{k\to\infty}\langle Ax, x - w_{n_k}\rangle = \liminf_{k\to\infty}\langle Ax, x - w_{n_k}\rangle \geq 0. \tag{4.20}$$

We obtain by Lemma 2.9 that $q \in VI(C, A)$. □

We now state and prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 3.4 to the solution of the underlying variational inequality.

Theorem 4.6. *Given Statement 3.1 and assume Assumptions 3.2 and 3.3 hold. Then, the sequence $\{x_n\}$ generated by Algorithm 3.4 converges strongly to the point $p = \Pi_{VI(C,A)}x_0$.*

Proof. Let $p \in VI(C, A)$. As in previous proofs, we maintain the notations u_n and v_n, that is $u_n = w_n - \gamma_n Ay_n$ and $v_n = \Pi_{Q_n}(u_n) = \Pi_{Q_n}(w_n - \gamma_n Ay_n)$. Then, from Algorithm 3.4, we obtain

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J^{-1}((1-\beta_n)Jw_n + \beta_n Jv_n)) \\
&= ||p||^2 - 2\langle p, (1-\beta_n)Jw_n + \beta_n Jv_n\rangle + ||(1-\beta_n)Jw_n + \beta_n Jv_n||^2 \\
&= ||p||^2 - 2(1-\beta_n)\langle p, Jwn\rangle - 2\beta_n\langle p, Jv_n\rangle + (1-\beta_n)||w_n||^2 + \beta_n||v_n||^2 - \beta_n(1-\beta_n)h(||Jw_n - Jv_n||) \\
&= (1-\beta_n)\phi(p, w_n) + \beta_n\phi(p, v_n) - \beta_n(1-\beta_n)h(||Jw_n - Jv_n||) \\
&\leq (1-\beta_n)\phi(p, w_n) + \beta_n\phi(p, w_n) - \beta_n(1-\beta_n)h(||Jw_n - Jv_n||) \\
&= \phi(p, w_n) - \beta_n(1-\beta_n)h(||Jw_n - Jv_n||), \qquad (4.21)
\end{aligned}$$

which implies, $\phi(p, x_{n+1}) \leq \phi(p, w_n)$. Hence,

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \phi(p, w_n) = \phi(p, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jx_n)) \\
&= V(p, \alpha_n Jx_0 + (1-\alpha_n)Jx_n) \\
&\leq V(p, \alpha_n Jx_0 + (1-\alpha_n)Jx_n - \alpha_n(Jx_0 - Jp)) + 2\alpha_n\langle J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jx_n) - p, Jx_0 - Jp\rangle \\
&\leq \alpha_n V(p, Jp) + (1-\alpha_n)V(p, Jx_n) + 2\alpha_n\langle x_{n+1} - p, Jx_0 - Jp\rangle \\
&\leq (1-\alpha_n)\phi(p, x_n) + 2\alpha_n\langle x_{n+1} - p, Jx_0 - Jp\rangle. \qquad (4.22)
\end{aligned}$$

We divide the rest of the proof into two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that the sequence $\{\phi(p, x_n)\}$ is monotone non-increasing. Since $\{\phi(p, x_n)\}$ is bounded, it is convergent and hence

$$\phi(p, x_n) - \phi(p, x_{n+1}) \to 0, \quad \text{as} \quad n \to \infty.$$

We obtain from (4.21) and simple estimation on $\phi(p, w_n)$, that

$$\beta_n(1-\beta_n)g(||Jw_n - Jv_n||) \leq \phi(p, Jx_0) + (1-\alpha_n)\phi(p, x_n) - \phi(p, x_{n+1}).$$

Using conditions B2, B3 and the property of h, we have $||Jv_n - Jw_n|| \to 0$ as $n \to \infty$. Thus, by the uniform continuity of J on the bounded subsets, we get

$$\lim_{n\to\infty} ||w_n - v_n|| = 0. \tag{4.23}$$

Furthermore,

$$||Jx_{n+1} - Jw_n|| = \beta_n ||Jv_n - Jw_n|| \to 0, \text{ as } n \to \infty. \tag{4.24}$$

