Received ; | Revised ; | Accepted
DOI: xxx/xxxx

ARTICLE TYPE

On the existence, uniqueness, and new analytic approximation of the modified error function in two-phase Stefan problems

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Summary

The existence and uniqueness of the solution is proved for a nonlinear boundary value problem for ODE subject to an infinite condition[1], which describes the study of two-phase Stefan problems on the semi-infinite line $[0, \infty)$. This result considerably extends the analysis of a recent work[4]. A highly accurate analytic approximate solution of this problem is also provided via the Adomian decomposition method.

KEYWORDS:
Boundary value problem, existence and uniqueness theorem, approximate solution, Modified error function, Stefan problem, Adomian decomposition method, Adomian polynomials.

1 | INTRODUCTION

In[1], the authors introduced the concept of the *modified error function* for the model conduction heat transfer with phase change into a semi-infinite slab $[0, \infty)$, where the thermal conductivities and specific heats of both phases were assumed to be a linear function of the temperature. This function is a solution of the following nonlinear boundary value problem

$$\begin{cases} [(1+\delta y)y']' + 2x(1+\gamma y)y' = 0, \ 0 < x < \infty, \\ y(0) = 0, \ y(\infty) = 1, \end{cases} \tag{1}$$

where the system parameters $\delta, \ \gamma \in (-1, \infty)$.
The modified error function for the case when $\gamma = 0$ and $\delta > 0$ was studied by the authors in[2,3], where the existence and uniqueness theorems were established by considering the fixed-point theorem. Also, explicit approximations solutions were provided in[3].
Recently, the authors[4] proved the existence and uniqueness of the solution of Pr.(1) under the following severely restrictive condition on δ and γ (Theorem 2.1[4])

$$\frac{\max(1,1+\delta)^{\frac{3}{2}} \max(1,1+\gamma)^{\frac{1}{2}}}{\max(1,1+\delta)^{\frac{5}{2}} \min(1,1+\gamma)^{\frac{1}{2}}}(2 \mid \delta \mid + \frac{\mid \delta - \gamma \mid \max(1,1+\delta)}{\min(1,1+\delta)\min(1,1+\gamma)}) < 1, \tag{2}$$

which is also rather quite complicated.
In this note, a result for the existence and uniqueness of the solution to Pr.(1) is proved without any conditions on

the parameters γ and δ, which considerably extends the result in[4]. It is often very difficult, if not impossible, to find explicit solutions of such problems. However, the Adomian decomposition method is the most important tool for finding solutions to this problem. Hence an exceptionally accurate approximate analytic solution of Pr.(1) is also provided. A comparison of our solution can be made with the useful result of our Theorem 3 that provides the lower and upper bounds of y and guarantees the existence of the solution of this problem in $[0,1]$ as well.

2 | EXISTENCE AND UNIQUENESS THEOREM

2.1 | An equivalent problem

Writing the nonlinear second-order ODE of Pr.(1) in an equivalent form as

$$\frac{1}{\delta}\left[(1+\delta y)(1+\delta y)'\right]'+\frac{2}{\gamma}x(1+\gamma y)'(1+\gamma y)=0,\ 0<x<\infty, \tag{3}$$

or

$$\frac{1}{\delta}\left((1+\delta y)^2\right)''+\frac{2\delta}{\gamma}x\left[(1+\gamma y)^2\right]'=0,\ 0<x<\infty, \tag{4}$$

or

$$\left((\frac{1}{\delta}+y)^2\right)''+\frac{2\gamma}{\delta}x\left((\frac{1}{\gamma}+y)^2\right)'=0,\ 0<x<\infty. \tag{5}$$

By the change of variable

$$z=\frac{1}{\delta}+y. \tag{6}$$

Eq. (5) becomes

$$\left(z^2\right)''+\frac{2\gamma}{\delta}x\left[(z+\frac{1}{\gamma}-\frac{1}{\delta})^2\right]'=0,\ 0<x<\infty, \tag{7}$$

or

$$\left(z^2\right)''+\frac{2\gamma}{\delta}x\left[z^2+2(\frac{1}{\gamma}-\frac{1}{\delta})z\right]'=0,\ 0<x<\infty, \tag{8}$$

Substituting the transformation $z^2=u$ into Eq. (8), we obtain

$$u''+\frac{2\gamma}{\delta}x\left(1+(\frac{1}{\gamma}-\frac{1}{\delta})u^{-\frac{1}{2}}\right)u'=0,\ 0<x<\infty. \tag{9}$$

Thus

Lemma 1. Pr.(1) can be converted to the nonlinear boundary value problem

$$\begin{cases} \frac{u''}{u'}+\frac{2\gamma}{\delta}x\left(1+\beta u^{-\frac{1}{2}}\right) = 0,\ \beta=\frac{1}{\gamma}-\frac{1}{\delta},\ \gamma\neq 0,\delta\neq 0,\ 0<x<\infty, \\ u(0)=\frac{1}{\delta^2},\ u(\infty) = (1+\frac{1}{\delta})^2, \end{cases} \tag{10}$$

where $u=(\frac{1}{\delta}+y)^2$.

