Monotonicity result for nabla fractional h-difference operators

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Abstract In this paper, we give a new method to show a monotonicity result for a function f satisfying $({}_a\nabla_h^\nu f)(t) \leq 0$ $(({}_a\nabla_{h,*}^\nu f)(t) \leq 0)$ with $\nu \in (0, 1]$, which has never been solved in other papers. In addition, we give an example to illustrate one of our main results.

Keywords Discrete fractional calculus; Nabla fractional h-difference operators; Monotonicity; Power rule.

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1 Introduction

Discrete fractional calculus and its applications has become an attractive topic in recent years, since Miller and Ross [1] initiated the discrete fractional calculus in 1988. For the basic theory of discrete fractional calculus, we could refer to [2–10] and the references therein. It is well known that monotonicity results play an important role in the study of discrete fractional calculus, and numerous monotonicity results about fractional calculus

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have been published. In [11–15], the authors obtain the monotonicity of f for $\nu \in (1,2)$. In [16], the authors have summarized the monotonicity and convexity results of f, for $(\Delta_a^\nu f)(t)$, $(\nabla_a^\nu f)(t)$ with $\nu \in (1,2)$. But there are few monotonicity results for a function f when $\nu \in (0,1]$. In [17, 18], the authors just presented the ν-increasing (or ν-decreasing) results for $\nu \in (0,1)$, but they do not guarantee that these results hold for $\nu = 1$.

In this paper, we give a new method to show some monotonicity results for $\nu \in (0,1]$, which are listed as follows:

Theorem 1.1. Assume $f : (h\mathbb{N})_a \to \mathbb{R}$, $({}_a\nabla_h^\nu f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0,1]$. If

$$f(a+kh) \geq \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} f(a)$$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$.

Theorem 1.2. Assume $f : (h\mathbb{N})_a \to \mathbb{R}$, $({}_a\nabla_{h,*}^\nu f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0,1]$. If

$$(1-\nu)f(a+kh) \geq \frac{h^\nu((k+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{l=0}^{k-2} (kh+h-lh)_h^{\overline{-\nu-1}} f(a+lh+h)$$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$.

2 Preliminaries

Let $\mathcal{F}_D$ denote the set of real valued functions defined on a domain D. We use the notation $(h\mathbb{N})_a := \{a, a+h, a+2h, \cdots\}$, $(h\mathbb{N})_{-\infty}^{+\infty} := \{\cdots, a-2h, a-h, a, a+h, a+2h, \cdots\}$, where $h > 0$, $a \in \mathbb{R}$. Let $\rho(t) := t-h$ for $t \in (h\mathbb{N})_{a+h}$. For the convenience of the reader, we recall some of the notation to be used here. For a function $f \in \mathcal{F}_{(h\mathbb{N})_a}$, the backward h-difference operator is defined as

$$(\nabla_h f)(t) := \frac{f(t) - f(t-h)}{h}, \quad t \in (h\mathbb{N})_{a+h}. \tag{2.1}$$

For arbitrary t, $\nu \in \mathbb{R}$, the h-factorial function is defined by

$$t_h^{\overline{\nu}} := h^\nu \frac{\Gamma(\frac{t}{h}+\nu)}{\Gamma(\frac{t}{h})}, \tag{2.2}$$

where Γ is the Euler gamma function with $\frac{t}{h} \notin \mathbb{Z}_- \cup \{0\}$, and we use the convention that $t_h^{\overline{\nu}} = 0$, when $\frac{t}{h}+\nu$ is a non-positive integer and $\frac{t}{h}$ is not a non-positive integer.

Definition 2.1. (See [8, Definition 2.2]). Let $f \in \mathcal{F}_{(h\mathbb{N})_a}$, and $\nu > 0$ be given. The fractional h-sum ${}_a\nabla_h^{-\nu} f$ is defined by

$$({}_a\nabla_h^{-\nu} f)(t) := \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{\nu-1}} f(sh), \quad t \in (h\mathbb{N})_a, \tag{2.3}$$

and $({}_a\nabla_h^0 f)(t) = f(t)$, $\rho(sh) = (s-1)h$, $({}_a\nabla_h^{-\nu} f)(a) = 0$.

