GLOBAL SOLUTION AND GLOBAL ORBIT TO REACTION-DIFFUSION EQUATION FOR FRACTIONAL DIRICHLET-TO-NEUMANN OPERATOR WITH SUBCRITICAL EXPONENT

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Abstract. We consider the reaction-diffusion equation for fractional Dirichlet-to-Neumann operator with subcritical exponent motivated by electrical impedance tomography (EIT) and a need to overcome the Non-locality of a fractional differential equation for modeling anomalous diffusion. We mainly deal with the asymptotic behavior of global solution and the boundedness of global orbit which allows us to show that any global solution is classical solution using Moser iteration technique.

Keywords: Fractional Dirichlet-to-Neumann operator, Subcritical exponent, Global solution, Global orbit.

1. Introduction

This work is motivated by electrical impedance tomography (EIT), which is a new medical imaging technology and classical Calder*ó*n problem, its basic principle is to inject a weak current to the electrodes on the surface of an imaging domain such as the human thorax, and measure induced boundary voltages on other electrodes, then according to the relationship between voltage and current, the change value of electric impedance or electrical impedance inside the imaging domain can be reconstructed. Unlike CT technology using X-ray or ultrasonic beam, EIT has no damage to human body, can measure repeatedly, and the imaging speed is fast, the cost is low, no special working environment is required, all these determine the broad application prospect of EIT, and the necessity of its research is self-evident. In modelling idealized EIT imaging problems, there are several premises as follows. (1) Due to the low permeability of biological organs and tissues, the magnetic field effect can be ignored, we only consider the electric field characteristics. (2) Assume the medium is isotropic, then conduction current density $\mathbf{J}$ and electric field intensity vector $\mathbf{E}$ satisfy

$$\mathbf{J} = \gamma \mathbf{E}, \tag{1.1}$$

where γ represents conductivity (scalar function). (3) According to Ampere's law and Faraday's induction law, conduction current density $\mathbf{J}$ and magnetic field intensity vector $\mathbf{H}$ satisfy $\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$, magnetic induction intensity $\mathbf{B}$ and electric field intensity $\mathbf{E}$ satisfy $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$. Since we employ a low frequency injection current (weak current excitation) in EIT, it follows that induced electric field is far less than Coulomb electric field, and displacement current is far less than conduction current, which indicates $\dfrac{\partial \mathbf{B}}{\partial t}$ and $\dfrac{\partial \mathbf{D}}{\partial t}$ are negligible, then we have

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{1.2}$$

$$\nabla \times \mathbf{E} = 0, \tag{1.3}$$

The basic equation (1.3) implies $\mathbf{E}$ is a irrotational field, which gives that electric field intensity $\mathbf{E}$ and potential function v satisfy

$$\mathbf{E} = -\nabla v. \tag{1.4}$$

Taking divergence in the both sides of (1.2), and substituting (1.4) into (1.1) gives

$$0 = \nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} = -\nabla \cdot (\gamma \nabla v),$$

which is the basic equation for the potential [1].

Given a potential function f on the surface of imaging domain and v solves

$$\begin{cases} -\nabla \cdot (\gamma \nabla v) = 0, \ x \in \Omega, \\ v = f, \ x \in \partial\Omega, \end{cases}$$

the Dirichlet-to-Neumann operator (DtN), or voltage-to-current map DN_γ is defined as

$$DN_\gamma(f) = \gamma \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega},$$

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$DN_\gamma(f)$ is exactly the current density flowing into the field, and ν represents the external normal vector to the surface of the medium. The DtN operator is well-known and has been widely studied in [2, 3, 4] and their references. It plays a fundamental role in EIT problem [5], one can recover γ from DtN operator DN_γ by measuring the current through the boundary caused by a family of potential functions f.

When $\gamma = y^{1-2s}(s \in (0,1))$, the corresponding Dirichlet problem is given by

$$\begin{cases} -div\left(y^{1-2s}\nabla v(x,y)\right) = 0, (x,y) \in \mathbb{R}^{N+1}_+, \\ \qquad v(x,y) = f, (x,y) \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $\mathbb{R}^{N+1}_+ = \{Z = (x,y) = (x_1, x_2, \cdots, x_n, y) \in \mathbb{R}^{N+1} | y > 0\}$. In such case, we introduce fractional DtN operator DN_s [6]. It has been showed in [7] and [8] that

$$DN_s(f) = A_s v = \partial^s_\nu v := k_s \lim_{y\to 0} y^{1-2s}\frac{\partial v}{\partial \nu} = -k_s \lim_{y\to 0} y^{1-2s}\frac{\partial v}{\partial y},$$

where the unit exterior normal vector $\nu = (0, \cdots, 0, -1) \in \mathbb{R}^{N+1}$, $k_s = \dfrac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$, and $A_s(0 < s < 1)$ represents the spectral fractional Laplacian operator.

Let Ω be a bounded smooth domain in $\mathbb{R}^N(N > 2s)$, denote half-cylinder $C = \{(x,y)|(x,y) \in \Omega \times (0, \infty)\}$, whose lateral boundary is $\partial_L C = \partial\Omega \times [0, \infty)$, and $p < 2^*_s$, here $2^*_s = \dfrac{2N}{N-2s}$ is critical exponent of Sobolev trace embedding inequality. We are concerned in this paper with the following nonlinear reaction-diffusion equation for the fractional DtN operator with critical exponent

$$\begin{cases} -div\left(y^{1-2s}\nabla v(x,y,t)\right) = 0, \ \ t \in \mathbb{R}^+, (x,y) \in C, \\ \qquad v(x,y,t) = 0, \ \ t \in \mathbb{R}^+, (x,y) \in \partial_L C, \\ \qquad \partial^s_\nu v(x,y,t) = -\dfrac{\partial v(x,y,t)}{\partial t} + |v|^{p-2}v, \ \ t \in \mathbb{R}^+, (x,y) \in \Omega \times \{0\}, \\ \qquad v(x,y,0) = v_0, \ \ (x,y) \in C. \end{cases} \tag{1.6}$$

The energy functional of (1.6) can be defined via

$$E(v(t)) = \frac{1}{2}\int_C k_s y^{1-2s}|\nabla v(t)|^2 dxdy - \frac{1}{p}\int_{\Omega\times\{0\}} |v(t)|^p dx, \ v \in H^s_{0,L}(C), \tag{1.7}$$

here

$$H^s_{0,L}(C) = \left\{ v \;\middle|\; v \in L^2(C): \ v = 0 \text{ a.e. on } \partial_L C, \ \int_C k_s y^{1-2s}|\nabla v|^2 dxdy < \infty \right\}, \tag{1.8}$$

with norm $\|v\| = \left(\int_C k_s y^{1-2s}|\nabla v|^2 dxdy\right)^{\frac{1}{2}}$.

