A new high-order accurate conservative finite difference scheme for the coupled nonlinear Schrödinger equations

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Abstract

In this paper, a fourth-order accurate conservative finite difference scheme for solving the coupled nonlinear Schrödinger (CNLS) equations is proposed. Conservation of the discrete energy and masses, priori estimates, existence and uniqueness of numerical solutions, convergence with second-order in time and fourth-order in space as well as stability of the present scheme are proved by discrete energy method. A convergent iterative method for the present scheme is developed. Numerical experiments are given to support the theoretical analysis.

Key words: CNLS equations, finite difference scheme, conservation, high accuracy

1. Introduction

The coupled nonlinear Schrödinger (CNLS) equations are one of the most important models in quantum mechanics. This system has been used in many other nonlinear problems such as optics, seismology, and plasma physics [1, 2]. Recently, numerical methods for various CNLS equations have become hot topics. Many researchers numerically investigated the CNLS equations by using the Galerkin method, the finite difference method, the symplectic geometry method, the spectral method, etc [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In this paper, we consider the following CNLS equations [2]:

$$iu_t + ku_{xx} + (|u|^2 + \beta|v|^2)u = 0, \quad x_l \le x \le x_r, \quad 0 < t \le T, \tag{1.1}$$

$$iv_t + kv_{xx} + (|v|^2 + \beta|u|^2)v = 0, \quad x_l \le x \le x_r, \quad 0 < t \le T, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x_l \le x \le x_r, \tag{1.3}$$

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$$u(x_l,t)=u(x_r,t)=0,\quad v(x_l,t)=v(x_r,t)=0,\quad 0<t\le T, \tag{1.4}$$

where $i=\sqrt{-1}$, $u_0(x)$ and $v_0(x)$ are two known smooth functions, $u(x,t)$ and $v(x,t)$ are two unknown complex-value functions, k describes the dispersion in the optic fiber, β is defined as birefringent optical fiber coupling parameter. When $k=\beta=1$, this system is called the Manakov equations, and when $\beta=0$, it is becomes the decoupled nonlinear Schrödinger equations. In the above cases, the system (1.1)-(1.4) is integrable. Furthermore, the system (1.1)-(1.4) satisfies the following mass and energy conservative properties [2]:

$$M_1(t)=\int_{x_l}^{x_r}|u^2|\mathrm{d}x=\int_{x_l}^{x_r}|u_0|^2\mathrm{d}x=M_1(0), \tag{1.5}$$

$$M_2(t)=\int_{x_l}^{x_r}|v^2|\mathrm{d}x=\int_{x_l}^{x_r}|v_0|^2\mathrm{d}x=M_2(0), \tag{1.6}$$

$$\begin{aligned}E(t)&=\int_{x_l}^{x_r}\left[\frac{k}{2}(|u_x|^2+|v_x|^2)-\frac{1}{4}(|u|^4+|v|^4)-\frac{\beta}{2}|u|^2|v|^2\right]\mathrm{d}x\\&=\int_{x_l}^{x_r}\left[\frac{k}{2}(|(u_0)_x|^2+|(v_0)_x|^2)-\frac{1}{4}(|u_0|^4+|v_0|^4)-\frac{\beta}{2}|u_0|^2|v_0|^2\right]\mathrm{d}x\\&=E(0),\quad t>0.\end{aligned} \tag{1.7}$$

For $k=1$, the problem (1.1)-(1.4) is strongly CNLS equations. Six point multi-symplectic formulations were proposed in [17, 18]. It has excellent long-time numerical behaviour and energy conservation property. Ismail and Taha [19] proposed a linearly implicit conservative method, which is second-order accurate in both time and space. In [20, 21], a symplectic difference scheme and a two-level finite difference scheme were developed. It was proved that both schemes are conservative, uniquely solvable, convergent with $O(\tau^2+h^2)$ and stable by the discrete energy method. Nonlinear and linear compact finite difference schemes were developed by Hu and Zhang [22]. These schemes are fourth-order accurate in space and second-order accurate in time. The existence and uniqueness, convergence and stability of the schemes were proved by using the matrix theory after transforming into matrix form.

For $k=1/2$ or other values, Ismail and Almari [23, 24] developed a finite difference scheme and an explicit Runge-Kutta scheme. These schemes are fourth-order accurate in space and second-order accurate in time, and were proved to be stable by using the von Neuman stability analysis. In [25], a splitting method from class of symplectic integrators and the multi-symplectic six-point scheme were considered for the integration of CNLS equations with periodic solutions. Galerkin method was proposed to solve the CNLS equations in [26]. It is unconditionally stable and second-order accurate both in time and in space. Nonlinear implicit difference schemes and linear implicit difference scheme were proposed in [27, 28, 29]. Uniqueness and existence of the difference solutions and second-order convergence in L_∞ norm were given. In [30, 31, 32], high order compact and non-compact schemes were constructed. All of them are conservative, convergent and stable. The Fourier psedospectral method, the Crank-Nicolson method and leap-frog method were developed in [33]. It is energy and mass conserved, uniquely solvable and unconditionally stable.

In this article, a new fourth-order accurate conservative finite difference scheme for the problem (1.1)-(1.4) is proposed. This scheme is coupled two-time level nonlinear, conserved, effective and high-order accurate. A priori estimates, convergence with second-order in time and fourth-order in space and stability by norm $\|\cdot\|_\infty$ are proved by the discrete energy method for the present scheme.

The rest of this paper are arranged as follows. In Section 2, a high-order accurate finite difference scheme is constructed and some lemmas are given. In Sections 3-5, discrete conservative laws, a priori estimates, existence and solvability, convergence with $O(\tau^2+h^4)$ and stability are proved by using the discrete energy method. In Section 6, we give a convergent iterative algorithm for the finite difference scheme. In Section 7, numerical examples are presented to show that computed results support our theoretical analysis. Finally, we give some concluding remarks in the last section.

2. Construction of finite difference scheme

In this section, we propose a coupled nonlinear two-time level, conservative, and high-order accurate finite difference scheme for the problem (1.1)-(1.4). Let $h = (x_r - x_l)/J$ and $\tau = T/N$ be the space-step and time-step, respectively, where J and N are given to be two positive integers. The point (x_j, t_n) is defined as $t_n = n\tau$, $n = 0, 1, \dots, N$ and $x_j = x_l + jh$, $j = -1, 0, \dots, J, J+1$. We denote $u_j^n \approx u(x_j, t_n)$, $v_j^n \approx v(x_j, t_n)$ and the space of complex discrete functions

$$\mathbb{Z}_h^0 = \{u = (u_j) | u_{-1} = u_0 = u_1 = u_{J-1} = u_J = u_{J+1} = 0, \quad j = -1, 0, \dots, J, J+1\}.$$

For any complex discrete functions $u^n, v^n \in \mathbb{Z}_h^0$, we define the difference operators, discrete inner product, L^p-norm and maximum-norm as follows:

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \quad (u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$

$$(u_j^n)_t = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2}, \quad \langle u^n, v^n \rangle = h \sum_{j=1}^{J-1} u_j^n \bar{v}_j^n,$$

$$\|u^n\|_p = \sqrt[p]{h \sum_{j=1}^{J-1} |u_j^n|^p}, \quad \|u^n\|_\infty = \max_{1 \le j \le J-1} |u_j^n|, \quad 1 < p < +\infty.$$

According to the Taylor expansion, we have the following formula [34]:

$$\frac{4}{3}(\psi_j)_{x\bar{x}} - \frac{1}{3}(\psi_j)_{\hat{x}\hat{x}} = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x_j) + O(h^4),$$

where ψ is a smooth function. Let C be a positive constant independing on τ and h, but may have different values at different occurrence. Based on the notations and expressions, we give the following finite difference scheme:

$$i(u_j^n)_t + \frac{4k}{3}(u_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(u_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + \frac{1}{2}\left[|u_j^{n+1}|^2 + |u_j^n|^2 + \beta(|v_j^{n+1}|^2 + |v_j^n|^2)\right]u_j^{n+\frac{1}{2}} = 0,$$

$$1 \leq j \leq J-1, \quad 0 \leq n \leq N-1, \tag{2.1}$$

$$i(v_j^n)_t + \frac{4k}{3}(v_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(v_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + \frac{1}{2}\left[|v_j^{n+1}|^2 + |v_j^n|^2 + \beta(|u_j^{n+1}|^2 + |u_j^n|^2)\right]v_j^{n+\frac{1}{2}} = 0,$$

$$1 \leq j \leq J-1, \quad 0 \leq n \leq N-1, \tag{2.2}$$

$$u_j^0 = u_0(x_j), \quad v_j^0 = v_0(x_j), \quad -1 \leq j \leq J+1, \tag{2.3}$$

$$u^n \in \mathbb{Z}_h^0, \quad v^n \in \mathbb{Z}_h^0, \quad 1 \leq n \leq N-1. \tag{2.4}$$

To analyze the discrete conservative property of the difference scheme, existence and uniqueness of the numerical solution, convergence and stability, we need the following lemmas.