Since J is norm-to-norm continuous on bounded subsets of E, we have $||x_{n+1} - w_n|| \to 0$ as $n \to \infty$. Therefore,

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - w_n|| + ||w_n - x_n|| \to 0, \text{ as } n \to \infty. \tag{4.25}$$

We now show that, the sequence $\{x_n\}$ converges strongly to $p = \Pi_{VI(C,A)} x_0$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty} \langle x_{n+1} - p, Jx_0 - Jp\rangle = \lim_{j\to\infty} \langle x_{n_j+1} - p, Jx_0 - Jp\rangle.$$

Since $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$, we have from Lemma 2.4 equation (2.8), that

$$\limsup_{n\to\infty} \langle x_{n+1} - p, Jx_0 - Jp\rangle = \lim_{j\to\infty} \langle q - p, Jx_0 - Jp\rangle. \tag{4.26}$$

It follows from Lemma 2.5, Lemma 4.2 (i) and (4.26), that $\phi(p, x_n) \to 0$ as $n \to \infty$. Therefore, by Lemma 2.3, that

$$\lim_{n\to\infty} ||p - x_n|| = 0.$$

This implies $\{x_n\}$ converges strongly to $p = \Pi_{VI(C,A)} x_0$.

Case 2: Suppose that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\phi(p, x_{n_j}) < \phi(p, x_{n_j+1})$, $\forall n \in \mathbb{N}$.

By Lemma 2.6, ther exists a non-decreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following inequality holds

$$\phi(p, x_{m_n}) \leq \phi(p, x_{m_n+1}) \text{ and } \phi(p, x_n) \leq \phi(p, x_{m_n+1}). \tag{4.27}$$

Note that from

$$\begin{aligned} \phi(p, x_{m_n}) &\leq \phi(p, x_{m_n+1}) \leq (1 - \beta_{m_n})\phi(p, w_{m_n}) + \beta_{m_n}\phi(p, v_{m_n}) - \beta_{m_n}(1 - \beta_{m_n})h(||Jv_{m_n} - Jw_{m_n}||) \\ &\leq \phi(p, w_{m_n}) - \beta_{m_n}(1 - \beta_{m_n})h(||Jv_{m_n} - Jw_{m_n}||) \\ &\leq \alpha_{m_n}\phi(p, x_0) + (1 - \alpha_{m_n})\phi(p, x_{m_n}) - \beta_{m_n}(1 - \beta_{m_n})h(||Jv_{m_n} - Jw_{m_n}||). \end{aligned}$$

Hence,

$$\begin{aligned} \beta_{m_n}(1 - \beta_{m_n})h(||Jv_{m_n} - Jw_{m_n}||) &\leq \; \leq \phi(p, x_{m_n}) - \phi(p, x_{m_n}) + \alpha_{m_n} M_1 \\ &\leq \alpha_{m_n} M_1 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.28}$$

By following the same argument as in **Case 1**, we obtain $||v_{m_n} - z_{m_n}|| \to 0$, $||w_{m_n} - x_{m_n}|| \to 0$, $||x_{m_n+1} - w_{m_n}|| \to 0$ and $||x_{m_n+1} - x_{m_n}|| \to 0$ as $n \to \infty$. Since $\{x_{m_n}\}$ is bounded, there exists a subsequence, still denoted $\{x_{m_n}\}$ such that $x_{m_n} \rightharpoonup q \in C$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle x_{m_n+1} - p, Jx_0 - Jp\rangle = \lim_{n\to\infty} \langle x_{m_n+1} - p, Jx_0 - Jp\rangle.$$