Definition 2.2. (See [8, Definition 2.3]). Let $f \in \mathcal{F}_{(h\mathbb{N})_a}$, $\nu \in (n-1, n)$ and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The Riemann-Liouville like fractional h-difference ${}_a\nabla_h^\nu f$ is defined by

$$({}_a\nabla_h^\nu f)(t) := (\nabla_h^n({}_a\nabla_h^{-\mu} f))(t) = \frac{h}{\Gamma(\mu)}\nabla_h^n\Big(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{\mu-1}}f(sh)\Big), \quad t \in (h\mathbb{N})_{a+nh}. \tag{2.4}$$

Remark 2.1. It is clear that Definition 2.2 is also true for $\nu = n$.

The following property is useful in this paper:

Property 2.1. The nabla difference of the h-rising factorial function satisfies

$${}_s\nabla_h(t-sh)_h^{\overline{\nu}} = -\nu(t-\rho(sh))_h^{\overline{\nu-1}}, \tag{2.5}$$

where ${}_s\nabla_h(t-sh)_h^{\overline{\nu}} = \frac{(t-sh)_h^{\overline{\nu}}-(t-sh+h)_h^{\overline{\nu}}}{h}$.

Proof. By formula (2.1), we have

$$\begin{aligned}
{}_s\nabla_h(t-sh)_h^{\overline{\nu}} &= \frac{1}{h}\big[{}_s\nabla_h(t-sh)_h^{\overline{\nu}} -_s \nabla_h(t-sh+h)_h^{\overline{\nu}}\big]\\
&= \frac{1}{h}\Big[\frac{h^\nu\Gamma(\frac{t}{h}-s+\nu)}{\Gamma(\frac{t}{h}-s)} - \frac{h^\nu\Gamma(\frac{t}{h}-s+1+\nu)}{\Gamma(\frac{t}{h}-s+1)}\Big]\\
&= h^{\nu-1}\Big(1-\frac{\frac{t}{h}-s+\nu}{\frac{t}{h}-s}\Big)\frac{\Gamma(\frac{t}{h}-s+\nu)}{\Gamma(\frac{t}{h}-s)}\\
&= -\nu h^{\nu-1}\frac{\Gamma(\frac{t}{h}-s+\nu)}{\Gamma(\frac{t}{h}-s+1)}\\
&= -\nu(t-\rho(sh))_h^{\overline{\nu-1}}.
\end{aligned}$$

The proof is complete. □

Lemma 2.1. Let $\nu \in (n-1, n]$ and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The following formula is equivalent to (2.4):

$$({}_a\nabla_h^\nu f)(t) = \begin{cases} \dfrac{h}{\Gamma(-\nu)}\displaystyle\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{-\nu-1}}f(sh), & \nu \in (n-1,n),\ t \in (h\mathbb{N})_{a+nh}, \\ (\nabla_h^n f)(t), & \nu = n,\ t \in (h\mathbb{N})_{a+nh}. \end{cases} \tag{2.6}$$

Proof. If $\nu = n$, then we have

$$({}_a\nabla_h^\nu f)(t) = (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t) = (\nabla_h^n({}_a\nabla_h^{-0} f))(t) = (\nabla_h^n f)(t).$$

If $\nu \in (n-1,n)$, then we have

$$\begin{aligned}({}_a\nabla_h^\nu f)(t) &= (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t)\\ &= \nabla_h^{n-1}\Big(\nabla_h\Big(\frac{h}{\Gamma(n-\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{n-\nu-1}}f(sh)\Big)\Big)\\ &= \nabla_h^{n-1}\Big(\frac{h}{\Gamma(n-\nu-1)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{n-\nu-2}}f(sh)\Big).\end{aligned}$$

Repeating the similar procedure $n-1$ times, we obtain

$$\begin{aligned}({}_a\nabla_h^\nu f)(t) &= (\nabla_h^n({}_a\nabla_h^{-(n-\nu)} f))(t)\\ &= \frac{h}{\Gamma(-\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{-\nu-1}}f(sh).\end{aligned}$$

