Convergence of approximate solutions for singular difference systems with maxima

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Abstract In this paper, by introducing a new singular fractional difference comparison theorem, the existence of maximal and minimal quasi-solutions are proved for the singular fractional difference system with maxima combined with the method of upper and lower solutions and the monotone iterative technique. Finally, we give an example to show the validity of the established results.

Keywords Singular fractional difference systems; Maxima; Extremal solutions; Iterative technique; Delay equations.

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1 Introduction

Since Rosenbrock [1] introduced the concept of singular systems in 1974, which have been widely used to describe some problems in practical fields, such as optimal control problems, constrained control problems, some population growth models and so on. Therefore, the theory of singular differential systems [2, 3] has more significance than the theory of ordinary differential equations. In this paper, we consider the following

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nonlinear singular fractional difference system with “maxima”

$$
\begin{cases} E_a\nabla^{\nu}_{h,*}x(t) = f\Big(t, x(t), \max\limits_{s\in(h\mathbb{N})^{t}_{t-\tau(t)}} x(s)\Big), & t \in (h\mathbb{N})^{a+n_0h}_{a+h}, \\ x(t) = \varphi(t), \quad t \in (h\mathbb{N})^{a}_{a-mh}, \end{cases} \tag{1.1}
$$

where E is a singular $n\times n$ matrix, $x \in \mathbb{R}^n$, $f : (h\mathbb{N})_a\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $\varphi : (h\mathbb{N})^a_{a-mh} \to \mathbb{R}^n$, $\tau : (h\mathbb{N})_{a+h} \to (h\mathbb{N})_h$, $a - mh := \min_{t\in(h\mathbb{N})_{a+h}}(t - \tau(t))$, m and n_0 are fixed positive constants. Moreover, in (1.1) the operator ${}_a\nabla^{\nu}_{h,*}$ is the Caputo-type fractional h-difference operator – see Section 2 for details. Due, therefore, to the fractionally of the operator problem (1.1) is nonlocal. Note that in (1.1) and throughout we use the notation

$$
(h\mathbb{N})_a := \{a, a + h, a + 2h\},
$$

for $h > 0$ and $a \in \mathbb{R}$.

The discrete fractional calculus in recent years has attracted much attention as a new area of research within the larger arena of difference calculus. One reason for this is due to the inherently nonlocal structure of the fractional difference and sum (no matter the underlying definition used). This nonlocal nature imparts significantly enhanced difficulty in the analysis of such difference operators.

For example, the analysis of initial and boundary value problems is more complicated (see, for example, representative papers by Atici and Eloe [4–6], Dahal and Goodrich [7], Ferreira [8], and Goodrich [9]), and, similarly, the connections between the sign of a fractional difference and the monotone or convex behavior of the underlying function is exquisitely complex (see, for example, representative papers by Abdeljawad and Abdalla [10], Du, Jia, Erbe, and Peterson [11], Goodrich and Lizama [12, 13], Goodrich, Lyons, and Velcsov [14], Goodrich, Lyons, Velcsov, and Scapellato [15], Goodrich and Muellner [16], and Jia, Erbe, and Peterson [17–19]). Consequently, there is mathematical value in continuing to develop our understanding of these types of nonlocal operators. We should also like to note that, increasingly, fractional difference operators have been used in applications such as tumor modeling [20–22], cryptography [23], and image processing [24].

The method of upper and lower solutions combined with monotone iterative [25] has been widely used to prove the existence of extremal solutions on nonlinear problems. Previous studies have mainly focused on the nonlinear differential systems [26–29] and nonlinear singular differential system [30–32]. However, we notice that this method for nonlinear singular difference system has been studied very rare, due to the memory of the fractional difference operator and the weak singularity of the kernel.

In the current paper, we extend this method to nonlinear singular difference system with “maxima”, we give a variable initial condition rather than fixed initial condition.

Further, we give a new singular fractional difference comparison principle, which will be used in our main results. Then, by means of upper and lower solutions and monotone iterative technique, existence of maximal and minimal quasi-solutions are proved. Finally, an example is given to illustrate Theorem 3.1 is attainable. While the first author has studied a problem similar to (1.1) recently [33, 34], in neither of these papers is the inclusion of a singular matrix E considered as in (1.1). In fact, in neither [33] nor [34] is a matrix coefficient considered. Therefore, this distinguishes the study here from earlier works and, moreover, requires a somewhat different approach.

2 Preliminaries

We begin with basic definitions which will be necessary for the following proof. The recent textbook by Goodrich and Peterson [35] can be consulted for a wealth of additional information on the discrete fractional calculus. We begin by recalling the definition of the Caputo h-difference operator – see, for example, [33, pp. 816–818] for additional details.

Definition 2.1. Let $\nu \in (n-1, n]$, where $n \in \mathbb{N}$, and put $\mu := n - \nu$. Then the **Caputo h-difference operator**, denoted ${}_a\nabla_{h,*}^{\nu}$ of order ν for a function $u : (h\mathbb{N})_a \to \mathbb{R}$ is defined by

$$\big({}_a\nabla_{h,*}^{\nu} u\big)(t) := \frac{h}{\Gamma(\mu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\mu} \big(t-(s+1)h\big)_h^{\overline{\mu-1}} \big(\nabla_h^n u\big)(sh),\ t \in (h\mathbb{N})_{a+\mu h},$$

where ∇_h^n is the n-fold composition of the backwards h-difference defined by

$$\big(\nabla_h u\big)(t) := \frac{u(t)-u(t-h)}{h},\ t \in (h\mathbb{N})_a,$$

and $t_h^{(\nu)}$ denotes the h-factorial function defined by

$$t_h^{\overline{\nu}} := h^{\nu} \cdot \frac{\Gamma\left(\frac{t}{h}+\nu\right)}{\Gamma\left(\frac{t}{h}\right)}$$

for any t, $\nu \in \mathbb{R}$ such that $\dfrac{t}{h} + 1 \notin \{\cdots, -2, -1, 0\}$.