Lemma 2.1. *(See [34]) For any discrete functions $u^n, v^n \in \mathbb{Z}_h^0$, we have*

$$Re\langle u_{x\bar{x}}^n, v^n\rangle = -Re\langle u_x^n, v_x^n\rangle = Re\langle u^n, v_{x\bar{x}}^n\rangle, \quad Re\langle u_{\hat{x}\hat{x}}^n, v^n\rangle = -Re\langle u_{\hat{x}}^n, v_{\hat{x}}^n\rangle = Re\langle u^n, v_{\hat{x}\hat{x}}^n\rangle.$$

Especially,

$$Re\langle u_{x\bar{x}}^n, u^n\rangle = -\|u_x^n\|^2, \quad Re\langle u_{\hat{x}\hat{x}}^n, u^n\rangle = -\|u_{\hat{x}}^n\|^2, \quad \|u_{\hat{x}}^n\|^2 \leq \|u_x^n\|^2.$$

Lemma 2.2. *For any discrete complex functions $u^n, v^n \in \mathbb{Z}_h^0$, we have*

$$Im\langle u_{x\bar{x}}^n, u^n\rangle = 0, \quad Im\langle u_{\hat{x}\hat{x}}^n, u^n\rangle = 0, \quad Im\langle |v^n|u^n, u^n\rangle = 0.$$

Proof. Setting $u^n = a^n + ib^n$, $a^n, b^n \in \mathbb{R}$, we have

$$Im\langle u_{x\bar{x}}^n, u^n\rangle$$

$$= Im\left\{h\sum_{j=1}^{J-1}\left[(a_j^n + ib_j^n)_{x\bar{x}}(a_j^n - ib_j^n)\right]\right\}$$

$$= Im\Big\{h\sum_{j=1}^{J-1}\big[\big((a_j^n)_{x\bar{x}}a_j^n + (b_j^n)_{x\bar{x}}b_j^n\big) + i\big((b_j^n)_{x\bar{x}}a_j^n - (a_j^n)_{x\bar{x}}b_j^n\big)\big]\Big\}$$

$$= Im\Big\{-h\sum_{j=1}^{J-1}\big[[(a_j^n)_x]^2 + [(b_j^n)_x]^2\big] + ih\sum_{j=1}^{J-1}\frac{(b_j^n)_x - (b_{j-1}^n)_x}{h}a_j^n - ih\sum_{j=1}^{J-1}\frac{(a_j^n)_x - (a_{j-1}^n)_x}{h}b_j^n\Big\}$$

$$= Im\Big\{i\sum_{j=1}^{J-1}(b_j^n)_x a_j^n - i\sum_{j=0}^{J-2}(b_j^n)_x a_{j+1}^n - i\sum_{j=1}^{J-1}(a_j^n)_x b_j^n + i\sum_{j=0}^{J-2}(a_j^n)_x b_{j+1}^n\Big\}$$

$$= -Im\Big\{ih\sum_{j=1}^{J-1}(b_j^n)_x\frac{a_{j+1}^n - a_j^n}{h} - ih\sum_{j=1}^{J-1}(a_j^n)_x\frac{b_{j+1}^n - b_j^n}{h}\Big\}$$

$$= -Im\Big\{ih\sum_{j=1}^{J-1}\big[(b_j^n)_x(a_j^n)_x - (a_j^n)_x(b_j^n)_x\big]\Big\} = 0.$$

Similarly, we have

$$Im\langle u_{\hat{x}\hat{x}}^n, u^n\rangle = 0, \quad Im\langle |v^n|u^n, u^n\rangle = 0.$$

This completes the proof.

3. Discrete conservative laws and priori estimates

Theorem 3.1. *The finite difference scheme (2.1)-(2.4) is conservative in the sense*

$$Q_1^n = Q_1^{n-1} = \ldots = Q_1^0, \quad n = 0, 1, \ldots, N \tag{3.1}$$

$$Q_2^n = Q_2^{n-1} = \ldots = Q_2^0, \quad n = 0, 1, \ldots, N, \tag{3.2}$$

$$E^n = E^{n-1} = \ldots = E^0, \quad n = 0, 1, \ldots, N, \tag{3.3}$$

where $Q_1^n = \|u^n\|^2$ *and* $Q_2^n = \|v^n\|^2$ *are discrete masses, and*

$$E^n = \frac{2k}{3}(\|u_x^n\|^2 + \|v_x^n\|^2) - \frac{k}{6}(\|u_{\hat{x}}^n\|^2 + \|v_{\hat{x}}^n\|^2) - \frac{h}{4}\sum_{j=1}^{J-1}(|u_j^n|^4 + |v_j^n|^4 + 2\beta|u_j^n|^2|v_j^n|^2)$$

is discrete energy.

Proof. Computing the inner product of Eq. (2.1) with $u^{n+1}+u^n$ and taking the imaginary part by Lemma 2.2, we obtain

$$\frac{1}{\tau}(\|u^{n+1}\|^2-\|u^n\|^2)+\frac{1}{2}Im\langle\big[|u^{n+1}|^2+|u^n|^2+\beta(|v^{n+1}|^2+|v^n|^2)\big]u^{n+\frac{1}{2}},u^{n+1}+u^n\rangle=0. \tag{3.4}$$

Noticing

$$\frac{1}{2}Im\langle\big[|u^{n+1}|^2+|u^n|^2+\beta(|v^{n+1}|^2+|v^n|^2)\big]u^{n+\frac{1}{2}},u^{n+1}+u^n\rangle$$

$$=Im\langle|u^{n+1}|^2+|u^n|^2+\beta(|v^{n+1}|^2+|v^n|^2),|u^{n+\frac{1}{2}}|^2\rangle=0,$$

we have

$$\frac{1}{\tau}(\|u^{n+1}\|^2-\|u^n\|^2)=0,$$

which implies $Q_1^n=Q_1^{n-1}=\ldots=Q_1^0$. Similarly, we can prove Eq. (3.2).

Computing the inner product of Eq. (2.1) with $u^{n+1}-u^n$, the inner product of Eq. (2.2) with $v^{n+1}-v^n$, and taking the real part by Lemma 2.1, we obtain

$$-\frac{2k}{3}(\|u_x^{n+1}\|^2-\|u_x^n\|^2)+\frac{k}{6}(\|u_{\hat{x}}^{n+1}\|^2-\|u_{\hat{x}}^n\|^2)+\frac{1}{2}Re\langle\big[|u^{n+1}|^2+|u^n|^2$$

$$+\beta(|v_x^{n+1}|^2+|v^n|^2)\big]u^{n+\frac{1}{2}},u^{n+1}-u^n\rangle=0, \tag{3.5}$$

$$-\frac{2k}{3}(\|v_x^{n+1}\|^2-\|v_x^n\|^2)+\frac{k}{6}(\|v_{\hat{x}}^{n+1}\|^2-\|v_{\hat{x}}^n\|^2)+\frac{1}{2}Re\langle\big[|v^{n+1}|^2+|v^n|^2$$

$$+\beta(|u^{n+1}|^2+|u^n|^2)\big]v^{n+\frac{1}{2}},v^{n+1}-v^n\rangle=0. \tag{3.6}$$

Noticing that

$$\frac{1}{2}Re\langle\big[|u^{n+1}|^2+|u^n|^2+\beta(|v^{n+1}|^2+|v^n|^2)\big]u^{n+\frac{1}{2}},u^{n+1}-u^n\rangle$$

$$=\frac{h}{4}\sum_{j=1}^{J-1}\big[|u_j^{n+1}|^2+|u_j^n|^2+\beta(|v_j^{n+1}|^2+|v_j^n|^2)\big](|u_j^{n+1}|^2-|u_j^n|^2)$$

$$=\frac{h}{4}\sum_{j=1}^{J-1}\big[|u_j^{n+1}|^4-|u_j^n|^4+\beta(|v_j^{n+1}|^2+|v_j^n|^2)(|u_j^{n+1}|^2-|u_j^n|^2)\big]. \tag{3.7}$$

Similarly, we have

$$\frac{1}{2}Re\langle\big[|v^{n+1}|^2+|v^n|^2+\beta(|u^{n+1}|^2+|u^n|^2)\big]v^{n+\frac{1}{2}},v^{n+1}-v^n\rangle$$

$$=\frac{h}{4}\sum_{j=1}^{J-1}\big[|v_j^{n+1}|^4-|v_j^n|^4+\beta(|u_j^{n+1}|^2+|u_j^n|^2)(|v_j^{n+1}|^2-|v_j^n|^2)\big]. \tag{3.8}$$

Adding Eq. (3.5) to Eq. (3.6), and considering Eqs. (3.7)-(3.8), we have

$$\frac{2k}{3}(\|u_x^{n+1}\|^2+\|v_x^{n+1}\|^2)-\frac{k}{6}(\|u_{\hat{x}}^{n+1}\|^2+\|v_{\hat{x}}^{n+1}\|^2)$$

$$-\frac{h}{4}\sum_{j=1}^{J-1}(|u_j^{n+1}|^4+|v_j^{n+1}|^4+2\beta|u_j^{n+1}|^2|v_j^{n+1}|^2)$$

$$=\frac{2k}{3}(\|u_x^n\|^2+\|v_x^n\|^2)-\frac{k}{6}(\|u_{\hat{x}}^n\|^2+\|v_{\hat{x}}^n\|^2)-\frac{h}{4}\sum_{j=1}^{J-1}(|u_j^n|^4+|v_j^n|^4+2\beta|u_j^n|^2|v_j^n|^2),$$

which implies $E^n=E^{n-1}=\ldots=E^0$. This completes the proof.