Actually, the Euler-Lagrange equation $E'(v) = 0$ by means of variational method is corresponding to stationary equation of (1.6). For every $v, \phi \in H^s_{0,L}(C)$, $\varepsilon \in \mathbb{R}$, we have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}(E(v + \varepsilon\phi))\Big|_{\varepsilon=0} \\ &= \left(\int_C k_s y^{1-2s}\nabla(v + \varepsilon\phi)\cdot\nabla\phi dxdy - \int_{\Omega\times\{0\}} |v + \varepsilon\phi|^{p-2}(v+\varepsilon\phi)\phi dx\right)\Big|_{\varepsilon=0} \\ &= \int_C k_s y^{1-2s}\nabla v \cdot \nabla\phi dxdy - \int_{\Omega\times\{0\}} |v|^{p-2}v\phi dx =: \langle E'(v), \ \phi\rangle. \end{aligned} \tag{1.9}$$

Since the domain C is unbounded and the solution maybe only is *Hölder* continuous near $y = 0$, and when $s > \dfrac{1}{2}$, $y = 0$ is probably a singularity of $y^{1-2s}\nabla v$, so we take a approximate domain $C^\delta := \Omega \times (\delta, \infty)(\delta > 0)$, where we have

$$\begin{aligned} \langle E'(v), \ \phi\rangle &= \lim_{\delta\to 0}\left(\int_{C^\delta} k_s y^{1-2s}\nabla v\cdot\nabla\phi dxdy - \int_{\Omega\times\{y=\delta\}} |v|^{p-2}v\phi dx\right) \\ &= \lim_{\delta\to 0}\left(k_s\int_{C^\delta} div\left(\phi y^{1-2s}\nabla v\right)dxdy - k_s\int_{C^\delta}\phi div(y^{1-2s}\nabla v)dxdy - \int_{\Omega\times\{y=\delta\}}|v|^{p-2}v\phi dx\right) \\ &= \lim_{\delta\to 0}\left(k_s\int_{\Omega\times\{y=\delta\}}\phi y^{1-2s}\frac{\partial v}{\partial\nu}dx - k_s\int_{C^\delta}\phi div(y^{1-2s}\nabla v)dxdy - \int_{\Omega\times\{y=\delta\}}|v|^{p-2}v\phi dx\right) \\ &= -k_s\int_C \phi div(y^{1-2s}\nabla v)dxdy - \int_{\Omega\times\{0\}}(|v|^{p-2}v - \partial^s_\nu v)\phi dx, \end{aligned}$$

by the arbitrariness of ϕ, it follows that

$$div\left((y^{1-2s}\nabla v\right) = 0, in\ C,$$

$$\partial_\nu^s v = |v|^{p-2}v, in\ \Omega\times\{0\}.$$

By the same token, taking a approximate domain C^δ, and then timing v_t and integrating in the domain the both sides of equation (1.6) in C^δ gives

$$\begin{aligned}
0 &= \int_{C^\delta} -div\left(y^{1-2s}\nabla v\right)v_t dxdy\\
&= -\int_{\partial C^\delta} v_t y^{1-2s}\nabla v\cdot n d\sigma + \int_{C^\delta} y^{1-2s}\nabla v\cdot\nabla v_t dxdy\\
&= -\int_{\partial_L C^\delta} v_t y^{1-2s}\nabla v\cdot\nu' d\sigma - \int_{\Omega\times\{y=\delta\}} v_t y^{1-2s}\nabla v\cdot\nu dx + \int_{C^\delta} y^{1-2s}\nabla v\cdot\nabla v_t dxdy\\
&= -\int_{\Omega\times\{y=\delta\}} v_t y^{1-2s}\nabla v\cdot\nu dx + \int_{C^\delta} y^{1-2s}\nabla v\cdot\nabla v_t dxdy,
\end{aligned}$$

where $\nu' = (\nu_\Omega, 0)$, ν_Ω is unit exterior normal vector to $\partial\Omega$, and $\nu = (0,\cdots,0,-1)$.

Letting $\delta\to 0$ leads to

$$0 = -\frac{1}{k_s}\int_{\Omega\times\{y=0\}} v_t\partial_\nu^s v dx + \int_C y^{1-2s}\nabla v\cdot\nabla v_t dxdy,$$

thus we obtain

$$\frac{dE(v(t))}{dt} = \int_C k_s y^{1-2s}\nabla v\cdot\nabla v_t dxdy - \int_{\Omega\times\{0\}} |v(t)|^{p-2}vv_t dx = -\int_{\Omega\times\{0\}} v_t^2 dx \le 0, \tag{1.10}$$

which implies functional $E(v)$ monotonically decreases in t, namely, $E(v)$ is Lyapunov functional.

Moreover, notice that

$$E(v(t) - E(v(t_0)) = \int_{t_0}^t \frac{dE(v)}{d\tau}d\tau = -\int_{t_0}^t\int_{\Omega\times\{0\}} v_\tau^2 dxd\tau,$$

we consequently arrive at **the energy inequality**

$$E(v(t_0)) = E(v(t)) + \int_{t_0}^t\int_{\Omega\times\{0\}} v_\tau^2 dxd\tau. \tag{1.11}$$

There is another motivation for us to observe problem (1.6). Using the extend method of spectral decomposition or the Caffarelli-Silvestre extension method [7] to let $v:\Omega\times(0,\ \infty)\to\mathbb{R}$ be an extension function of $u:\Omega\to\mathbb{R}$, we can see that equations (1.6) equivalent to the following problem with fractional Laplacian operator

$$\begin{cases}
A_s u = -\dfrac{\partial u}{\partial t} + |u|^{p-2}u, & (x,t)\in\Omega\times(0,\infty),\\
u = 0, & (x,t)\in(\mathbb{R}^N\setminus\Omega)\times[0,\infty),\\
u(x,\ 0) = u_0, & x\in\Omega,
\end{cases} \tag{1.12}$$

In recent years, on account of the better accuracy in describing practical problems compared with the classical Laplacian operator, the spectral fractional Laplacian operator A_s has attracted considerable attention of mathematics and physicists. The fractional Laplacian operator was first proposed in observation of Levy stationary diffusion process in physics, later also used to describe the phenomena such as plasma anomalous diffusion, stochastic analysis and fluid dynamics, etc. There have been extensive studies of the nonlinear fractional Laplacian problem, especially the semilinear elliptic problem with fractional Laplacian, e.g. [9, 10, 11]. While the fractional parabolic equation is more complicated than the elliptic case, the literature involved is relatively limited. The pioneering result about parabolic equation is obtained by Sugitani [12]. He investigated the heat equation $\partial_t u + A_s u = u^{p-1}(0 < s \le 1)$ in the whole space $\mathbb{R}^N$, here $A_s(0 < s < 1)$ denotes spectral fractional Laplacian operator, $p \le 2 + p_F^* < 2_s^*$, $p_F^* = \dfrac{2s}{N}$ is Fujita exponent of the corresponding fractional equations. The study produced that nonnegative solution of the equation blows up in finite time. A. Fino and G. Karch [13] observed that the large time asymptotic behavior with relation to the equation's system mass $M(t) = \int_{\mathbb{R}^N} u(x,t)dx$. Specifically, if $p \le 2 + p_F^*$, then $\lim\limits_{t\to\infty} M(t) = 0$, while if $p > 2 + P_F^*$, then $\lim\limits_{t\to\infty} M(t) = M_\infty > 0$. We can further consider the case of subcritical growth $p < 2_s^*$. The main difficulty of fractional problem is owing to the fact that the fractional Laplacian operator is nonlocal, to overcome this, we turn to investigate its equivalent problem (1.6) so that we can use variational technique.

In fact, let $\{\lambda_k, \phi_k\}_{k=1}^{\infty}$ be the eigenvalues and its corresponding normal eigenfunctions of negative Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary condition, i.e.

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k, & \text{in } \Omega, \\ \phi_k = 0, & \text{on } \Omega, \end{cases} \tag{1.13}$$

$\{\phi_k\}$ constitute the complete orthonormal basis of $L^2(\Omega)$, which gives the spectral decomposition of operator $-\Delta$.

Set

$$\mathcal{V}_0(\Omega) = \Big\{u(t) = u(t)(x) = u(x,t) = \sum_{k=1}^{\infty} c_k(t)\phi_k(x) \in L^2(\Omega) \Big| \sum_{k=1}^{\infty} c_k^2(t)\lambda_k^s < +\infty\Big\},$$

then for each $u(t) \in \mathcal{V}_0(\Omega)$, A_s can be defined as

$$A_s : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$$
$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)\phi_k(x) \mapsto A_s u = \sum_{k=1}^{\infty} c_k(t)\lambda_k^s\phi_k(x),$$

here $c_k(t) = \int_\Omega u(x,t)\phi_k(x)dx$, $\mathcal{V}_0^*(\Omega)$ is dual space of $\mathcal{V}_0(\Omega)$.