Remark 2.1. Let $\mathbb{Z}^-$ denote the nonpositive integers. Then as is the standard convention we take $t_h^{\overline{\nu}} := 0$ whenever $\dfrac{t}{h} + \nu \in \mathbb{Z}^-$ and $\dfrac{t}{h} \notin \mathbb{Z}^-$.

Throughout this paper, we will use the following notation (See [29])

$$(z_i, [x]_{p_i}, [y]_{q_i}) = \begin{cases} (\underbrace{x_1, x_2, \cdots, x_{i-1}, z_i, x_{i+1}, \cdots, x_{p_i+1}}_{p_i}, \underbrace{y_{p_i+2}, \cdots, y_n}_{q_i}) & \text{for } p_i > i, \\ (\underbrace{x_1, x_2, \cdots, x_{p_i}}_{p_i}, \underbrace{y_{p_i+1}, \cdots, y_{i-1}, z_i, y_{i+1}, \cdots, y_n}_{q_i+1}) & \text{for } p_i < i, \end{cases}$$

where p_i and q_i are nonnegative integers such that $p_i + q_i = n - 1$, $i = 1, 2, \cdots, n$. It follows from [36] that the pair (E, A) is regular and impulse-free. Then there exist two nonsingular matrices M, N such that

$$MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the system (1.1) is equivalent to

$$\begin{cases} {}_a\nabla^{\nu}_{h,*} x^{(1)}(t) = M_1 f\Big(t, x(t), \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big) = g^{(1)}\Big(t, x(t), \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big), \\ 0 = M_2 f\Big(t, x(t), \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big) = g^{(2)}\Big(t, x(t), \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big), \quad t \in (h\mathbb{N})^{a+n_0h}_{a+h}, \\ x(t) = \varphi(t), \quad t \in (h\mathbb{N})^a_{a-mh}, \end{cases} \tag{2.1}$$

where $N^{-1}x(t) = col(x^{(1)}(t), x^{(2)}(t))$, $x^{(1)} \in \mathbb{R}^r$, $x^{(2)} \in \mathbb{R}^{n-r}$. And $M = (M_1^T, M_2^T)$, $M_1 \in \mathbb{R}^{r\times n}$, $M_2 \in \mathbb{R}^{(n-r)\times n}$, $N = (N_1^T, N_2^T)$, $N_1 \in \mathbb{R}^{n\times r}$, $N_2 \in \mathbb{R}^{n\times(n-r)}$. According to the above notations, the system (2.1) can be written as

$$\begin{cases} {}_a\nabla^{\nu}_{h,*} x_i(t) = g_i\Big(t, x_i, [x]_{p_i}, [x]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_i(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big]_{q_i}\Big), \\ \hfill i = 1, \cdots, r, \\ 0 = g_i\Big(t, x_i, [x]_{p_i}, [x]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_i(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big]_{q_i}\Big), \\ \hfill i = r+1, \cdots, n, \quad t \in (h\mathbb{N})^{a+n_0h}_{a+h}, \\ x_i(t) = \varphi_i(t), \quad t \in (h\mathbb{N})^a_{a-mh}. \end{cases} \tag{2.2}$$

Definition 2.2. The function α_0, $\beta_0 : (h\mathbb{N})^{a+n_0h}_{a-mh} \to \mathbb{R}^n$ is called a couple of lower and upper quasi-solutions of the system (2.1) if the following inequalities are satisfied:

$$\begin{cases} {}_a\nabla^{\nu}_{h,*} \alpha_{0i}(t) \le g_i\Big(t, \alpha_{0i}, [\alpha_0]_{p_i}, [\beta_0]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_{0i}(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_0(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_0(s)\Big]_{q_i}\Big), \\ \hfill i = 1, \cdots, r, \\ 0 \le g_i\Big(t, \alpha_{0i}, [\alpha_0]_{p_i}, [\beta_0]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_{0i}(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_0(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_0(s)\Big]_{q_i}\Big), \\ \hfill i = r+1, \cdots, n, \quad t \in (h\mathbb{N})^{a+n_0h}_{a+h}, \\ {}_a\nabla^{\nu}_{h,*} \beta_{0i}(t) \ge g_i\Big(t, \beta_{0i}, [\beta_0]_{p_i}, [\alpha_0]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_{0i}(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_0(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_0(s)\Big]_{q_i}\Big), \\ \hfill i = 1, \cdots, r, \\ 0 \ge g_i\Big(t, \beta_{0i}, [\beta_0]_{p_i}, [\alpha_0]_{q_i}, \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_{0i}(s), \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_0(s)\Big]_{p_i}, \Big[\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_0(s)\Big]_{q_i}\Big), \\ \hfill i = r+1, \cdots, n, \quad t \in (h\mathbb{N})^{a+n_0h}_{a+h}, \\ \alpha_{0i}(t) \le \varphi_i(t), \quad t \in (h\mathbb{N})^a_{a-mh} \\ \beta_{0i}(t) \ge \varphi_i(t), \quad t \in (h\mathbb{N})^a_{a-mh}. \end{cases} \tag{2.3}$$

Remark 2.2. When the system (2.1) is a scalar case, i.e., $n = 1$, $p_1 = q_1 = 0$, the couple of lower and upper quasi-solutions of the system (2.1) are said to be lower and upper solutions of the system (2.1).

