Improved reproducing kernel method to solve space-time fractional advection-dispersion equation

Tofigh Allahviranloo *[1], Hussein Sahihi†[2], Soheil Salahshour‡[3] and Dumitru Baleanu§[4]

[1,2,3]Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey
[4]Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara, Turkey
[4]Institute of Space Sciences, Magurele–Bucharest, Romania

December 22, 2020

Abstract

In this paper we consider, Space-Time Fractional Advection-Dispersion equation on a finite domain with variable coefficients. Fractional Advection- Dispersion equation as a model for transporting heterogeneous subsurface media as one approach to the modeling of the generally non-Fickian behavior of transport. We use a semi-analytical method as Reproducing kernel Method to solve Space-Time Fractional Advection-Dispersion equation so that we can get better approximate solutions than the methods with which this problem has been solved. The main obstacle to solve this problem is the existence of a Gram-Schmidt orthogonalization process in the general form of reproducing kernel method, that is very time consuming. So, we introduce the Improved Reproducing Kernel Method, which is a different implementation for the general form of the reproducing kernel method. In this method, the Gram-Schmidt orthogonalization process is eliminated to significantly reduce the CPU-time. Also, present method, increases the accuracy of approximate solutions. Due to the increasing accuracy of approximate solutions, we will be able to provide a valid error

*tofigh.allahviranloo@eng.bau.edu.tr
†hu.sahihi@gmail.com
‡soheil.salahshour@eng.bau.edu.tr
§dumitru@cankaya.edu.tr

analysis for this technique. The accuracy of the theoretical results illustrated by solving two numerical examples and we show that present method could provide better approximations than the general form of reproducing kernel method. One of the main drawbacks of the present method is that it is not able to solve the fractional problems that have no boundary conditions. The Improved Reproducing Kernel Method is applicable to solve the fractional problems when the problem has conditions at the beginning and end of its definition interval or region, even if these conditions are nonlocal or any mixed form. It is important to note that the proposed method can be implemented on problems with fractional derivatives, despite the boundary conditions at the beginning and end of their definition interval or region.

Keyword: Space-time fractional advection dispersion equation; Fractional derivative; Reproducing kernel method; Error analysis.

1 Introduction

We consider the following (S-TFA-D) equation on a finite domain with variable coefficients. The existence and uniqueness of the sufficiently smooth solution have been studied in [21].

$$\begin{aligned}
&\tfrac{\partial^{\mu_1} y(x,t)}{\partial t^{\mu_1}} = -p(x,t) D_x^{\mu_2} y(x,t) + q(x,t) D_x^{\mu_3} y(x,t) + f(x,t),\\
&0 < t \leq 1, \qquad 0 < x < 1,\\
&y(x,0) = \psi(x), \quad y(0,t) = 0, \quad y(1,t) = \varphi(x).
\end{aligned} \tag{1}$$

Fractional derivatives are employed in physics, hydrology, polymer physic, biophysics, thermodynamics, and chaotic dynamics [5–14]. Various authors have introduced the fractional advection-dispersion equation as a model for transporting heterogeneous subsurface media as one approach to the modeling of the generally non-Fickian behavior of transport [15, 16, 27, 28]. Fractional derivative relative to t (time) is the Caputo of order $(0 < \mu_1 \leq 1)$ and fractional derivatives relative to x (space) are the Riemann-Liouville of order $(1 < \mu_2 \leq 2)$ and $(0 < \mu_3 \leq 1)$, which are physical considerations [1–4]. Average fluid velocity and the dispersion coefficient are $p(x,t)$ and $q(x,t)$ respectively. The function $f(x,t)$ can be used to represent sources and sinks. Flow is from left to right by assuming $p(x,t) \geq 0$ and $q(x,t) \geq 0$ and $y(x,t)$ is the solute concentration. General form of the RKM is fully introduced in [18] and has already been used to solve Eq. (1), [17]. Since the general form of the RKM has a Gram-Schmidt orthogonalization process, CPU time can increased. However, the presence of the Gram-Schmidt orthogonalization process in the general form of the RKM reduces the accuracy of the approximate solutions; furthermore, it is not possible to provide a correct error analysis. If the accuracy of the approximate solutions is not appropriate, then, we cannot provide a suitable approximation for the derivative of the solution, and the same can be said about the error analysis for the derivative of the approximate solution. Therefore, we seek to implement RKM without using the orthogonalization process so that we can obtain the results of solving the problem in less time and increase the accuracy of the approximate solutions, if possible. The proposed method in this study makes it unnecessary to use the Gram-Schmidt orthogonalization process. We introduce IRKM that is a different implementation of the RKM in general form with a very low CPU time. However, we show that the IRKM improves the approximate solutions compared to the general form of the RKM, significantly. We employ the idea presented in [19] by Wang et al. The convergence and error analysis theorems are proved, and numerical examples are presented to compare the accuracy of IRKM in comparison with the other methods. Some applications of the RKM to solve fractional problems are studied by Ali Akgul et al. [29, 30] who investigated the electrodynamic flow and solutions of strongly non-linear equations via the reproducing kernel model. Omar Abu Arqub et al. [31] applied a fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-fractional Volterra integrodifferential equations. Also, in [32], the authors have considered Dirichlet time-fractional Diffusion-Gordon types equations in porous media using reproducing kernel approach. Beside, Dehghan et al. [33] have used the element free Galerkin approach based on the reproducing kernel particle method for solving the 2D fractional Tricomi-type equation with Robin boundary condition. Some numerical methods for solving fractional problems by Omid Nikan et al. are presented in [35–39].