The proof is complete. □

Definition 2.3. Let $\nu \in (n-1,n]$, and set $\mu = n-\nu$, where $n \in \mathbb{N}_1$. The Caputo like h-difference operator ${}_a\nabla_{h,*}^\nu f$ of order ν for a function $f \in \mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$({}_a\nabla_{h,*}^\nu f)(t) := ({}_a\nabla_h^{-\mu}(\nabla_h^n f))(t) = \frac{h}{\Gamma(\mu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{\mu-1}}(\nabla_h^n f)(sh), \quad t \in (h\mathbb{N})_{a+nh}. \tag{2.7}$$

Definition 2.4. (See [8, Definition 2.3]). Let $\nu \neq -1,-2,\cdots$. Then we define the ν-th order nabla fractional h-Taylor monomial $\hat{H}_\nu(t,a)$ by

$$\hat{H}_\nu(t,a) := \frac{(t-a)_h^{\overline{\nu}}}{\Gamma(\nu+1)} = h^\nu\frac{\Gamma(\frac{t-a}{h}+\nu)}{\Gamma(\nu+1)\Gamma(\frac{t-a}{h})}, \tag{2.8}$$

where $t \in (h\mathbb{N})_a$.

Lemma 2.2. Assume the functions f, $g : (h\mathbb{N})_a \to \mathbb{R}$ and b, $c \in (h\mathbb{N})_a$, $b < c$. Then we have the following summation by parts formulas:

$$\sum_{s=\frac{b}{h}+1}^{\frac{c}{h}} f(sh)(\nabla_h g)(sh) = \frac{f(sh)g(sh)}{h}\Big|_{s=\frac{b}{h}}^{\frac{c}{h}} - \sum_{s=\frac{b}{h}+1}^{\frac{c}{h}} g(\rho(sh))(\nabla_h f)(sh), \tag{2.9}$$

$$\sum_{s=\frac{b}{h}+1}^{\frac{c}{h}} f(\rho(sh))(\nabla_h g)(sh) = \frac{f(sh)g(sh)}{h}\Big|_{s=\frac{b}{h}}^{\frac{c}{h}} - \sum_{s=\frac{b}{h}+1}^{c} g(sh)(\nabla_h f)(sh). \tag{2.10}$$

Proof. By formula (2.1), we have

$$\begin{aligned}\nabla_h(fg)(t) &= \frac{1}{h}[f(t)g(t) - f(\rho(t))g(\rho(t))] \\ &= \frac{1}{h}[f(t)g(t) - f(t)g(\rho(t)) + f(t)g(\rho(t)) - f(\rho(t))g(\rho(t))] \\ &= f(t)(\nabla_h g)(t) + g(\rho(t))(\nabla_h f)(t),\end{aligned} \tag{2.11}$$

or, alternatively, we have

$$\begin{aligned}\nabla_h(fg)(t) &= \frac{1}{h}[f(t)g(t) - f(\rho(t))g(\rho(t))] \\ &= \frac{1}{h}[f(t)g(t) - f(\rho(t))g(t) + f(\rho(t))g(t) - f(\rho(t))g(\rho(t))] \\ &= g(t)(\nabla_h f)(t) + f(\rho(t))(\nabla_h g)(t).\end{aligned} \tag{2.12}$$

If we take the summation on both sides of (2.11), (2.12), we can obtain the formulas (2.9), (2.10), respectively. This completes the proof. □

Lemma 2.3. Assume $f : (h\mathbb{N})_a \to \mathbb{R}$, and $\nu \in (0, 1)$. Then the following formula holds:

$$({}_a\nabla_h^\nu f)(t) = \frac{(t-a)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) + ({}_a\nabla_{h,*}^\nu f)(t), \tag{2.13}$$

Proof. According to the summation by parts (2.10) and Definition 2.3, we have

$$\begin{aligned}({}_a\nabla_{h,*}^\nu f)(t) &= ({}_a\nabla_h^{-(1-\nu)}(\nabla_h f))(t) \\ &= \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu}} (\nabla_h f)(sh) \\ &= \frac{h}{\Gamma(1-\nu)} \Big[\frac{(t-sh)_h^{\overline{-\nu}} f(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}} - \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \nu(t-\rho(sh))_h^{\overline{-\nu-1}} f(sh) \Big] \\ &= -\frac{(t-a)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu-1}} f(sh) \\ &= -\frac{(t-a)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) + ({}_a\nabla_h^\nu f)(t).\end{aligned}$$

The proof is complete. □

Lemma 2.4. Let $f \in \mathcal{F}_{(h\mathbb{N})_a}$, and $\nu > 0$. Then

$$({}_a\nabla_h^{-\nu} f)(t) = h^\nu \left({}_{\frac{a}{h}}\nabla_1^{-\nu} \widetilde{f} \right) \left(\frac{t}{h} \right), \tag{2.14}$$

where $t \in (h\mathbb{N})_{a+h}$, and $\widetilde{f}(s) = f(sh)$.