Remark 2.3. The function of α_0, $\beta_0 : (h\mathbb{N})_{a-mh}^{a+n_0h} \to \mathbb{R}^n$ is said to be a couple of quasi-solutions of the system (2.1), if the inequalities (2.3) are satisfied as equalities.

We give the following sets for convenience.

$S(\alpha_0, \beta_0) = \{u : (h\mathbb{N})_{a-mh}^{a+n_0h} \to \mathbb{R}^n \mid \alpha_0(t) \le u(t) \le \beta_0(t),\ t \in (h\mathbb{N})_{a-mh}^{a+n_0h}\}$.

Lemma 2.1. (See [37]) Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point in X.

In our further investigations, we will need some results on linear singular difference inequalities and systems. Consider the following singular fractional difference system with "maxima"

$$\begin{cases} {}_a\nabla_{h,*}^{\nu} x_i(t) + A_i x_i(t) + B_i \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) = \sigma_i(t),\ i = 1, \cdots, r, \\ A_i x_i(t) + B_i \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) = \sigma_i(t),\ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})_{a+h}^{a+n_0h}, \\ x_i(t) = \varphi_i(t), \quad t \in (h\mathbb{N})_{a-mh}^{a}, \end{cases} \tag{2.4}$$

where A_i, B_i are positive constants.

Lemma 2.2. Assume that the function $x_i : (h\mathbb{N})_{a-mh}^{a+n_0h} \to \mathbb{R}$, and $\Lambda = \max\left\{\frac{(n_0h)_h^{\overline{\nu}}}{\Gamma(\nu+1)}(A_i + B_i), \frac{B_i}{A_i}\right\} < 1$. Then the system (2.4) has a unique solution for $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$.

Proof. Let X be the space of real-valued functions defined on $(h\mathbb{N})_{a-mh}^{a+n_0h}$. Then we define a norm $\|\cdot\|$ on X by $\|x\| = \max\{|x(t)| : t \in (h\mathbb{N})_{a-mh}^{a+n_0h}\}$ so that the pair $(X, \|\cdot\|)$ is a Banach space. Now we define the map $T : X \to X$ by $Tx = (T_1x, T_2x)$, where T_1 and T_2 are defined as follows

$$\begin{cases} T_1 x_i(t) = x_i(a) + {}_a\nabla_h^{-\nu}\Big[- A_i x_i(t) - B_i \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) + \sigma_i(t)\Big],\ i = 1, \cdots, r, \\ T_2 x_i(t) = -\dfrac{B_i}{A_i} \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) + \dfrac{1}{A_i}\sigma_i(t),\ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})_{a+h}^{a+n_0h}, \\ x_i(t) = \varphi_i(t), \quad t \in (h\mathbb{N})_{a-mh}^{a}. \end{cases} \tag{2.5}$$

Next, we will show that T is a contraction map. For $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$, and x_i, $y_i \in X$, we have

$$|T_1x_i - T_1y_i| \ = \Big|{}_a\nabla_h^{-\nu}\Big[- A_i(x_i(t) - y_i(t)) - B_i(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) - \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s))\Big]\Big|$$

$$
\begin{aligned}
&\le {}_a\nabla_h^{-\nu}(A_i+B_i)\|x_i-y_i\| \\
&\le \frac{(n_0h)_h^{\overline{\nu}}}{\Gamma(\nu+1)}(A_i+B_i)\|x_i-y_i\|,\ i=1,\cdots,r,
\end{aligned}
$$

$$
\begin{aligned}
|T_2x_i-T_2y_i| &= \Big|-\frac{B_i}{A_i}\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s)+\frac{B_i}{A_i}\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s))\Big| \\
&\le \frac{B_i}{A_i}\|x_i-y_i\|,\ i=r+1,\cdots,n,\quad t\in(h\mathbb{N})_{a+h}^{a+n_0h}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d(Tx_i,Ty_i) &= d\big((T_1x_i,T_2x_i),(T_1y_i,T_2y_i)\big) \\
&\le \Lambda d(x_i,y_i),\ t\in(h\mathbb{N})_{a+h}^{a+n_0h}.
\end{aligned}
$$

Hence, by Lemma 2.1, the operator T has a unique fixed point – that is, the system (2.1) has a unique solution. The proof is complete. □

Lemma 2.3. Assume that the function $m:(h\mathbb{N})_{a-mh}^{a+n_0h}\to\mathbb{R}^n$ satisfy the inequalities

$$
\begin{cases}
{}_a\nabla_{h,*}^{\nu}m_i(t)\le -A_im_i(t)-B_i\displaystyle\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s),\ i=1,\cdots,r,\\
0\le -A_im_i(t)-B_i\displaystyle\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s),\ i=r+1,\cdots,n,\quad t\in(h\mathbb{N})_{a+h},\\
m_i(t)\le 0,\quad t\in(h\mathbb{N})_{a-mh}^{a},
\end{cases}
\tag{2.6}
$$

where A_i, B_i are positive constants, and $A_i>B_i$. Then the inequality $m_i(t)\le 0$ for $t\in(h\mathbb{N})_{a-mh}^{a+n_0h}$.

