ON OSTROWSKI-MERCER INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTION

MUHAMMAD AAMIR ALI[1]*, IFRA BASHIR SIAL[2], AND HÜSEYIN BUDAK[3]

Abstract. In this note, for differentiable convex functions, we prove some new Ostrowski-Mercer inequalities. These inequalities generalize an Ostrowski inequality and related inequalities proved in [3, 5]. Some applications to special means are also given.

1. Introduction

The study of different forms of fundamental inequality has been the subject of great interest for well over a century. A variety of mathematicians, interested in both pure and applied mathematics. One of the various ones mathematical basic discoveries of A. M. Ostrowski [15] is the following classical integral inequality:

Theorem 1. *Let $f:[1,\infty)\to\mathbb{R}$ is differentiable functions on $(1,\infty)$ and $f\in L[a,b]$, where $a,b\in[1,\infty)$ with $a<b$. If $|f'(x)|\le M$, then we have following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{M}{(b-a)}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]. \tag{1.1}$$

Ostrowski inequality has applications in quadrature, theory of probability and optimization, stochastic, statistics, information and the theory of integral operator. A number of scientists have concentrated over the last few years on Ostrowski type inequalities for bounded variation functions, see for example [4, 6, 8, 9, 17, 18]. Until now, a significant number of research papers and books have been published on Ostrowski inequalities and their numerous applications.

In literature, the well-known Jensen inequality [13] states that if f is a convex function on an interval contains in x_n, then

$$f\left(\sum_{j=1}^{n}\lambda_j x_j\right)\le\sum_{j=1}^{n}\lambda_j f(x_j). \tag{1.2}$$

In convex functions theory, Hermite-Hadamard inequality is very important which was discovered by C. Hermite and J. Hadamard independently (see, also [10], and [16, p.137])

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2} \tag{1.3}$$

where $f:I\to\mathbb{R}$ is a convex function over I and $a,b\in I$ with $a<b$. In the case of concave mappings, the above inequality satisfies in reverse order.

The following variant of Jensen inequality, known as the Jensen-Mercer, was demonstrated by Mercer [12]:

1991 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Ostrowski inequality, Jensen-Mercer inequality, convex functions.

[1]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China.
Email: mahr.muhammad.aamir@gmail.com.
[2]School of Sciences, Jiangsu University, Zhenjinag, China.
Email: ifrabashir92@gmail.com.
[3]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY.
Email: hsyn.budak@gmail.com .

Theorem 2. *If f is a convex function on $[a,b]$, then the following inequality is true:*

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j x_j\right)\leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(x_j) \tag{1.4}$$

for all $x_j\in[a,b]$ and $\lambda_j\in[0,1]$ with $\sum_{j=1}^{n}\lambda_j=1$.

In [11], the idea of Jensen-Mercer inequality has been used by Kian and Moslehian, and the following Hermite-Hadamard-Mercer inequality was demonstrated:

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y f(a+b-t)\,dt \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{1.5}$$

where f is convex function on $[a,b]$. For some recent studies linked to Jensen-Mercer inequality, one can consult [1,2,7,14].

Inspired by this ongoing studies, we develop some new Ostrowski type inequalities by using the Jensen-Mercer inequalities for differentiable convex functions.

2. Ostrowski-Mercer Inequalities

New Ostrowski-Mercer inequalities are obtained for differentiable convex functions in this section. For this, we first give a new integral identity that will serve as an auxiliary to produce subsequent results for advancement.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f\in L[a,b]$, then for all $x,u_1,u_2,v\in[a,b]$ and $t\in[0,1]$, the following equality satisfies:*

$$\begin{aligned} &(v-u_1)^2\int_0^1 tf'(x+a-(tu_1+(1-t)v))\,dt-(u_2-v)^2\int_0^1 tf'(x+b-(tu_2+(1-t)v))\,dt \\ &= (v-u_1)f(x+a-u_1)+(u_2-v)f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1}f(t)\,dt+\int_{x+b-u_2}^{x+b-v}f(t)\,dt\right]. \end{aligned} \tag{2.1}$$