Remark 1.1. *One of the main drawbacks of the proposed method in this paper and also the general form of the RKM is that it is not able to solve the fractional problems that have no boundary conditions. The RKM is applicable to solve the fractional problems when the problem has conditions at the beginning and end of its definition interval or region, even if these conditions are in nonlocal form, differential-difference form, integral, or any mixed form. In general, even if there are boundary conditions as a nonlocal or differential-difference or integral form for a solution and its derivative at the boundary point of the fractional differential equation, the approximate solution can be calculated with appropriate accuracy for the fractional differential equation by RKM. This result is due to the implementation of the RKM to* *overcome various fractional problems.*

2 Preliminaries

We define the reproducing kernel space $W^m[0,1]$ such that $y^{(m-1)}(x)$ is absolutely continuous and $y^{(m)}(x) \in L^2[0,1]$ and equipped with conditions $y(0) = y(1) = 0$. The inner product and norm are as follows,

$$\begin{aligned} &< y_1(x), y_2(x) >_{W^m} = \\ &\qquad \sum_{i=0}^{m-1} {y_1}^{(i)}(0){y_2}^{(i)}(0) + \int_0^1 {y_1}^{(m-1)}(x){y_2}^{(m-1)}(x)dx, \\ &\|y(x)\|_{W^m} = \sqrt{< y, y >_{W^m}}, \qquad y_1(x), y_2(x) \in W^m, \end{aligned}$$

and we consider the reproducing kernels functions for spaces $W^2[0,1]$ and $W^3[0,1]$ in the following form [18],

$$R_\eta(x) = \tfrac{3\eta x}{13} - \tfrac{5\eta^2 x}{26} - \tfrac{5\eta^3 x}{78} - \tfrac{\eta^5 x}{156} - \tfrac{5\eta x^2}{26} + \tfrac{21\eta^2 x^2}{104} + \tfrac{5\eta^4 x^2}{624} - \tfrac{\eta^5 x^2}{624} - \tfrac{5\eta x^3}{78} - \tfrac{5\eta^3 x^3}{936} + \tfrac{5\eta^4 x^3}{1872} - \tfrac{\eta^5 x^3}{1872} + \tfrac{5\eta^2 x^4}{624} + \tfrac{5\eta^3 x^4}{1872} - \tfrac{5\eta^4 x^4}{3744} + \tfrac{\eta^5 x^4}{3744} - \tfrac{\eta x^5}{156} - \tfrac{\eta^2 x^5}{624} - \tfrac{\eta^3 x^5}{1872} + \tfrac{\eta^4 x^5}{3744} - \tfrac{\eta^5 x^5}{18720} + \frac{\begin{cases} 13x^5 + 105x^3\eta^2 - 15x^4\eta + 50x\eta^4 - 25x^2\eta^3 & x \le \eta \\ 50\eta x^4 + 105x^2\eta^3 - 25x^3\eta^2 + 13\eta^5 - 15x\eta^4 & \eta < x \end{cases}}{1560},$$

$$Q_\xi(t) = \xi t + \frac{\begin{cases} 3t^2\xi - t^3 & t \le \xi \\ 3t\xi^2 - \xi^3 & \xi < t \end{cases}}{6}.$$

We define the reproducing kernel space $\mathcal{W}^{(3,2)}(D)$ such that $\frac{\partial^3 y(x,t)}{\partial x^2 \partial t^1}$ is completely continuous on $D \equiv [0,1] \times [0,1]$ and $\frac{\partial^5 y(x,t)}{\partial x^3 \partial t^2} \in L^2(D)$ and equipped with conditions $y(0,t) = y(1,t) = y(x,0) = 0$. The inner product and norm are as follows,

$$\begin{aligned} &< y_1(x,t), y_2(x,t) >_{\mathcal{W}^{(3,2)}(D)} = \\ &\qquad \sum_{i=0}^{2} \int_0^1 \left[\frac{\partial^2}{\partial t^2} \frac{\partial^i}{\partial x^i} y_1(0,t) \frac{\partial^2}{\partial t^2} \frac{\partial^i}{\partial x^i} y_2(0,t) \right] dt \\ &\qquad + \sum_{j=0}^{1} \left\langle \frac{\partial^j}{\partial t^j} y_1(x,0), \frac{\partial^j}{\partial t^j} y_2(x,0) \right\rangle_{\mathcal{W}^3[0,1]} \\ &\qquad + \int_0^1 \int_0^1 \frac{\partial^3}{\partial x^3} \frac{\partial^2}{\partial t^2} y_1(x,t) \frac{\partial^3}{\partial x^3} \frac{\partial^2}{\partial t^2} y_2(x,t) dx dt, \end{aligned}$$

$$\|y(x,t)\|_{\mathcal{W}^{(3,2)}(D)} = \sqrt{< y, y >_{\mathcal{W}^{(3,2)}(D)}}.$$

and reproducing kernels functions for space $\mathcal{W}^{(3,2)}(D)$ is $K_{\eta,\xi}(x,t) = R_\eta(x)Q_\xi(t)$, see [18].

Definition 2.1. *The Caputo and Riemann-Liouville fractional differential operator* $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ *and* $D_x^\mu u(x,t)$ *are defined as,*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, \qquad \alpha = m \in \mathbb{N}, \end{cases}$$

$$D_x^\mu u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \dfrac{\partial^m}{\partial x^m} \displaystyle\int_0^x (x-\theta)^{m-\mu-1} u(\theta,t) d\theta, \\ m-1 < \mu < m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m}, \qquad \mu = m \in \mathbb{N}. \end{cases}$$

We can see the properties of operators $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ *and* $D_x^\mu u(x,t)$ *in [1–4].*

3 Construction of the numerical method

Suppose $L(y(x,t)) \equiv \frac{\partial^{\mu_1} y(x,t)}{\partial t^{\mu_1}} + p(x,t) D_x^{\mu_2} y(x,t) - q(x,t)$
$D_x^{\mu_3} y(x,t)$ is a bounded linear operator and $L : \mathcal{W}^{(3,2)}(D) \longrightarrow$
$\mathcal{W}^{(1,1)}(D)$, and $k_{\eta,\xi}(x,t) = r_\eta(x) q_\xi(t)$ is reproducing kernel function for $\mathcal{W}^{(1,1)}(D)$. We choose a dense set $\{(x_i,t_j)\}_{i,j=1}^\infty$ in D and define, $\phi_{i,j}(x,t) = k_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$ and $\varphi_{i,j}(x,t) =$
$L^*\phi_{ij}(x,t)$, where L^* is adjoint operator of L and L^{-1} exists.

Theorem 3.1. $\Psi_{ij}(x,t) = K_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$ *are complete function system in* $\mathcal{W}^{(3,2)}(D)$*.*

Proof 3.1. *See [19–21].*

Now, using the following theorem, we introduce the IRKM.