Proof. Since the operator ${}_a\nabla_h^{-\nu}$ is positive, it follows that the inequality (2.6) is equivalent to

$$
\begin{cases}
m_i(t)-m_i(a)\le {}_a\nabla_h^{-\nu}\Big[-A_im_i(t)-B_i\displaystyle\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s)\Big],\ i=1,\cdots,r,\\
0\le -A_im_i(t)-B_i\displaystyle\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s),\ i=r+1,\cdots,n,\quad t\in(h\mathbb{N})_{a+h},\\
m_i(t)\le 0,\quad t\in(h\mathbb{N})_{a-mh}^{a}.
\end{cases}
\tag{2.7}
$$

By induction, when $t=a+h$, from the assumption $A_i>B_i$, we obtain

$$
\begin{cases}
m_i(a+h)\le {}_a\nabla_h^{-\nu}\Big[-A_i\big(m_i(a+h)-\displaystyle\min_{s\in(h\mathbb{N})_{a+h-\tau(t)}^{a+h}} m_i(s)\big)\Big],\ i=1,\cdots,r,\\
0\le -A_i\Big(m_i(a+h)-\displaystyle\min_{s\in(h\mathbb{N})_{a+h-\tau(t)}^{a+h}} m_i(s)\Big),\ i=r+1,\cdots,n,\quad t\in(h\mathbb{N})_{a+h},\\
m_i(t)\le 0,\quad t\in(h\mathbb{N})_{a-mh}^{a}.
\end{cases}
\tag{2.8}
$$

Since $m_i(a+h) - \min_{s\in(h\mathbb{N})_{a+h-\tau(t)}^{a+h}} m_i(s) > 0$, so we have $m(a+h) \le 0$. Now, we assume that $m_i(t) \le 0$ for $t \in (h\mathbb{N})_{a-mh}^{a+nh}$. We will show that $m_i(a+(n+1)h) \le 0$. From the inequality (2.6), we have

$$\begin{cases} m_i(a+(n+1)h) \le {}_a\nabla_h^{-\nu}\Big[-A_i\Big(m_i(a+(n+1)h) - \min_{s\in(h\mathbb{N})_{a+(n+1)h-\tau(t)}^{a+(n+1)h}} m_i(s)\Big)\Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \cdots, r, \\ 0 \le -A_i\Big(m_i(a+(n+1)h) - \min_{s\in(h\mathbb{N})_{a+(n+1)h-\tau(t)}^{a+(n+1)h}} m_i(s)\Big), \ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})_{a+h}, \\ m_i(t) \le 0, \quad t \in (h\mathbb{N})_{a-mh}^{a}. \end{cases} \tag{2.9}$$

Similarly, we have $m_i(a+(n+1)h) \le 0$. Thus, we conclude that $m_i(t) \le 0$ for $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$. The proof is complete. $\square$

3 Monotone iterative technique

Here we develop the monotone iterative technique for nonlinear singular fractional difference system with maxima which yields monotone sequences converging to the extremal solutions of the system (2.1).

Theorem 3.1. Assume that the following conditions hold.

$(A_{3.1})$ The functions α_0, $\beta_0 : (h\mathbb{N})_{a-mh}^{a+n_0h} \to \mathbb{R}^n$ are a couple of lower and upper quasi-solutions of the system (2.1) respectively, and $\alpha_0(t) \le \beta_0(t)$ for $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$.

$(A_{3.2})$ There exists a function $g:\ (h\mathbb{N})_{a+h}^{a+n_0h} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $g_j(t,x,y) = g_j(t, x_j, [x]_{p_j}, [x]_{q_j}, x_j, [x]_{p_j}, [x]_{q_j})$ is monotone nondecreasing with respect to $[x]_{p_j}$, $[y]_{p_j}$, and monotone nonincreasing with respect to $[x]_{q_j}$, $[y]_{q_j}$, and for x, $y \in S(\alpha_0, \beta_0)$, $y(t) \le x(t)$ the following inequality holds:

$$\begin{aligned} &g_i\Big(t, x_i, [x]_{p_i}, [x]_{q_i}, \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s), \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x(s)\Big]_{p_i}, \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x(s)\Big]_{q_i}\Big) \\ &\quad - g_i\Big(t, y_i, [x]_{p_i}, [x]_{q_i}, \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s), \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x(s)\Big]_{p_i}, \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x(s)\Big]_{q_i}\Big) \\ &\qquad \ge -A_i(x_i(t) - y_i(t)) - B_i\Big(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) - \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s)\Big), \quad i = 1, 2, \cdots, n, \end{aligned} \tag{3.1}$$

where A_i and B_i are positive constants, and $B_i < A_i \le \dfrac{\Gamma(\nu+1)}{2(n_0h)_h^{\overline{\nu}}}$.

Then there exist sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ which converge uniformly and monotonically to $\rho(t)$ and $r(t)$, where ρ and r are minimal and maximal quasi-solutions of the system (2.1) respectively. Moreover, if $x(t)$ is a solution of the system (2.1) such that $\alpha_0(t) \le x(t) \le \beta_0(t)$, then $\rho(t) \le x(t) \le r(t)$ for $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$.