Proof. It is enough to remember that

$$\begin{aligned} I &= (v-u_1)^2\int_0^1 tf'(x+a-(tu_1+(1-t)v))\,dt \\ &\quad -(u_2-v)^2\int_0^1 tf'(x+b-(tu_2+(1-t)v))\,dt \\ &= (v-u_1)^2 I_1-(u_2-v)^2 I_2. \end{aligned} \tag{2.2}$$

Using the integration by parts, we get the equalities

$$\begin{aligned} I_1 &= \int_0^1 tf'(x+a-(tu_1+(1-t)v))\,dt \\ &= \frac{f(x+a-u_1)}{v-u_1}-\frac{1}{(v-u_1)^2}\int_{x+a-v}^{x+a-u_1}f(t)\,dt \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_2 &= \int_0^1 tf'(x+b-(tu_2+(1-t)v))\,dt \\ &= -\frac{f(x+b-u_2)}{(u_2-v)}+\frac{1}{(u_2-v)^2}\int_{x+b-u_2}^{x+b-v}f(t)\,dt. \end{aligned} \tag{2.4}$$

We obtain the resulting equality (2.1) by placing the equalities (2.3) and (2.4) in (2.2). □

Remark 1. *If we set $u_1 = a$, $u_2 = b$ and $v = x$ in Lemma 1, then Lemma 1 reduces to [5, Lemma 1].*

Theorem 3. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|$ is convex on $[a,b]$, then we have the following inequality*

$$\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \tag{2.5}$$
$$\leq \frac{1}{6}\Big[(v-u_1)^2\{3(|f'(x)|+|f'(a)|)-2|f'(u_1)|-|f'(v)|\}$$
$$+(u_2-v)^2\{3(|f'(x)|+|f'(b)|)-2|f'(u_2)|-|f'(v)|\}\Big].$$

Proof. Taking modulus in Lemma 1 and from Jensen-Mercer inequality, we have the inequality

$$\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right|$$
$$\leq (v-u_1)^2\int_0^1 t\,|f'(x+a-(tu_1+(1-t)v))|\,dt$$
$$+(u_2-v)^2\int_0^1 t\,|f'(x+b-(tu_2+(1-t)v))|\,dt$$
$$\leq (v-u_1)^2\int_0^1 t\,[|f'(x)|+|f'(a)|-t|f'(u_1)|-(1-t)|f'(v)|]\,dt$$
$$+(u_2-v)^2\int_0^1 t\,[|f'(x)|+|f'(b)|-t|f'(u_2)|-(1-t)|f'(v)|]\,dt$$
$$= \frac{1}{6}\Big[(v-u_1)^2\{3(|f'(x)|+|f'(a)|)-2|f'(u_1)|-|f'(v)|\}$$
$$+(u_2-v)^2\{3(|f'(x)|+|f'(b)|)-2|f'(u_2)|-|f'(v)|\}\Big]$$

which ends the proof. □

Corollary 1 (Ostrowski-Mercer Inequality)**.** *In Theorem 3, if we choose $|f'(t)| \leq M$ for all $t \in [a,b]$, then we have the following Ostrowski-Mercer inequality*

$$\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \tag{2.6}$$
$$\leq \frac{M}{2}\left((v-u_1)^2+(u_2-v)^2\right).$$

Proof. The result can be easily obtained by using $|f'(x+a-(tu_1+(1-t)v))| \leq M$ and $|f'(x+b-(tu_2+(1-t)v))| \leq M$. □

Remark 2. *If we consider $u_1 = a$, $u_2 = b$ and $v = x$ in Corollary 1, then inequality (2.6) reduces to (1.1).*

Remark 3. *If we consider $u_1 = a$, $u_2 = b$ and $v = x$ in Theorem 3, then inequality (2.1) reduces to [5, Theorem 3].*

Theorem 4. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|^q$, $q > 1$ is convex on $[a,b]$, then we have the following inequality*

$$\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \tag{2.7}$$
$$\leq \frac{1}{2(1+p)^{\frac{1}{p}}}\Big[(v-u_1)^2\left(2|f'(x)|^q+2|f'(a)|^q-|f'(u_1)|^q-|f'(v)|^q\right)^{\frac{1}{q}}$$
$$+(u_2-v)^2\left(2|f'(x)|^q+2|f'(b)|^q-|f'(u_2)|^q-|f'(v)|^q\right)^{\frac{1}{q}}\Big]$$

where $\frac{1}{r}+\frac{1}{p}=1$.