Denote

$$\mathcal{I} = \Big\{q(y) \,\Big|\, q(y) \text{ satisfies } \int_0^\infty (|q'(y)|^2 + |q(y)|^2)y^{1-2s}dy < \infty, \text{ and } q(0) = 1,\ q(\infty) = 0\Big\},$$

Consider following minimum functional problem in the function space $\mathcal{I}$,

$$J(q) = \inf\Big\{\int_0^\infty (|q'(y)|^2 + |q(y)|^2)y^{1-2s}dy\Big\}.$$

We can show that, for each $u(t) \in \mathcal{V}_0(\Omega)$, there is a unique extension

$$v(x,y,t) = \sum_{k=1}^{\infty} c_k(t)\phi_k(x)q(\sqrt{\lambda_k}y), \quad (x,y) \in C,$$

such that

$$\int_C y^{1-2s}|\nabla v|^2dxdy = \int_0^\infty y^{1-2s}\int_\Omega (|\frac{\partial v}{\partial y}|^2 + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^2)dxdy = k_s\sum_{k=1}^{\infty} c_k^2(t)\lambda_k^s.$$

In fact, assume function q is a minimizer of functional $J(q)$, we have

$$\int_0^\infty (|\varphi'(y)|^2 + |\varphi(y)|^2)y^{1-2s}dy < \infty, \text{ and } q(0) = q(\infty) = 0,$$

for every function φ.

Taking derivative of following one variable function

$$j(\varepsilon) = J(q+\varepsilon\varphi) = \int_0^\infty (|q' + \varepsilon\varphi'|^2 + |q+\varepsilon\varphi|^2)y^{1-2s}dy,$$

one has

$$\begin{aligned} j'(\varepsilon)\Big|_{\varepsilon=0} &= \int_0^\infty [2(q'+\varepsilon\varphi')\varphi' + 2(q+\varepsilon\varphi)\varphi]y^{1-2s}dy\Big|_{\varepsilon=0} \\ &= \int_0^\infty (2q'\varphi' + 2q\varphi)y^{1-2s}dy \\ &= -\int_0^\infty 2\varphi y^{1-2s}(q'' + \frac{1-2s}{y}q' - q)dy = 0. \end{aligned}$$

By the arbitrariness of function φ, it follows that the minimizer function q solves exactly the following Bessel equation [14, 15]

$$\begin{cases} q'' + \dfrac{1-2s}{y}q' - q = 0, \\ q(0) = 1,\ q(\infty) = 0. \end{cases} \tag{1.14}$$

Conversely, if function q is a solution to equation (1.14), then for all $h(y) \in \mathcal{I}$,

$$\begin{aligned} j(h) &= \int_0^\infty y^{1-2s}(|h'|^2 + h^2)dy \\ &\geq \int_0^\infty y^{1-2s}(|q'|^2 + 2q'(h'-q') + q^2 + 2q(h-q))dy \end{aligned}$$

$$= \int_0^\infty y^{1-2s}(|q'|^2 + q^2)dy,$$

which indicates function q is a minimizer of functional J.

In the subcritical case, the coerciveness and weak lower semi-continuity of functional J guarantee the uniqueness of q.

We can now verify that

$$\begin{aligned} div(y^{1-2s}\nabla v) &= \Delta_x v + v_{yy} + \frac{1-2s}{y}v_y \\ &= c_k(t)\Delta\phi_k(x)q(\sqrt{\lambda_k}y) + c_k(t)\phi_k(x)(\lambda_k q''(\sqrt{\lambda_k}y) + \frac{1-2s}{y}\sqrt{\lambda_k}q'(\sqrt{\lambda_k}y)) \\ &= c_k(t)\phi_k(x)\lambda_k\Big[-q(\sqrt{\lambda_k}y) + q''(\sqrt{\lambda_k}y) + \frac{1-2s}{y}\sqrt{1-2s}\sqrt{\lambda_k}yq'(\sqrt{\lambda_k}y)\Big] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty y^{1-2s}\int_\Omega |\nabla v|^2 dxdy &= \int_0^\infty y^{1-2s}\int_\Omega \Big[c_k^2(t)|\nabla\varphi_k(x)|^2 q^2(\sqrt{\lambda_k}y) + c_k^2(t)\varphi_k^2(x)(q'(\sqrt{\lambda_k}y))^2\Big]dxdy \\ &= c_k^2(t)\lambda_k\int_0^\infty y^{1-2s}\Big[q^2(\sqrt{\lambda_k}y) + \big|q'(\sqrt{\lambda_k}y)\big|^2\Big]dy \\ &= c_k^2(t)\lambda_k\int_0^\infty (\frac{z}{\sqrt{\lambda_k}})^{1-2s}\Big[q^2(z) + \big|q'(z)\big|^2\Big]\frac{1}{\sqrt{\lambda_k}}dz \\ &= c_k^2(t)\lambda_k^s\int_0^\infty y^{1-2s}\Big(q^2(y) + \big|q'(y)\big|^2\Big)dy < +\infty \end{aligned}$$

in light of

$$\int_\Omega |\varphi_k(x)|^2 dx = 1, \quad \int_\Omega |\nabla\varphi_k(x)|^2 dx = \lambda_k.$$

In addition, from [14] we get

$$\begin{aligned} A_s v = A_s u &= c_k(t)\lambda_k^s\phi_k(x) \\ \partial_\nu^s v &:= -k_s\lim_{y\to 0} y^{1-2s}\frac{\partial v}{\partial y} \\ &= k_s c_k(t)\varphi_k(x)\lim_{y\to 0} -y^{1-2s}\sqrt{\lambda_k}q'(\sqrt{\lambda_k}y) \\ &= k_s c_k(t)\varphi_k(x)\frac{1}{k_s}\lambda_k^s = c_k(t)\lambda_k^s\phi_k(x) \end{aligned} \tag{1.15}$$

in $\Omega \times \{0\}$. We complete the proof of the equivalence between the equation (1.12) and (1.6).

It should be remarked that A_s is just the fractional Laplacian operator $(-\Delta)^s$ when $\Omega = \mathbb{R}^N$.

Next, we are going to explain following notations and definitions for the rest of the article.

Denote norm of space L^p and strong(weak) convergency in the relate space as $\|\cdot\|_p$ and $\to$ ($\rightharpoonup$) respectively.

Definition 1.1. *The function $v = v(x, y, t)$ is referred to as a weak solution of equation (1.6) in $C_T = C \times (0, T)$, iff*

$$v \in L^2([0, T]; H^s_{0,L}(C)),$$
$$v_t \in L^2(\Omega_T) = L^2([0, T]; L^2(\Omega)),$$

and satisfies the equation (1.6) in the distributional sense, namely

$$\int_0^T\int_C k_s y^{1-2s}\nabla v\cdot\nabla\varphi dxdydt + \int_0^T\int_{\Omega\times\{0\}}\Big(v_t - |v|^{p-2}v\Big)\varphi dxdt = 0,$$

for every test function $\varphi \in C_0^1([0, T]; H^s_{0,L}(C))$.

Definition 1.2. *If sequence $\{v_n\}$ satisfies $\sup_n |E(v_n)| < \infty (E(v_n) \to c)$, $E'(v_n) \to 0 (n \to \infty)$, in the function space $H^{-s}_{0,L}(C)$, then it's referred to as $Palais - Smale (PS)((PS)_c)$ sequence; And the energy functional of a stationary equation satisfies $(PS)((PS)_c)$ condition, which means every $(PS)((PS)_c)$ sequence $\{v_n\}$ has (strong) convergent subsequence.*

We focus on blow-up behavior of local solution, the asymptotic profile of global solution, namely the relationship between global solution and stationary solution. Furthermore, we deal with the boundedness of global orbit in function space $H^s_{0,L}(C)$, based upon this result, it can be shown that any global solution is classical solution with Moser iteration technique. The main results in this paper are as follows.