Lemma 3.1. *(See [28]) For any discrete function $u^n\in\mathbb{Z}_h^0$, we have*

$$\|u^n\|_p\le C(\|u_x^n\|^\alpha\|u^n\|^{1-\alpha}+\|u^n\|),$$

where $\alpha=\frac{1}{2}-\frac{1}{p}$, $p\ge 2$, C is a constant independing on p and h.

Lemma 3.2. *(See [28]) For any $x\ge 0$, $y\ge 0$ and $p\ge 1$, we have*

$$(x+y)^p\le 2^{p-1}(x^p+y^p).$$

Lemma 3.3 (Discrete Sobolev's inequality). *(See [31]) For any discrete function $u^n\in\mathbb{Z}_h^0$, there exist two positive constants C_1 and C_2 such that*

$$\|u^n\|_\infty\le C_1\|u^n\|+C_2\|u_x^n\|.$$

Theorem 3.2. *The finite difference scheme (2.1)-(2.4) satisfies*

$$\|u^n\|\le C,\quad \|u_x^n\|\le C,\quad \|u^n\|_\infty\le C,\quad \|v^n\|\le C,\quad \|v_x^n\|\le C,\quad \|v^n\|_\infty\le C.$$

Proof. It follows from Eq. (3.1) and Eq. (3.2) that $\|u^n\|\le C$, $\|v^n\|\le C$. According to Eq. (3.3) and Lemma 2.1, we have

$$\frac{k}{2}(\|u_x^n\|^2+\|v_x^n\|^2)\le\frac{2k}{3}(\|u_x^n\|^2+\|v_x^n\|^2)-\frac{k}{6}(\|u_{\hat{x}}^n\|^2+\|v_{\hat{x}}^n\|^2)$$

$$= \frac{h}{4}\sum_{j=1}^{J-1}(|u_j^n|^4 + |v_j^n|^4 + 2\beta|u_j^n|^2|v_j^n|^2) + E^0. \tag{3.9}$$

Applying Lemmas 3.1-3.2 with $p = 4$, we have

$$\begin{aligned}
&\frac{h}{4}\sum_{j=1}^{J-1}(|u_j^n|^4 + |v_j^n|^4 + 2\beta|u_j^n|^2|v_j^n|^2) \\
&\leq \frac{(1+2\beta)h}{4}\sum_{j=1}^{J-1}(|u_j^n|^4 + |v_j^n|^4) \\
&\leq \frac{(1+2\beta)C^4}{4}\left[(\|u_x^n\|^{\frac{1}{4}}\|u^n\|^{\frac{3}{4}} + \|u^n\|)^4 + (\|v_x^n\|^{\frac{1}{4}}\|v^n\|^{\frac{3}{4}} + \|v^n\|)^4\right] \\
&\leq 2(1+2\beta)C^4\left(\|u_x^n\|\|u^n\|^3 + \|u^n\|^4 + \|v_x^n\|\|v^n\|^3 + \|v^n\|^4\right) \\
&\leq (1+2\beta)C^4\left[\epsilon_1(\|u_x^n\|^2 + \|v_x^n\|^2) + \frac{1}{\epsilon_1}(\|u^n\|^6 + \|v^n\|^6) + 2(\|u^n\|^4 + \|v^n\|^4)\right] \\
&\leq (1+2\beta)C^4\left[\epsilon_1(\|u_x^n\|^2 + \|v_x^n\|^2) + \frac{1}{\epsilon_1}(\|u^0\|^6 + \|v^0\|^6) + 2(\|u^0\|^4 + \|v^0\|^4)\right]
\end{aligned} \tag{3.10}$$

for any positive constant ϵ_1. Letting $\epsilon_1 = k/[4(1+2\beta)C^4]$, substituting Eq. (3.10) to Eq. (3.9), we have

$$\|u_x^n\|^2 + \|v_x^n\|^2 \leq \frac{16(1+2\beta)^2C^4}{k^2}(\|u^0\|^6 + \|v^0\|^6) + \frac{8(1+2\beta)C^4}{k}(\|u^0\|^4 + \|v^0\|^4) + \frac{4E^0}{k},$$

which implies $\|u_x^n\| \leq C$, $\|v_x^n\| \leq C$. Hence, $\|u^n\|_\infty \leq C$ and $\|v^n\|_\infty \leq C$ by Lemma 3.3. This completes the proof.

4. Existence

To show the existence of the numerical solution for the difference scheme (2.1)-(2.4), we shall use the following fixed point theorem of Brouwer.

Lemma 4.1 (Brouwer theorem). *(See [20, 28]) Suppose that $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is a finite dimensional Hilbert space with the inner product, $\|\cdot\|$ is the associated norm, and $\mathscr{F} : \mathscr{H} \mapsto \mathscr{H}$ is continuous. Moreover, assume that*

$$\exists\lambda > 0, \quad \forall\chi \in \mathscr{H}, \quad \|\chi\| = \lambda, \quad Re\langle\mathscr{F}(\chi), \chi\rangle \geq 0.$$

Then there exists $\chi^* \in \mathscr{H}$ such that $\mathscr{F}(\chi^*) = 0$ and $\|\chi^*\| \leq \lambda$.

Theorem 4.1. *The solutions of the finite difference scheme (2.1)-(2.4) exist.*

Proof. Suppose that $u^0, u^1, \ldots, u^n$ and $v^0, v^1, \ldots, v^n$ satisfy the finite difference scheme (2.1)-(2.4), then we prove that there exist u^{n+1} and ρ^{n+1} satisfying Eqs. (2.1)-(2.4). For a fixed n, we rewrite Eq. (2.1) and Eq. (2.2) in the term of

$$2((W_1)_j - u_j^n) - \frac{4ik\tau}{3}((W_1)_j)_{x\bar{x}} + \frac{ik\tau}{3}((W_1)_j)_{\hat{x}\hat{x}} - \frac{i\tau}{2}\big[|2(W_1)_j - u_j^n|^2 + |u_j^n|^2$$

$$+ \beta(|2(W_2)_j - v_j^n|^2 + |v_j^n|^2)\big](W_1)_j = 0, \tag{4.1}$$

$$2((W_2)_j - v_j^n) - \frac{4ik\tau}{3}((W_2)_j)_{x\bar{x}} + \frac{ik\tau}{3}((W_2)_j)_{\hat{x}\hat{x}} - \frac{i\tau}{2}\big[|2(W_2)_j - v_j^n|^2 + |v_j^n|^2$$

$$+ \beta(|2(W_1)_j - u_j^n|^2 + |u_j^n|^2)\big](W_2)_j = 0, \tag{4.2}$$

for $1 \le j \le J-1, 0 \le n \le N-1$, where $(W_1, W_2) = (u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}})$. Let $\mathbb{Z}_{\Delta h} = \{W = (W_1, W_2) | W_1, W_2 \in \mathbb{Z}_h^0\}$ and define

$$\langle W, W' \rangle = \langle (W_1, W_2), (W_1', W_2') \rangle = \langle W_1, W_1' \rangle + \langle W_2, W_2' \rangle, \quad \|W\|^2 = \|W_1\|^2 + \|W_2\|^2.$$

Define the map $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) : \mathbb{Z}_{\Delta h} \mapsto \mathbb{Z}_{\Delta h}$ by

$$\mathscr{F}_1(W_1) = 2(W_1 - u^n) - \frac{4ik\tau}{3}(W_1)_{x\bar{x}} + \frac{ik\tau}{3}(W_1)_{\hat{x}\hat{x}} - \frac{i\tau}{2}\big[|2W_1 - u^n|^2 + |u^n|^2$$

$$+ \beta(|2W_2 - v^n|^2 + |v^n|^2)\big]W_1, \tag{4.3}$$

$$\mathscr{F}_2(W_2) = 2(W_2 - v^n) - \frac{4ik\tau}{3}(W_2)_{x\bar{x}} + \frac{ik\tau}{3}(W_2)_{\hat{x}\hat{x}} - \frac{i\tau}{2}\big[|2W_2 - v^n|^2 + |v^n|^2$$

$$+ \beta(|2W_1 - u^n|^2 + |u^n|^2)\big]W_2, \quad \forall W \in Z_{\Delta h}. \tag{4.4}$$

Computing the inner product of Eqs. (4.3)-(4.4) with $W = (W_1, W_2)$ and taking the real part, we have

$$Re\langle \mathscr{F}(W), W \rangle$$

$$= Re\langle \mathscr{F}_1(W_1), W_1 \rangle + Re\langle \mathscr{F}_2(W_2), W_2 \rangle$$

$$= 2(\|W_1\|^2 + \|W_2\|^2) - 2(Re\langle u^n, W_1 \rangle + Re\langle v^n, W_2 \rangle)$$

$$+ Re\Big\{\frac{4ik\tau}{3}\sum_{j=1}^{J-1}\big[|((W_1)_j)_x|^2+|((W_2)_j)_x|^2\big]-\frac{ik\tau}{3}\sum_{j=1}^{J-1}\big[|((W_1)_j)_{\hat{x}}|^2+|((W_2)_j)_{\hat{x}}|^2\big]\Big\}$$

$$- Re\Big\{\frac{i\tau h}{2}\sum_{j=1}^{J-1}\big[|2(W_1)_j-u_j^n|^2+|u_j^n|^2+\beta(|2(W_2)_j-v_j^n|^2+|v_j^n|^2)\big]|(W_1)_j|^2\Big\}$$

$$- Re\Big\{\frac{i\tau h}{2}\sum_{j=1}^{J-1}\big[|2(W_2)_j-v_j^n|^2+|v_j^n|^2+\beta(|2(W_1)_j-u_j^n|^2+|u_j^n|^2)\big]|(W_2)_j|^2\Big\}$$

$$\geq 2\|W\|^2-(\|u^n\|^2+\|W_1\|^2+\|v^n\|^2+\|W_2\|^2)$$

$$=\|W\|^2-(\|u^n\|^2+\|v^n\|^2), \tag{4.5}$$

which implies $Re\langle\mathscr{F}(W),W\rangle\geq 0$ for $\forall W\in\mathbb{Z}_{\Delta h}$ when $\|W\|^2=\|u^n\|^2+\|v^n\|^2+1$. By Lemma 4.1, we conclude that there exists $W^*\in Z_{\Delta h}$ such that $\mathscr{F}(W^*)=0$. Hence, there exist $u^{n+1}=2W_1^*-u^n$ and $v^{n+1}=2W_2^*-v^n$ satisfying the difference scheme (2.1)-(2.4). This completes the proof.