Theorem 3.2. *The following function*

$$y(x,t) = \sum_{i=1}^\infty \sum_{j=1}^\infty c_{ij} \Psi_{ij}(x,t), \tag{2}$$

is exact solution of Eq. (1) where $\{(x_i,t_j)\}_{i,j=1}^\infty$ *are dense points on* D*.*

Proof 3.2. *See [19, 22].*

Consider approximate solution of Eq. (1) as,

$$y_n(x,t) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}\Psi_{ij}(x,t), \tag{3}$$

where n is number of collocation points to apply the IRKM on D and $i = 1, 2, \ldots, n_1$, $j = 1, 2, \ldots, n_2$ and $n = n_1 \times n_2$. For determining the unknown coefficients c_{ij}, we define residual function $\mathcal{R}$ as,

$$\mathcal{R}_n(x,t) = L(y_n(x,t)) - f(x,t), \tag{4}$$

and we obtain c_{ij} such that $\langle \mathcal{R}_n(x,t), \Psi_{ij}(x,t)\rangle_{\mathcal{W}^{(3,2)}(D)} = 0$. Using Eq. (3) and Eq. (4) we have,

$$\begin{aligned} &\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij} L\Psi_{ij}(x,t)|_{(x,t)=(x_\iota,t_\ell)} = f(x_\iota,t_\ell), \\ &\iota = 1,2,\ldots,n_1 \quad , \quad \ell = 1,2,\ldots,n_2 \quad , \quad i = 1,2,\ldots,n_1 \quad , \\ &j = 1,2,\ldots,n_2 \quad , \quad \Psi_{ij}(x,t) = K_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)} \quad , \end{aligned} \tag{5}$$

where $L\Psi_{ij}(x,t)|_{(x,t)=(x_\iota,t_\ell)}$ is matrix $n \times n$. Eq (5) is a system of algebraic equations to obtain c_{ij}.

3.1 Convergence analysis

Theorem 3.3. *Let $\mathcal{B} = \{y_n(x,t)|\ \|y_n(x,t)\|_{\mathcal{W}^{(3,2)}(D)} \leq \rho\}$ where is compact set in $C(D)$ and ρ is a constant, then, $y_n(x,t)$ and its derivatives $\partial_x\partial_t y_n(x,t)$ and $\partial_x^2\partial_t y_n(x,t)$ are uniformly convergent to $y(x,t)$, $\partial_x\partial_t y(x,t)$ and $\partial_x^2\partial_t y(x,t)$ respectively.*

Proof 3.3. *We need to show that $\|y_n(x,t) - y(x,t)\|_{\mathcal{W}^{(3,2)}} \longrightarrow 0$ when $n \longrightarrow \infty$. From Eqs. (3),(4),(5) we have,*

$$\begin{aligned} &\mathcal{L}(y_n(x_i,t_j)) = f(x_i,t_j), \\ &i = 1,2,\ldots,n_1 \quad , \quad j = 1,2,\ldots,n_2 \quad , \quad n_1 \times n_2 = n. \end{aligned}$$

Since $\mathcal{B}$ is compact and $\{y_n(x,t)\}_{n=1}^{\infty} \in \mathcal{B}$, it is clear that $y(x,t) \in \mathcal{B}$ and convergent subsequence $\{y_{n_l}(x,t)\}_{l=1}^{\infty}$ exists such that converge to an element of $\mathcal{B}$. Suppose $y_{n_l}(x,t) \longrightarrow y(x,t)$ when $l \to \infty, n \to \infty$ and we have,

$$\begin{aligned} &\mathcal{L}(y_{n_l}(x_i,t_j)) = f(x_i,t_j), \\ &i = 1,2,\ldots,n_1 \quad , \quad j = 1,2,\ldots,n_2 \quad , \quad n_1 \times n_2 = n. \end{aligned} \tag{6}$$

Table 1
Max absolute error for Example 4.1 with different value of n

$\mathcal{W}^{(3,2)}$	PM	PM	[17]	[17]	[26]
	n=36	$n=144$	$n=36$	$n=144$	$n=400$
E_n	3.0×10^{-7}	3.0×10^{-9}	2.1×10^{-4}	4.1×10^{-5}	2.0×10^{-2}
Time (Sec)	2290.67	9845.89	–	–	–

Suppose $y_{n_l}(x,t) \longrightarrow y(x,t)$. Since $\mathcal{L}$ is a bounded linear operator, therefore we have,

$\mathcal{L}(y_{n_l}(x,t)) \longrightarrow \mathcal{L}(y(x,t))$,

and from the Eq. (6) $\mathcal{R}_n(x,t) \longrightarrow 0$ and $\|y_n(x,t)-y(x,t)\|_{\mathcal{W}^{(3,2)}} \longrightarrow 0$ when $n \longrightarrow \infty$. Now we prove that $y_n(x,t)$ uniformly convergent to $y(x,t)$ when, $n\to\infty$, from the reproducing properties we have,

$$\begin{aligned}|y_n(x,t)-y(x,t)| =& |<y_n(\eta,\xi)-y(\eta,\xi),K_{x,t}(\eta,\xi)>_{\mathcal{W}^{(3,2)}}|\\ \le& \|y_n(\eta,\xi)-y(\eta,\xi)\|_{\mathcal{W}^{(3,2)}}\|K_{x,t}(\eta,\xi)\|_{\mathcal{W}^{(3,2)}}\\ \le& \mathring{C}\|y_n(\eta,\xi)-y(\eta,\xi)\|_{\mathcal{W}^{(3,2)}}\end{aligned}$$

therefore $y_n(x,t) \longrightarrow y(x,t)$ when, $n\to\infty$. Prove the Convergence of $\partial_x\partial_t y_n(x,t)$ and $\partial_x^2\partial_t y_n(x,t)$ are similar to $y_n(x,t)$.

3.2 Stability

Theorem 3.4. *If the Eq. (1) has solution $y(x,t)$, then the present method on the solution $y(x,t)$ from $y_n(x,t)$ is stable in the reproducing kernel space $\mathcal{W}_2^{(3,2)}(D)$.*

Proof 3.4. *See [34].*

3.3 Error analysis

Theorem 3.5. *Suppose $y_n(x,t)$ is the approximate solution of the Eq. (1) in space $\mathcal{W}^{(3,2)}(D)$ and $y(x,t)$ is the exact solution. $(x,t)\in D$ and $\|y(x,t)-y_n(x,t)\|_\infty = \max_{(x,t)\in D}|y(x,t)-y_n(x,t)|$, $\|\partial_x\partial_t y(x,t)-\partial_x\partial_t y_n(x,t)\|_\infty = \max_{(x,t)\in D}$
$|\partial_x\partial_t y(x,t)-\partial_x\partial_t y_n(x,t)|$ and C_1,C_2,C_3,C_4,C_5,C_6 are positive constants, $h_x = max_{1\le i\le n_1}|x_{i+1}-x_i|$, and $h_t =$*