Remark 1. For $\gamma=\delta=0$, Pr.(1) becomes a simple linear second-order ODE. For $\delta>0,\gamma=0$, Pr.(1) can be converted into a special case of Pr.(10):

$$\begin{cases} u''+2xu^{-\frac{1}{2}}u' = 0,\ 0<x<\infty, \\ u(0)=1,\ u(\infty) = (1+\delta)^2, \end{cases} \tag{11}$$

where $u=(1+\delta y)^2$.

2.2 | Existence

Now we prove the double inequalities for the lower and upper bounds of the solution $y(x)$ for different values of δ and γ that guarantee the existence of the solution of Pr.(1).

Theorem 1. There is at least one solution $y(x)$ of Pr.(1) such that

1. For $0<\gamma<\delta$ or $-1<\gamma<0<\delta$ or $-1<\gamma<\delta<0$, we have
$$y_1(x) \le y(x) \le y_2(x), \tag{12}$$
where
$$y_1 = -\frac{1}{\delta} + \sqrt{\frac{1}{\delta^2} + \frac{\delta+2}{\delta}\,\mathrm{erf}\left(\sqrt{\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})}x\right)} \tag{13}$$
and
$$y_2 = -\frac{1}{\delta} + \sqrt{\frac{1}{\delta^2} + \frac{\delta+2}{\delta}\,\mathrm{erf}(x)}. \tag{14}$$

2. For $0<\delta<\gamma$ or $-1<\delta<0<\gamma$ or $-1<\delta<\gamma<0$, we have
$$y_2(x) \le y(x) \le y_1(x). \tag{15}$$

Proof. 1. Since $\frac{1}{\delta^2} \le u(x) \le (1+\frac{1}{\delta})^2$, that is $\frac{\delta+1}{\delta} \le \frac{1}{\sqrt{u(x)}} \le \delta$ and in view of $0<\gamma<\delta$ and $\beta = \frac{1}{\gamma}-\frac{1}{\delta}>0$, we have
$$2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})x \le \frac{2\gamma}{\delta}x\left(1+\beta u^{-\frac{1}{2}}(x)\right) \le 2x. \tag{16}$$
If $u'(x)\ge 0$. Then
$$2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})xu'(x) \le \frac{2\gamma}{\delta}x\left(1+\beta u^{-\frac{1}{2}}(x)\right)u'(x) \le 2xu'(x). \tag{17}$$
Similarly, if $u'(x)\le 0$. Then
$$2xu'(x) \le \frac{2\gamma}{\delta}x\left(1+\beta u^{-\frac{1}{2}}(x)\right)u'(x) \le 2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})xu'(x). \tag{18}$$
Let
$$G_1(x,u,u') = \begin{cases} 2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})xu', & \text{if } u'(x)\ge 0,\\ 2xu', & \text{if } u'(x)\le 0\end{cases} \tag{19}$$
and
$$G_2(x,u,u') = \begin{cases} 2xu', & \text{if } u'(x)\ge 0,\\ 2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})xu', & \text{if } u'(x)\le 0.\end{cases} \tag{20}$$
For comparison purposes, we have the following linear boundary value problems:
For $u'(x)\ge 0$,
$$\begin{cases} u_1''+2x\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})u_1' = 0,\ 0<x<\infty,\\ u_1(0)=\frac{1}{\delta^2},\ u_1(\infty) = (1+\frac{1}{\delta})^2\end{cases} \tag{21}$$
and
$$\begin{cases} u_2''+2xu_2' = 0,\ 0<x<\infty,\\ u_2(0)=\frac{1}{\delta^2},\ u_2(\infty) = (1+\frac{1}{\delta})^2.\end{cases} \tag{22}$$
For $u'(x)\le 0$,
$$\begin{cases} u_1''+2xu_1' = 0,\ 0<x<\infty,\\ u_1(0)=\frac{1}{\delta^2},\ u_1(\infty) = (1+\frac{1}{\delta})^2\end{cases} \tag{23}$$
and
$$\begin{cases} u_2''+2x\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})u_2' = 0,\ 0<x<\infty,\\ u_2(0)=\frac{1}{\delta^2},\ u_2(\infty) = (1+\frac{1}{\delta})^2.\end{cases} \tag{24}$$
Then the solutions of these BVPs can be immediately obtained as follows
$$u_1(x) = \frac{1}{\delta^2} + \frac{\delta+2}{\delta}\,\mathrm{erf}\left(\sqrt{\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})}x\right) \tag{25}$$

and

$$u_2(x) = \frac{1}{\delta^2} + \frac{\delta+2}{\delta}\,\mathrm{erf}(x). \tag{26}$$