Proof. For functions μ, $\nu \in S(\alpha_0, \beta_0)$ and consider the following singular fractional difference system with "maxima"

$$\begin{cases} {}_a\nabla_{h,*}^{\nu} x_i(t) + A_i x_i(t) + B_i \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) = \sigma_i(t,\mu,\nu), \ i = 1, \cdots, r, \\ A_i x_i(t) + B_i \max\limits_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) = \sigma_i(t,\mu,\nu), \ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})_{a+h}^{a+n_0h}, \\ x_i(t) = \varphi_i(t), \quad t \in (h\mathbb{N})_{a-mh}^{a}, \end{cases} \tag{3.2}$$

where

$$\sigma_i(t,\mu,\nu) = g_i\Big(t, \mu_i, [\mu]_{p_i}, [\nu]_{q_i}, \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} \mu_i(s), \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} \mu(s)\Big]_{p_i}, \Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} \nu(s)\Big]_{q_i}\Big)$$
$$+ A_i\mu_i(t) + B_i \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} \mu_i(s).$$

By Lemma 2.2, the system (3.2) has a unique solution. Suppose that for the couple μ, $\nu \in S(\alpha_0, \beta_0)$ there exist two distinct solutions $x(t)$ and $y(t)$ of the system (3.2). Define a function $m(t) = x(t) - y(t)$ for $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$, $m(t) = \big(m_1(t), m_2(t), \cdots, m_n(t)\big)$. Then $m_j(t)$ $(j = 1, 2, \cdots, n)$ satisfy the inequalities

$$\begin{aligned} {}_a\nabla_{h,*}^{\nu} m_i(t) &= {}_a\nabla_{h,*}^{\nu} x_i(t) - {}_a\nabla_{h,*}^{\nu} y_i(t) \\ &= -A_i(x_i(t) - y_i(t)) - B_i\Big(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) - \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s)\Big) \\ &\le -A_i m_i(t) - B_i \min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s), \ i = 1, \cdots, r, \\ 0 &= -A_i(x_i(t) - y_i(t)) - B_i\Big(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_i(s) - \max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} y_i(s)\Big) \\ &\le -A_i m_i(t) - B_i \min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} m_i(s), \ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})_{a+h}^{a+n_0h}, \\ m_i(t) &\le 0, \quad t \in (h\mathbb{N})_{a-mh}^{a}. \end{aligned}$$

In view of Lemma 2.3, we have $x_i(t) \le y_i(t)$, $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$. Similarly, we can show that $y_i(t) \le x_i(t)$, $t \in (h\mathbb{N})_{a-mh}^{a+n_0h}$. So, we obtain that the system (3.2) has a unique solution.

Define the map $\mathscr{A} : S(\alpha_0, \beta_0) \times S(\alpha_0, \beta_0) \to S(\alpha_0, \beta_0)$ by $\mathscr{A}(\mu, \nu) = x$, where $x = (x_1, x_2, \cdots, x_n)$ and $x_i(t)$ is the unique solution of system (3.2) for the functions μ, $\nu \in S(\alpha_0, \beta_0)$. We shall now show that $\alpha_0(t) \le \mathscr{A}(\alpha_0(t), \beta_0(t))$. Define $\alpha_1(t) =$

$\mathscr{A}(\alpha_0(t),\beta_0(t))$, $p(t)=\alpha_0(t)-\alpha_1(t)$ for $t\in(h\mathbb{N})_{a-mh}^{a+n_0h}$, $p(t)=(p_1(t),p_2(t),\cdots,p_n(t))$. Using the assumption $(A_{3.2})$, we have

$$
\begin{aligned}
&{}_a\nabla_{h,*}^{\nu}p_i(t)\\
&={}_a\nabla_{h,*}^{\nu}\alpha_{0i}(t)-{}_a\nabla_{h,*}^{\nu}\alpha_{1i}(t)\\
&\le g_i\Big(t,\alpha_{0i},[\alpha_0]_{p_i},[\beta_0]_{q_i},\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{0i}(s),\Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_0(s)\Big]_{p_i},\Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\beta^{(0)}(s)\Big]_{q_i}\Big)\\
&\quad+A_i\alpha_{1i}(t)+B_i\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{1i}(s)-\sigma_i(t,\alpha_0,\beta_0),\\
&\le -A_i(\alpha_{0i}(t)-\alpha_{1i}(t))-B_i\Big(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{0i}(s)-\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{1i}(s)\Big)\\
&\le -A_ip_i(t)-B_i\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}p_i(s),\ i=1,\cdots,r,
\end{aligned}
$$

and that

$$
\begin{aligned}
0\ &\le g_i\Big(t,\alpha_{0i},[\alpha_0]_{p_i},[\beta_0]_{q_i},\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{0i}(s),\Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_0(s)\Big]_{p_i},\Big[\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\beta_0(s)\Big]_{q_i}\Big)\\
&\quad+A_i\alpha_{1i}(t)+B_i\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{1i}(s)-\sigma_i(t,\alpha_0,\beta_0),\\
&=-A_i(\alpha_{0i}(t)-\alpha_{1i}(t))-B_i\Big(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{0i}(s)-\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}\alpha_{1i}(s)\Big)\\
&\le -A_ip_i(t)-B_i\min_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}}p_i(s),\ i=r+1,\cdots,n,\quad t\in(h\mathbb{N})_{a+h}^{a+n_0h},
\end{aligned}
$$

and that

$$p_i(t)\le 0,\quad t\in(h\mathbb{N})_{a-mh}^{a}.$$

Consequently, by Lemma 2.3, we have $\alpha_0(t)\le\alpha_1(t)$ for $t\in(h\mathbb{N})_{a-mh}^{a+n_0h}$. Similarly, we can show that $\alpha_1(t)\le\beta_0(t)$ for $t\in(h\mathbb{N})_{a-mh}^{a+n_0h}$.

