SOME PARAMETERIZED HERMITE-HADAMARD AND SIMPSON TYPE INEQUALITIES FOR CO-ODINATED CONVEX FUNCTIONS

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Abstract. In this paper, we first obtain an identity for twice partially differentiable mappings involving some parameters. Moreover, by utilizing this identity and functions whose twice partially derivatives in absolute value are co-ordinated convex, we establish some inequalities which generalize several inequalities, such as trapezoid, midpoint and Simpson's inequalities.

1. Introduction

The inequality which is known as Hermite-Hadamard inequality offered by C. Hermite and J. Hadamard independently (see, e.g., [9], [21, p.137]). This is one of the most well proved inequalities in the theory of convex functions with a geometrical interpretation and having many applications. These inequalities can be stated as: If the mapping $\mathcal{F}: I \to \mathbb{R}$ is convex on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1.1}$$

If $\mathcal{F}$ is concave mapping, then the above inequality is reversed. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied.

On the other hand, the following inequality is well known in the literature as Simpson's inequality.

Theorem 1. *Let $\mathcal{F}: [\kappa_1, \kappa_2] \to \mathbb{R}$ be a four times continuously differentiable mapping on (κ_1, κ_2) and $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup\left|\mathcal{F}^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} + 2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right| \leq \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty (\kappa_2-\kappa_1)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities (see, [4, 6, 10, 11, 13–15, 18, 22, 23, 25–28, 30–32, 34, 36]).

In [12], Du et al. gave the following generalized identity to find the estimations of Simpson's type inequalities for differentiable extended (s,m)-convex functions.

Lemma 1. *[12]Consider the mapping $\mathcal{F}: I \subseteq [0,\infty] \to \mathbb{R}$ is differentiable on I° (interior of I), where $\kappa_1, \kappa_2 \in I^\circ$ such that $0 < \kappa_1 < \kappa_2$. If $\mathcal{F}'$ is integrable on κ_1, κ_2 and $\lambda, \mu \in \mathbb{R}$, then for all $x \in [m\kappa_1, \kappa_2]$, where $m \in (0,\infty]$ is fixed, the following equality holds:*

$$\begin{aligned}
&\lambda\mathcal{F}(m\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{m\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-m\kappa_1}\int_{m\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx \\
=\;& (\kappa_2-m\kappa_1)\left[\int_0^{\frac{1}{2}}(\tau-\lambda)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau + \int_{\frac{1}{2}}^1(\tau-\mu)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau\right].
\end{aligned}$$

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Theorem 2. *[12] We assume that the conditions of Lemma 1 hold. If $|\mathcal{F}'|$ is an extended (s,m)-convex function on $[\kappa_1,\kappa_2]$, for some fixed $s,m \in [-1,1]\times(0,1]$ and $0 \le \lambda,\mu \le 1$, then for all $x \in [m\kappa_1,\kappa_2]$, the following inequality of Simpson's type holds*

$$\begin{aligned}&\left|\lambda\mathcal{F}(m\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{m\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-m\kappa_1}\int_{m\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right|\\ \le\;&(\kappa_2-m\kappa_1)\left[\nu_1(\lambda,\mu,s)\left|\mathcal{F}'(\kappa_2)\right|+m\nu_2(\lambda,\mu,s)\left|\mathcal{F}'(\kappa_1)\right|\right]\end{aligned}$$

where

$$\nu_1(\lambda,\mu,s)=\frac{2\lambda^{s+1}+2\mu^{s+1}+\frac{1}{2^{s+1}}\left[2(s+1)-2(s+2)(\mu+\lambda)\right]+(s+1-\mu s+2\mu)}{(s+1)(s+2)}$$

and

$$\nu_2(\lambda,\mu,s)=\frac{2(1-\lambda)^{s+2}+2(1-\mu)^{s+2}+\frac{1}{2^{s+1}}\left[2(\mu+\lambda)(s+2)-2(s+3)\right]+(\lambda s+2\lambda-1)}{(s+1)(s+2)}.$$

From Lemma 1 and Theorem 2, we obtain the following results.