5. Convergence, stability and Solvability

Lemma 5.1 (Discrete Gronwall inequality). *(See [34]) Suppose that $\{G^n\}_{n=0}^{\infty}$ is non-negative sequences and satisfies*

$$G^0\leq A,\quad G^n\leq A+B\tau\sum_{i=0}^{n-1}G^i,\quad n=1,2,\dots,$$

where A and B are non-negative constants. Then G satisfies

$$G^n\leq Ae^{Bn\tau},\quad n=0,1,2,\dots.$$

Lemma 5.2. *(See [28]) For any discrete complex functions $U^n,V^n,u^n,v^n\in\mathbb{Z}_h^0$, we have*

$$\big||U^n|^2V^n-|u^n|^2v^n\big|\leq\big(\max\{|U^n|,|V^n|,|u^n|,|v^n|\}\big)^2\cdot\big(2|U^n-u^n|+|V^n-v^n|\big).$$

Theorem 5.1. *Suppose that $u(x,t),v(x,t)\in C_{x,t}^{6,3}$, then the solutions of the finite difference scheme (2.1)-(2.4) converge to the solutions of the problem (1.1)-(1.4) and the rate of the convergence is $O(\tau^2+h^4)$ by norm $\|\cdot\|_\infty$.*

Proof. Let $e_j^n = U_j^n - u_j^n$, $f_j^n = V_j^n - v_j^n$, where $U_j^n = u(x_j, t_n)$ and $V_j^n = v(x_j, t_n)$ are the solutions of the problem (1.1)-(1.4). Then we have the following error equations as

$$i(e_j^n)_t + \frac{4k}{3}(e_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(e_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + (g_1)_j^n = r_j^n, \quad 1 \le j \le J-1, \quad 0 \le n \le N-1, \tag{5.1}$$

$$i(f_j^n)_t + \frac{4k}{3}(f_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(f_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + (g_2)_j^n = r_j^n, \quad 1 \le j \le J-1, \quad 0 \le n \le N-1, \tag{5.2}$$

$$e_j^0 = 0, \quad f_j^0 = 0, \quad -1 \le j \le J+1, \tag{5.3}$$

$$e^n \in \mathbb{Z}_h^0, \quad f^n \in \mathbb{Z}_h^0, \quad 1 \le n \le N, \tag{5.4}$$

where

$$(g_1)_j^n = \frac{1}{2}\big\{\big[|U_j^{n+1}|^2 + |U_j^n|^2 + \beta(|V_j^{n+1}|^2 + |V_j^n|^2)\big]U_j^{n+\frac{1}{2}}$$

$$- \big[|u_j^{n+1}|^2 + |u_j^n|^2 + \beta(|v_j^{n+1}|^2 + |v_j^n|^2)\big]u_j^{n+\frac{1}{2}}\big\},$$

$$(g_2)_j^n = \frac{1}{2}\big\{\big[|V_j^{n+1}|^2 + |V_j^n|^2 + \beta(|U_j^{n+1}|^2 + |U_j^n|^2)\big]V_j^{n+\frac{1}{2}}$$

$$- \big[|v_j^{n+1}|^2 + |v_j^n|^2 + \beta(|u_j^{n+1}|^2 + |u_j^n|^2)\big]v_j^{n+\frac{1}{2}}\big\}.$$

Using the Taylor expansion, we get $|r_j^n| \le C(\tau^2 + h^4)$ and $|s_j^n| \le C(\tau^2 + h^4)$, where C is a positive constant independing on τ and h.

Computing the inner product of Eq. (5.1) with $e^{n+1} + e^n$ and the inner product of Eq. (5.2) with $f^{n+1} + f^n$, and taking the imaginary part, we obtain

$$\frac{1}{\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) + Im\langle (g_1)^n, e^{n+1} + e^n\rangle = Im\langle r^n, e^{n+1} + e^n\rangle, \tag{5.5}$$

$$\frac{1}{\tau}(\|f^{n+1}\|^2 - \|f^n\|^2) + Im\langle (g_2)^n, f^{n+1} + f^n\rangle = Im\langle s^n, f^{n+1} + f^n\rangle. \tag{5.6}$$

Applying Lemma 5.2, we have

$$|(g_1)_j^n| \le \frac{1}{2}\Big(\big||U_j^{n+1}|^2 U_j^{n+\frac{1}{2}} - |u_j^{n+1}|u_j^{n+\frac{1}{2}}\big| + \big||U_j^n|^2 U_j^{n+\frac{1}{2}} - |u_j^n|u_j^{n+\frac{1}{2}}\big|\Big)$$

$$+\frac{\beta}{2}\Big(\big||V_j^{n+1}|^2U_j^{n+\frac{1}{2}}-|v_j^{n+1}|u_j^{n+\frac{1}{2}}\big|+\big||V_j^{n}|^2U_j^{n+\frac{1}{2}}-|v_j^{n}|u_j^{n+\frac{1}{2}}\big|\Big)$$

$$\leq\frac{1}{2}\Big(\big(\max\big\{|U_j^{n+1}|,|U_j^{n+\frac{1}{2}}|,|u_j^{n+1}|,|u_j^{n+\frac{1}{2}}|\big\}\big)^2\cdot(2|e_j^{n+1}|+|e_j^{n+\frac{1}{2}}|)$$

$$+\big(\max\big\{|U_j^{n}|,|U_j^{n+\frac{1}{2}}|,|u_j^{n}|,|u_j^{n+\frac{1}{2}}|\big\}\big)^2\cdot(2|e_j^{n}|+|e_j^{n+\frac{1}{2}}|)\Big)$$

$$+\frac{\beta}{2}\Big(\big(\max\big\{|V_j^{n+1}|,|U_j^{n+\frac{1}{2}}|,|v_j^{n+1}|,|u_j^{n+\frac{1}{2}}|\big\}\big)^2\cdot(2|f_j^{n+1}|+|e_j^{n+\frac{1}{2}}|)$$

$$+\big(\max\big\{|V_j^{n}|,|U_j^{n+\frac{1}{2}}|,|v_j^{n}|,|u_j^{n+\frac{1}{2}}|\big\}\big)^2\cdot(2|f_j^{n}|+|e_j^{n+\frac{1}{2}}|)\Big),$$

then there exists a constant C such that

$$\|(g_1)^n\|^2\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2), \tag{5.7}$$

$$\|(g_1)_x^n\|^2\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2$$
$$+\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|f_x^{n+1}\|^2+\|f_x^n\|^2). \tag{5.8}$$

Then we obtain

$$\big|Im\langle(g_1)^n,e^{n+1}+e^n\rangle\big|\leq C(\|(g_1)^n\|^2+\|e^{n+1}\|^2+\|e^n\|^2)$$
$$\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2). \tag{5.9}$$

Similarly, we have

$$\|(g_2)^n\|^2\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2), \tag{5.10}$$

$$\|(g_2)_x^n\|^2\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2$$
$$+\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|f_x^{n+1}\|^2+\|f_x^n\|^2) \tag{5.11}$$

and

$$\big|Im\langle(g_2)^n,f^{n+1}+f^n\rangle\big|\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2). \tag{5.12}$$

Note that

$$Im\langle r^n, e^{n+1}+e^n\rangle \leq \|r^n\|^2+\frac{1}{2}\|e^{n+1}\|^2+\frac{1}{2}\|e^n\|^2, \tag{5.13}$$

$$Im\langle s^n, f^{n+1}+f^n\rangle \leq \|s^n\|^2+\frac{1}{2}\|f^{n+1}\|^2+\frac{1}{2}\|f^n\|^2. \tag{5.14}$$

Substituting Eqs. (5.9), (5.12)-(5.14) into Eqs. (5.5)-(5.6) and then adding Eq. (5.5) and Eq. (5.6) together, we have

$$\begin{aligned}&\|e^{n+1}\|^2+\|f^{n+1}\|^2-(\|e^n\|^2+\|f^n\|^2)\\ &\leq C\tau(\|e^{n+1}\|^2+\|e^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2)+\tau(\|r^n\|^2+\|s^n\|^2)\end{aligned} \tag{5.15}$$

Summing up Eq. (5.15) from 0 to $n-1$, we have

$$\begin{aligned}&(1-C\tau)(\|e^{n+1}\|^2+\|f^{n+1}\|^2)\\ &\leq \|e^0\|^2+\|f^0\|^2+C\tau\sum_{l=0}^{n-1}(\|e^l\|^2+\|f^l\|^2)+\tau\sum_{l=0}^{n-1}(\|r^l\|^2+\|s^l\|^2).\end{aligned} \tag{5.16}$$

If τ is sufficiently small such that $1-C\tau>0$, we have

$$\|e^n\|^2+\|f^n\|^2 \leq C\tau\sum_{l=0}^{n-1}(\|e^l\|^2+\|f^l\|^2)+C(\tau^2+h^4)^2, \tag{5.17}$$

where

$$\tau\sum_{l=0}^{n-1}(\|r^l\|^2+\|s^l\|^2) \leq n\tau\cdot\max_{0\leq l\leq n-1}(\|r^l\|^2+\|s^l\|^2) \leq CT(\tau^2+h^4)^2.$$

Applying Lemma 5.1, we obtain

$$\|e^n\|^2+\|f^n\|^2 \leq C(\tau^2+h^4)^2\cdot e^{CT} \leq C(\tau^2+h^4)^2,$$

which implies $\|e^n\| \leq C(\tau^2+h^4)$, $\|f^n\| \leq C(\tau^2+h^4)$.