Let μ, $\nu\in S(\alpha_0,\beta_0)$ be such that $\mu(t)\le\nu(t)$ for $t\in(h\mathbb{N})_{a-mh}^{a+n_0h}$. From the definition of $\mathscr{A}$ and Lemma 2.3, it follows that the inequality $\mathscr{A}(\mu,\nu)\le\mathscr{A}(\nu,\mu)$ is valid.

Define the sequences of functions $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ by the equalities

$$\alpha_{n+1}(t)=\mathscr{A}(\alpha_n(t),\beta_n(t)),\quad \beta_{n+1}(t)=\mathscr{A}(\beta_n(t),\alpha_n(t)).$$

By induction, we have

$$\alpha_0(t)\le\alpha_1(t)\le...\le\alpha_n(t)\le\beta_n(t)\le...\le\beta_1(t)\le\beta_0(t),\quad\text{for }\ t\in(h\mathbb{N})_{a-mh}^{a+n_0h}.$$

Using the Arzelà-Ascoli theorem, thus both sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are uniformly convergent on $(h\mathbb{N})_{a-mh}^{a+n_0h}$. Then there exist functions $\rho(t)$ and $r(t)$ such that

$$\lim_{n\to\infty}\alpha_n(t)=\rho(t),\ \lim_{n\to\infty}\beta_n(t)=r(t).$$

From the uniform convergence and the definition of the functions $\alpha_n(t)$ and $\beta_n(t)$, it follows that

$$\begin{cases} {}_a\nabla^{\nu}_{h,*}\alpha_{n,i}(t) + A_i\alpha_{n,i}(t) + B_i \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_{n,i}(s) = \sigma_i(t,\alpha_{n-1},\beta_{n-1}),\ i=1,\cdots,r, \\ A_i\alpha_{n,i}(t) + B_i \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_{n,i}(s) = \sigma_i(t,\alpha_{n-1},\beta_{n-1}),\ i=r+1,\cdots,n, \quad t\in(h\mathbb{N})^{a+n_0h}_{a+h}, \\ \alpha_{n,i}(t) = \varphi_i(t), \quad t\in(h\mathbb{N})^a_{a-mh}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} {}_a\nabla^{\nu}_{h,*}\beta_{n,i}(t) + A_i\beta_{n,i}(t) + B_i \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_{n,i}(s) = \sigma_i(t,\beta_{n-1},\alpha_{n-1}),\ i=1,\cdots,r, \\ A_i\beta_{n,i}(t) + B_i \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \beta_{n,i}(s) = \sigma_i(t,\beta_{n-1},\alpha_{n-1}),\ i=r+1,\cdots,n, \quad t\in(h\mathbb{N})^{a+n_0h}_{a+h}, \\ \beta_{n,i}(t) = \varphi_i(t), \quad t\in(h\mathbb{N})^a_{a-mh}, \end{cases} \tag{3.4}$$

From the systems (3.3) and (3.4) as n approaches infinity we obtain that the functions (ρ, r) is a couple of quasi-solutions of the system (2.1).

In the next step, we will show that (ρ, r) are minimal and maximal quasi-solutions of the system (2.1). Let $(z,u)\in S(\alpha_0,\beta_0)$ be quasi-solutions of the system (2.1). Suppose that for some n, $\alpha_n(t)\le z(t)\le \beta_n(t)$, $\alpha_n(t)\le u(t)\le\beta_n(t)$, $t\in(h\mathbb{N})^{a+n_0h}_{a-mh}$. Put $p(t)=\alpha_{n+1}(t)-z(t)$ so that $\alpha_{n+1}(t)\le z(t)$ on $(h\mathbb{N})^a_{a-mh}$. For $t\in(h\mathbb{N})^{a+n_0h}_{a+h}$, we wish to prove that $\alpha_{n+1}(t)\le z(t)$ on $(h\mathbb{N})^{a+n_0h}_{a+h}$. From the condition $(A_{3.2})$, we deduce that

$$\begin{aligned} {}_a\nabla^{\nu}_{h,*}p_i(t) &= {}_a\nabla^{\nu}_{h,*}\alpha_{n+1,i}(t) - {}_a\nabla^{\nu}_{h,*}z_i(t) \\ &= -A_i\alpha_{n+1,i}(t) - B_i\max_{s\in[t-h,t]}\alpha_{n+1,i}(s) + \sigma_i\Big(t,\alpha_n(t),\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\beta_n(s)\Big) \\ &\quad - g_i\Big(t,z_i(t),[z]_{p_i},[u]_{q_i},\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}z_i(s),\Big[\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}z(s)\Big]_{p_i},\Big[\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}u(s)\Big]_{q_i}\Big) \\ &= -A_i(\alpha_{n+1,i}(t)-\alpha_{n,i}(t)) - B_i\Big(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_{n+1,i}(s) - \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_{n,i}(s)\Big) \\ &\quad + A_i(z_i(t)-\alpha_n(t)) + B_i\Big(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}z_i(s) - \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_n(s)\Big) \\ &= -A_i(\alpha_{n+1,i}(t)-z_i(t)) - B_i\Big(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_{n+1,i}(s) - \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}z_i(s)\Big) \\ &\le -A_ip_i(t) - B_i\min_{s\in(h\mathbb{N})^t_{t-\tau(t)}}p_i(s),\ i=1,\cdots,r, \end{aligned}$$