Theorem 1.1. *If $v = v(x, y, t; v_0)$ is a global solution of equation (1.6), then the ω-limit set of v_0, which is defined as*

$$\omega(v_0) = \{\omega \in H^s_{0,L}(C) : \exists t_n \to +\infty, v_n = v(x, y, t_n, u_0) \rightharpoonup w \text{ in } H^s_{0,L}(C)\},$$

contains a stationary solution w.

Theorem 1.2. *If $v = v(x, y, t; v_0)$ is a global solution of equation (1.6), and uniformly bounded in the function space $H^S_{0,L}(C)$ with respect to t, then for every sequence $t_n \to \infty$, there exists a stationary solution w, such that $v(x, y, t_n; v_0) \rightharpoonup w$ in $H^s_{0,L}(C)$.*

Theorem 1.3. *If $v(x, y, t; v_0)$ is a global solution of equation (1.6), then the global orbit is bounded in $H^s_{0,L}(C)$, that is*

$$\sup_{t\geq 0} \int_C k_s y^{1-2s} |\nabla v(x, y)|^2 dxdy < \infty.$$

Theorem 1.4. *If $v(x, y, t; v_0)$ is a global solution of equation (1.6), then for every $q(1 \leq q < \infty), t_0 > 0$, one has*

$$v \in L^q(\Omega \times \{0\} \times [t_0, \infty))$$

and

$$\|v\|_{L^q(\Omega\times\{0\}\times(t_0,\infty))} \leq C,$$

where C depends on N, q and t_0. Moreover, $v(x, y, t; v_0)$ is a classical solution.

We conclude this section with presenting the inequality of fractional Sobolev trace embedding $H^s_{0,L}(C) \subset L^p(\Omega \times \{0\})(1 \leq p \leq 2^*_s)$ [16, 17, 18]:

Suppose $v \in H^s_{0,L}(C)$, then there is a constant $C = C(p, N, s, |\Omega|) > 0$, such that $\|v(x, 0)\|_p \leq C\|v\|,\ 1 \leq p \leq 2^*_s$, i.e.

$$\left(\int_{\Omega\times\{0\}} |v(x, 0)|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_C k_s y^{1-2s} |\nabla v(x, y)|^2 dxdy \right)^{\frac{1}{2}}. \tag{1.16}$$

And we refer to S as the best constant for the inequality above means that

$$S = \inf \left\{ \frac{\int_C k_s y^{1-2s} |\nabla v(x, y)|^2 dxdy}{\left(\int_{\Omega\times\{0\}} |v(x, 0)|^{2^*_s} dx \right)^{\frac{2}{2^*_s}}} : v \in H^s_{0,L}(C) \right\}. \tag{1.17}$$

2. Proof of theorems

Before performing the proof of Theorem 1.1, let us first show the following lemma.

Lemma 2.1. *Provided there exists $t_0 \geq 0$ such that $E(v(t_0)) \leq 0$, then function v blows up in finite time.*

Proof of Lemma 2.1. There are several methods to prove the finite time blowup of solutions: comparison method, eigenfunction method, energy method, and concave function method (see [19, 20]) which we adopt here.

The proof is by contradiction. Assume that $T = t_{\max} = \infty$, put $f(t) = \frac{1}{2} \int_{t_0}^t \int_{\Omega\times\{0\}} v^2 dxd\tau$, taking derivative directly gives

$$f'(t) = \frac{1}{2} \int_{\Omega\times\{0\}} v^2 dx = \frac{1}{2} \|v(t_0)\|_2^2 + \int_{t_0}^t \int_{\Omega\times\{0\}} vv_\tau dxd\tau,$$

$$f''(t) = \int_{\Omega\times\{0\}} vv_t dx.$$

On the other hand, timing v and then integrating both sides of equation (1.6) in C can also lead to the first and second derivative of function $f(t)$ as follows

$$f'(t) = \frac{1}{2} \|v(t_0)\|_2^2 + \int_{t_0}^t \int_C -k_s y^{1-2s} |\nabla v|^2 dxdyd\tau + \int_{t_0}^t \int_{\Omega\times\{0\}} |v|^p dxd\tau, \tag{2.1}$$

$$f''(t) = \int_C -k_s y^{1-2s} |\nabla v|^2 dxdy + \int_{\Omega\times\{0\}} |v|^p dx, \tag{2.2}$$

Multiplying both sides of the energy inequality (1.11) by p and adding it to (2.2) derives

$$f''(t) = \left(\frac{p}{2} - 1\right) \int_C k_s y^{1-2s} |\nabla v|^2 dxdy + p \int_{t_0}^t \int_{\Omega\times\{0\}} v_\tau^2 dxd\tau - pE(v(t_0)). \tag{2.3}$$

By the assumption $E(v(t_0)) \leq 0$, and $\frac{p}{2} - 1 > 0$, it follows that

$$\left(\frac{p}{2} - 1\right) \int_C k_s y^{1-2s} |\nabla v|^2 dxdy - pE(v(t_0)) > 0, \tag{2.4}$$

for all $t \geq t_0$. Then we get

$$f''(t) > p \int_{t_0}^{t} \int_{\Omega\times\{0\}} v_\tau^2 dx d\tau, \tag{2.5}$$

Further applying the Hӧider inequality, we have

$$\begin{aligned} f(t)f''(t) >& \frac{p}{2} \int_{t_0}^{t} \|v(\tau)\|_2^2 d\tau \cdot \int_{t_0}^{t} \|v_\tau(\tau)\|_2^2 d\tau \\ \geq& \frac{p}{2} \left(\int_{t_0}^{t} \int_{\Omega\times\{0\}} v v_\tau dx d\tau \right)^2 \\ =& \frac{p}{2} \left(\int_{t_0}^{t} f''(t) d\tau \right)^2 \\ =& \frac{p}{2} (f'(t) - f'(t_0))^2. \end{aligned}$$

Set $\alpha = \dfrac{p}{2} - 1 > 0$, from (2.5), it is clear that $f''(t) > 0$, so we have $f'(t) > f'(t_1) > f'(t_0)$ for $t > t_1 > t_0$, and

$$f(t) - f(t_1) = \int_{t_1}^{t} f'(\tau) d\tau \geq f'(t_1)(t - t_1) \to 0,$$

as $t \to \infty$, and accordingly obtain

$$f'(t)f''(t) > (1 + 2\alpha)(f'(t) - f'(t_0))^2 \frac{f'(t)}{f(t)} > (1 + 2\alpha)(f'(t_1) - f'(t_0))^2 \frac{f'(t)}{f(t)}$$

for $t > t_1$. Therefore, integrating both sides yields

$$\frac{1}{2}(f'(t))^2|_{t_1}^{t} > (1 + 2\alpha)(f'(t_1) - f'(t_0))^2 \ln f(t)|_{t_1}^{t} \to \infty,$$

which implies $\lim\limits_{t\to\infty} f'(t) = \infty$, then there is a $t_2 > t_1$, such that

$$f(t)f''(t) > (1 + \alpha)(f'(t))^2$$

for $t > t_2$.

Let $J(t) = f(t)^{-\alpha}$, we have $J''(t) = -\alpha f(t)^{-\alpha-2} \left((1 + \alpha)(f'(t))^2 - f(t)f''(t) \right) < 0$ $(t > t_2)$. Clearly, $\lim\limits_{t\to\infty} J(t) = 0$, which means there is a $t_3 > t_2$, such that $J(t) < J(t_3) < 0$ for $t > t_3$, then we further have

$$0 < J(t) < J(t_3) + J'(t_3)(t - t_3) \to -\infty (t \to \infty),$$

a contradiction£Thereby we conclude v blows up in finite time.