Computing the inner product of Eq. (5.2) with $e^{n+1}-e^n$ and the inner product of Eq. (5.3) with $f^{n+1}-f^n$, and taking the real part, we have

$$\begin{aligned}&\frac{2k}{3}(\|e_x^{n+1}\|^2-\|e_x^n\|^2)-\frac{k}{6}(\|e_{\hat{x}}^{n+1}\|^2-\|e_{\hat{x}}^n\|^2)\\ &= Re\langle (g_1)^n, e^{n+1}-e^n\rangle - Re\langle r^n, e^{n+1}-e^n\rangle,\end{aligned} \tag{5.18}$$

$$\frac{2k}{3}(\|f_x^{n+1}\|^2 - \|f_x^n\|^2) - \frac{k}{6}(\|f_{\hat{x}}^{n+1}\|^2 - \|f_{\hat{x}}^n\|^2)$$

$$= Re\langle (g_2)^n, f^{n+1} - f^n\rangle - Re\langle s^n, f^{n+1} - f^n\rangle. \tag{5.19}$$

From Eq. (5.1) and Lemma 2.1, we have

$$\begin{aligned}
&\big|Re\langle (g_1)^n, e^{n+1} - e^n\rangle\big| \\
&= \big|Re\langle (g_1)^n, \tau(e^n)_t\rangle\big| \\
&= \Big|Re\langle (g_1)^n, \frac{4ik\tau}{3}e_{x\bar{x}}^{n+\frac{1}{2}} - \frac{ik\tau}{3}e_{\hat{x}\hat{x}}^{n+\frac{1}{2}} + i\tau(g_1)^n - i\tau r^n\rangle\Big| \\
&= \Big| - \frac{4k\tau}{3}Im\langle (g_1)_x^n, e_x^{n+\frac{1}{2}}\rangle + \frac{k\tau}{3}Im\langle (g_1)_{\hat{x}}^n, e_{\hat{x}}^{n+\frac{1}{2}}\rangle - \tau Im\langle (g_1)^n, r^n\rangle\Big|.
\end{aligned} \tag{5.20}$$

Substituting Eqs. (5.7)-(5.9) into Eq. (5.20), we have

$$\begin{aligned}
&\big|Re\langle (g_1)^n, e^{n+1} - e^n\rangle\big| \\
&\leq C\tau(\|(g_1)_x^n\|^2 + \|(g_1)_{\hat{x}}^n\|^2 + \|(g_1)^n\|^2 + \|e_x^{n+\frac{1}{2}}\|^2 + \|e_{\hat{x}}^{n+\frac{1}{2}}\|^2 + \|r^n\|^2) \\
&\leq C\tau(\|(g_1)_x^n\|^2 + \|(g_1)^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|r^n\|^2) \\
&\leq C\tau(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|f^{n+1}\|^2 + \|f^n\|^2 \\
&\quad + \|f_x^{n+1}\|^2 + \|f_x^n\|^2 + \|r^n\|^2).
\end{aligned} \tag{5.21}$$

Similarly, we have

$$\begin{aligned}
\big|Re\langle (g_2)^n, f^{n+1} - f^n\rangle\big| \leq C\tau(&\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|f^{n+1}\|^2 \\
&+ \|f^n\|^2 + \|f_x^{n+1}\|^2 + \|f_x^n\|^2 + \|s^n\|^2).
\end{aligned} \tag{5.22}$$

Noticing that

$$\big|Re\langle r^n, e^{n+1} - e^n\rangle\big| \leq \|r^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2), \tag{5.23}$$

$$\big|Re\langle s^n, f^{n+1} - f^n\rangle\big| \leq \|s^n\|^2 + \frac{1}{2}(\|f^{n+1}\|^2 + \|f^n\|^2). \tag{5.24}$$

Substituting Eqs. (5.21)-(5.24) into Eqs. (5.18)-(5.19) and then adding Eq. (5.18) and Eq. (5.19), we obtain

$$\frac{2k}{3}\big(\|e_x^{n+1}\|^2-\|e_x^n\|^2+\|f_x^{n+1}\|^2-\|f_x^n\|^2\big)-\frac{k}{6}\big(\|e_{\hat{x}}^{n+1}\|^2-\|e_{\hat{x}}^n\|^2+\|f_{\hat{x}}^{n+1}\|^2-\|f_{\hat{x}}^n\|^2\big)$$

$$\le C\tau(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|f^{n+1}\|^2+\|f^n\|^2+\|f_x^{n+1}\|^2+\|f_x^n\|^2$$

$$+\|r^n\|^2+\|s^n\|^2). \tag{5.25}$$

Let

$$A^n=\frac{2k}{3}(\|e_x^n\|^2+\|f_x^n\|^2)-\frac{k}{6}(\|e_{\hat{x}}^n\|^2+\|f_{\hat{x}}^n\|^2).$$

It follows from Lemma 2.1 that

$$0\le\frac{k}{2}(\|e_x^n\|^2+\|f_x^n\|^2)\le A^n.$$

Summing up Eq. (5.25) from 0 to $n-1$, we have

$$A^n\le A^0+C\tau\sum_{l=0}^{n}(\|e^l\|^2+\|f^l\|^2+\|e_x^l\|^2+\|f_x^l\|^2)+C\tau\sum_{l=0}^{n-1}(\|r^l\|^2+\|s^l\|^2). \tag{5.26}$$

Noticing that

$$C\tau\sum_{l=0}^{n}(\|e^l\|^2+\|f^l\|^2)\le C(n+1)\tau\cdot\max_{0\le l\le n}(\|e^l\|^2+\|f^l\|^2)\le C(T+\tau)(\tau^2+h^4)^2,$$

$$C\tau\sum_{l=0}^{n-1}(\|r^l\|^2+\|s^l\|^2)\le Cn\tau\cdot\max_{0\le l\le n-1}(\|r^l\|^2+\|s^l\|^2)\le CT(\tau^2+h^4)^2,$$

$$A^0=\frac{2k}{3}(\|e_x^0\|^2+\|f_x^0\|^2)-\frac{k}{6}(\|e_{\hat{x}}^0\|^2+\|f_{\hat{x}}^0\|^2)\le C(\tau^2+h^4)^2,$$

then we have

$$(\frac{k}{2}-C\tau)(\|e_x^n\|^2+\|f_x^n\|^2)\le C\tau\sum_{l=0}^{n-1}(\|e_x^l\|^2+\|f_x^l\|^2)+C(\tau^2+h^4)^2. \tag{5.27}$$

If τ is sufficiently small such that $\frac{k}{2}-C\tau>0$, we obtain

$$\|e_x^n\|^2+\|f_x^n\|^2\le C\tau\sum_{l=0}^{n-1}(\|e_x^l\|^2+\|f_x^l\|^2)+C(\tau^2+h^4)^2. \tag{5.28}$$

Applying Lemma 5.1, we have

$$\|e_x^n\|^2 + \|f_x^n\|^2 \le C(\tau^2 + h^4)^2 \cdot e^{CT} \le C(\tau^2 + h^4)^2,$$

which implies $\|e_x^n\| \le C(\tau^2 + h^4)$, $\|f_x^n\| \le C(\tau^2 + h^4)$. Hence, we obtain $\|e^n\|_\infty \le C(\tau^2 + h^4)$, $\|f^n\|_\infty \le C(\tau^2 + h^4)$ by Lemma 3.3. This completes the proof.

Using similar proof for Theorem 5.1, we can conclude Theorem 5.2 and Theorem 5.3.