Table 2
Max absolute error E_n and error ratio for Example 4.1

PM						
$n=12$	$n=24$	$\frac{E_{24}}{E_{12}}$	$n=48$	$\frac{E_{48}}{E_{24}}$	$n=96$	$\frac{E_{96}}{E_{48}}$
2×10^{-5}	1.4×10^{-6}	0.07	1.10×10^{-7}	0.08	1.1×10^{-8}	0.1
Time (Sec)						
804.11	1509.11	–	3040.69	–	6241.66	–

Table 3
Max absolute error E'_n and error ratio for Example 4.1

PM						
$n=12$	$n=24$	$\frac{E'_{24}}{E'_{12}}$	$n=48$	$\frac{E'_{48}}{E'_{24}}$	$n=96$	$\frac{E'_{96}}{E'_{48}}$
1.7×10^{-4}	1.1×10^{-5}	0.06	9.00×10^{-7}	0.08	$9.\times10^{-8}$	0.1

Table 4
Max absolute error E''_n and error ratio for Example 4.1

PM						
$n=12$	$n=24$	$\frac{E''_{24}}{E''_{12}}$	$n=48$	$\frac{E''_{48}}{E''_{24}}$	$n=96$	$\frac{E''_{96}}{E''_{48}}$
3.6×10^{-4}	4×10^{-5}	0.11	7×10^{-6}	0.17	1.2×10^{-6}	0.17

Table 5
Stability of the present method for Example 4.1 with $n=12$

PM			
	${}^{\varepsilon}E_n$	${}^{\varepsilon}E'_n$	${}^{\varepsilon}E''_n$
$\epsilon=10^{-4}$	1.75×10^{-4}	6.5×10^{-4}	2.8×10^{-3}
	$\lvert E_n-{}^{\varepsilon}E_n\rvert$	$\lvert E'_n-{}^{\varepsilon}E'_n\rvert$	$\lvert E''_n-{}^{\varepsilon}E''_n\rvert$
	1.5×10^{-4}	4.8×10^{-4}	2.4×10^{-3}

Table 6
Max absolute error E_n and error ratio for Example 4.2

PM						
$n=12$	$n=24$	$\frac{E_{24}}{E_{12}}$	$n=48$	$\frac{E_{48}}{E_{24}}$	$n=96$	$\frac{E_{96}}{E_{48}}$
4.0×10^{-5}	2.4×10^{-6}	0.06	8.0×10^{-8}	0.03	3.6×10^{-9}	0.04
Time (Sec)						
1519.25	2819.56	–	5002.38	–	9563.05	–

Table 7
Max absolute error E'_n and error ratio for Example 4.2

PM						
$n = 12$	$n = 24$	$\frac{E'_{24}}{E'_{12}}$	$n = 48$	$\frac{E'_{48}}{E'_{24}}$	$n = 96$	$\frac{E'_{96}}{E'_{48}}$
4.4×10^{-4}	4.4×10^{-5}	0.1	2.4×10^{-6}	0.05	1.6×10^{-7}	0.06

Table 8
Max absolute error E''_n and error ratio for Example 4.2

PM						
$n = 12$	$n = 24$	$\frac{E''_{24}}{E''_{12}}$	$n = 48$	$\frac{E''_{48}}{E''_{24}}$	$n = 96$	$\frac{E''_{96}}{E''_{48}}$
2.4×10^{-3}	4.4×10^{-4}	0.18	4.5×10^{-5}	0.1	6.5×10^{-6}	0.14

Table 9
Stability of the present method for Example 4.2 with $n = 12$

PM			
	${}^{\varepsilon}E_n$	${}^{\varepsilon}E'_n$	${}^{\varepsilon}E''_n$
$\epsilon = 10^{-6}$	4.3×10^{-5}	4.45×10^{-4}	2.45×10^{-3}
	$\lvert E_n - {}^{\varepsilon}E_n \rvert$	$\lvert E'_n - {}^{\varepsilon}E'_n \rvert$	$\lvert E''_n - {}^{\varepsilon}E''_n \rvert$
	3×10^{-6}	5×10^{-6}	5×10^{-5}

$max_{1\le j\le n_2}|t_{j+1}-t_j|$. $n=n_1\times n_2$ *where is number of collocation points in region* D. *If* $\partial_x^3\partial_t y(x,t)$, $\partial_x^2\partial_t^2 y(x,t)\in C([0,1]\times[0,1])$ *and* $\|\partial_x^3\partial_t y_n(x,t)\|_\infty$, $\|\partial_x^2\partial_t^2 y_n(x,t)\|_\infty$ *are bonded then,*

$$\begin{aligned}
&\|y(x,t)-y_n(x,t)\|_\infty\le C_2h_x^2h_t+C_1h_x^3,\\
&\|\partial_x\partial_t y(x,t)-\partial_x\partial_t y_n(x,t)\|_\infty\le C_4h_xh_t+C_3h_x^2,\\
&\|\partial_x^2\partial_t y(x,t)-\partial_x^2\partial_t y_n(x,t)\|_\infty\le C_5h_x+C_6h_t.
\end{aligned}$$

Proof 3.5. *From [16–19, 24], in each* $[x_i,x_{i+1}]\times[t_j,t_{j+1}]\subset D$ *we have*

$$\begin{aligned}
&\partial_x^2\partial_t y(x,t)-\partial_x^2\partial_t y_n(x,t)=\partial_x^2\partial_t y(x,t)-\partial_x^2\partial_t y(x_i,t_j)+\\
&\partial_x^2\partial_t y_n(x_i,t_j)-\partial_x^2\partial_t y_n(x,t)+\partial_x^2\partial_t y(x_i,t_j)-\partial_x^2\partial_t y_n(x_i,t_j).
\end{aligned}\tag{7}$$