Hence from (2.8), we have

$$\limsup_{n\to\infty} \langle x_{m_n+1} - p, Jx_0 - Jp\rangle = \lim_{n\to\infty} \langle q - p, Jx_0 - Jp\rangle \leq 0.$$

From (4.27), we have

$$\begin{aligned} 0 &\leq \phi(p, x_{m_n+1}) - \phi(p, x_{m_n}) \\ &\leq (1 - \alpha_{m_n})\phi(p, x_{m_n}) + 2\alpha_{m_n}\langle x_{m_n+1} - p, Jx_0 - Jp\rangle - \phi(p, x_{m_n}). \end{aligned}$$

Since $\alpha_{m_n} > 0$, we get

$$\phi(p, x_{m_n}) \to 0 \text{ as } n \to \infty \tag{4.29}$$

and by Lemma 2.3, we obtain $\lim_{n\to\infty} ||x_{m_n} - p|| = 0$. Consequently, we obtain $||x_n - p|| \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $p = \Pi_{VI(C,A)} x_0$. □

We obtain a result given in the real Hilbert space as a direct consequence of our main theorem:

Corollary 4.7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $g_i : H \to \mathbb{R}$, $i = 1, 2 \ldots, m$, be family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, $A : C \to H$ be a bounded, uniformly continuous pseudomonotone operator. Suppose $VI(C, A) \neq \emptyset$ and Assumption 3.3 is satisfied. Then the sequence $\{x_n\}$ given by the following Algorithm 4.8 converges strongly to a unique solution $p = P_{VI(C,A)}x_0$, where $P_{VI(C,A)}$ is the metric projection of C onto $VI(C, A)$.*

Algorithm 4.8. *Explicit extragradient algorithm*

Step I: Choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying Assumption 3.3, take $\eta, \mu \in (0, 1)$ and $\lambda \in (0, 1)$. Let x_0 be a given starting point. Set $n = 1$.

Step II: For $i = 1, 2 \cdots, m$ and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : g_i(w_n) + \langle g_i'(w_n), z - w_n\rangle \le 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := (\alpha_n x_0 + (1 - \alpha_n)x_n)$ and compute

$$z_n = P_{C_n}(w_n - \lambda A w_n).$$

If $r_\lambda(w_n) = 0$: STOP.

Step III: Compute $y_n = w_n - \theta_n r_\lambda(w_n)$, $\quad n \ge 1$, where $\theta_n = \eta^{m_n}$ and m_n is the smallest positive whole number m such that

$$\langle Ay_n, r_\lambda(w_n)\rangle \ge \frac{\mu}{2}||w_n - z_n||.$$

Step IV: Define

$$x_{n+1} = ((1 - \beta_n)Jw_n + \beta_n PC_n(w_n - \gamma_n Ay_n))$$

where

$$\gamma_n = \frac{\langle Ay_n, w_n - y_n\rangle}{||Ay_n||^2}.$$

Step V: Set $n := n + 1$ and go to step I.

Remark 4.9. *We remark that Algorithm 4.8 coincides with Algorithm 1.1 and Corollary 4.7 coincides with Theorem 4.4 of* [28] *when the half space C_n is replaced by a feasible set C and the pseudomontone A is reduced to a monotone operator.*

5. A practical model and computational results

In this section, we provide an application of our main result in the form of an equilibrium-optimization which can be regarded as an extension of a Nash-Cournot oligopolistic equilibrium in electricity markets. The equilibrium-optimization model has been investigated in some research articles (see for example [14, 42]). In this equilibrium model, we assume that there are n companies, with each company i possessing generating units I_i. Let x denote the vector whose entry x_j stands for the power generated by unit j. Following [14], we suppose that the $P_i(s)$ is a decreasing affine function of s where $s = \sum_{j=1}^N x_j$ and N is the number of generating units, that is $P_i(s) = \alpha - \beta_i(s)$. The profit made by company i is given by $f_i(x) = P_i(s)(\sum_{j\in I_i} x_j) - \sum_{j\in I_i} c_j x_j$, where $c_j x_j$ is the cost of generating x_j from generating unit j. Suppose that K_i is the strategy set of company i, that is the condition $\sum_{j\in I_i} x_j \in K_i$ must be satisfied for every i. Then the strategy set of the model is

$$K := K_1 \times K_2 \times \cdots \times K_n.$$

In other for each company to maximize its profit, a commonly used approach by choosing the corresponding production level under the presumption that the production of other companies are parametric input is the Nash equilibrium concept.