Corollary 1. *Consider the mapping $\mathcal{F}: I := [\kappa_1,\kappa_2] \to \mathbb{R}$ is differentiable on I° (interior of I). If $\mathcal{F}'$ is integrable on κ_1,κ_2 and $\lambda,\mu \in \mathbb{R}$, then for all $x \in [\kappa_1,\kappa_2]$, the following equality holds:*

$$\begin{aligned}&\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\\ =\;&(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}(\tau-\lambda)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau+\int_{\frac{1}{2}}^{1}(\tau-\mu)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau\right].\end{aligned}$$

Corollary 2. *We assume that the conditions of Corollary 1 hold. If $|\mathcal{F}'|$ is a convex function on $[\kappa_1,\kappa_2]$, for $0 \le \lambda,\mu \le 1$, then for all $x \in [\kappa_1,\kappa_2]$, the following inequality holds*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right|\\ \le\;&(\kappa_2-m\kappa_1)\left[\nu_1(\lambda,\mu,1)\left|\mathcal{F}'(\kappa_2)\right|+\nu_2(\lambda,\mu,1)\left|\mathcal{F}'(\kappa_1)\right|\right].\end{aligned}$$

For the other concepts used in Lemma 1 and Theorem 2, one can read [12].
A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $\mathcal{F}: \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v) \in \Delta$ and $\tau,\sigma \in [0,1]$, if it satisfies the following inequality:*

$$\begin{aligned}&\mathcal{F}(\tau x+(1-\tau)\,y,\sigma u+(1-\sigma)\,v)\\ \le\;&\tau\sigma\;\mathcal{F}(x,u)+\tau(1-\sigma)\mathcal{F}(x,v)+\sigma(1-\tau)\mathcal{F}(y,u)+(1-\tau)(1-\sigma)\mathcal{F}(y,v).\end{aligned}\tag{1.2}$$

The mapping $\mathcal{F}$ is a co-ordinated concave on Δ if the inequality (1.2) holds in reversed direction for all $\tau,\sigma \in [0,1]$ and $(x,u),(y,v) \in \Delta$.

In [8], Dragomir proved the following inequality which is Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose that $\mathcal{F}:\Delta\to\mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$
\begin{aligned}
(1.3)\qquad & \mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
\leq\ & \frac{1}{2}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx+\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy\right]\\
\leq\ & \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)dydx\\
\leq\ & \frac{1}{4}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_3)dx+\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_4)dx\right.\\
+\ & \left.\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_1,y)dy+\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_2,y)dy\right]\\
\leq\ & \frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_4)}{4}.
\end{aligned}
$$

The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping $\mathcal{F}$ is a co-ordinated concave mapping.

Over the years, many papers are dedicated on the generalizations and new versions of the inequalities (1.3) using the different type convex functions. For the other Hermite-Hadamard type inequalities for co-ordinated convex functions, please refer to [1–3, 5, 7, 16, 17, 19, 24, 29, 33, 35, 37].

In [20], Özdemir et al. gave the following identity and using the this identity, the authors established some Simpson type inequalities for double integrals:

Lemma 2. *$\mathcal{F}:\Delta:=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]\to\mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\dfrac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\in L(\Delta)$, then we have the following equality*

$$
\begin{aligned}
& \frac{\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)}{9}\\
& +\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_4)}{36}\\
& -\frac{1}{6(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left[\mathcal{F}(x,\kappa_3)+4\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}(x,\kappa_4)\right]dx\\
& -\frac{1}{6(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left[\mathcal{F}(\kappa_1,y)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)+\mathcal{F}(\kappa_2,y)\right]dy\\
& +\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx\\
=\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 q(\tau,\sigma)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1\tau+(1-\tau)\kappa_2,\kappa_3\sigma+(1-\sigma)\kappa_4)\,d\sigma d\tau
\end{aligned}
$$