It follows that $u_i(x),\ i=1,2$ satisfy the following conditions:

$$u_1'' + G_1(x, u_1, u_1') \geq 0, \tag{27}$$

$$u_2'' + G_2(x, u_2, u_2') \leq 0 \tag{28}$$

and $u_1(x) < u_2(x),\ x \in (0,\infty)$, where the ranges of $u_i,\ i=1,2$ are $[\frac{1}{\delta^2}, (1+\frac{1}{\delta})^2]$. Also, the following condition is satisfied

$$G_1(x, u_1-u_2, u_1'-u_2') \leq f(x, u_1, u_1') - f(x, u_2, u_2') \leq G_2(x, u_1-u_2, u_1'-u_2'), \tag{29}$$

where $f(x,u,u') = \frac{2\gamma}{\delta} x \left(1+\beta u^{-\frac{1}{2}}(x)\right) u'(x)$. Thus, we can conclude from Theorem 7.3 (see pp. 110-111[5]) that Pr.(10) has at least one solution $u(x)$ such that

$$u_1(x) \leq u(x) \leq u_2(x). \tag{30}$$

If we set $u(x) = \left(\frac{1}{\delta} + y(x)\right)^2$, then we obtain (12).

The proofs of all cases are similar. □

2.3 | Uniqueness

We are now ready to prove the uniqueness of the solution.

Theorem 2. The given boundary problem Pr.(1) has only one solution y that satisfies (12) or (15).

Proof. Let u and v be two solutions of Pr. (10). Thus

$$0 = (u-v)'' + f(x,u,u') - f(x,v,v'). \tag{31}$$

Since $f(x,u,u') - f(x,v,v') \leq G_2(x, u-v,\ u'-v')$. If we let $w(x) = u(x) - v(x)$, then for $w'(x) \geq 0$, we have

$$w''(x) + 2xw'(x) \geq 0 \tag{32}$$

and for $w'(x) \leq 0$, we have

$$w''(x) + 2\frac{\gamma}{\delta}(1+\beta\frac{\delta+1}{\delta})xw'(x) \geq 0. \tag{33}$$

Multiplying both sides of Eq. (32) by w, and integrating from 0 to ∞ and taking into account that $w(0) = w(\infty) = 0$, we obtain

$$\int_0^\infty w^2(x)dx + \int_0^\infty (w')^2(x)dx \leq 0. \tag{34}$$

This implies $w(x) = 0$. Similarly for Eq. (33), we obtain $w(x) = 0$. Thus the solution is unique. □

3 | NUMERICAL RESULTS - ADOMIAN DECOMPOSITION METHOD

Integrating the nonlinear second-order ODE of Pr.(10) from 0 to x, we obtain

$$u' = C_1(\gamma,\delta)\exp\left(-\frac{2\gamma}{\delta}\int_0^x \xi\left(1+\beta u^{-\frac{1}{2}}(\xi)\right)d\xi\right),\ 0 < x < \infty, \tag{35}$$

where $C_1(\gamma,\delta) = u'(0)$ is an unknown constant.

Hence

$$u = \frac{1}{\delta^2} + C_1(\gamma,\delta)\int_0^x \exp\left(-\frac{2\gamma}{\delta}\int_0^\eta \xi\left(1+\beta u^{-\frac{1}{2}}(\xi)\right)d\xi\right)d\eta,\ 0 \leq x < \infty. \tag{36}$$

The constant $C_1(\gamma,\delta)=u'(0)$ can be determined by using the second boundary condition $u(\infty)=(1+\frac{1}{\delta})^2$, from which we can readily obtain

$$C_1(\gamma,\delta)=u'(0)=\frac{1+\frac{2}{\delta}}{\int_0^\infty \exp\left(-\frac{2\gamma}{\delta}\int_0^x \xi\left(1+\beta u^{-\frac{1}{2}}(\xi)\right)d\xi\right)dx}. \tag{37}$$

We now propose to solve the second-order nonlinear boundary value problem Pr. (10) by the Adomian decomposition method (ADM)[6,7,8,9,10?,12].