Proof. From Lemma 1 and Hölder's inequality, we have the inequality

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \quad (2.8)\\
\leq\;& (v-u_1)^2 \int_0^1 t\left|f'(x+a-(tu_1+(1-t)v))\right| dt \\
&+(u_2-v)^2 \int_0^1 t\left|f'(x+b-(tu_2+(1-t)v))\right| dt \\
\leq\;& (v-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(x+a-(tu_1+(1-t)v))\right|^q dt\right)^{\frac{1}{q}} \\
&+(u_2-v)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(x+b-(tu_2+(1-t)v))\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From Jensen-Mercer inequality, we have the inequality

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\;& (v-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[|f'(x)|^q+|f'(a)|^q-t|f'(u_1)|^q-(1-t)|f'(v)|^q\right] dt\right)^{\frac{1}{q}} \\
&+(u_2-v)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[|f'(x)|^q+|f'(b)|^q-t|f'(u_2)|^q-(1-t)|f'(v)|^q\right] dt\right)^{\frac{1}{q}} \\
=\;& \frac{1}{2(1+p)^{\frac{1}{p}}}\left[(v-u_1)^2\left(2|f'(x)|^q+2|f'(a)|-|f'(u_1)|^q-|f'(v)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+(u_2-v)^2\left(2|f'(x)|^q+2|f'(b)|-|f'(u_2)|^q-|f'(v)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which finished the proof. □

Corollary 2. *In Theorem 4, if we choose $|f'(t)| \leq M$ for all $t \in [a,b]$, then we have the following Ostrowski-Mercer inequality*

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\;& \frac{M}{(p+1)^{\frac{1}{p}}}\left((v-u_1)^2+(u_2-v)^2\right).
\end{aligned}$$

Proof. The result can be easily obtained by using $|f'(x+a-(tu_1+(1-t)v))| \leq M$ and $|f'(x+b-(tu_2+(1-t)v))| \leq M$. □

Remark 4. *If we consider $u_1=a$, $u_2=b$ and $v=x$ in Corollary 2, then Corollary 2 reduces to [3, Theorem 3 (for $s=1$)].*

Remark 5. *If we consider $u_1=a$, $u_2=b$ and $v=x$ in Theorem 4, then we have the following inequality*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\;& \frac{1}{2(b-a)(1+p)^{\frac{1}{p}}}\left[(x-a)^2\left(|f'(x)|^q+|f'(a)|^q\right)^{\frac{1}{q}}+(b-x)^2\left(|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 5. *We assume that the conditions of Lemma 1 hold. If the mapping* $|f'|^q$, $q \geq 1$ *is convex on* $[a,b]$, *then we have the following inequality*

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\ & \frac{1}{2}\left[(v-u_1)^2\left(\frac{3\left(|f'(x)|^q+|f'(a)|^q\right)-2|f'(u_1)|^q-|f'(v)|^q}{3}\right)^{\frac{1}{q}}\right. \\
&\left.+(u_2-v)^2\left(\frac{3\left(|f'(x)|^q+|f'(b)|^q\right)-2|f'(u_2)|^q-|f'(v)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.9}$$

Proof. From Lemma 1 and well-known power mean inequality, we obtain the inequality

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\ & (v-u_1)^2\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(x+a-(tu_1+(1-t)v))\right|^q dt\right)^{\frac{1}{q}} \\
&+(u_2-v)^2\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(x+b-(tu_2+(1-t)v))\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From Jensen-Mercer inequality, we obtain that