which the mapping $q:[0,1]\times[0,1]\to\mathbb{R}$ is defined by

$$
q(\tau,\sigma)=\begin{cases}
\left(\tau-\frac{1}{6}\right)\left(\sigma-\frac{1}{6}\right) & 0\leq\tau\leq\frac{1}{2},\quad 0\leq\sigma\leq\frac{1}{2}\\[2mm]
\left(\tau-\frac{1}{6}\right)\left(\sigma-\frac{5}{6}\right) & 0\leq\tau\leq\frac{1}{2},\quad \frac{1}{2}\leq\sigma\leq 1\\[2mm]
\left(\tau-\frac{5}{6}\right)\left(\sigma-\frac{1}{6}\right) & \frac{1}{2}\leq\tau\leq 1,\quad 0\leq\sigma\leq\frac{1}{2}\\[2mm]
\left(\tau-\frac{5}{6}\right)\left(\sigma-\frac{5}{6}\right) & \frac{1}{2}\leq\tau\leq 1,\quad \frac{1}{2}\leq\sigma\leq 1.
\end{cases}
$$

Inspired by the ongoing studies, we give the refinements of the inequalities proved by Du et al. in [12] for partially differentiable co-ordinated convex functions which generalize the results given in [16, 20, 24].

2. New Parameterized Inequalities

In this section, we first define the following mapping

$$\begin{aligned}
&\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\\
=\;& (\mu_1-\lambda_1)(\mu_2-\lambda_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&+\lambda_1(\mu_2-\lambda_2)\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+(1-\mu_1)(\mu_2-\lambda_2)\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\\
&+(\mu_1-\lambda_1)\lambda_2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+(\mu_1-\lambda_1)(1-\mu_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&+\lambda_1\lambda_2\mathcal{F}(\kappa_1,\kappa_3)+(1-\mu_1)\lambda_2\mathcal{F}(\kappa_2,\kappa_3)+\lambda_1(1-\mu_2)\mathcal{F}(\kappa_1,\kappa_4)+(1-\mu_1)(1-\mu_2)\mathcal{F}(\kappa_2,\kappa_4)\\
&-\frac{\lambda_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_3)\,dx-\frac{\mu_2-\lambda_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx\\
&-\frac{1-\mu_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_4)\,dx-\frac{\lambda_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_1,y)\,dy\\
&-\frac{\mu_1-\lambda_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy-\frac{1-\mu_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_2,y)\,dy\\
&+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx
\end{aligned}$$

Now, we give the following equality.

Lemma 3. *Let $\mathcal{F}:\Delta:=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]\to\mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\in L(\Delta)$, then we have the following equality*

$$\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 w(\tau,\sigma)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\sigma d\tau$$

where the mapping $w:[0,1]\times[0,1]\to\mathbb{R}$ is defined by

$$w(\tau,\sigma)=\begin{cases}(\tau-\lambda_1)(\sigma-\lambda_2) & 0\le\tau\le\frac{1}{2},\ 0\le\sigma\le\frac{1}{2}\\ (\tau-\lambda_1)(\sigma-\mu_2) & 0\le\tau\le\frac{1}{2},\ \frac{1}{2}\le\sigma\le 1\\ (\tau-\mu_1)(\sigma-\lambda_2) & \frac{1}{2}\le\tau\le 1,\ 0\le\sigma\le\frac{1}{2}\\ (\tau-\mu_1)(\sigma-\mu_2) & \frac{1}{2}\le\tau\le 1,\ \frac{1}{2}\le\sigma\le 1.\end{cases}$$