We recall that $\bar{x} \in K = K_1 \times K_2 \times \cdots \times K_n$ is an equilibrium point of the model if $f_i(\bar{x}) \ge f_i(\bar{x}[x_i])$, $\quad \forall\, x_i \in K_i$, $\quad \forall\, 1, 2 \ldots, n$, where vector $\bar{x}[x_i]$ stands for the vector obtained from $\bar{x}$ by replacing $\bar{x}_i$ with x_i. By taking $f(x, y) = \phi(x, y) - \phi(x, x)$ with

$$\phi(x, y) = -\sum_{i=1}^n f_i(x[y_i]).$$

Table 1. Algorithm 3.4

Size N	CPU time (seconds)	Number of Iter. (n)
10	0.3383	16
20	0.4741	14
50	2.4429	14
80	5.8955	17
100	11.6723	21

Then the problem of finding a Nash equilibrium point of the model can be formulated as

$$\bar{x} \in K : f(\bar{x}, x) \geq 0, \forall\, x \in K. \tag{5.1}$$

We suppose for every j, the cost c_j for production and the environmental fee f are increasingly convex functions. The convexity assumption here means that both the cost and fee for producing a unit production increases as the quantity of the production gets larger. Under this convexity assumption, it is easy to see [53] that (5.1) is equivalent to

$$x \in C : \langle Bx - a + \nabla\varphi(x), y - x\rangle \geq 0, \quad \forall\, y \in C,$$

where $a := (\alpha, \alpha \cdots, \alpha)^T$,

$$B_1 = \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \beta_n \end{pmatrix}, B = \begin{pmatrix} 0 & \beta_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \cdots & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n & \beta_n & \beta_n & \beta_n & 0 \end{pmatrix}$$

and $\varphi(x) := x^T B_1 x + \sum_{j=1}^{N} c_j x_j$. This holds when c_j is differentiable convex for every j.

We test the proposed algorithm with the cost function given by

$$c_j(x_j) = \frac{1}{2}Px_j^2 + q_j x_j,\ P_j \geq 0.$$

The algorithm was coded in MATLAB 2019a on a Dell i7 Dual core 8.00GB(7.78 GB usable) RAM laptop. The computational results are shown in Table 1. The parameters β_j for all $j = 1, 2 \ldots, n$, matrix P and vector q were generated randomly in the interval $(0, 1]$, $[1, 50]$ and $[1, 50]$ respectively. We perform our Algorithm 3.4 by varying the choices of N, different initial choices x_0 generated randomly in the interval $[1, 50]$ and $n = 10$ with $\frac{||x_{n+1} - x_n||}{||x_1 - x_0||} < 10^{-6}$ our stopping criterion.

6. Conclusion

In this paper, we proposed an explicit extragradient method for solving variational inequality problems. Our proposed algorithm empolys an Armijo linesearch rule which does not depend on a Lipschitz constant of the underlining pseudomotone operator. We established a strong convergence theorem of the algorithm under some mild conditions in the framework of Banach space. As a numerical experiment we give an application of our result to a model in electricity production. Our result extends the results of [28], [44] and other corresponding results in this direction.

Declaration
The authors declare that they have no competing interests.