and that

$$\begin{aligned} 0 &= -A_i\alpha_{n+1,i}(t) - B_i\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_{n+1,i}(s) + \sigma_i(t,\alpha_n,\beta_n) \\ &\quad - g_i\Big(t,\alpha_{n,i},[\alpha_n]_{p_i},[\beta_n]_{q_i},\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_{n,i}(s),\Big[\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\alpha_n(s)\Big]_{p_i},\Big[\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}}\beta_n(s)\Big]_{q_i}\Big), \end{aligned}$$

$$= -A_i(\alpha_{n+1,i}(t) - z_i(t)) - B_i\Big(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} \alpha_{n+1,i}(s) - \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} z_i(s)\Big)$$
$$\le -A_i p_i(t) - B_i \min_{s\in(h\mathbb{N})^t_{t-\tau(t)}} p_i(s),\ i = r+1, \cdots, n, \quad t \in (h\mathbb{N})^{a+n_0h}_{a+h},$$

and that

$$p_i(t) \le 0, \quad t \in (h\mathbb{N})^a_{a-mh}.$$

By Lemma 2.3, we have $p(t) \le 0$, showing that $\alpha_{n+1}(t) \le z(t)$ for $t \in (h\mathbb{N})^{a+n_0h}_{a+h}$. This proves that $\alpha_{n+1}(t) \le z(t)$, $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$. Using a similar argument we can prove that $z(t) \le \beta_{n+1}(t)$ and $\alpha_{n+1}(t) \le u(t) \le \beta_{n+1}(t)$ for $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$. Thus we conclude that $\alpha_n(t) \le z(t), u(t) \le \beta_n(t)$ on $(h\mathbb{N})^{a+n_0h}_{a-mh}$ for all $n \in \mathbb{N}$. Now as $n \to \infty$ yields $\rho(t) \le z(t), u(t) \le r(t)$, $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$, which shows that (ρ, r) are minimal and maximal quasi-solutions of the system (2.1) respectively.

Let $x \in S(\alpha_0, \beta_0)$ be a solution of the system (2.1). The functions (x, x) is a couple of quasi-solutions of the system (2.1), from the above discussion, we have $\rho(t) \le x(t) \le r(t)$ for $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$. The proof is complete. □

When $n = 1$ and $p_1 = q_1 = 0$, we could obtain the following corollary.

Corollary 3.1. Assume that the following conditions hold.

$(A_{3.3})$ The functions α_0, β_0 : $(h\mathbb{N})^{a+n_0h}_{a-mh} \to R$ are lower and upper solutions of the system (2.1) respectively, and $\alpha_0(t) \le \beta_0(t)$ for $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$.

$(A_{3.4})$ There exists a function g : $(h\mathbb{N})^{a+n_0h}_{a+h} \times R \times R \to R$ such that for $x, y \in S(\alpha_0, \beta_0)$, $y(t) \le x(t)$ the following inequality holds

$$\begin{aligned} &g\Big(t, x, \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s)\Big) - g\Big(t, y, \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} y(s)\Big) \\ &\quad \ge -A(x(t) - y(t)) - B\Big(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x(s) - \max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} y(s)\Big), \end{aligned} \tag{3.5}$$

where A and B are positive constants, and $B_i < A_i \le \dfrac{\Gamma(\nu+1)}{2(n_0h)^{\overline{\nu}}_h}$.

Then there exist sequences $\{\alpha_n\}$, $\{\beta_n\}$ which converge uniformly and monotonically to $\rho(t)$ and $r(t)$, where ρ and r are minimal and maximal quasi-solutions of the system (2.1) respectively. Moreover, if $x(t)$ is a solution of the system (2.1) such that $\alpha_0(t) \le x(t) \le \beta_0(t)$, then $\rho(t) \le x(t) \le r(t)$ for $t \in (h\mathbb{N})^{a+n_0h}_{a-mh}$.

4 Application

Now we will give an example to illustrate that the above suggested method is attainable.

Example 4.1. Consider the following singular fractional difference system with "maxima"

$$\begin{cases} {}_a\nabla^{\nu}_{h,*}x_1(t) = \dfrac{1}{1-x_1(t)} - 2\max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s) - 1, \\ \qquad 0 = x_2^2(t) - \max\limits_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s), \quad t \in (h\mathbb{N})^5_1, \\ \quad x_1(t) = x_2(t) = 0, \quad t \in (h\mathbb{N})^0_{-1}, \end{cases} \tag{4.1}$$

where $\nu = 0.5$, $h = 1$, $a = 0$, $\tau(t) = 1$ for $t \in (h\mathbb{N})^5_{-1}$. It is easy to check that the system (4.1) has a zero solution. $\alpha_0(t) = \left(-\frac{1}{4}, -\frac{1}{4}\right)^T$ is a lower solution and $\beta_0(t) = \left(\frac{1}{4}, \frac{1}{4}\right)^T$ is an upper solution of the system (4.1). We will construct sequences of functions that converge uniformly to 0.