Proof. From the definition of the mapping $w(\tau,\sigma)$, we have

$$\begin{aligned}
& (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 w(\tau,\sigma)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\sigma d\tau \\
= \; & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_0^{\frac12}(\tau-\lambda_1)(\sigma-\lambda_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
& +(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_{\frac12}^1(\tau-\lambda_1)(\sigma-\mu_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
& +(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^1\int_0^{\frac12}(\tau-\mu_1)(\sigma-\lambda_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
& +(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^1\int_{\frac12}^1(\tau-\mu_1)(\sigma-\mu_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
= \; & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)[J_1+J_2+J_3+J_4].
\end{aligned}$$

Integration by parts, one can easily obtain

$$\begin{aligned}
J_1 = \; & \int_0^{\frac12}\int_0^{\frac12}(\tau-\lambda_1)(\sigma-\lambda_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
= \; & \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)} \\
& \times\Bigg[\left(\frac12-\lambda_1\right)\left(\frac12-\lambda_2\right)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\lambda_1\left(\frac12-\lambda_2\right)\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right) \\
& -\left(\frac12-\lambda_2\right)\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx+\left(\frac12-\lambda_1\right)\lambda_2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right) \\
& +\lambda_1\lambda_2\mathcal{F}(\kappa_1,\kappa_3)-\frac{\lambda_2}{\kappa_2-\kappa_1}\int_0^{\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}(x,\kappa_3)\,dx \\
& -\left(\frac12-\lambda_1\right)\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy-\frac{\lambda_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}(\kappa_1,y)\,dy \\
& +\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}(x,y)\,dydx\Bigg],
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
J_2 = \; & \int_0^{\frac12}\int_{\frac12}^1(\tau-\lambda_1)(\sigma-\mu_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
= \; & \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\Bigg[\left(\frac12-\lambda_1\right)(1-\mu_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\lambda_1(1-\mu_2)\mathcal{F}(\kappa_1,\kappa_4) \\
& -\frac{1-\mu_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}(x,\kappa_4)\,dx-\left(\frac12-\lambda_1\right)\left(\frac12-\mu_2\right)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
& -\lambda_1\left(\frac12-\mu_2\right)\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\left(\frac12-\mu_2\right)\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx \\
& -\left(\frac12-\lambda_1\right)\frac{1}{\kappa_4-\kappa_3}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy-\frac{\lambda_1}{\kappa_4-\kappa_3}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}(\kappa_1,y)\,dy \\
& +\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}(x,y)\,dydx\Bigg],
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
(2.3)\quad J_3 &= \int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}(\tau-\mu_1)(\sigma-\lambda_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
&= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)} \\
&\quad\times\Bigg[(1-\mu_1)\left(\frac{1}{2}-\lambda_2\right)\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad-\left(\frac{1}{2}-\mu_1\right)\left(\frac{1}{2}-\lambda_2\right)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad-\left(\frac{1}{2}-\lambda_2\right)\frac{1}{\kappa_2-\kappa_1}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx+(1-\mu_1)\lambda_2\mathcal{F}(\kappa_2,\kappa_3) \\
&\quad-\left(\frac{1}{2}-\mu_1\right)\lambda_2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)-\frac{\lambda_2}{\kappa_2-\kappa_1}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\mathcal{F}(x,\kappa_3)\,dx \\
&\quad-\frac{1-\mu_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}(\kappa_2,y)\,dy+\left(\frac{1}{2}-\mu_1\right)\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy \\
&\quad+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathcal{F}(x,y)\,dydx\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
(2.4)\quad J_4 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(\tau-\mu_1)(\sigma-\mu_2)\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\,d\tau d\sigma \\
&= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)} \\
&\quad\times\Bigg[(1-\mu_1)(1-\mu_2)\mathcal{F}(\kappa_2,\kappa_4)-\left(\frac{1}{2}-\mu_1\right)(1-\mu_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right) \\
&\quad-\frac{1-\mu_2}{\kappa_2-\kappa_1}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\mathcal{F}(x,\kappa_4)\,dx-(1-\mu_1)\left(\frac{1}{2}-\mu_2\right)\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad+\left(\frac{1}{2}-\mu_1\right)\left(\frac{1}{2}-\mu_2\right)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad+\left(\frac{1}{2}-\mu_2\right)\frac{1}{\kappa_2-\kappa_1}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx \\
&\quad-\frac{1-\mu_1}{\kappa_4-\kappa_3}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}(\kappa_2,y)\,dy+\left(\frac{1}{2}-\mu_1\right)\frac{1}{\kappa_4-\kappa_3}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy \\
&\quad+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}\mathcal{F}(x,y)\,dydx\Bigg].
\end{aligned}$$