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\ & (v-u_1)^2\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[|f'(x)|^q+|f'(a)|^q-t|f'(u_1)|^q-(1-t)|f'(v)|^q\right] dt\right)^{\frac{1}{q}} \\
&+(u_2-v)^2\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[|f'(x)|^q+|f'(b)|^q-t|f'(u_2)|^q-(1-t)|f'(v)|^q\right] dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{2}\left[(v-u_1)^2\left(\frac{3\left(|f'(x)|^q+|f'(a)|^q\right)-2|f'(u_1)|^q-|f'(v)|^q}{3}\right)^{\frac{1}{q}}\right. \\
&\left.+(u_2-v)^2\left(\frac{3\left(|f'(x)|^q+|f'(b)|^q\right)-2|f'(u_2)|^q-|f'(v)|^q}{3}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which finishes the proof. □

Remark 6. *In Theorem 5, if we choose* $|f'(t)| \leq M$ *for all* $t \in [a,b]$, *then we recapture the inequality (2.6).*

Remark 7. *If we consider* $u_1 = a$, $u_2 = b$ *and* $v = x$ *in Theorem 5, then we have the following inequality*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \frac{1}{2(b-a)}\left[(x-a)^2\left(\frac{2|f'(x)|^q+|f'(a)|^q}{3}\right)^{\frac{1}{q}}+(b-x)^2\left(\frac{2|f'(x)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 8. *In the previous inequalities, by setting* $x = \frac{a+b}{2}$, *one can acquire multiple midpoint type inequalities. Furthermore, it leaves the specifics to the interested reader.*

Theorem 6. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|^q$, $q>1$ is concave on $[a,b]$, then we have the following inequality*

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[(v-u_1)^2\left|f'\left(x+a-\frac{u_1+v}{2}\right)\right|+(u_2-v)^2\left|f'\left(x+b-\frac{u_2+v}{2}\right)\right|\right]
\end{aligned} \tag{2.10}$$

where $\frac{1}{r}+\frac{1}{p}=1$.

Proof. From Lemma 1 and Hölder's inequality, we have the inequality

$$\begin{aligned}
&\left|(v-u_1) f(x+a-u_1)+(u_2-v) f(x+b-u_2)-\left[\int_{x+a-v}^{x+a-u_1} f(t)\,dt+\int_{x+b-u_2}^{x+b-v} f(t)\,dt\right]\right| \\
\leq\ & (v-u_1)^2\int_0^1 t\left|f'(x+a-(tu_1+(1-t)v))\right|dt \\
&+(u_2-v)^2\int_0^1 t\left|f'(x+b-(tu_2+(1-t)v))\right|dt \\
\leq\ & (v-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(x+a-(tu_1+(1-t)v))\right|^q dt\right)^{\frac{1}{q}} \\
&+(u_2-v)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(x+b-(tu_2+(1-t)v))\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}$$

Since $|f'|^q$ is concave mapping, therefore from inequality (1.5), we have

$$\int_0^1 \left|f'(x+a-(tu_1+(1-t)v))\right|^q dt \leq \left|f'\left(x+a-\frac{u_1+v}{2}\right)\right|^q \tag{2.12}$$

and

$$\int_0^1 \left|f'(x+b-(tu_2+(1-t)v))\right|^q dt \leq \left|f'\left(x+b-\frac{u_2+v}{2}\right)\right|^q. \tag{2.13}$$

We obtain the resulting inequality (2.10) by placing the inequalities (2.12) and (2.13) in (2.11). □

Remark 9. *If we consider $u_1=a$, $u_2=b$ and $v=x$ in Theorem 6, then Theorem 6 becomes [3, Theorem 5 (for $s=1$)].*

3. Application to special means

For arbitrary positive numbers a,b $(a\neq b)$, we consider the means as follows:

(1) The arithmatic mean
$$A(a,b)=\frac{a+b}{2}.$$
(2) The generalize logarithmic mean
$$L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right]^{\frac{1}{p}},\ p\in\mathbb{R}\backslash\{-1,0\}.$$
(3) The identric mean
$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a\neq b,\\ a, & \text{if } a=b,\end{cases}\quad a,b>0.$$