Indeed, the equation $f(t)f''(t) - (1 + \alpha)(f'(t))^2 > 0$ can be rewritten to $\dfrac{d}{dt}\left(\dfrac{f'(t)}{f^{\alpha+1}(t)}\right) > 0$, which leads to

$$\frac{f'(t)}{f^{\alpha+1}(t)} > \frac{f'(0)}{f^{\alpha+1}(0)} := A,$$

integrating both sides with respect to t from 0 to t gives

$$\frac{1}{\alpha}\left(\frac{1}{f^\alpha(0)} - \frac{1}{f^\alpha(t)}\right) > At \Rightarrow f^\alpha(t) > \frac{f^\alpha(0)}{1 - f^\alpha(0)\alpha A t} \to \infty \ (t \to \frac{1}{f^\alpha(0)\alpha A}).$$

□

In the same manner used above, we can carry out the proof of following corollary,

Corollary 2.1. *If* $\int_C k_s y^{1-2s} |\nabla v|^2 dx dy \to \infty$ *when* $t \to t_{\max}$, *then* $t_{\max}(v_0) < \infty$.

We are now in a position to prove theorems.

Proof of Theorem 1.1 . According to Lemma 2.1, if there is a $t_0 \geq 0$ such that $E(v(t_0)) \leq 0$, then function v blows up in finite time, so v is a global solution implies $E(v(t)) > 0$ for all $t \geq 0$. Combining with the energy inequality (1.11), we have

$$0 < E(v(t)) \leq E(v_0), \tag{2.6}$$

and

$$\int_0^\infty \int_{\Omega\times\{0\}} v_\tau^2 dx d\tau \leq C < \infty, \tag{2.7}$$

then it can be seen that there is a time sequence $\{t_n\}$ satisfying $t_n \to \infty$ as $n \to \infty$, such that

$$\int_{\Omega\times\{0\}} v_\tau(x, y, t_n; v_0)^2 dx \to 0, \tag{2.8}$$

if not, suppose $\int_{\Omega\times\{0\}} v_\tau(x,y,t_n;v_0)^2 dx \to a \neq 0$, then there would get

$$\int_0^\infty \int_{\Omega\times\{0\}} v_\tau^2 dx d\tau = \int_0^\infty a d\tau = \infty,$$

which contradicts (2.7).

The formula (2.6) and (2.8) indicates $E(v_n)$ is bounded and $E'(v_n) \to 0$ respectively, which because

$$\frac{dE(v(t))}{dt} = \int_C k_s y^{1-2s} \nabla v \cdot \nabla v_t dxdy - \int_{\Omega\times\{0\}} |v(t)|^{p-2} v v_t dx = \langle E'(v), v_t \rangle = -\int_{\Omega\times\{0\}} v_t^2 dx,$$

it follows that $\{v_n : t_n \to \infty\}$ is PS sequence of the stationary equation corresponding to equation (1.6). Further, we can show that $\|v_n\|$ is bounded, namely, there is a constant $C < +\infty$ such that

$$\int_C |\nabla v_n|^2 dx \leq C.$$

In view of the weak compactness of reflexive space, that is bounded sequence must has convergent subsequence, there exists a subsequence of v_n (not relabeled) and a function w such that

$$v_n \rightharpoonup w, \text{ in } H^s_{0,L}(C),$$
$$v_n \to w, \text{ in } L^p(\Omega\times\{0\})\ (2 \leq p < 2^*_s).$$

□

Proof of Theorem 1.2 . For every sequence $t_n \to \infty$, let $v_n = v(x,y,t_n;v_0)$, since $\{v_n\}$ is uniformly bounded under the norm of $H^s_{0,L}(C)$, using the same argument as in the proof of Theorem 1.1, there is a subsequence $\{v_n\}$ and a function w such that

$$v_n \rightharpoonup v \text{ in } H^s_{0,L}(C),$$
$$v_n \to v \text{ in } L^p(\Omega\times\{0\}),$$
$$v_n \to v \text{ in } a.e.\ \Omega\times\{0\}.$$

Now we choose a test function

$$\varphi(x,y,t) = \begin{cases} \rho(t-t_n)\psi(x,y), & \text{for } t > t_n, x \in \bar{\Omega}, \\ 0, & \text{for } 0 \leq t \leq t_n, x \in \bar{\Omega}, \end{cases} \tag{2.9}$$

where $\psi \in H^s_{0,L}(C),\ \rho \in C^2_0(0,1),\ \rho \geq 0,\ \int_0^1 \rho(s)ds = 1$.

From the Definition 1.1, we have

$$\int_0^T \int_C k_s y^{1-2s} \nabla v \cdot \nabla\varphi dxdydt + \int_0^T \int_{\Omega\times\{0\}} \left(v_t - |v|^{p-2}v\right)\varphi dxdt = 0,$$

plugging the chosen test function back into the above equation gives

$$\int_{t_n}^{t_n+1} \left[\int_C -\rho(t-t_n) k_s y^{1-2s} \nabla v \cdot \nabla\psi dxdy - \int_{\Omega\times\{0\}} v_t \rho(t-t_n)\psi dx + \int_{\Omega\times\{0\}} |v|^{p-2} v\rho(t-t_n)\psi dx\right] dt = 0,$$

where

$$\begin{aligned} \int_{t_n}^{t_n+1} v_t \rho(t-t_n)\psi dt &= v\rho(t-t_n)\psi|_{t_n}^{t_n+1} - \int_{t_n}^{t_n+1} v\rho'(t-t_n)\psi dt \\ &= -\int_{t_n}^{t_n+1} v\rho'(t-t_n)\psi dt, \end{aligned}$$

which follows from $\rho(0) = \rho(1) = 0$. Then we obtain

$$\int_{t_n}^{t_n+1} \left[\int_C -\rho(t-t_n) k_s y^{1-2s} \nabla v \cdot \nabla\psi dxdy + \int_{\Omega\times\{0\}} \left(v\rho'(t-t_n)\psi + |v|^{p-2} v\rho(t-t_n)\psi\right) dx\right] dt = 0.$$

Performing a variable substitution $\delta = t - t_n$ yields

$$\begin{aligned} \int_0^1 \Big[\int_C &-\rho(\delta) k_s y^{1-2s} \nabla v(t_n+\delta) \cdot \nabla\psi dxdy \\ &+ \int_{\Omega\times\{0\}} \left(v(t_n+\delta)\rho'(\delta)\psi + |v(t_n+\delta)|^{p-2} v(t_n+\delta)\rho(\delta)\psi\right) dx\Big] d\delta = 0. \end{aligned} \tag{2.10}$$

Owing to the uniform boundedness of $v(t_n+\delta)(0 \leq \delta \leq 1)$ in $H^s_{0,L}(C)$, we can choose the same subsequence of $\{t_n\}$, a function w_δ and w satisfying

$$\|v(t_n+\delta) - w_\delta\|_{L^p(\Omega\times\{0\})} \to 0,$$
$$\|v(t_n) - w\|_{L^p(\Omega\times\{0\})} \to 0.$$

The following is to show that $w_\delta = w$ a.e. in $\Omega\times\{0\}$. By the energy inequality (1.11), we know
$$\int_0^t\int_{\Omega\times\{0\}} v_\tau^2 dxd\tau < \infty, \tag{2.11}$$
and then employing Hölder's inequality, we acquire
$$\begin{aligned}\int_{\Omega\times\{0\}} |v(t_n+\delta)-v(t_n)|^2dx &= \int_{\Omega\times\{0\}} |\int_{t_n}^{t_n+\delta} v_\tau d\tau|^2 dx\\ &\le \int_{\Omega\times\{0\}} |\left(\int_{t_n}^{t_n+\delta} 1^2 d\tau\right)^{\frac{1}{2}}\left(\int_{t_n}^{t_n+\delta} v_\tau^2 d\tau\right)^{\frac{1}{2}}|^2dx\\ &= \delta\int_{t_n}^{t_n+\delta}\int_{\Omega\times\{0\}}|v_\tau|^2dxd\tau \to 0, t_n\to\infty, 0\le\delta\le 1.\end{aligned}$$
Consequently,
$$\|v(t_n+\delta)-v(t_n)\|_{L^2(\Omega\times\{0\})}\to 0, t_n\to\infty$$
for $0\le\delta\le 1$, namely, w_δ equals w a.e. in $\Omega\times\{0\}$.