Theorem 5.2. *Suppose that $u(x,t), v(x,t) \in C_{x,t}^{6,3}$, then the finite difference scheme (2.1)-(2.4) is stable by norm $\|\cdot\|_\infty$.*

Theorem 5.3. *The finite difference scheme (2.1)-(2.4) is uniquely solvable.*

6. Iterative algorithm

For fixed n, we rewrite Eq. (2.1) and Eq. (2.2) in the following form

$$\frac{2i}{\tau}(u_j^{n+\frac{1}{2}} - u_j^n) + \frac{4k}{3}(u_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(u_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + \frac{1}{2}\big[|2u_j^{n+\frac{1}{2}} - u_j^n|^2 + |u_j^n|^2$$

$$+ \beta\big(|2v_j^{n+\frac{1}{2}} - v_j^n|^2 + |v_j^n|^2\big)\big]u_j^{n+\frac{1}{2}} = 0, \tag{6.1}$$

$$\frac{2i}{\tau}(v_j^{n+\frac{1}{2}} - v_j^n) + \frac{4k}{3}(v_j^{n+\frac{1}{2}})_{x\bar{x}} - \frac{k}{3}(v_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + \frac{1}{2}\big[|2v_j^{n+\frac{1}{2}} - v_j^n|^2 + |v_j^n|^2$$

$$+ \beta\big(|2u_j^{n+\frac{1}{2}} - u_j^n|^2 + |v_j^n|^2\big)\big]v_j^{n+\frac{1}{2}} = 0. \tag{6.2}$$

Then we provide the following iterative method to compute the solution of the finite difference scheme (2.1)-(2.4):

$$\frac{2i}{\tau}(u_j^{n+\frac{1}{2}(s+1)} - u_j^n) + \frac{4k}{3}(u_j^{n+\frac{1}{2}(s+1)})_{x\bar{x}} - \frac{k}{3}(u_j^{n+\frac{1}{2}(s+1)})_{\hat{x}\hat{x}} + \frac{1}{2}\big[|2u_j^{n+\frac{1}{2}(s)} - u_j^n|^2 + |u_j^n|^2$$

$$+ \beta\big(|2v_j^{n+\frac{1}{2}(s)} - v_j^n|^2 + |v_j^n|^2\big)\big]u_j^{n+\frac{1}{2}(s)} = 0, \tag{6.3}$$

$$\frac{2i}{\tau}(v_j^{n+\frac{1}{2}(s+1)} - v_j^n) + \frac{4k}{3}(v_j^{n+\frac{1}{2}(s+1)})_{x\bar{x}} - \frac{k}{3}(v_j^{n+\frac{1}{2}(s+1)})_{\hat{x}\hat{x}} + \frac{1}{2}\big[|2v_j^{n+\frac{1}{2}(s)} - v_j^n|^2 + |v_j^n|^2$$

$$+ \beta\big(|2u_j^{n+\frac{1}{2}(s)} - u_j^n|^2 + |v_j^n|^2\big)\big]v_j^{n+\frac{1}{2}(s)} = 0, \tag{6.4}$$

for $s = 0, 1, \ldots$, where

$$u_j^{n+\frac{1}{2}(0)} = \begin{cases} u_j^n, & n = 0, \\ \frac{3}{2}u_j^n - \frac{1}{2}u_j^{n-1}, & n \ge 1, \end{cases} \qquad v_j^{n+\frac{1}{2}(0)} = \begin{cases} v_j^n, & n = 0, \\ \frac{3}{2}v_j^n - \frac{1}{2}v_j^{n-1}, & n \ge 1. \end{cases}$$

Theorem 6.1. *Suppose that $u(x,t), v(x,t) \in C^{6,3}_{x,t}$, if τ and h are sufficiently small, then the solutions of the iterative method (6.3)-(6.4) converge to the solutions of the finite difference scheme (6.1)-(6.2) by norm $\|\cdot\|_\infty$.*

Proof. Let $\varepsilon_j^{(s)} = u_j^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}(s)}$, $\eta_j^{(s)} = v_j^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}(s)}$. Then when $n=0$, we have

$$\varepsilon_j^{(0)} = u_j^{\frac{1}{2}} - u_j^0 = \frac{1}{2}(u_j^1 - u_j^0) = \frac{1}{2}[(u_j^1 - U_j^1) + (U_j^1 - U_j^0) + (U_j^0 - u_j^0)]$$

$$\leq C(\tau^2 + h^4) + C(\tau) + 0 \leq C(\tau + h^4),$$

and when $n \geq 1$,

$$\varepsilon_j^{(0)} = u_j^{n+\frac{1}{2}} - (\frac{3}{2}u_j^n - \frac{1}{2}u_j^{n-1}) = \frac{1}{2}(u_j^{n+1} - 2u_j^n + u_j^{n-1})$$

$$= \frac{1}{2}[(u_j^{n+1} - U_j^{n+1}) - 2(u_j^n - U_j^n) + (u_j^{n-1} - U_j^{n-1}) + (U_j^{n+1} - 2U_j^n + U_j^{n-1})]$$

$$\leq C(\tau^2 + h^4) + C(\tau^2 + h^4) + C(\tau^2 + h^4) + C(\tau^2) \leq C(\tau^2 + h^4),$$

where $U_j^n = u(x_j, t_n)$. Similarly, we have $\eta_j^{(0)} \leq C(\tau + h^4)$ when $n = 0$ and $\eta_j^{(0)} \leq C(\tau^2 + h^4)$ when $n \geq 1$. This yield

$$\|\varepsilon^{(0)}\|_\infty \leq \begin{cases} C(\tau + h^4), & n = 0, \\ C(\tau^2 + h^4), & n \geq 1, \end{cases} \qquad \|\eta^{(0)}\|_\infty \leq \begin{cases} C(\tau + h^4), & n = 0, \\ C(\tau^2 + h^4), & n \geq 1. \end{cases}$$

If τ and h are sufficiently small, we have $\|\varepsilon^{(0)}\|_\infty \leq C$, $\|\eta^{(0)}\|_\infty \leq C$ for $n = 0, 1, \ldots, N$. Suppose $\|\varepsilon^{(s)}\|_\infty \leq C$, $\|\eta^{(s)}\|_\infty \leq C$, then

$$\|u^{n+\frac{1}{2}(s)}\|_\infty \leq \|u^{n+\frac{1}{2}}\|_\infty + \|\varepsilon^{(s)}\|_\infty \leq C, \quad \|v^{n+\frac{1}{2}(s)}\|_\infty \leq \|v^{n+\frac{1}{2}}\|_\infty + \|\eta^{(s)}\|_\infty \leq C.$$

Substituting Eqs. (6.3)-(6.4) from Eqs. (6.1)-(6.2), we have

$$\frac{2i}{\tau}\varepsilon_j^{(s+1)} + \frac{4k}{3}(\varepsilon_j^{(s+1)})_{x\bar{x}} - \frac{k}{3}(\varepsilon_j^{(s+1)})_{\hat{x}\hat{x}} + p_j^{(s)} = 0, \tag{6.5}$$

$$\frac{2i}{\tau}\eta_j^{(s+1)} + \frac{4k}{3}(\eta_j^{(s+1)})_{x\bar{x}} - \frac{k}{3}(\eta_j^{(s+1)})_{\hat{x}\hat{x}} + q_j^{(s)} = 0, \tag{6.6}$$

where

$$p_j^{(s)} = \frac{1}{2}(|2u_j^{n+\frac{1}{2}} - u_j^n|^2 + \beta|2v_j^{n+\frac{1}{2}} - v_j^n|^2)u_j^{n+\frac{1}{2}} - \frac{1}{2}(|2u_j^{n+\frac{1}{2}(s)} - u_j^n|^2$$

$$+\beta|2v_j^{n+\frac{1}{2}(s)}-v_j^n|^2)u_j^{n+\frac{1}{2}(s)},$$

$$q_j^{(s)}=\frac{1}{2}(|2v_j^{n+\frac{1}{2}}-v_j^n|^2+\beta|2u_j^{n+\frac{1}{2}}-u_j^n|^2)v_j^{n+\frac{1}{2}}-\frac{1}{2}(|2v_j^{n+\frac{1}{2}(s)}-v_j^n|^2$$

$$+\beta|2u_j^{n+\frac{1}{2}(s)}-u_j^n|^2)v_j^{n+\frac{1}{2}(s)}.$$

Applying Lemma 5.2, we have

$$|p_j^{(s)}|\le\frac{1}{2}\Big||2u_j^{n+\frac{1}{2}}-u_j^n|^2u_j^{n+\frac{1}{2}}-|2u_j^{n+\frac{1}{2}(s)}-u_j^n|^2u_j^{n+\frac{1}{2}(s)}\Big|$$

$$+\frac{\beta}{2}\Big||2v_j^{n+\frac{1}{2}}-v_j^n|^2u_j^{n+\frac{1}{2}}-|2v_j^{n+\frac{1}{2}(s)}-v_j^n|^2u_j^{n+\frac{1}{2}(s)}\Big|$$

$$\le\frac{1}{2}\big(\max\{|2u_j^{n+\frac{1}{2}}-u_j^n|,|u_j^{n+\frac{1}{2}}|,|2u_j^{n+\frac{1}{2}(s)}-u_j^n|,|u_j^{n+\frac{1}{2}(s)}|\}\big)^2\cdot\big(2|2\varepsilon_j^{(s)}|+|\varepsilon_j^{(s)}|\big)$$

$$+\frac{\beta}{2}\big(\max\{|2v_j^{n+\frac{1}{2}}-v_j^n|,|u_j^{n+\frac{1}{2}}|,|2v_j^{n+\frac{1}{2}(s)}-v_j^n|,|u_j^{n+\frac{1}{2}(s)}|\}\big)^2\cdot\big(2|2\eta_j^{(s)}|+|\varepsilon_j^{(s)}|\big)$$

$$\le C(|\varepsilon_j^{(s)}|+|\eta_j^{(s)}|).$$

Similarly, we have

$$|q_j^{(s)}|\le C(|\varepsilon_j^{(s)}|+|\eta_j^{(s)}|).$$

Computing the inner product of Eq. (6.5) with $\varepsilon^{(s+1)}$ and the inner product of Eq. (6.6) with $\eta^{(s+1)}$, and taking the imaginary part, we obtain

$$\frac{2}{\tau}\|\varepsilon^{(s+1)}\|^2=-Im\langle p^{(s)},\varepsilon^{(s+1)}\rangle, \tag{6.7}$$

$$\frac{2}{\tau}\|\eta^{(s+1)}\|^2=-Im\langle q^{(s)},\eta^{(s+1)}\rangle. \tag{6.8}$$