By the equalities (2.1)-(2.4), we have

$$\begin{aligned}
&(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)[J_1+J_2+J_3+J_4] \\
=\;&(\mu_1-\lambda_1)(\mu_2-\lambda_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&+\lambda_1(\mu_2-\lambda_2)\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+(1-\mu_1)(\mu_2-\lambda_2)\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right) \\
&+(\mu_1-\lambda_1)\lambda_2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+(\mu_1-\lambda_1)(1-\mu_2)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right) \\
&+\lambda_1\lambda_2\mathcal{F}(\kappa_1,\kappa_3)+(1-\mu_1)\lambda_2\mathcal{F}(\kappa_2,\kappa_3)+\lambda_1(1-\mu_2)\mathcal{F}(\kappa_1,\kappa_4)+(1-\mu_1)(1-\mu_2)\mathcal{F}(\kappa_2,\kappa_4) \\
&-\frac{\lambda_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_3)\,dx-\frac{\mu_2-\lambda_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1-\mu_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x,\kappa_4)\,dx-\frac{\lambda_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_1,y)\,dy\\
&-\frac{\mu_1-\lambda_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy-\frac{1-\mu_1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\kappa_2,y)\,dy\\
&+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx
\end{aligned}
$$

which completes the proof. □

Remark 1. *If we choose $\lambda_1=\lambda_2=\frac{1}{6}$ and $\mu_1=\mu_2=\frac{5}{6}$ in Lemma 3, then Lemma 3 reduces to Lemma 2 which was proved in [20].*

Remark 2. *If we choose $\lambda_1=\lambda_2=0$ and $\mu_1=\mu_2=1$ in Lemma 3, then Lemma 3 reduces to [16, Lemma 1].*

Remark 3. *If we choose $\lambda_1=\lambda_2=\mu_1=\mu_2=\frac{1}{2}$ in Lemma 3, then Lemma 3 reduces to [24, Lemma 1].*

Theorem 4. *We assume that the conditions of Lemma 3 hold. If $\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\right|$ is a co-ordinated convex function on Δ, then we have the following inequality*

$$
\begin{aligned}
&|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|\\
\leq\ &(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\\
&\times\left[\Psi_1\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|+\Psi_2\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|+\Psi_3\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|+\Psi_3\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|\right]
\end{aligned}
$$

where

$$
\Psi_1=\left[\frac{\lambda_1^3+\mu_1^3}{3}-\frac{\lambda_1}{8}-\frac{5\mu_1}{8}+\frac{5}{12}\right]\left[\frac{\lambda_2^3+\mu_2^3}{3}-\frac{\lambda_2}{8}-\frac{5\mu_2}{8}+\frac{5}{12}\right],
$$

$$
\Psi_2=\left[\frac{\lambda_1^3+\mu_1^3}{3}-\frac{\lambda_1}{8}-\frac{5\mu_1}{8}+\frac{5}{12}\right]\left[-\frac{\lambda_2^3+\mu_2^3}{3}+\lambda_2^2+\mu_2^2-\frac{7\mu_2+3\lambda_2}{8}+\frac{1}{3}\right],
$$

$$
\Psi_3=\left[-\frac{\lambda_1^3+\mu_1^3}{3}+\lambda_1^2+\mu_1^2-\frac{7\mu_1+3\lambda_1}{8}+\frac{1}{3}\right]\left[\frac{\lambda_2^3+\mu_2^3}{3}-\frac{\lambda_2}{8}-\frac{5\mu_2}{8}+\frac{5}{12}\right]
$$