Now rearrange (2.10) to get
$$\begin{aligned}&\int_0^1\left[\int_C -\rho(\delta)k_s y^{1-2s}\nabla v(t_n)\cdot\nabla\psi dxdy + \int_{\Omega\times\{0\}}\left(v(t_n)\rho'(\delta)\psi + |v|^{p-2}v(t_n)\rho(\delta)\psi\right)dx\right]d\delta\\ &-\int_0^1\int_C k_s y^{1-2s}\left(\nabla v(t_n+\delta)-\nabla v(t_n)\right)\rho(\delta)\cdot\nabla\psi dxdyd\delta\\ &+\int_0^1\int_{\Omega\times\{0\}}\left(v(t_n+\delta)-v(t_n)\right)\rho'(\delta)\psi dxd\delta\\ &+\int_0^1\int_{\Omega\times\{0\}}\left(|v|^{p-2}v(t_n+\delta)-v^{p-2}v(t_n)\right)\rho(\delta)\psi dxd\delta = 0.\end{aligned}\tag{2.12}$$
The last three terms of the left side of the above equation approach $0(t_n\to\infty)$ by Lebesgue dominated convergent theorem. The second term $\int_0^1\int_{\Omega\times\{0\}} v(t_n)\rho'(\delta)\psi dxd\delta$ also approaches $0(t_n\to\infty)$, the reason is that
$$\begin{aligned}\int_0^1\int_{\Omega\times\{0\}} v(t_n)\rho'(\delta)\psi dxd\delta &\to \int_0^1\int_{\Omega\times\{0\}} w\rho'(\delta)\psi dxd\delta\\ &= \int_{\Omega\times\{0\}} w\psi\int_0^1\rho'(\delta)d\delta dx = \int_{\Omega\times\{0\}} w\psi(\rho(1)-\rho(0))dx = 0.\end{aligned}$$
Hence we obtain
$$\int_0^1\rho(\delta)d\delta\left(\int_C k_s y^{1-2s}\nabla v(t_n)\cdot\nabla\psi dxdy - |v|^{p-2}v(t_n)\psi dx\right) = o(1)(n\to\infty).$$
Note that $\int_0^1\rho(\delta)d\delta = 1$, we are finally led to
$$\int_C k_s y^{1-2s}\nabla v(t_n)\cdot\nabla\psi dxdy - |v|^{p-2}v(t_n)\psi dx = o(1)(n\to\infty),$$
which indicates $v(t_n)$ approaches a solution of stationary equation in the weak sense. □

Before proceeding to prove Theorem 1.3, we first establish several lemmas.

Lemma 2.2. *If $v(x,y,t;v_0)$ is a global solution of the equation (1.6), then for each $0<A<B$, there exists $\tau=\tau(A,B)>0$, and if $\int_{\Omega\times\{0\}}|v_0|^pdx\le A$, then as $t\in[0,\tau]$, one has $\int_{\Omega\times\{0\}}|v(x,y,t;v_0)|^pdx\le B$.*

Proof of Lemma 2.2 . For convenience, denote $v(x,y,t;v_0)$ as v. Assume $S(v_0)=\{t>0, \int_{\Omega\times\{0\}}|v(x,y,t;v_0)|^pdx = B\}$ is not empty, set $\sigma(v_0)=\inf S(v_0)$, it suffices to show that there is $\tau>0$ such that $\sigma(v_0)\ge\tau$ for every v_0 satisfying $\int_{\Omega\times\{0\}}|v|^pdx\le A$. Put the test function φ in Definition 1.1 be v^{p-1}, namely,
$$\int_C k_s y^{1-2s}\nabla v\cdot\nabla(v^{p-1})dxdy + \int_{\Omega\times\{0\}}\left(v_t - |v|^{p-2}v\right)v^{p-1}dx = 0,$$
hereby we arrive at
$$\frac{1}{p}\frac{d}{dt}\int_{\Omega\times\{0\}} v^pdx = -\int_C k_s y^{1-2s}\nabla v\cdot\nabla(v^{p-1})dxdy + \int_{\Omega\times\{0\}} v^{2p-2}dx$$

$$= -\frac{4(p-1)}{p^2}\int_C k_s y^{1-2s}|\nabla(v^{\frac{p}{2}})|^2 dxdy + \int_{\Omega\times\{0\}} v^{2p-2}dx,$$

for almost all $t \in (0,\sigma)$. Using Hölder's inequality produces

$$\begin{aligned}\int_{\Omega\times\{0\}} v^{2p-2}dx &= \int_{\Omega\times\{0\}} v^{2p-2-\frac{N(p-2)}{2s}}\cdot v^{\frac{N(p-2)}{2s}}dx\\ &\le \left(\int_{\Omega\times\{0\}} v^p dx\right)^{\frac{\theta_1}{p}}\left(\int_{\Omega\times\{0\}} v^\gamma dx\right)^{\frac{\theta_2}{\gamma}} \qquad (2.13)\\ &= \|v(x,0)\|_p^{\theta_1}\|v(x,0)\|_\gamma^{\theta_2},\end{aligned}$$

here

$$\theta_1 = 2p-2-\frac{N(p-2)}{2s},\ \theta_2 = \frac{N(p-2)}{2s},\ \gamma = \frac{Np}{N-2s}.$$

According to the continuity of fractional Sobolev trace embedding $H^s_{0,L}(C) \subset L^{2^*_s}(\Omega\times\{0\})$, it follows that

$$\|v(x,0)^{\frac{p}{2}}\|_{2^*_s} \le C\left(\int_C k_s y^{1-2s}|\nabla(v^{\frac{p}{2}})|^2 dxdy\right)^{\frac{1}{2}},$$

rewriting the above inequality gives

$$\|v(x,0)\|_\gamma \le C\left(\int_C k_s y^{1-2s}|\nabla(v^{\frac{p}{2}})|^2 dxdy\right)^{\frac{1}{p}}. \qquad (2.14)$$

The condition $p < \dfrac{2N}{N-2s}$ ensures $\dfrac{\theta_2}{p} < 1$, then combine (2.13), (2.14) and Young inequality, we see that

$$\frac{d}{dt}\int_{\Omega\times\{0\}} v^p dx \le C(B)$$

for almost all $t \in (0,\sigma)$, the conclusion follows immediately. □

Lemma 2.3. *If $v(x,y,t;v_0)$ is a global solution of the equation (1.6), and satisfies*

$$\liminf_{t\to\infty}\|v(x,y,t;v_0)\|_p = k < \infty,$$

$$\limsup_{t\to\infty}\|v(x,y,t;v_0)\|_p = \infty,$$

then for each $B > K$, there is a stationary solution $w \in \omega(v_0)$, such that $\|w\|_p = B$.