Proof. By Definition 2.1, we have

$$\begin{aligned}({}_a\nabla_h^{-\nu}f)(t) &= \frac{h}{\Gamma(\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{\nu-1}}f(sh)\\ &= \frac{h^\nu}{\Gamma(\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(\frac{t}{h}-\rho(s))^{\overline{\nu-1}}\widetilde{f}(s)\\ &= h^\nu\left({}_{\frac{a}{h}}\nabla_1^{-\nu}\widetilde{f}\right)\left(\frac{t}{h}\right).\end{aligned}$$

The proof is complete. □

Lemma 2.5. (See [2, Theorem 3.93]). Assume $\nu\in\mathbb{R}^+$ and $\mu\in\mathbb{R}$ such that μ, $\mu+\nu$, and $\mu-\nu$ are nonnegative integers. Then we have that

(i) $\nabla_a^{-\nu}(t-a)^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\overline{\mu+\nu}},\ \ t\in\mathbb{N}_a,$

(ii) $\nabla_a^{\nu}(t-a)^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}},\quad t\in\mathbb{N}_{a+2}.$

Lemma 2.6. Assume $\nu\in\mathbb{R}^+$ and $\mu\in\mathbb{R}$ such that μ, $\mu+\nu$, and $\mu-\nu$ are nonnegative integers. Then we have that

(i) ${}_a\nabla_h^{-\nu}(t-a)_h^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)_h^{\overline{\mu+\nu}},\quad t\in(h\mathbb{N})_a,$

(ii) ${}_a\nabla_h^{\nu}(t-a)_h^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)_h^{\overline{\mu-\nu}},\quad t\in(h\mathbb{N})_{a+2h}.$

Proof. (i) Let $f(s)=(sh-a)_h^{\overline{\mu}}=h^\mu(s-\frac{a}{h})^{\overline{\mu}}$. Then, using Lemmas 2.4, 2.5, we have

$$\begin{aligned}{}_a\nabla_h^{-\nu}(t-a)_h^{\overline{\mu}} &= h^\nu({}_{\frac{a}{h}}\nabla_1^{-\nu}f)(\frac{t}{h})\\ &= h^{\nu+\nu}({}_{\frac{a}{h}}\nabla_1^{-\nu}(s-\frac{a}{h})^{\overline{\mu}})(\frac{t}{h})\\ &= h^{\mu+\nu}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\Big(\frac{t-a}{h}\Big)^{\overline{\mu+\nu}}\\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)_h^{\overline{\mu+\nu}}.\end{aligned}$$

(ii) By Definition 2.2, we have

$$\begin{aligned}{}_a\nabla_h^{\nu}(t-a)_h^{\overline{\mu}} &= \nabla_h^n\left({}_a\nabla_h^{-(n-\nu)}(t-a)_h^{\overline{\mu}}\right)\\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\nu+1)}\nabla_h^n\left((t-a)_h^{\overline{\mu+n-\nu}}\right)\\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)_h^{\overline{\mu-\nu}}.\end{aligned}$$

This completes the proof. □

Remark 2.2. In Lemma 2.6, when $\mu+\nu$ and $\mu-\nu$ are negative, the conclusion still holds true.

3 Main Results

In this part, we give a new method to show the monotonicity results for $\nu \in (0,1]$.

Theorem 3.1. Assume $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$, $({}_a\nabla_h^\nu f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0,1]$. If

$$f(a+kh) \ge \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} f(a+h) \tag{3.1}$$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}$.