and

$$
\Psi_4=\left[-\frac{\lambda_1^3+\mu_1^3}{3}+\lambda_1^2+\mu_1^2-\frac{7\mu_1+3\lambda_1}{8}+\frac{1}{3}\right]\left[-\frac{\lambda_2^3+\mu_2^3}{3}+\lambda_2^2+\mu_2^2-\frac{7\mu_2+3\lambda_2}{8}+\frac{1}{3}\right].
$$

Proof. Taking the modulus in Lemma 3, we have

$$
\begin{aligned}
&|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|\\
\leq\ &(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1|w(\tau,\sigma)|\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\right|d\sigma d\tau.
\end{aligned}
$$

Since $\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\right|$ is a co-ordinated convex function on Δ, we have

$$
\begin{aligned}
&\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\right|\\
\leq\ &\tau\sigma\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|+\tau(1-\sigma)\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|+(1-\tau)\sigma\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|+(1-\tau)(1-\sigma)\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|.
\end{aligned}
$$

Then it follows

$$\begin{aligned}
& |\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \\
\leq\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3) \\
& \left[\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|\int_0^1\int_0^1 |w(\tau,\sigma)|\,\tau\sigma d\sigma d\tau+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|\int_0^1\int_0^1 |w(\tau,\sigma)|\,\tau(1-\sigma)\,d\sigma d\tau\right. \\
& \left.+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|\int_0^1\int_0^1 |w(\tau,\sigma)|\,(1-\tau)\,\sigma d\sigma d\tau+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|\int_0^1\int_0^1 |w(\tau,\sigma)|\,(1-\tau)(1-\sigma)\,d\sigma d\tau\right] \\
\leq\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\Psi_1\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|+\Psi_2\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|+\Psi_3\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|+\Psi_3\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|\right].
\end{aligned}$$

This completes the proof. □

Remark 4. *If we choose $\lambda_1=\lambda_2=\frac{1}{6}$ and $\mu_1=\mu_2=\frac{5}{6}$ in Theorem 4, then Theorem 4 reduces to [20, Theorem 3].*

Remark 5. *If we choose $\lambda_1=\lambda_2=0$ and $\mu_1=\mu_2=1$ in Theorem 4, then Theorem 4 reduces to [16, Theorem 2].*

Remark 6. *If we choose $\lambda_1=\lambda_2=\mu_1=\mu_2=\frac{1}{2}$ in Theorem 4, then Theorem 4 reduces to [24, Theorem 2].*

Theorem 5. *We assume that the conditions of Lemma 3 hold. If $\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, then we have the following inequality*

$$\begin{aligned}
|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \leq\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left(A_p(\lambda,\mu)\right)^{\frac{1}{p}} \\
& \times\left(\frac{\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}
& A_p(\lambda,\mu) \\
=\ & \frac{1}{(p+1)^2}\left[\lambda_1^{p+1}+\left(\frac{1}{2}-\lambda_1\right)^{p+1}+\left(\mu_1-\frac{1}{2}\right)^{p+1}+(1-\mu_1)^{p+1}\right] \\
& \times\left[\lambda_2^{p+1}+\left(\frac{1}{2}-\lambda_2\right)^{p+1}+\left(\mu_2-\frac{1}{2}\right)^{p+1}+(1-\mu_2)^{p+1}\right].
\end{aligned}$$

Proof. Taking the modulus in Lemma 3 and using the Hölders inequality, we have

$$\begin{aligned}
(2.5)\quad & |\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \\
\leq\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 |w(\tau,\sigma)|\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\right|d\sigma d\tau \\
\leq\ & (\kappa_2-\kappa_1)(\kappa_4-\kappa_3) \\
& \times\left(\int_0^1\int_0^1 |w(\tau,\sigma)|^p\,d\sigma d\tau\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\right|^q d\sigma d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