Proof of Lemma 2.2 . Select a sequence $\{t_n\}$ with $\int_{\Omega\times\{0\}}|v(x,y,t_n;v_0)|^p dx = B$. By $E(v(x,y,t_n;v_0)) \le E(v_0)$, it can easily be seen that $\int_C k_s y^{1-2s}|\nabla v(x,y,t_n;v_0)|^2 dxdy$ is bounded. And compact trace embedding $H^s_{0,L}(C) \subset L^p(\Omega\times\{0\})(1 \le p < 2^*_s)$ indicates that there is a subsequence $v(x,y,t_n;v_0)$ (not relabelled), such that $v(x,y,t_n;v_0) \to w$ in $L^p(\Omega\times\{0\})$, similar to the proof of Theorem 1.2, it can immediately achieve w is a stationary solution. □

Lemma 2.4. *Suppose $v(x,y,t;v_0)$ is a global solution to the equation (1.6), and $w \in \omega(v_0)$, then there is a positive constant $K = K(v_0)$, such that $\int_C k_s y^{1-2s}|\nabla w|^2 dxdy \le K$.*

Proof of Lemma 2.2 . We know $E(w) \le E(v_0)$ from the energy inequality, on account of $w \in \omega(v_0)$, namely w is a stationary solution, we have

$$\int_C k_s y^{1-2s}|\nabla w|^2 dxdy = \int_{\Omega\times\{0\}}|w|^p dx,$$

further we obtain

$$\int_C k_s y^{1-2s}|\nabla w|^2 dxdy \le CE(v_0).$$

□

With the help of the proceeding lemmas, we now can prove Theorem 1.3.

Proof of Theorem 1.3 . Assume

$$\limsup_{t\to\infty}\int_C k_s y^{1-2s}|\nabla v(x,y,t;v_0)|^2 dxdy = \infty,$$

by the energy inequality, we see that

$$\limsup_{t\to\infty}\|v(x,y,t;v_0)\|_p = \infty,$$

If $\|v(x,y,t;v_0)\|_p \to \infty$ when $t \to \infty$, then Sobolev trace embedding theorem implies that it contradicts Corollary 2.1. If $\liminf_{t\to\infty} \|v(x,y,t;v_0)\|_p$ is finite, then Lemma 2.3 yields $\omega(v_0)$ contains a stationary solution with arbitrarily large $L^p(\Omega\times\{0\})-$ norm, however from Sobolev trace embedding theorem, we can derive that it contradicts Lemma 2.4, which completes the proof. □

Proof of Theorem 1.4. As to the L^q estimate of solution, we adopt Moser iteration, which has been used to establish the regularity for the weak solutions of semilinear heat equation with critical exponent and semilinear elliptic equation with fractional Laplacian respectively [19, 21].

For arbitrary fixed $t_0 > 0$ and $T > 0$, we choose a suitable cut-off function $\eta \in C^\infty(0,T)$ satisfying

$$\begin{aligned}
&0 \le \eta(t) \le 1, t \in (0,T),\\
&\eta(t) = 1, t \in [t_0, T],\\
&\eta(t) = 0, t \in [0, \frac{t_0}{2}],\\
&|\eta_t| \le \frac{1}{t_0}.
\end{aligned}$$

Substituting $\phi = v^{2\rho+1}\eta^2$ ($\rho > 0$, to be determined later) for the test function in Definition 1.1 gives

$$\int_0^T \left(\int_{\Omega\times\{0\}} \left[\phi v_t - \phi|v|^{p-2}v\right] dx + \int_C k_s y^{1-2s} \nabla v \cdot \nabla\phi dxdy \right) dt = 0. \tag{2.15}$$

Suppose $v \in L^{2\rho+2}(C_T)$, integrating the first term on the left-hand part of (2.15) by parts leads to

$$\begin{aligned}
\int_0^T \int_{\Omega\times\{0\}} \phi v_t dxdt &= \int_0^T \int_{\Omega\times\{0\}} v_t v^{2\rho+1}\eta^2 dxdt\\
&= \frac{1}{2(\rho+1)} \int_0^T \int_{\Omega\times\{0\}} (v^{2\rho+2})_t \eta^2 dxdt\\
&= \frac{1}{2(\rho+1)} \int_0^T \int_{\Omega\times\{0\}} (v^{2\rho+2}\eta^2)_t dxdt - \frac{1}{\rho+1} \int_0^T \int_{\Omega\times\{0\}} v^{2\rho+2}\eta\eta_t dxdt. \qquad (2.16)
\end{aligned}$$

Concerning the third term on the left-hand part of (2.15), we have the following estimates

$$\begin{aligned}
\int_0^T \int_C k_s y^{1-2s} \nabla v \cdot \nabla(v^{2\rho+1}\eta^2) dxdydt &= \int_0^T \int_C k_s y^{1-2s} \nabla v \cdot \nabla(v^{2\rho+1})\eta^2 dxdydt \qquad (2.17)\\
&= \frac{2\rho+1}{(\rho+1)^2} \int_0^T \int_C k_s y^{1-2s} |\nabla v^{\rho+1}|^2 \eta^2 dxdydt\\
&\ge \frac{1}{\rho+1} \int_0^T \int_C k_s y^{1-2s} |\nabla v^{\rho+1}|^2 \eta^2 dxdydt. \qquad (2.18)
\end{aligned}$$

With regard to the second term on the left-hand part of (2.15), using Hölder's inequality, we arrive at

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\{0\}} \phi v^{p-2} v dxdt = \int_0^T \int_{\Omega\times\{0\}} \eta^2 v^{2\rho+2} v^{p-2} v dxdt\\
&= \int_0^T \eta^2 \left(\int_{\{v^{p-2}<M\}\cap\Omega\times\{0\}} v^{2\rho+2} v^{p-2} v dx + \int_{\{v^{p-2}\ge M\}\cap\Omega\times\{0\}:=\mathcal{M}} v^{2\rho+2} v^{p-2} v dx \right) dt\\
&\le M \int_0^T \int_{\Omega\times\{0\}} \eta^2 v^{2\rho+2} dxdt + \int_0^T \eta^2 \left(\int_{\mathcal{M}} (v^{p-2})^{\frac{1}{1-\frac{2}{p}}} dx \right)^{1-\frac{2}{p}} \left(\int_{\mathcal{M}} (v^{2\rho+2})^{\frac{p}{2}} dx \right)^{\frac{2}{p}} dt\\
&\le M \int_0^T \int_{\Omega\times\{0\}} \eta^2 v^{2\rho+2} dxdt + \int_0^T \eta^2 \left(\int_{\mathcal{M}} v^p dx \right)^{1-\frac{2}{p}} \left(\int_{\Omega\times\{0\}} (v^{\rho+1})^p dx \right)^{\frac{2}{p}} dt\\
&\le M \int_0^T \int_{\Omega\times\{0\}} \eta^2 v^{2\rho+2} dxdt + \frac{1}{S} \sup_t \left(\int_{\mathcal{M}} v^p dx \right)^{1-\frac{2}{p}} \int_0^T \eta^2 \int_C k_s y^{1-2s} |\nabla v^{\rho+1}|^2 dxdydt. \qquad (2.19)
\end{aligned}$$

Then by (2.15)–(2.19), we get

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega\times\{0\}} v_t \phi dxdt - \int_0^T \int_{\Omega\times\{0\}} |v|^{p-2} v\phi dxdt + \int_0^T \int_C k_s y^{1-2s} \nabla v \cdot \nabla\phi dxdydt\\
&\ge \frac{1}{2(\rho+1)} \int_0^T \int_{\Omega\times\{0\}} (v^{2\rho+2}\eta^2)_t dxdt - \frac{1}{\rho+1} \int_0^T \int_{\Omega\times\{0\}} v^{2\rho+2}\eta\eta_t dxdt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho+1}\int_0^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt - M\int_0^T\int_{\Omega\times\{0\}}\eta^2 v^{2\rho+2}dxdt\\
&-\frac{1}{S}\sup_t\left(\int_M v^p dx\right)^{1-\frac{2}{p}}\int_0^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt
\end{aligned}
$$

where it follows

$$
\begin{aligned}
\int_0^T\int_{\Omega\times\{0\}}(v^{2\rho+2}\eta^2)_t dxdt &= \int_0^T \frac{d}{dt}\left(\int_{\Omega\times\{0\}} v^{2\rho+2}\eta^2 dx\right)dt\\
&= \int_{\Omega\times\{0\}} v^{2\rho+2}\eta^2 dx|_{t=0}^{t=T} = \int_{\Omega\times\{0\}} v(x,y,T)^{2\rho+2}dx,
\end{aligned}
$$

from $\eta(0)=\eta(T)=0$. Letting $\frac{1}{\beta}=1-\frac{2}{p}$, $\varepsilon(M)=\frac{1}{S}\sup_t\left(\int_M v^p dx\right)^{\frac{1}{\beta}}$, Theorem 1.3 implies $\varepsilon(M)$ is finite. Arranging the above, we see that

$$
\begin{aligned}
&\frac{1}{2(\rho+1)}\int_{\Omega\times\{0\}} v(T)^{2\rho+2}dx + \frac{1}{\rho+1}\int_0^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt\\
&\le \frac{1}{\rho+1}\int_0^T\int_{\Omega\times\{0\}}\eta|\eta_t|v^{2\rho+2}dxdt + M\int_0^T\int_{\Omega\times\{0\}}\eta^2 v^{2\rho+2}dxdt\\
&+\varepsilon(M)\int_0^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt. \qquad (2.20)
\end{aligned}
$$