Proposition 1. *Let $a, b > 0$, then we have the following inequality*

$$\begin{aligned} & \left|(v-u_1)\left(2A(x,a)-u_1\right)^n+(u_2-v)\left(2A(x,b)-u_2\right)^n\right. \\ & \left.-\left[(v-u_1)L_n^n(x+a-u_1,x+a-v)+(u_2-v)L_n^n(x+b-v,x+b-u_2)\right]\right| \\ \leq\; & \frac{M}{2}\left((v-u_1)^2+(u_2-v)^2\right). \end{aligned}$$

Proof. The result can be directly obtained by applying Corollary 1 to the convex function $f(x)=x^n$, $x>0$.There are omitted the information. □

Proposition 2. *Let $a, b > 0$, then we have the following inequality*

$$\begin{aligned} & \Big|\ln\left(2A(x,a)-u_1\right)^{(v-u_1)}+\ln\left(2A(x,b)-u_2\right)^{(u_2-v)} \\ & -\left[\ln I(x+a-u_1,x+a-v)^{(v-u_1)}+\ln I(x+b-v,x+b-u_2)^{(u_2-v)}\right]\Big| \\ \leq\; & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{(v-u_1)^2}{x+a-\frac{u_1+v}{2}}+\frac{(u_2-v)^2}{x+b-\frac{u_2+v}{2}}\right]. \end{aligned}$$

Proof. The result can be directly obtained by applying Theorem 6 to the concave function $f(x)=\ln x$. There are omitted the information. □

Acknowledgment

We thankful to editor and referees for their careful reading and valuable suggestions to make the article friendly readable.

Funding

The work was supported by the Natural Science Foundation of China (Grant No. 11971241).

Availability of Data Materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Author's Contribution

The study was carried out in collaboration of all authors. All authors read and approved the final manuscript

Competing Interests

It is declared that authors has no competing interests.

References

[1] T. Abdeljawad, M. A. Ali, P. O. Mohammed and A. Kashuri, *On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals,* AIMS Mathematics, 6(1), 2021, 712-725, doi: 10.3934/math.2021043.

[2] M. M. Ali and A. R. Khan, *Generalized integral Mercer's inequality and integral means*, Journal of Inequalities and Special Functions, 10 (1) (2019), pp. 60-76.

[3] M. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense.* Applied Mathematics Letters, 23(9), 2010, 1071-1076.

[4] H. Budak, M. Z. Sarikaya and S. S. Dragomir, *Some perturbed Ostrowski type inequalities for twice differentiable functions*, In Advances in Mathematical Inequalities and Applications (pp. 279-294). Birkhäuser, Singapore, 2018.

[5] P. Cerone and S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstratio Math.37(2), 2004, 299–308.

[6] P. Cerone, S. S. Dragomir and J. Roumeliotis, *An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications*, East Asian mathematical journal, 15(1), pp. 1-9, 1999.

[7] H. H. Chu, S. Rashid, H. Hammouch and Y-M. Chu, *New fractional estimates for Hermite-Hadamard-Mercer's type inequalities,* Alexadria Engineering Journal, https://doi.org/10.1016/j.aej.2020.06.040.

[8] S.S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. Appl. 1 (2) (1998).

[9] S.S. Dragomir, *The Ostrowski integral inequality for Lipschitzian mappings and applications*, Comput. Math. Appl. 38 (1999) 33–37.

[10] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[11] M. Kian and M.S. Moslehian, *Refinements of the operator Jensen-Mercer inequality*, Electron. J. Linear Algebra, 26:742-753, 2013.

[12] A. McD. Mercer, *A Variant of Jensenís Inequality*, J. Ineq. Pure and Appl. Math 4 (4) (2003), Art 73.

[13] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[14] M. Niezgoda, *A generalization of Mercer's result on convex functions*, Nonlinear Analysis 71 (2009), 277.

[15] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[16] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[17] M. Z. Sarikaya H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proceedings of the American Mathematical Society, 145(4), 1527-1538., 2017.

[18] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, 63(7), 2012, 1147–1154.