We write two terms of Taylor series expansion of $\partial_x^2\partial_t y(x,t)$ *at the point* (x_i,t_j) *as,*

$$\begin{aligned}
\partial_x^2\partial_t y(x,t)=\partial_x^2\partial_t y(x_i,t_j)+\Big[&(x-x_i)\partial_x^3\partial_t y(x_i,t_j)\\
&+(t-t_j)\partial_x^2\partial_t^2 y(x_i,t_j)\Big],
\end{aligned}$$

since $\partial_x^3\partial_t y(x,t)$, $\partial_x^2\partial_t^2 y(x,t)\in C([0,1]\times[0,1])$ *constants* $\tilde{C}_1,\tilde{C}_2$ *exist such that* $\forall(x,t)\in D$, $|\frac{\partial^4 y(x,t)}{\partial x^2\partial t^2}|\le\tilde{C}_2$ *and* $|\frac{\partial^4 y(x,t)}{\partial x^3\partial t}|$
$\le\tilde{C}_1$,

$$\|\partial_x^2\partial_t y(x,t)-\partial_x^2\partial_t y(x_i,t_j)\|_\infty\le\tilde{C}_1h_x+\tilde{C}_2h_t.\tag{8}$$

Moreover, we can write

$$\begin{aligned}
\partial_x^2\partial_t y_n(x_i,t_j)-\partial_x^2\partial_t y_n(x,t)&=-\int_{x_i}^{x}\partial_s^3\partial_t y_n(s,t_j)ds\\
&\quad-\int_{t_j}^{t}\partial_x^2\partial_w^2 y_n(x,w)dw,\\
|\partial_x^2\partial_t y_n(x_i,t_j)-\partial_x^2\partial_t y_n(x,t)|&\le\int_{x_i}^{x}|\partial_s^3\partial_t y_n(s,t_j)|ds\\
&\quad+\int_{t_i}^{t}|\partial_x^2\partial_w^2 y_n(x,w)|dw,
\end{aligned}$$

since $\|\partial_x^3\partial_t y_n(x,t)\|_\infty$, $\|\partial_x^2\partial_t^2 y_n(x,t)\|_\infty$ *are bonded, we have*

$$\|\partial_x^2\partial_t y_n(x_i,t_j)-\partial_x^2\partial_t y_n(x,t)\|_\infty\le\hat{C}_1h_x+\hat{C}_2h_t.\tag{9}$$

For any $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$, by using Theorem 3.3 for the approximate solution in $\mathcal{W}^{(3,2)}(D)$ exists n sufficiently large such that,

$$\begin{aligned}
&|\partial_x^2\partial_t y(x_i,t_j) - \partial_x^2\partial_t y_n(x_i,t_j)| \leq \epsilon_1,\\
&|\partial_x\partial_t y(x_i,t) - \partial_x\partial_t y_n(x_i,t)| \leq \epsilon_2,\\
&|y(x_i,t) - y_n(x_i,t)| \leq \epsilon_3,
\end{aligned} \tag{10}$$

by combining above equations we have,

$$\|\partial_x^2\partial_t y(x,t) - \partial_x^2\partial_t y_n(x,t)\|_\infty \leq C_5 h_x + C_6 h_t.$$

We know have,

$$\begin{aligned}
\partial_x\partial_t y(x,t) - \partial_x\partial_t y_n(x,t) = \partial_x\partial_t y(x_i,t) - \partial_x\partial_t y_n(x_i,t) +&\\
\int_{x_i}^{x} (\partial_\xi^2\partial_t y(\xi,t) - \partial_\xi^2\partial_t y_n(\xi,t))d\xi&,\\
y(x,t) - y_n(x,t) = y(x_i,t) - y_n(x_i,t) +&\\
\int_{x_i}^{x} (\partial_\xi\partial_t y(\xi,t) - \partial_\xi\partial_t y_n(\xi,t))d\xi&,
\end{aligned} \tag{11}$$

by combining Eqs. (10) and (11) we have,

$$\begin{aligned}
&\|\partial_x\partial_t y(x,t) - \partial_x\partial_t y_n(x,t)\|_\infty \leq C_4 h_x h_t + C_3 h_x^2,\\
&\|y(x,t) - y_n(x,t)\|_\infty \leq C_2 h_x^2 h_t + C_1 h_x^3.
\end{aligned}$$

Remark 3.1. *In Theorem 3.5, we proved the least degree of convergence. If there are more smooth conditions, the maximum error bound can be reduced, and the convergence order can be increased. Suppose $\partial_x^4\partial_t y(x,t)$, $\partial_x^3\partial_t^2 y(x,t) \in C([0,1]\times[0,1])$ and $\|\partial_x^4\partial_t y_n(x,t)\|_\infty$, $\|\partial_x^3\partial_t^2 y_n(x,t)\|_\infty$ are bonded and in each $[x_i,x_{i+1}]\times[t_j,t_{j+1}] \subset D$, then*

$$\begin{aligned}
&\partial_x^3\partial_t y(x,t) - \partial_x^3\partial_t y_n(x,t) = \partial_x^3\partial_t y(x,t) - \partial_x^3\partial_t y(x_i,t_j) +\\
&\partial_x^3\partial_t y_n(x_i,t_j) - \partial_x^3\partial_t y_n(x,t) + \partial_x^3\partial_t y(x_i,t_j) - \partial_x^3\partial_t y_n(x_i,t_j).
\end{aligned} \tag{12}$$

We write two terms of Taylor series expansion of $\partial_x^3\partial_t y(x,t)$ at the point (x_i,t_j) as,

$$\begin{aligned}
\partial_x^3\partial_t y(x,t) = \partial_x^3\partial_t y(x_i,t_j) + \Big[&(x-x_i)\partial_x^4\partial_t y(x_i,t_j)\\
&+ (t-t_j)\partial_x^3\partial_t^2 y(x_i,t_j)\Big],
\end{aligned}$$

since $\partial_x^4\partial_t y(x,t),\ \partial_x^3\partial_t^2 y(x,t) \in C([0,1]\times[0,1])$ *constants* $\tilde{M}_1, \tilde{M}_2$ *exist such that* $\forall (x,t) \in D$, $|\frac{\partial^5 y(x,t)}{\partial x^3 \partial t^2}| \leq \tilde{M}_2$ *and* $|\frac{\partial^5 y(x,t)}{\partial x^4 \partial t}| \leq \tilde{M}_1$,

$$\|\partial_x^3\partial_t y(x,t) - \partial_x^3\partial_t y(x_i,t_j)\|_\infty \leq \tilde{M}_1 h_x + \tilde{M}_2 h_t. \tag{13}$$