By definition of $w(\tau,\sigma)$, we can write

$$
\begin{aligned}
(2.6)\qquad & \int_0^1\int_0^1 |w(\tau,\sigma)|^p\, d\sigma d\tau \\
= & \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |\tau-\lambda_1|^p |\sigma-\lambda_2|^p\, d\tau d\sigma + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 |\tau-\lambda_1|^p |\sigma-\mu_2|^p\, d\tau d\sigma \\
& + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} |\tau-\mu_1|^p |\sigma-\lambda_2|^p\, d\tau d\sigma + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 |\tau-\mu_1|^p |\sigma-\mu_2|^p\, d\tau d\sigma \\
= & \frac{\lambda_1^{p+1}+\left(\frac{1}{2}-\lambda_1\right)^{p+1}}{p+1}\frac{\lambda_2^{p+1}+\left(\frac{1}{2}-\lambda_2\right)^{p+1}}{p+1} \\
& + \frac{\lambda_1^{p+1}+\left(\frac{1}{2}-\lambda_1\right)^{p+1}}{p+1}\frac{\left(\mu_2-\frac{1}{2}\right)^{p+1}+(1-\mu_2)^{p+1}}{p+1} \\
& + \frac{\left(\mu_1-\frac{1}{2}\right)^{p+1}+(1-\mu_1)^{p+1}}{p+1}\frac{\lambda_2^{p+1}+\left(\frac{1}{2}-\lambda_2\right)^{p+1}}{p+1} \\
& + \frac{\left(\mu_1-\frac{1}{2}\right)^{p+1}+(1-\mu_1)^{p+1}}{p+1}\frac{\left(\mu_2-\frac{1}{2}\right)^{p+1}+(1-\mu_2)^{p+1}}{p+1} \\
= & \frac{1}{(p+1)^2}\left[\lambda_1^{p+1}+\left(\frac{1}{2}-\lambda_1\right)^{p+1}+\left(\mu_1-\frac{1}{2}\right)^{p+1}+(1-\mu_1)^{p+1}\right] \\
& \times\left[\lambda_2^{p+1}+\left(\frac{1}{2}-\lambda_2\right)^{p+1}+\left(\mu_2-\frac{1}{2}\right)^{p+1}+(1-\mu_2)^{p+1}\right]
\end{aligned}
$$

Since $\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, we obtain

$$
\begin{aligned}
(2.7)\qquad & \int_0^1\int_0^1 \left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\left(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3\right)\right|^q d\sigma d\tau \\
\leq & \frac{\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|^q}{4}.
\end{aligned}
$$

By substituting (2.6) and (2.7) in (2.5), we can obtain the desired result. □

Corollary 3. *If we choose $\lambda_1=\lambda_2=\frac{1}{6}$ and $\mu_1=\mu_2=\frac{5}{6}$ in Theorem 5, then we have the following Simpson's type inequality*

$$
\begin{aligned}
& \left|\frac{4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)}{9}\right. \\
& +\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_4)}{36} \\
& -\frac{1}{6(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left[\mathcal{F}(x,\kappa_3)+4\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{F}(x,\kappa_4)\right]dx \\
& -\frac{1}{6(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left[\mathcal{F}(\kappa_1,y)+4\int_{\kappa_3}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)+\mathcal{F}(\kappa_2,y)\right]dy \\
& \left.+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx\right|
\end{aligned}
$$

$$\leq \left(\frac{4}{9}\right)^{\frac{1}{p}} \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{9(p+1)^{\frac{2}{p}}} \left(\frac{2^{p+1}+1}{2^{p+1}}\right)^{\frac{2}{p}}$$
$$\times \left(\frac{\left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_3)\right|^q + \left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(\kappa_1,\kappa_4)\right|^q + \left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_3)\right|^q + \left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(\kappa_2,\kappa_4)\right|^q}{4}\right)^{\frac{1}{q}}.$$