Adding Eqs. (6.7)-(6.8) and applying the Cauchy-Schwarz inequality, we have

$$\frac{2}{\tau}(\|\varepsilon^{(s+1)}\|^2+\|\eta^{(s+1)}\|^2)\le\big|Im\langle p^{(s)},\varepsilon^{(s+1)}\rangle\big|+\big|Im\langle q^{(s)},\eta^{(s+1)}\rangle\big|$$

$$\le\Big|h\sum_{j=1}^{J-1}p_j^{(s)}\varepsilon_j^{(s+1)}\Big|+\Big|h\sum_{j=1}^{J-1}q_j^{(s)}\eta_j^{(s+1)}\Big|$$

$$\leq Ch \sum_{j=1}^{J-1} \big(|\varepsilon_j^{(s+1)}| + |\eta_j^{(s+1)}|\big)(|\varepsilon_j^{(s)}| + |\eta_j^{(s)}|)$$

$$\leq C\epsilon_2(\|\varepsilon^{(s+1)}\|^2 + \|\eta^{(s+1)}\|^2) + \frac{C}{\epsilon_2}(\|\varepsilon^{(s)}\|^2 + \|\eta^{(s)}\|^2) \tag{6.9}$$

for any positive constant ϵ_2. Let $\epsilon_2 = 1/(C\tau)$, we have

$$\|\varepsilon^{(s+1)}\|^2 + \|\eta^{(s+1)}\|^2 \leq C^2\tau^2(\|\varepsilon^{(s)}\|^2 + \|\eta^{(s)}\|^2) \leq (C\tau)^{2s}(\|\varepsilon^{(0)}\|^2 + \|\eta^{(0)}\|^2). \tag{6.10}$$

Computing the inner product of Eq. (6.5) with $-\varepsilon^{(s+1)}$ and the inner product of Eq. (6.6) with $-\eta^{(s+1)}$, and taking the real part, we have

$$\frac{4k}{3}\|\varepsilon_x^{(s+1)}\|^2 - \frac{k}{3}\|\varepsilon_{\hat{x}}^{(s+1)}\|^2 = Re\langle p^{(s+1)}, \varepsilon^{(s+1)}\rangle, \tag{6.11}$$

$$\frac{4k}{3}\|\eta_x^{(s+1)}\|^2 - \frac{k}{3}\|\eta_{\hat{x}}^{(s+1)}\|^2 = Re\langle q^{(s+1)}, \eta^{(s+1)}\rangle, \tag{6.12}$$

Adding Eqs. (6.11)-(6.12) and applying Lemma 2.1, we obtain

$$k(\|\varepsilon_x^{(s+1)}\|^2 + \|\eta_x^{(s+1)}\|^2) \leq \big|Re\langle p^{(s)}, \varepsilon^{(s+1)}\rangle\big| + \big|Re\langle q^{(s)}, \eta^{(s+1)}\rangle\big|.$$

Similar to Eq. (6.9), we have

$$k(\|\varepsilon_x^{(s+1)}\|^2 + \|\eta_x^{(s+1)}\|^2) \leq C\epsilon_3(\|\varepsilon^{(s+1)}\|^2 + \|\eta^{(s+1)}\|^2) + \frac{C}{\epsilon_3}(\|\varepsilon^{(s)}\|^2 + \|\eta^{(s)}\|^2)$$

$$\leq \left(C^3\tau^2\epsilon_3 + \frac{C}{\epsilon_3}\right)(\|\varepsilon^{(s)}\|^2 + \|\eta^{(s)}\|^2),$$

which is obtained based on Eq. (6.10). Lettiing $\epsilon_3 = 1/(C\tau)$, we obtain

$$\|\varepsilon_x^{(s+1)}\|^2 + \|\eta_x^{(s+1)}\|^2 \leq \frac{2C^2\tau}{k}(\|\varepsilon^{(s)}\|^2 + \|\eta^{(s)}\|^2) \leq \frac{2C(C\tau)^{2s}}{k}(\|\varepsilon^{(0)}\|^2 + \|\eta^{(0)}\|^2). \tag{6.13}$$

According to Lemma 3.3, if τ is sufficiently small such that $\tau \leq \frac{1}{2C}$, we have

$$\|\varepsilon^{(s+1)}\|_\infty^2 + \|\eta^{(s+1)}\|_\infty^2 \leq \frac{C}{k}\left(\frac{1}{4}\right)^s(\|\varepsilon^{(0)}\|^2 + \|\eta^{(0)}\|^2) \to 0,$$

when $s \to \infty$. This completes the proof.

7. Numerical examples

In this section, numerical results are presented to test the error estimate and the conservation laws for the finite difference scheme (2.1)-(2.4). For convenience, denote the maximum error norms and convergence orders as

$$\|e^n\|_\infty = \|U^n - u^n\|_\infty, \quad \|f^n\|_\infty = \|V^n - v^n\|_\infty, \quad \text{order } 1 = \log_2\left(\frac{\|e^n(h,\tau)\|_\infty}{\|e^n(\frac{h}{2},\frac{\tau}{2})\|_\infty}\right),$$

$$\text{order } 2 = \log_2\left(\frac{\|e^n(h,\tau)\|_\infty}{\|e^n(h,\frac{\tau}{2})\|_\infty}\right), \quad \text{order } 3 = \log_2\left(\frac{\|f^n(h,\tau)\|_\infty}{\|f^n(h,\frac{\tau}{2})\|_\infty}\right),$$

$$\text{order } 4 = \log_2\left(\frac{\|e^n(h,\tau)\|_\infty}{\|e^n(\frac{h}{2},\frac{\tau}{4})\|_\infty}\right), \quad \text{order } 5 = \log_2\left(\frac{\|f^n(h,\tau)\|_\infty}{\|f^n(\frac{h}{2},\frac{\tau}{4})\|_\infty}\right),$$

where U^n and V^n are the solutions of the problem (1.1)-(1.4).

7.1. Single soliton

Consider the parameter $k = \frac{1}{2}$ and the initial conditions [29]:

$$u_0(x) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\big(\sqrt{2\alpha}x\big)\exp\big(i\nu x\big),$$

$$v_0(x) = -\sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\big(\sqrt{2\alpha}x\big)\exp\big(i\nu x\big),$$

then the problem (1.1)-(1.4) has the following exact solutions

$$u(x,t) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\Big(\sqrt{2\alpha}(x-\nu t)\Big)\exp\left(i\nu x - i\Big(\frac{\nu^2}{2}-\alpha\Big)t\right),$$

$$v(x,t) = -\sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\Big(\sqrt{2\alpha}(x-\nu t)\Big)\exp\left(i\nu x - i\Big(\frac{\nu^2}{2}-\alpha\Big)t\right),$$

where α and ν are two known constants. In this example, the parameters were chosen as

$$x_l = -20, \quad x_r = 60, \quad \alpha = 1, \quad \nu = 1, \quad \beta = 2/3.$$

The comparison of maximum error norms and convergence orders were reported in Tables 1-3. It is easy to see that the present scheme has much higher convergence and smaller error than the schemes in [20, 28, 29]. At the same time, the present scheme (2.1)-(2.4) is second-order accurate in time and fourth-order accurate in space as seen in Table 2 and Table 3. Furthermore, Q_1^n, Q_2^n and E^n at different times were listed in Table 4, which support that the scheme (2.1)-(2.4) preserves the discrete conservative properties very well. Numerical traveling solitons were showed in Fig. 1. We see that numerical solutions agree with the exact solutions.

7.2. Collision of two solitons

Consider the parameter $k = 1$ and the initial conditions [4]:

$$u_0(x) = \sqrt{2}r_1 \mathrm{sech}(r_1 x + x_0/2)\exp(i\nu_1 x),$$

$$v_0(x) = \sqrt{2}r_2 \mathrm{sech}(r_2 x - x_0/2)\exp(i\nu_2 x),$$

where $\nu_1 = -\nu_2 = \nu/4$. As we know in [22], the collision of the solitary wave is elastic when $\beta = 0$ or $\beta = 1$ and is inelastic when $\beta \neq 0$ and $\beta \neq 1$. We analyze the collision of two solitons by taking $x_l = -40$, $x_r = 40$, $x_0 = 18$, $h = 0.125$, $\tau = h^2$ and by considering two scenarios for simulating the dynamics and for simulating three cases were used to simulate the dynamics of the two solitons:

Case 1. The elastic and inelastic collision of two solitons. We take $T = 50$, $r_1 = r_2 = 1$, $\nu = 1, 1.3$, and $\beta = 0, 1, 2/3$.

Case 2. The fusion and creation of new vector solitons. We take $T = 30$, $r_1 = 1.2$, $r_2 = 1$, $\nu = 1, 1.5$ and $\beta = 0.5, 3, 5$.

Case 3. The trapped and reflected solitons. We take $T = 100$, $r_1 = 1.2$, $r_2 = 1$, $\nu = 1.05, 1.15$ and $\beta = 2/3$.