Moreover, we can write

$$\begin{aligned}\partial_x^3\partial_t y_n(x_i,t_j) - \partial_x^3\partial_t y_n(x,t) = &-\int_{x_i}^{x} \partial_s^4\partial_t y_n(s,t_j)ds \\ &-\int_{t_j}^{t} \partial_x^3\partial_w^2 y_n(x,w)dw,\end{aligned}$$

$$\begin{aligned}|\partial_x^3\partial_t y_n(x_i,t_j) - \partial_x^3\partial_t y_n(x,t)| \leq &\int_{x_i}^{x} |\partial_s^4\partial_t y_n(s,t_j)|ds \\ &+\int_{t_i}^{t} |\partial_x^3\partial_w^2 y_n(x,w)|dw,\end{aligned}$$

since $\|\partial_x^4\partial_t y_n(x,t)\|_\infty$, $\|\partial_x^3\partial_t^2 y_n(x,t)\|_\infty$ *are bonded, we have,*

$$\|\partial_x^3\partial_t y_n(x_i,t_j) - \partial_x^3\partial_t y_n(x,t)\|_\infty \leq \hat{M}_1 h_x + \hat{M}_2 h_t. \tag{14}$$

If we assume that $\partial_x^m\partial_t y_n(x,t)$ *uniformly convergent to* $\partial_x^m\partial_t y(x,t)$ *where* $m = 0,1,2,3$ *and for any* $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$, *exists* n *sufficiently large such that,*

$$\begin{aligned}&|\partial_x^3\partial_t y(x_i,t_j) - \partial_x^3\partial_t y_n(x_i,t_j)| \leq \epsilon_1, \\ &|\partial_x^2\partial_t y(x_i,t) - \partial_x^2\partial_t y_n(x_i,t)| \leq \epsilon_2, \\ &|\partial_x\partial_t y(x_i,t) - \partial_x\partial_t y_n(x_i,t)| \leq \epsilon_3, \\ &|y(x_i,t) - y_n(x_i,t)| \leq \epsilon_4,\end{aligned} \tag{15}$$

and therefore we have,

$$\|\partial_x^3\partial_t y(x,t) - \partial_x^3\partial_t y_n(x,t)\|_\infty \leq M_1 h_x + M_2 h_t.$$

We know have,

$$\partial_x^2\partial_t y(x,t) - \partial_x^2\partial_t y_n(x,t) = \partial_x^2\partial_t y(x_i,t) - \partial_x^2\partial_t y_n(x_i,t) + \int_{x_i}^{x} (\partial_\xi^3\partial_t y(\xi,t) - \partial_\xi^3\partial_t y_n(\xi,t))d\xi,$$

$$\partial_x\partial_t y(x,t) - \partial_x\partial_t y_n(x,t) = \partial_x\partial_t y(x_i,t) - \partial_x\partial_t y_n(x_i,t) + \int_{x_i}^{x} (\partial_\xi^2\partial_t y(\xi,t) - \partial_\xi^2\partial_t y_n(\xi,t))d\xi,$$

$$y(x,t)-y_n(x,t)=y(x_i,t)-y_n(x_i,t)+$$
$$\int_{x_i}^{x}(\partial_\xi\partial_t y(\xi,t)-\partial_\xi\partial_t y_n(\xi,t))d\xi,$$

by combining above equations, similar to Theorem 3.5 we have,

$$\|\partial_x^2\partial_t y(x,t)-\partial_x^2\partial_t y_n(x,t)\|_\infty \le M_3 h_x h_t + M_4 h_x^2,$$
$$\|\partial_x\partial_t y(x,t)-\partial_x\partial_t y_n(x,t)\|_\infty \le M_5 h_x^2 h_t + M_6 h_x^3,$$
$$\|y(x,t)-y_n(x,t)\|_\infty \le M_7 h_x^3 h_t + M_8 h_x^4.$$

4 Numerical implementations

In this section, presented method is illustrated by solving some numerical examples. The following definitions and notations are setup for all examples. $E_n = Max_{(x,t)\in D}\big|y_n(x,t)-y(x,t)\big|$, $E'_n = Max_{(x,t)\in D}\big|\partial_x\partial_t y_n(x,t) - \partial_x\partial_t y(x,t)\big|$ and $E''_n = Max_{(x,t)\in D}\big|\partial_x^2\partial_t y_n(x,t)-\partial_x^2\partial_t y(x,t)\big|$ are maximum absolute errors where n is number of collocation points. All present numerical results are obtained by using Mathematica 12.1. In Table 1 maximum absolute errors are compared with methods [17] and [26]. Error ratios for Examples 4.1 and 4.2 are provided in Tables 2, 3, 4 and Tables 6, 7, 8, respectively. Stability analysis of the present method are shown in Tables 5 and 9 for Examples 4.1 and 4.2, respectively. Figures 1, 2, 3, 4, 5 and 6 are shown absolute errors for Examples 4.1 and 4.2 and their derivatives.

Algorithm

1. Choose n collocation points in the region $[0,1]\times[0,1]$ where $n=n_1\times n_2$;
2. Set $\Psi_{ij}(x,t)=K_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$ where $i=1,2,\ldots,n_1$ and $j=1,2,\ldots,n_2$;
3. Set $G=[L\Psi_{ij}(x,t)|_{(x,t)=(x_\iota,t_\ell)}]_{i=1,2,\ldots,n_1,j=1,2,\ldots,n_2}^{\iota=1,2,\ldots,n_1,\ell=1,2,\ldots,n_2}$, $n\times n$ matrix;
4. Compute $B=[f(x_\iota,t_\ell)]^T_{\iota=1,2,\ldots,n_1,\ell=1,2,\ldots,n_2}$;
5. solve system of the algebraic equations $GC=B$, where $C=[c_{ij}]^T_{i=1,2,\ldots,n_1,j=1,2,\ldots,n_2}$;
6. Set $y_n(x,t)=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}c_{ij}\Psi_{ij}(x,t)$;

Example 4.1. *[17, 26]*

$$\frac{\partial^{0.6}y(x,t)}{\partial t^{0.6}} = x^{0.6}t^{1.4}D_x^{0.6}y(x,t)+\frac{5\Gamma(1.4)x^{1.6}t^{1.4}}{\Gamma(2.4)}\quad D_x^{1.6}y(x,t) +f(x,t),$$

$$0<t\le 1,\qquad 0<x<1,$$
$$y(x,0)=x^2,\quad y(0,t)=0,\quad y(1,t)=4t^2+1,$$

where $y(x,t)=x^2(4t^2+1)$ is exact solution.