Now we can construct a increasing sequence, which converge to 0. It is easy to see that the matrices L_1 and L_2 can be chosen as

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 0 & \frac{29}{4} \end{pmatrix},$$

respectively. Choose $k_{01} = \frac{4}{5}$, $k_{02} = \frac{4}{5}$, and consider the following singular fractional difference system

$$\begin{aligned} {}_a\nabla^{\nu}_{h,*}x_1(t) &= \frac{1}{1+\frac{1}{4}} + 2\times\frac{1}{4} - 1 + 2\left(x_1(t) + \frac{1}{4}\right) - 8\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s) + \frac{1}{4}\right) \\ &= 2x_1(t) - 8\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s) - \frac{6}{5}, \\ 0 &= \left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right) + \left(x_2(t) + \frac{1}{4}\right) - \frac{29}{4}\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s) + \frac{1}{4}\right) \\ &= x_2(t) - \frac{29}{4}\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s) - \frac{5}{4}, \quad t \in (h\mathbb{N})^5_1, \\ x_1(t) &= -\frac{k_{01}}{4}, \ x_2(t) = -\frac{k_{02}}{4}, \quad t \in (h\mathbb{N})^0_{-1}. \end{aligned} \tag{4.2}$$

Then the system (4.2) has an exact solution $\alpha_1(t) = \left(-\frac{1}{5}, -\frac{1}{5}\right)^T$.

Choose $k_{11} = \frac{29}{36}$, $k_{12} = \frac{101}{125}$ and consider the following singular fractional difference system

$$
\begin{aligned}
{}_a\nabla^{\nu}_{h,*}x_1(t) &= \frac{1}{1+\frac{1}{5}} + 2\times\frac{1}{5} - 1 + 2\left(x_1(t)+\frac{1}{5}\right) - 8\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s)+\frac{1}{5}\right)\\
&= 2x_1(t) - 8\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s) - \frac{29}{30},\\
0 &= \left(-\frac{1}{5}\right)^2 - \left(-\frac{1}{5}\right) + \left(x_2(t)+\frac{1}{5}\right) - \frac{29}{4}\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s)+\frac{1}{5}\right)\\
&= x_2(t) - \frac{29}{4}\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s) - \frac{101}{100},\quad t\in(h\mathbb{N})^5_1,\\
x_1(t) &= -\frac{k_{11}}{5},\ x_2(t) = -\frac{k_{12}}{5},\quad t\in(h\mathbb{N})^0_{-1}.
\end{aligned}
\tag{4.3}
$$

Then the system (4.3) has an exact solution $\alpha_2(t) = \left(-\frac{29}{180}, -\frac{101}{625}\right)^T$.

Analogously, we will construct a decreasing sequence that uniformly converges to 0. Choose $p_{01} = \frac{8}{9}$, $p_{02} = \frac{44}{50}$, and consider the following singular fractional difference system

$$
\begin{aligned}
{}_a\nabla^{\nu}_{h,*}x_1(t) &= \frac{1}{1-\frac{1}{4}} - 2\times\frac{1}{4} - 1 + 2\left(x_1(t)-\frac{1}{4}\right) - 8\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s)-\frac{1}{4}\right)\\
&= 2x_1(t) - 8\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_1(s) + \frac{4}{3},\\
0 &= \left(\frac{1}{4}\right)^2 - \frac{1}{4} + \left(x_2(t)-\frac{1}{4}\right) - \frac{29}{4}\left(\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s)-\frac{1}{4}\right)\\
&= x_2(t) - \frac{29}{4}\max_{s\in(h\mathbb{N})^t_{t-\tau(t)}} x_2(s) + \frac{11}{8},\quad t\in(h\mathbb{N})^5_1,\\
x_1(t) &= \frac{p_{01}}{4},\ x_2(t) = \frac{p_{02}}{4},\quad t\in(h\mathbb{N})^0_{-1}.
\end{aligned}
\tag{4.4}
$$

Then the system (4.4) has an exact solution $\beta_1(t) = \left(\frac{2}{9}, \frac{11}{50}\right)^T$.

Choose $p_{11} = \frac{37}{42}$, $p_{12} = \frac{6017}{6875}$, and consider the following singular fractional difference system

$$
\begin{aligned}
{}_a\nabla_{h,*}^{\nu} x_1(t) &= \frac{1}{1-\frac{2}{9}} - 2\times\frac{2}{9} - 1 + 2\left(x_1(t)-\frac{2}{9}\right) - 8\left(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_1(s) - \frac{2}{9}\right)\\
&= 2x_1(t) - 8\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_1(s) + \frac{74}{63},\\
0 &= \left(\frac{11}{50}\right)^2 - \frac{11}{50} + \left(x_2(t)-\frac{11}{50}\right) - \frac{29}{4}\left(\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_2(s) - \frac{11}{50}\right)\\
&= x_2(t) - \frac{29}{4}\max_{s\in(h\mathbb{N})_{t-\tau(t)}^{t}} x_2(s) + \frac{6017}{5000}, \quad t\in(h\mathbb{N})_1^5,\\
x_1(t) &= \frac{2}{9}p_{11},\ x_2(t) = \frac{11}{50}p_{12}, \quad t\in(h\mathbb{N})_{-1}^0.
\end{aligned}
\tag{4.5}
$$

Then the system (4.5) has an exact solution $\beta_2(t) = \left(\frac{37}{189}, \frac{6017}{31250}\right)^T$.

Obviously, we can see that $\alpha_0(t) < \alpha_1(t) < \alpha_2(t) < x(t) = 0 < \beta_2(t) < \beta_1(t) < \beta_0(t)$ on $(h\mathbb{N})_{-1}^5$. So, Example 4.1 illustrates that the Theorem 3.1 is feasible.

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