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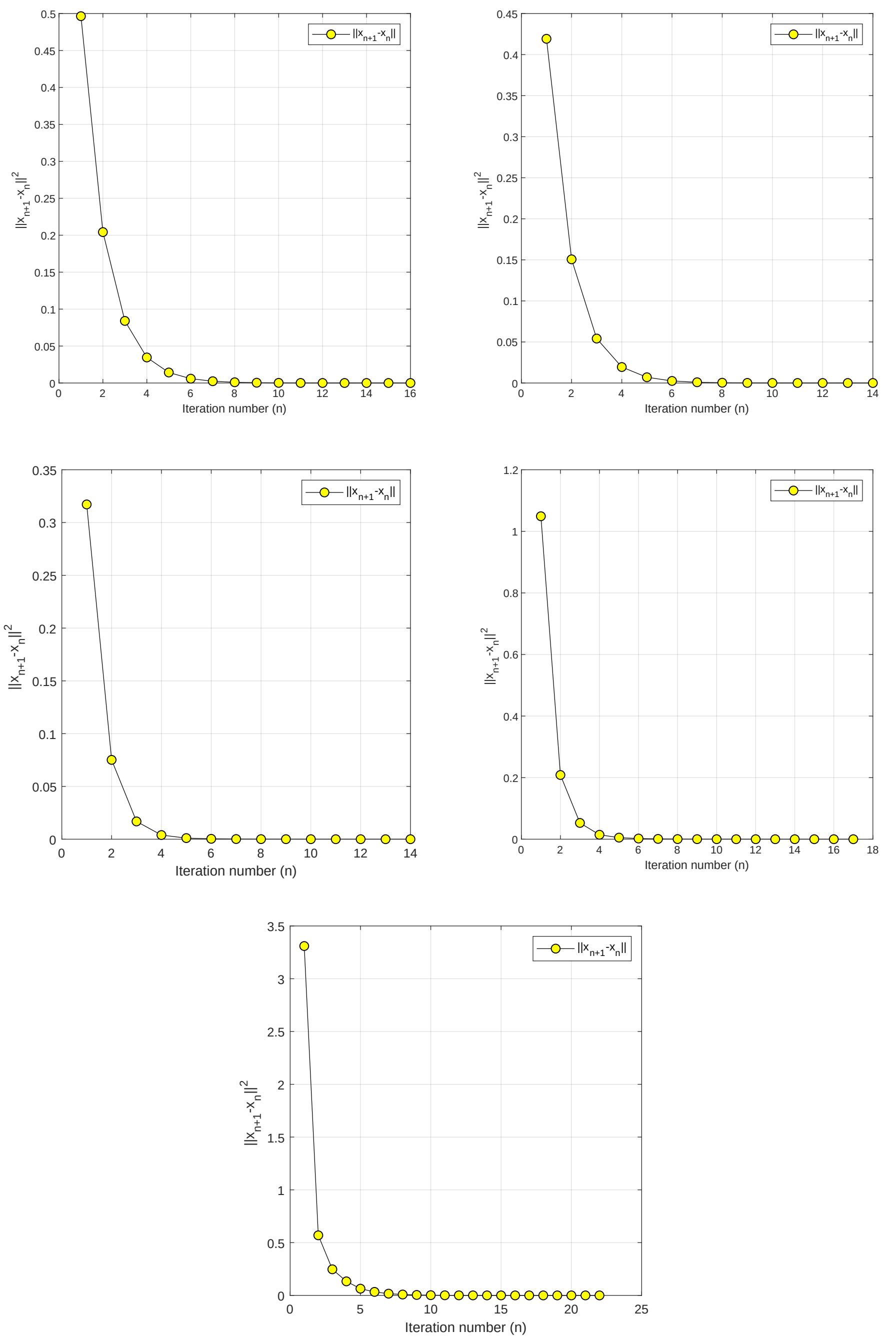

FIGURE 1. Top Left: N=10; Top Right: N= 20; Middle left: N= 50; Middle right: N=80; Bottom: N=100.

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