Example 4.2.

$$\frac{\partial^{0.4}y(x,t)}{\partial t^{0.4}} = \frac{0.3\Gamma(0.7)x^{1.5}t^{2.3}}{\Gamma(5.6)} D_x^{0.8} y(x,t) \quad + \frac{2\Gamma(0.2)x^{0.4}t^{1.2}}{\Gamma(3.4)} D_x^{1.3} y(x,t) + f(x,t),$$

$$0 < t \leq 1, \qquad 0 < x < 1,$$
$$y(x,0) = xe^{x-1}, \quad y(0,t) = 0, \quad y(1,t) = t^2 + 1,$$

where $y(x,t) = xe^{x-1}(t^2+1)$ *is exact solution.*

5 Conclusion

In this paper, we solved the space-time fractional advection-dispersion equation with variable coefficients using IRKM, such that makes us unnecessary to use the Gram-Schmidt orthogonalization process. Moreover, we provided the convergence and error analysis. According to the theoretical results of the error analysis, the error ratio for the approximate solution $y_n(x,t)$ and its derivative $\partial_x\partial_t y_n(x,t)$ and $\partial_x^2\partial_t y_n(x,t)$ must be at most 0.125, 0.25 and 0.5 respectively. However, according to Remark 3.1 the error ratio for approximate solution $y_n(x,t)$ and its derivative $\partial_x\partial_t y_n(x,t)$ and $\partial_x^2\partial_t y_n(x,t)$ must be at most 0.0625, 0.125 and 0.25 respectively. However, numerical examples showed that the results of the present IRKM could provide better approximations than RKM what was introduced in [18]. Finally, it is important to note that the proposed method can be implemented on problems with fractional derivatives, despite the boundary conditions at the beginning and end of their definition interval.

Conflict of interest The authors declare that they have no conflict of interest.

Acknowledgments
We thank the anonymous reviewers for helpful comments, which lead to definite improvement in the manuscript.

References

[1] K.S. Miller, B. Ross, An Introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, Inc, New York, 1993.

[2] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[3] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin Heidelberg, 2010.

[4] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integral and Derivatives, Theory and Applications, Gordon and Breach, Switzerland, 1993.

[5] M.M. Meerschaert, H.P. Scheffler, Semistable Lévy motion, Frac. Calc. Appl. Anal. 5 (2002) 27-54.

[6] M. Meerschaert, D. Benson, H.P. Scheffler, B. Baeumer, Stochastic solution of space–time fractional diffusion equations, Phys. Rev. E 65 (2002) 1103-1106.

[7] B. West, M. Bologna, P. Grigolini, Physics of Fractal Operators, Springer, New York, 2003.

[8] G. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, Oxford, 2005.

[9] B. Baeumer, M.M. Meerschaert, D.A. Benson, S.W. Wheatcraft, Subordinated advection–dispersion equation for contaminant transport, Water Resource Res. 37 (2001) 1543–1550.

[10] R. Schumer, D.A. Benson, M.M. Meerschaert, S.W. Wheatcraft, Eulerian derivation of the fractional advection–dispersion equation, J. Contaminant Hydrol. 48 (2001) 69-88.

[11] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Multiscaling fractional advection–dispersion equations and their solutions, Water Resource Res. 39 (2003) 1022-1032.

[12] P. Reimus, G. Pohll, T. Mihevc, J. Chapman, M. Haga, B. Lyles, S. Kosinski, R. Niswonger, P. Sanders, Testing and parameterizing a conceptual model for solute transport in a fractured granite using multiple tracers in a forced-gradient test, Water Resour. Res. 39 (2003) 1356-1370.

[13] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[14] G. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, Phys. Rep. 371 (2002) 461-580.

[15] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1-77.

[16] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection–dispersion equation, Water Resour. Res. 36 (6) (2000) 1403-1412.

[17] W. Jiang, Y. Lin, Approximate solution of the fractional advection–dispersion equation, Comput. Phys. Commun. 181 (2010) 557-561.

[18] M. Cui, Y. Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science, Hauppauge, New York, United States, 2009.

[19] Y. Wang, M. Du, F. Tan, Z. Li, T. Nie, Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions, Appl. Math. Comput. 219 (2013) 5918-5925.

[20] H. Sahihi, S. Abbasbandy, T. Allahviranloo, Reproducing kernel method for solving singularly perturbed differential-difference equations with boundary layer behavior in Hilbert space, J. Comput. Appl. Math. 328 (2018) 30-43.

[21] H. Sahihi, S. Abbasbandy, T. Allahviranloo, Computational method based on reproducing kernel for solving singularly perturbed differential -difference equations with a delay, Appl. Math. Comput. 361 (2019) 583-598.

[22] Y. Wang, L. Su, X. Cao, X. Li, Using reproducing kernel for solving a class of singularly perturbed problems, Comput. Math. Appl. 61 (2011) 421-430.

[23] K. Atkinson, W. Han, Theoretical Numerical Analysis A Functional Analysis Framework, Third Edition, Springer Science, 2009.

[24] M. Fardi, M. Ghasemi, Solving nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations in reproducing kernel Hilbert space, Wiley. Online. Library. 33 (2017) 174-198.

[25] M. Al-Smadi, O.A. Arqub, Solving Computational algorithm for solving fredholm time-fractional partial integro differential equations of dirichlet functions type with error estimates, Appl. Math. Comput. 342 (2019) 280-294.

[26] F. Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage, Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation, Appl. Math. Comput. 191 (2007) 12-20.

[27] F. Liu, P. Zhuang, K. Burrage, Numerical methods and analysis for a class of fractional advection–dispersion models, Comput. Math. Appl 64 (2012) 2990-3007.

[28] M.L. Kavvas, T. Tu, A. Ercan, J. Polsinelli, Fractional governing equations of transient groundwater flow in confined aquifers with multi-fractional dimensions in fractional time, Earth. Syst. Dynam. 8 (2017) 921-929.

[29] A. Akgül, D. Baleanu, M. Inc, F. Tchier, On the solutions of electro hydrodynamic flow with fractional differential equations by reproducing kernel method, Open. Physics. 14 (2016) 685-689.

[30] A. Akgül, E.K. Akgül, D. Baleanu, Reproducing kernel method for strongly nonlinear equation, American. Institute. of Physics. 1978 (2018).

[31] O.A. Arqub, B. Maayah, Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-Fractional Volterra integro-differential equations, Chaos. Solit. Frac. 126 (2019) 394-402.

[32] O.A. Arqub, N. Shawagfeh, Application of Reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-gordon types equations in porous media, J. Porous. Media. 22 (2019) 411-434.