Proof. If $\nu = 1$, it is easy to show that $(\nabla_h f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}$. If $\nu \in (0,1)$, by Lemma 2.1, we have

$$({}_a\nabla_h^\nu f)(t) = \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu-1}} f(sh).$$

We prove that $f(a+nh) \le \frac{\Gamma(\nu+n-1)}{\Gamma(\nu)\Gamma(n)} f(a+h)$ for $n \in \mathbb{N}_1$, by the principle of strong induction. When $t = a+2h$, we have

$$\begin{aligned}({}_a\nabla_h^\nu f)(a+2h) &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+2} (a+2h-\rho(sh))_h^{\overline{-\nu-1}} f(sh) \\ &= h^{-\nu}(-\nu f(a+h) + f(a+2h)) \le 0,\end{aligned}$$

that is,

$$f(a+2h) \le \nu f(a+h).$$

So, we have

$$f(a+2h) - f(a+h) \le (\nu-1) f(a+h) \le 0.$$

Now, we assume $f(a+kh) \le \frac{\Gamma(\nu+k-1)}{\Gamma(\nu)\Gamma(k)} f(a+h)$ for $k = 1, 2, \cdots, n$. When $t = a+(n+1)h$, we have

$$\begin{aligned}({}_a\nabla_h^\nu f)(a+(n+1)h) &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+n+1} (a+(n+1)h-\rho(sh))_h^{\overline{-\nu-1}} f(sh) \\ &\xlongequal{s-\frac{a}{h}-1=k} \frac{h}{\Gamma(-\nu)} \sum_{k=0}^{n} (nh+h-kh)_h^{\overline{-\nu-1}} f(a+kh+h) \\ &= \frac{h}{\Gamma(-\nu)} \sum_{k=0}^{n-1} (nh+h-kh)_h^{\overline{-\nu-1}} f(a+kh+h) \\ &\quad + h^{-\nu} f(a+(n+1)h) \le 0,\end{aligned}$$

By Lemma 2.6, we have

$$ {}_a\nabla_h^{\nu} \frac{(t-a)_h^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{(t-a)_h^{\overline{-1}}}{\Gamma(0)} = 0, $$

that is,

$$ \begin{aligned} &\frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{-\nu-1}} \frac{(sh-a)_h^{\overline{\nu-1}}}{\Gamma(\nu)} \\ &\overset{t=a+nh+h}{\underset{n\in\mathbb{N}_1}{=\!=\!=}} \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+n+1} (a+nh+2h-sh)_h^{\overline{-\nu-1}} \frac{h^{\nu-1}\Gamma(s-\frac{a}{h}+\nu-1)}{\Gamma(\nu)\Gamma(s-\frac{a}{h})} \\ &\overset{k=s-(\frac{a}{h}+1)}{=\!=\!=} \frac{h}{\Gamma(-\nu)} \sum_{k=0}^{n} (nh+h-kh)_h^{\overline{-\nu-1}} \frac{h^{\nu-1}\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} = 0. \end{aligned} $$

Then, we obtain

$$ -\frac{h^{\nu}}{\Gamma(-\nu)} \sum_{k=0}^{n-1} (nh+h-kh)_h^{\overline{-\nu-1}} \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} = \frac{h^{-1}\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)}. $$

So, we have

$$ \begin{aligned} f(a+(n+1)h) &\le -\frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{k=0}^{n-1} (nh+h-kh)_h^{\overline{-\nu-1}} f(a+kh+h) \\ &\le -\frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{k=0}^{n-1} (nh+h-kh)_h^{\overline{-\nu-1}} \frac{\Gamma(\nu+k)}{\Gamma(\nu)\Gamma(k+1)} f(a+h) \\ &= \frac{\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)} f(a+h). \end{aligned} $$

Hence, by the condition (3.1), we have

$$ f(a+(n+1)h) - f(a+nh) \le \frac{\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)} f(a+h) - f(a+nh) \le 0. $$

Therefore, we conclude $(\nabla_h f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}$. The proof is complete. □

Corollary 3.1. Assume $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$, $({}_a\nabla_h^{\nu} f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0,1]$. If

$$ f(a+kh) \le \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} f(a+h) \tag{3.2} $$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$.