Based on the classical Adomian decomposition method, we decompose the solution into the solution components $u_n(x)$

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{38}$$

and the composite nonlinearity in Eq. (36) is decomposed in terms of the Adomian polynomials as follows[?]

$$\widetilde{N}_2u(x)=\exp\left(-\frac{2\gamma}{\delta}\int\limits_0^x \xi\left(1+\beta u^{-\frac{1}{2}}(\xi)\right)\right)=\sum_{n=0}^{\infty}A_n(x)=N_0(N_1x). \tag{39}$$

Let $N_0u^0=e^{-u^0}=\sum_{n=0}^{\infty}A_n^0(u_0^0,u_1^0,...,u_n^0)$ and $N_1u^1=\frac{2\gamma}{\delta}\int_0^x\xi\left(1+\beta u^{-\frac{1}{2}}(\xi)\right)d\xi$. Thus the Adomian polynomials A_n^0 for the $N_0u^0=e^{-u^0}$ are given by

$$\begin{cases} A_0^0 = e^{-u_0^0},\\ A_1^0 = (-u_1^0)e^{-u_0^0},\\ A_2^0 = (-u_2^0+\frac{1}{2}(u_1^0)^2)e^{-u_0^0},\\ \quad ... \end{cases} \tag{40}$$

Now calculating the A_n for N_1x, we have

$$\begin{cases} A_0^1 = \frac{2\gamma}{\delta}\int_0^x\xi\left(1+\beta(u_0^1)^{-\frac{1}{2}}(\xi)\right)d\xi,\\ A_1^1 = \frac{2\gamma}{\delta}\int_0^z\xi\left(1+\beta(u_0^1)^{-\frac{3}{2}}(\xi)u_1^1(\xi)\right)d\xi,\\ A_2^1 = \frac{2\gamma}{\delta}\int_0^z\xi\left(1+\beta[\frac{3}{4}(u_0^1)^{-\frac{5}{2}}(\xi)(u_1^1)^2(\xi)-\frac{1}{2}u_2^1(\xi)(u_0^1)^{-\frac{1}{2}}(\xi)]\right)d\xi,\\ \quad ... \end{cases} \tag{41}$$

Upon substitution of these into Eq. (36), we establish the Adomian recursion scheme for the function $u(x)$ as follows

$$\begin{cases} u_0(x) = \frac{1}{\delta^2},\\ u_{n+1}(x) = C_1(\delta,\gamma)\int_0^x A_n(\xi)d\xi,\ n\geq 0. \end{cases} \tag{42}$$

The first components of the solution $u(x)$ are

$$\begin{cases} u_0(x) = \frac{1}{\delta^2},\\ u_1(x) = C_1\int_0^x A_0(\xi)d\xi = C_1(\delta,\gamma)\int_0^x\exp\left(-\frac{\gamma}{\delta}(1+\beta\delta)\xi^2\right)d\xi = C_1(\delta,\gamma)\int_0^x\exp(-\xi^2)d\xi,\\ \quad ... . \end{cases} \tag{43}$$

Hence

$$u(x)=u_0(x)+u_1(x)+.... \tag{44}$$

Since the Adomian decomposition method usually converges quite rapidly, we first consider only the first component of this solution, then

$$u(x) = \frac{1}{\delta^2} + C_1(\delta, \gamma) \int_0^x \exp(-\xi^2) d\xi. \tag{45}$$

Now imposing the boundary condition at infinity $u(\infty) = (1 + \frac{1}{\delta})^2$, we obtain

$$C_1(\delta, \gamma) = \frac{\delta + 2}{\delta} \frac{2}{\sqrt{\pi}}. \tag{46}$$

If so desired to further refine our approximation of this constant, we only need calculate additional higher-order terms in (42), but as we shall soon see in the sequel, our new approximate solution is already quite good and indeed exceptionally accurate. Therefore

$$u(x) = \frac{1}{\delta^2} + \frac{\delta + 2}{\delta} \operatorname{erf}(x). \tag{47}$$

Consequently, in view of $u(x) = (\frac{1}{\delta} + y(x))^2$ the approximate solution of Pr.(1) is given by

$$y = -\frac{1}{\delta} + \sqrt{\frac{1}{\delta^2} + \frac{\delta + 2}{\delta} \operatorname{erf}(x)}, \tag{48}$$

which is exactly the upper and lower bounds of the exact solution (see Theorem 3). Another observation is that this approximate solution is independent of gamma and can be regarded as a new approximate analytic solution of Pr.(1) in the semi-infinite interval. Figures 1 and 2 have been drawn to show the upper and lower bounds of the solution $y(x)$. The maximum difference between the upper and lower bounds of the exact solution $\| y_1 - y_2 \|_\infty$ is negligible and displayed in Figure 3. The residual Error $ER(x)$ is also considered with different values of δ and γ in Figure 4 for small and large values of x by substituting $y(x)$ in Pr.(1).

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