[33] M. Dehghan, M. Abbaszadeh, Element free Galerkin approach based on the reproducing kernel particle method for solving 2D fractional Tricomi-type equation with Robin boundary condition, Comput. Math. Appl. 73 (2017) 1270-1285.

[34] T. Allahviranloo, H. Sahihi, Reproducing kernel method to solve parabolic partial differential equations with nonlocal conditions, Nume. Method. Parti. Diffe. Equ. 36 (2020) 1758-1772.

[35] O. Nikan, J.A. Tenreiro Machado, A. Golbabai, T. Nikazad, Numerical approach for modeling fractal mobile/immobile transport model in porous and fractured media, Inter. Commun. Heat. Mass. Trans. 111 (2020) 104443.

[36] O. Nikan, H. Jafari, A. Golbabai, Numerical analysis of the fractional evolution model for heat flow in materials with memory, Alexan. Engine. J. 59 (2020) 2627-2637.

[37] O. Nikan, J.A. Tenreiro Machado, Z. Avazzadeh, H. Jafari, Numerical evaluation of fractional Tricomi-type model arising from physical problems of gas dynamics, J. Advanc. Resear. 25 (2020) 205-216.

[38] O. Nikan, J.A. Tenreiro Machado, T. Nikazad, Numerical solution of the fractional Rayleigh–Stokes model arising in a heated generalized second-grade fluid, Engine. with. Comput. 1 January 2020.

[39] O. Nikan, A. Golbabai, J.A. Tenreiro Machado, T. Nikazad, Numerical approximation of the time fractional cable model arising in neuronal dynamics, Engine. with. Comput. 25 May 2020.

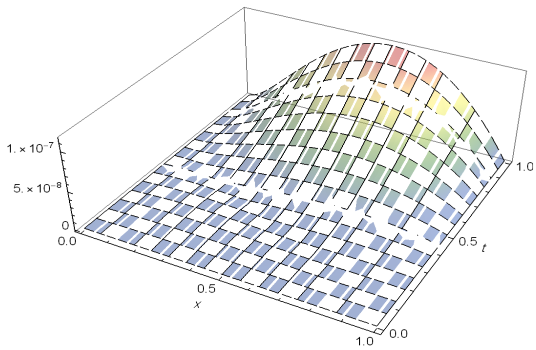

Fig. 1. Approximate solution for Example 4.1, with $n = 48$ ($|y_n(x,t) - y(x,t)|$).

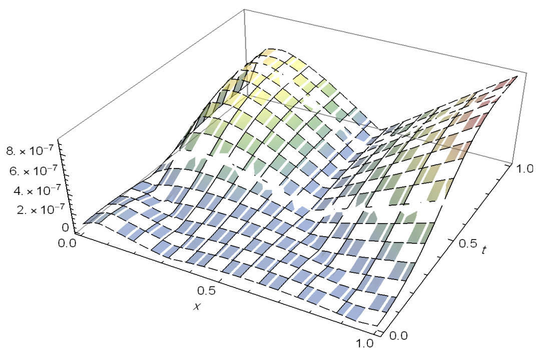

Fig. 2. Approximate solution for Example 4.1, with $n = 48$ ($|\partial_x \partial_t y_n(x,t) - \partial_x \partial_t y(x,t)|$).

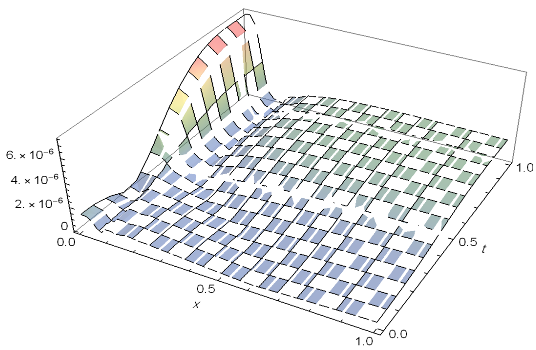

Fig. 3. Approximate solution for Example 4.1, with $n = 48$ ($|\partial_x^2 \partial_t y_n(x,t) - \partial_x^2 \partial_t y(x,t)|$).

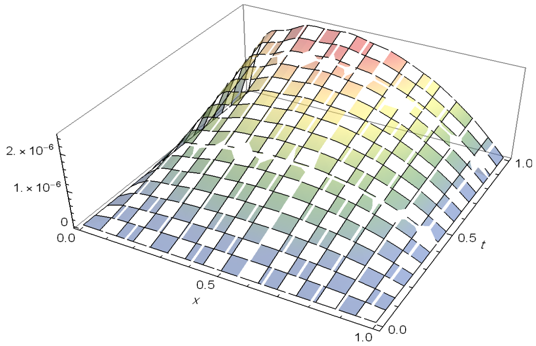

Fig. 4. Approximate solution for Example 4.2, with $n = 24$ ($|y_n(x,t) - y(x,t)|$).

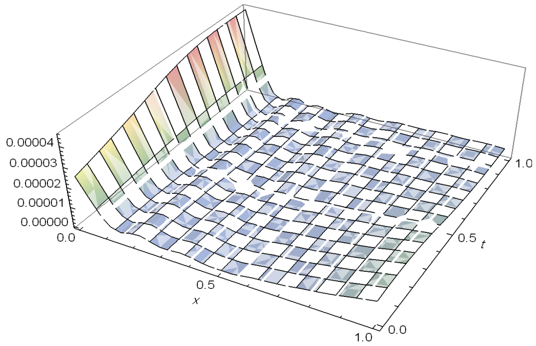

Fig. 5. Approximate solution for Example 4.2, with $n = 24$ ($|\partial_x \partial_t y_n(x,t) - \partial_x \partial_t y(x,t)|$).

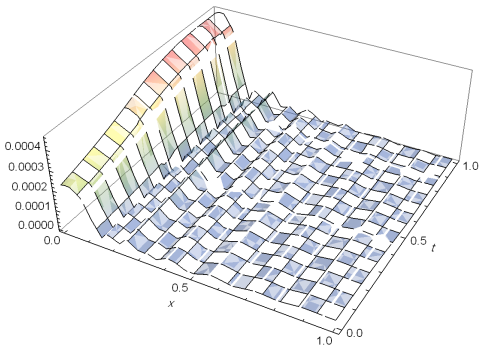

Fig. 6. Approximate solution for Example 4.2, with $n = 24$ ($|\partial_x^2 \partial_t y_n(x,t) - \partial_x^2 \partial_t y(x,t)|$).