Proof. Put $g = -f$. Then $g(a+kh) \ge \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} g(a+h)$. Consequently, each of the hypotheses of Theorem 3.1 is satisfied. Therefore, we conclude that $(\nabla_h g)(t) \le 0$, whence $(\nabla_h f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$. The proof is complete. □

Theorem 3.2. Assume $f : (h\mathbb{N})_a \to \mathbb{R}$, $({}_a\nabla^{\nu}_{h,*}f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0, 1]$. If

$$(1-\nu)f(a+kh) \geq \frac{h^{\nu}((k+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)}\sum_{l=0}^{k-2}(kh+h-lh)_h^{\overline{-\nu-1}}f(a+lh+h) \quad (3.3)$$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \leq 0$ for $t \in (h\mathbb{N})_{a+2h}$.

Proof. If $\nu = 1$, clearly, the conclusion is true. If $\nu \in (0, 1)$, by the formula (2.13), and Lemma 2.1, we have

$$({}_a\nabla^{\nu}_{h,*}f)(t) = -\frac{(t-a)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) + \frac{h}{\Gamma(-\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{-\nu-1}}f(sh).$$

When $t = a + 2h$, we have

$$\begin{aligned}({}_a\nabla^{\nu}_{h,*}f)(a+h) &= -\frac{(2h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) + \frac{h}{\Gamma(-\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+2}(a+2h-\rho(sh))_h^{\overline{-\nu-1}}f(sh)\\ &= -h^{-\nu}(1-\nu)f(a) - h^{-\nu}\nu f(a+h) + h^{-\nu}f(a+2h) \leq 0,\end{aligned}$$

that is,

$$f(a+2h) \leq \nu f(a+h) + (1-\nu)f(a).$$

So, according to the condition (3.3), we have

$$f(a+2h) - f(a+h) \leq 0.$$

When $t = a + (n+1)h$, we have

$$\begin{aligned}&({}_a\nabla^{\nu}_{h,*}f)(a+(n+1)h)\\ &= -\frac{((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) + \frac{h}{\Gamma(-\nu)}\sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+n+1}(a+(n+1)h-\rho(sh))_h^{\overline{-\nu-1}}f(sh)\\ &\overset{s-\frac{a}{h}-1=k}{=\!=\!=} -\frac{((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) + \frac{h}{\Gamma(-\nu)}\sum_{k=0}^{n}(nh+h-kh)_h^{\overline{-\nu-1}}f(a+kh+h)\\ &= -\frac{((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) + \frac{h}{\Gamma(-\nu)}\sum_{k=0}^{n-1}(nh+h-kh)_h^{\overline{-\nu-1}}f(a+kh+h)\\ &\quad + h^{-\nu}f(a+(n+1)h) \leq 0.\end{aligned}$$

So, we obtain

$$\begin{aligned}f(a+(n+1)h) &\leq \frac{h^{\nu}((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)}\sum_{k=0}^{n-1}(nh+h-kh)_h^{\overline{-\nu-1}}f(a+kh+h)\\ &= \frac{h^{\nu}((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)}f(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)}\sum_{k=0}^{n-2}(nh+h-kh)_h^{\overline{-\nu-1}}f(a+kh+h)\\ &\quad + \nu f(a+nh).\end{aligned}$$

Hence, by the condition (3.3), we have

$$\begin{aligned} f(a+(n+1)h)-f(a+nh) &\le \frac{h^{\nu}((n+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) \\ &- \frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{k=0}^{n-2} (nh+h-kh)_h^{\overline{-\nu-1}} f(a+kh+h) \\ &+ (\nu-1) f(a+nh) \le 0. \end{aligned}$$

Thus, we conclude $(\nabla_h f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}$. The proof is complete. □

Corollary 3.2. Assume $f : (h\mathbb{N})_a \to \mathbb{R}$, $({}_a\nabla_{h,*}^{\nu} f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$, $\nu \in (0,1]$. If

$$(1-\nu)f(a+kh) \le \frac{h^{\nu}((k+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} f(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{l=0}^{k-2} (kh+h-lh)_h^{\overline{-\nu-1}} f(a+lh+h) \quad (3.4)$$

for $k \in \mathbb{N}_1$, then $(\nabla_h f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$.