In Case 1, we can see that the two solitons move forward without any changes in shape and velocity after collision when $\nu = 1$, $\beta = 0$ and $\nu = 1$, $\beta = 1$ as shown in Fig. 2 (a) and Fig. 2 (b), respectively. The two solitons are transient after colliding when $\nu = 1.3$, $\beta = 2/3$ as seen in Fig. 2 (c). In Case 2, it can be seen that the two solitons are fused into one after colliding when $\nu = 1$, $\beta = 0.5$ as shown in Fig. 3 (a). The two solitons are created a new vector soliton when $\nu = 1.5$, $\beta = 3$ and two new vector solitons when $\nu = 1$, $\beta = 5$ after colliding as seen in Fig. 3 (b) and Fig. 3 (c), respectively. In Case 3, we can see that two solitons are trapped after colliding when $\nu = 1.05$, $\beta = 2/3$ as shown in Fig. 4 (a). Furthermore, when $\nu = 1.15$, $\beta = 2/3$, the two solitons are reflected and then transient after their collision as seen in Fig. 4 (b). In addition, Fig. 2 (b), Fig. 3, and Fig. 4 show the inelastic collision of two solitons. We can see that a small number of radiation waves are developed after colliding. The results of the simulations agree with those obtained in [4, 22].

8. Conclusion

In this paper, a high-order accurate conservative finite difference scheme for the coupled nonlinear Schrödinger (CNLS) equations has been developed. It is proved that the present scheme is conservative, uniquely solvable, stable, convergent with fourth-order in space and second-order in time. Convergent iterative method for the present scheme is developed and proved. Numerical experiments for the present scheme support the theoretical analysis. Some study cases are given to investigate the collision of two solitons.

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Table 1: The comparison of the maximum error norms and the convergence orders at $T = 10$ with $h = \tau = 0.2$.

Scheme		h, τ	$h/2, \tau/2$	$h/4, \tau/4$	$h/8, \tau/8$
Scheme in [20]	$\|e^n\|_\infty$	1.9106e-01	4.5983e-02	1.1392e-02	2.8400e-03
	order 1	–	2.0548	2.0131	2.0041
Scheme in [28]	$\|e^n\|_\infty$	1.9028e-01	4.5853e-02	1.1367e-02	2.8347e-03
	order 1	–	2.0530	2.0121	2.0036
Scheme in [29]	$\|e^n\|_\infty$	8.6086e-01	1.9455e-01	4.6981e-02	1.1655e-02
	order 1	–	2.1456	2.0500	2.0111
Present scheme	$\|e^n\|_\infty$	2.8396e-02	5.5521-03	1.1912e-03	2.1679e-04
	order 1	–	2.3546	2.2206	2.4581

Table 2: Errors and the temporal convergence orders of the present scheme at $T = 10$ with $h = 0.025$, $\tau = 0.2$.

	h, τ	$h, \tau/2$	$h, \tau/4$	$h, \tau/8$
$\|e^n\|_\infty$	2.1553e-02	5.1320e-03	1.2661e-03	3.1679e-04
order 2	–	2.0703	2.0190	1.9988
$\|f^n\|_\infty$	2.1553e-02	5.1320e-03	1.2661e-03	3.1679e-04
order 3	–	2.0703	2.0190	1.9988

Table 3: Errors and the space convergence orders of the present scheme at $T = 10$ with $h = 0.25$, $\tau = h^2$.

	h, τ	$h/2, \tau/4$	$h/4, \tau/16$	$h/8, \tau/64$
$\|e^n\|_\infty$	1.9890e-02	1.2735e-03	8.1044e-05	5.0836e-06
order 4	–	3.9652	3.9740	3.9948
$\|f^n\|_\infty$	1.9890e-02	1.2735e-03	8.1044e-05	5.0836e-06
order 5	–	3.9652	3.9740	3.9948

Table 4: Q_1^n, Q_2^n and E^n computed by the present scheme at different times with $h = 0.25$, $\tau = h^2$.

T	Q_1^n	Q_2^n	E^n
0	1.69705627499030	1.69705627499030	0.281091622315605
10	1.69705627497490	1.69705627497490	0.281091622325967
20	1.69705627495949	1.69705627495949	0.281091622336329
30	1.69705627494404	1.69705627494404	0.281091622346729
40	1.69705627492861	1.69705627492861	0.281091622357112

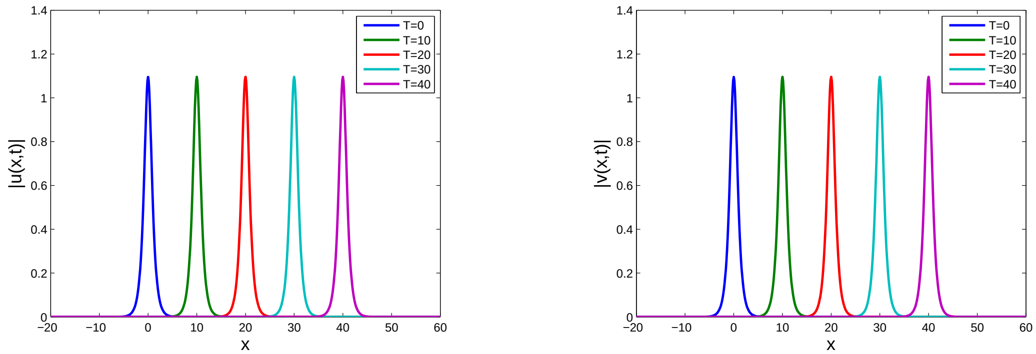

Fig. 1: Traveling solitons of the present scheme at different times with $h = 0.25$, $\tau = h^2$.

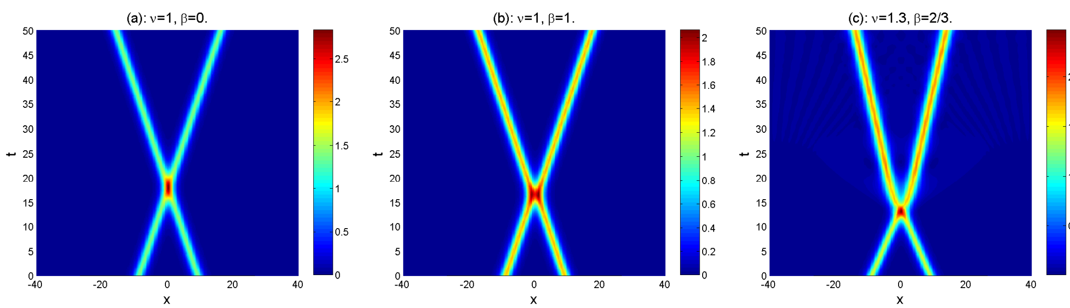

Fig. 2: The elastic and inelastic collision of the two solitons with various ν, β.

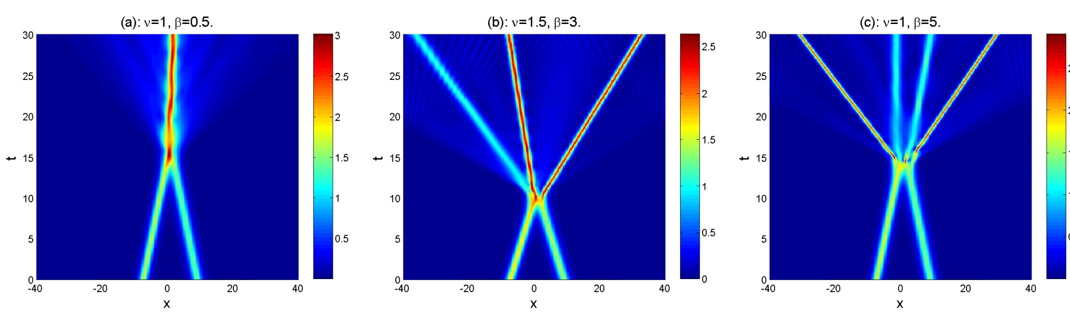

Fig. 3: The funsion and creation of new vector soliton with various ν, β.

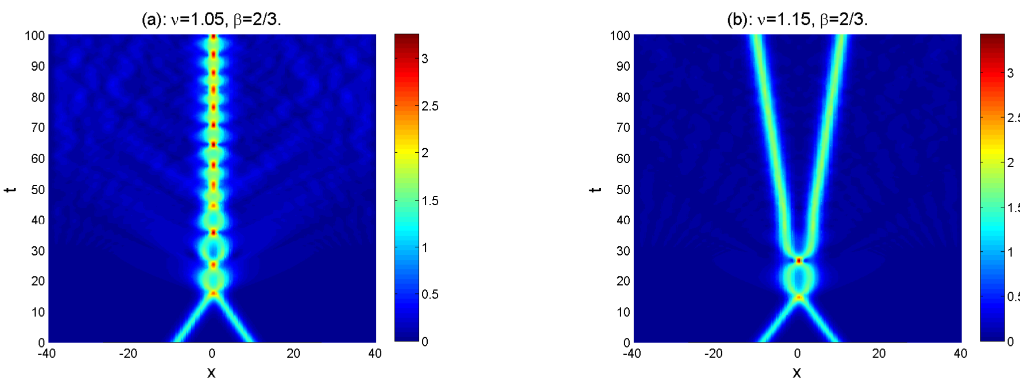

Fig. 4: The trapped and reflected solitons with various ν.