Remark 7. *If we choose $\lambda_1=\lambda_2=0$ and $\mu_1=\mu_2=1$ in Theorem 5, then Theorem 5 reduces to [16, Theorem 3].*

Remark 8. *If we choose $\lambda_1=\lambda_2=\mu_1=\mu_2=\frac{1}{2}$ in Theorem 5, then Theorem 5 reduces to [24, Theorem 3].*

Theorem 6. *We assume that the conditions of Lemma 3 hold. If $\dfrac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}$ is bounded on Δ, i.e.*

$$\left\|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}\right\|_\infty = \sup_{(x,y)\in\Delta}\left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(x,y)\right| < \infty,$$

then we have the following inequality

$$|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{4}\left[\lambda_1^2+\left(\frac{1}{2}-\lambda_1\right)^2+\left(\mu_1-\frac{1}{2}\right)^2+(1-\mu_1)^2\right]$$
$$\times\left[\lambda_2^2+\left(\frac{1}{2}-\lambda_2\right)^2+\left(\mu_2-\frac{1}{2}\right)^2+(1-\mu_2)^2\right]\left\|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}\right\|_\infty.$$

Proof. From Lemma 3, we have

$$|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \leq (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 |w(\tau,\sigma)|\left|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}(\tau\kappa_2+(1-\tau)\kappa_1,\sigma\kappa_4+(1-\sigma)\kappa_3)\right| d\sigma d\tau.$$

Since $\dfrac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}$ is bounded on Δ, we obtain

$$|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)| \leq (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left\|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}\right\|_\infty\int_0^1\int_0^1 |w(\tau,\sigma)|\, d\sigma d\tau$$
$$= \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{4}\left[\lambda_1^2+\left(\frac{1}{2}-\lambda_1\right)^2+\left(\mu_1-\frac{1}{2}\right)^2+(1-\mu_1)^2\right]$$
$$\times\left[\lambda_2^2+\left(\frac{1}{2}-\lambda_2\right)^2+\left(\mu_2-\frac{1}{2}\right)^2+(1-\mu_2)^2\right]\left\|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}\right\|_\infty$$

which completes the proof. □

Remark 9. *If we choose $\lambda_1=\lambda_2=\frac{1}{6}$ and $\mu_1=\mu_2=\frac{5}{6}$ in Theorem 6, then Theorem 6 reduses to [20, Theorem 4].*

Corollary 4. *If we choose $\lambda_1=\lambda_2=0$ and $\mu_1=\mu_2=1$ in Theorem 6, then we have the following Midpoint type inequality*

$$\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}\left(x,\frac{\kappa_3+\kappa_4}{2}\right)dx\right.$$
$$\left.-\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2},y\right)dy+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{16}\left\|\frac{\partial^2 \mathcal{F}}{\partial\tau\partial\sigma}\right\|_\infty.$$

Corollary 5. *If we choose $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{2}$ in Theorem 6, then we have the following Trapezoid type inequality*

$$\begin{aligned}
&\left| \frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_4)}{4} \right. \\
&-\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left[\mathcal{F}(x,\kappa_3)+\mathcal{F}(x,\kappa_4)\right]dx-\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left[\mathcal{F}(\kappa_1,y)+\mathcal{F}(\kappa_2,y)\right]dy \\
&\left. +\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(x,y)\,dydx \right| \\
\leq\ & \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{16}\left\|\frac{\partial^2\mathcal{F}}{\partial\tau\partial\sigma}\right\|_{\infty}.
\end{aligned}$$

3. Conclusion

In this work, We proved the generalized version of Simpson's inequalities for twice partially differentiable co-ordinated convex functions. We obtained several new and existing inequalities of Simpson's type, midpoint type, and trapezoidal type in special cases of newly proved inequalities. It is an interesting and new problem that the forthcoming researcher can prove similar inequalities for different kinds of convexity in their future research.

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