Proof. Let $g = -f$. Then $(1-\nu)g(a+kh) \ge \frac{h^{\nu}((k+1)h)_h^{\overline{-\nu}}}{\Gamma(1-\nu)} g(a) - \frac{h^{\nu+1}}{\Gamma(-\nu)} \sum_{l=0}^{k-2} (kh+h-lh)_h^{\overline{-\nu-1}} g(a+lh+h)$. Consequently, each of the hypotheses of Theorem 3.2 is satisfied. Therefore, we conclude that $(\nabla_h g)(t) \le 0$, whence $(\nabla_h f)(t) \ge 0$ for $t \in (h\mathbb{N})_{a+2h}$. The proof is complete. □

4 Example

Now, we give an example to illustrate Theorem 3.1.

Example 4.1. Consider $f(t) = (\frac{3}{8})^t$, $t \in (h\mathbb{N})_{a+h}$, $({}_a\nabla_h^{\nu} f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}^{a+3h}$, $\nu = \frac{1}{2}$, $a = 0$, $h = 1$. If

$$f(a+kh) \ge \frac{\Gamma(k+\nu)}{\Gamma(\nu)\Gamma(k+1)} f(a+h) \quad (4.1)$$

for $k \in \mathbb{N}_1^2$, then $(\nabla_h f)(t) \le 0$ for $t \in (h\mathbb{N})_{a+2h}^{a+3h}$.

When $t = a+2h$, we have

$$\begin{aligned} ({}_a\nabla_h^{\nu} f)(a+2h) &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+2} (a+2h-\rho(sh))_h^{\overline{-\nu-1}} f(sh) \\ &= \frac{h}{\Gamma(-\nu)} \left[(2h)_h^{\overline{-\nu-1}} f(a+h) + (h)_h^{\overline{-\nu-1}} f(a+2h) \right] \\ &= -\nu h^{-\nu} f(a+h) + h^{-\nu} f(a+2h) \\ &= -\frac{1}{2} \cdot \frac{3}{8} + \left(\frac{3}{8}\right)^2 \end{aligned}$$

$$= -\frac{3}{64} \leq 0,$$

which implies

$$f(a+2h) \leq \nu f(a+h).$$

When $t = a+3h$, we have

$$\begin{aligned}
({}_{a}\nabla_{h}^{\nu} f)(a+3h) &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}+1}^{\frac{a}{h}+3} (a+3h-\rho(sh))_{h}^{\overline{-\nu-1}} f(sh) \\
&= \frac{h}{\Gamma(-\nu)} \left[(3h)_{h}^{\overline{-\nu-1}} f(a+h) + (2h)_{h}^{\overline{-\nu-1}} f(a+2h) + (h)_{h}^{\overline{-\nu-1}} f(a+3h) \right] \\
&= -\frac{\nu(1-\nu)}{2} f(a+h) - \nu h^{-\nu} f(a+2h) + h^{-\nu} f(a+3h) \\
&= -\left(\frac{1}{2}\right)^{3} \cdot \frac{3}{8} - \frac{1}{2} \cdot \left(\frac{3}{8}\right)^{2} + \left(\frac{3}{8}\right)^{3} \\
&= -\frac{33}{512} \leq 0,
\end{aligned}$$

this yields

$$f(a+3h) \leq \frac{\nu(\nu+1)}{2} f(a+h).$$

Further, when $n = 1$, we obtain

$$\frac{3}{8} \geq \frac{\Gamma(1+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2)} \cdot \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{8},$$

that is,

$$f(a+h) \geq \frac{\Gamma(1+\nu)}{\Gamma(\nu)\Gamma(1+1)} f(a+h) = \nu f(a+h).$$

When $n = 2$, we have

$$\left(\frac{3}{8}\right)^{2} \geq \frac{\Gamma(2+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(3)} \cdot \frac{3}{8} = \frac{3}{8} \cdot \frac{3}{8},$$

that is,

$$f(a+2h) \geq \frac{\Gamma(2+\nu)}{\Gamma(\nu)\Gamma(2+1)} f(a+h) = \frac{\nu(\nu+1)}{2} f(a+h).$$

Hence, we conclude that

$$f(a+3h) \leq \frac{\nu(\nu+1)}{2} f(a+h) \leq f(a+2h) \leq \nu f(a+h) \leq f(a+h).$$

Therefore, we conclude that $f(t)$ is nonincreasing on $(h\mathbb{N})_{a+2h}^{a+3h}$.

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