Let $\rho_0=0, \rho_i+1=(\rho_{i-1}+1)(1+\frac{1}{\beta})$, $i\ge 1$. It's easy to see that for arbitrary fixed q $(1\le q<\infty)$, there is always a i_0 such that $2(\rho_{i_0-1}+1)<q\le 2(\rho_{i_0}+1)=2(\rho_{i_0-1}+1)(1+\frac{1}{\beta})$. Take suitable M to satisfy

$$
\varepsilon(M)=\frac{1}{q}<\frac{1}{2(\rho_{i_0-1}+1)}.
$$

Then by (2.20), we have

$$
\begin{aligned}
&\frac{1}{2(\rho+1)}\int_{\Omega\times\{0\}} v(T)^{2\rho+2}dx + \left(\frac{1}{\rho+1}-\frac{1}{q}\right)\int_0^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt\\
&\le \frac{1}{2(\rho+1)}\int_0^T\int_{\Omega\times\{0\}}\eta|\eta_t|v^{2\rho+2}dxdt + M\int_0^T\int_{\Omega\times\{0\}}\eta^2 v^{2\rho+2}dxdt. \qquad (2.21)
\end{aligned}
$$

And in view of $\eta(t)=1(t\in[t_0,T]), \eta(t)=0(t\in[0,\frac{t_0}{2}])$, along with Hölder's inequality and Sobolev trace inequality, we obtain

$$
\begin{aligned}
&\int_{t_0}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)(1+\frac{1}{\beta})}\eta^{2(1+\frac{1}{\beta})}dxdt\\
&\le \int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)(1+\frac{1}{\beta})}\eta^{2(1+\frac{1}{\beta})}dxdt\\
&\le \left(\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}}\left(v^{(2\rho+2)}\eta^2\right)^{\frac{1}{\beta}\beta}dxdt\right)^{\frac{1}{\beta}}\left(\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}}\left(v^{(2\rho+2)}\eta^2\right)^{\frac{1}{1-\frac{1}{\beta}}}dxdt\right)^{1-\frac{1}{\beta}}\\
&= \left(\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)}\eta^2 dxdt\right)^{\frac{1}{\beta}}\left(\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}}\left(v^{(2\rho+2)}\eta^2\right)^{p}dxdt\right)^{\frac{2}{p}}\\
&\le C\left(\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)}\eta^2 dxdt\right)^{\frac{1}{\beta}}\int_{\frac{t_0}{2}}^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt.
\end{aligned}
$$

Combining (2.21) and $|\eta_t|\le\frac{1}{t_0}, \eta(t)\le 1$, we derive

$$
\begin{aligned}
&\left(\frac{1}{\rho+1}-\frac{1}{q}\right)\int_{\frac{t_0}{2}}^T\int_C k_s y^{1-2s}|\nabla v^{\rho+1}|^2\eta^2 dxdydt\\
&\le \frac{1}{t_0(\rho+1)}\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)}\eta dxdt + M\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)}\eta dxdt\\
&\le \left(\frac{1}{t_0(\rho+1)}+M\right)\int_{\frac{t_0}{2}}^T\int_{\Omega\times\{0\}} v^{(2\rho+2)}dxdt.
\end{aligned}
$$

From the above, it turns out that

$$\begin{aligned}
&\int_{t_0}^{T}\int_{\Omega\times\{0\}} v^{(2\rho+2)(1+\frac{1}{\beta})}dxdt\\
&\le C\frac{\dfrac{1}{t_0(\rho+1)}+M}{\dfrac{1}{\rho+1}-\dfrac{1}{q}}\left(\int_{\frac{t_0}{2}}^{T}\int_{\Omega\times\{0\}} v^{2\rho+2}dxdt\right)^{1+\frac{1}{\beta}}\\
&= C\left(\int_{\frac{t_0}{2}}^{T}\int_{\Omega\times\{0\}} v^{2\rho+2}dxdt\right)^{1+\frac{1}{\beta}}, \qquad (2.22)
\end{aligned}$$

where $C = C(N, s, q, t_0)$.

Plugging $\rho = \rho_{i-1}$ back into (2.22),

$$\left(\int_{t_0}^{T}\int_{\Omega\times\{0\}} v^{2(\rho_{i-1}+1)(1+\frac{1}{\beta})}dxdt\right)^{\frac{1}{(\rho_{i-1}+1)(1+\frac{1}{\beta})}} \le C\left(\int_{\frac{t_0}{2}}^{T}\int_{\Omega\times\{0\}} v^{2(\rho_{i-1}+1)}dxdt\right)^{\frac{1}{\rho_{i-1}+1}},$$

that is

$$\begin{aligned}
&\left(\int_{t_0}^{T}\int_{\Omega\times\{0\}} v^{2(\rho_i+1)}dxdt\right)^{\frac{1}{(\rho_i+1)}} \le C\left(\int_{\frac{t_0}{2}}^{T}\int_{\Omega\times\{0\}} v^{2(\rho_{i-1}+1)}dxdt\right)^{\frac{1}{\rho_{i-1}+1}}\\
&\le C^i\left(\int_{\frac{t_0}{2^i}}^{T}\int_{\Omega\times\{0\}} v^{2(\rho_0+1)}dxdt\right)^{\frac{1}{\rho_0+1}}\\
&= C^i\int_{\frac{t_0}{2^i}}^{T}\int_{\Omega\times\{0\}} v^2dxdt \to C_0\int_0^T\int_{\Omega\times\{0\}} v^2dxdt(i\to\infty).
\end{aligned}$$

Therefore $u \in W_q^{2s,1}(Q_\infty) = L_q([t_0,\infty); W_q^{2s}(\Omega\times\{0\}))\bigcap W_q^1([t_0,\infty); L_q(\Omega\times\{0\}))$ for any $1 \le q < \infty$, where $Q_\infty = \Omega\times\{0\}\times[t_0,\infty)$, and by the embedding of anisotropic spaces [22, 23], it follows that $u \in BUC([t_0,\infty); BUC^\gamma(\Omega\times\{0\}))$, where $0 < \gamma \le 2s - \dfrac{N+2s}{q}$, and $BUC^\gamma(\Omega\times\{0\})$ is the Banach space of bounded Hölder continuous functions of order γ on $\Omega\times\{0\}$ for $\gamma \in \mathbb{R}^+\backslash\mathbb{N}$, while for $\gamma \in \mathbb{N}_0$, $BUC^\gamma(\Omega\times\{0\})$ is the Banach space of γ-times bounded uniformly continuously differentiable functions on $\Omega\times\{0\}$ (see [23] A.4). Furthermore, applying the standard bootstrap argument [24] we finally obtain $u(x,t)$ is a classical solution for all $t \ge t_0 > 0$, which completes the proof of Theorem 1.4.

□

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