Uniqueness of the weak solution to the fractional anisotropic Navier-stokes equations

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Abstract

In this work, we demonstrate uniqueness of the weak solution to the fractional anisotropic Navier-Stokes system with only horizontal dissipation.

Keywords: fractional anisotropic Navier-Stokes equations; the trilinear estimate; weak solution; anisotropic dyadic decomposition.

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1. Introduction

In this paper, we are interested in the following fractional anisotropic Navier-stokes system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + \nu_h(-\Delta_h)^\alpha u + (u\cdot\nabla)u + \nabla p = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u=(u^1,u^2,u^3)$ designates the velocity of the fluid, p is the scalar pressure,the constants $\nu_h>0$ and $\alpha>0$, represent respectively the horizonal viscosity and dissipation, more, the fractional power of the horizontal laplacian can be defined in terms of the Fourier transform:

$$\mathcal{F}[(-\Delta_h)^\alpha u](t,\xi) = |\xi_h|^{2\alpha}\mathcal{F}u(t,\xi),$$

Meanwhile, let's note down

$$\Lambda_h = (-\Delta_h)^{1/2}.$$

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In particular, (1.1) becomes the anisotropic Navier-stokes equations in the case of $\alpha = 1$. We proved the uniqueness of weak solutions for fractional Navier-Stokes equations in [9], and also established the global existence and uniqueness of regular solutions in spatial variable for the higher-order elliptic Navier-Stokes system in [10]. We also proved the time-local existence of the mild solution to the fractional Navier-Stokes-Coriolis system in Sobolev space [11].
In[5], the author investigate the equations of anisotropic incompressible viscous fluids in $\mathbb{R}^3$, rotating around an inhomogeneous vector $\mathbb{B}(t, x_1, x_2)$. The author prove the global existence of strong solutions in suitable anisotropic Sobolev spaces for small initial data as well as uniform local existence result to the Rossby number in the same spaces under assumption that $\mathbb{B}(t, x_1)$, or $\mathbb{B}(t, x_2)$. In[8], the authors proved that as long as the velocity $u \in L^q(0, T, \dot{B}_{r,q}^{\frac{d}{r}+\frac{2\alpha}{q}-2\alpha+1})$ such that $\frac{d}{r} + \frac{2\alpha}{q} > \alpha$ then the generalized Navier-Stokes equations has a uniqueness solution. Hence, we investigate the equations of fractional anisotropic incompressible viscous fluids in $\mathbb{R}^3$ in this paper, we prove this type of result to the $3D$ fractional anisotropic Navier-Stokes system with only horizontal dissipation.
we are going to designate that u is a weak solution of the fractional anisotropic Navier-Stokes equations if u such that
$(H1)$ u is weakly continuous $\mathbb{R}^+ \to L^2$;
$(H2)$ $u \in \mathcal{L} = L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+, \dot{H}^{\alpha,0})$;
$(H3)$ For all Φ in $D'(\mathbb{R}^+ \times \mathbb{R}^3)$ such that $\mathrm{div}\Phi = 0$,

$$\int_0^t (u, \partial_t \Phi) d\tau + \int_0^t (u, (-\Delta_h)^\alpha \Phi) d\tau + \int_0^\infty (u, (u \cdot \nabla)\Phi) d\tau + (u_0, \Phi(0)) = 0,$$

with $\forall \Phi \in D'(\mathbb{R}^+ \times \mathbb{R}^3)$,

$$\int_0^\infty (u, \nabla \Phi) d\tau = 0.$$

Proposition 1.1. *Let $\alpha > \frac{1}{2}$, $2 \leq r < \infty, 2 < q < \infty$, then $\forall u_0 \in \dot{B}_{(r,\infty),q}^{\frac{2}{r}+2\alpha+2,0}$ we have the unique solution u associated with u_0 such that $\forall T < T^*$ and $\forall p \in [q, \infty)$,*

$$u \in L^p(0, T; \dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{p}-2\alpha+2,0}).$$

The proof of Proposition 1.1 is straightforward. It suffices to modify the proof of Proposition 1.1 in [4] slightly, and so it is omitted here.

Theorem 1.1. *Let u and v be two weak solutions associated with $u_0 \in L^2 \cap \dot{B}_{(r,\infty),q}^{\frac{2}{r}-2\alpha+2,0}$ and $v_0 \in L^2$ such that $\mathrm{div}u_0 = 0, \mathrm{div}v_0 = 0$. Let $2 \leq r < \infty, 2 < q < \infty$, and $\frac{2}{r} + \frac{2\alpha}{q} > \alpha$,*

$v \in \mathcal{L}$ is any weak solutions associated with v_0, u is the unique solution associated with u_0 with $u \in L^q(0,T;\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}) \cap \mathcal{L}$. For $\forall T>0$, if $w=u-v$, then $\forall t \le T$,we have:

$$\|w(t)\|^2_{L^2} \le \|v_0-u_0\|^2_{L^2} \exp\Big(C\int_0^t \|u(s)\|^q_{\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2}_{(r,\infty),q}} ds\Big).$$

In particular, $u_0=v_0$, we get $u=v$.

Considering the $3D$ fractional anisotropic Navier-Stokes system with only horizontal dissipation, it is reasonable to use anisotropic functional spaces which distinguish horizontal derivatives from the vertical one. Let us recall the definition of those spaces, it requries an anisotropic dyadic decomposition of the Fourier space, so let us start by recalling the definition opertors of localization in Fourier space:

$$\Delta^h_j a = \mathcal{F}^{-1}(\Psi(2^{-j}|\xi_h|)\hat{a}) \quad \text{for all}\, j \in \mathbb{Z},$$

$$\dot{\Delta}^v_l a = \mathcal{F}^{-1}(\Psi(2^{-l}|\xi_v|)\hat{a}) \quad \text{for all } l \in \mathbb{Z},$$
$$\dot{S}^h_j a = \mathcal{F}^{-1}(\chi(2^{-j}|\xi_h|)\hat{a}) \quad \text{for all}\, j \in \mathbb{Z},$$
$$\dot{S}^v_l a = \mathcal{F}^{-1}(\chi(2^{-l}|\xi_v|)\hat{a}) \quad \text{for all } l \in \mathbb{Z},$$

where $\xi_h=(\xi_1,\xi_2)$, $\mathcal{F}(a)$ or $\hat{a}$ denotes the Fourier transform of a, while $\mathcal{F}^{-1}(a)$ designates the inverse Fourier transform of a, $\chi(\tau)$ and $\Psi(\tau)$ are smooth function such that

$$\text{supp}\Psi \subset \{\tau \in \mathbb{R} : \frac{3}{4} \le |\tau| \le \frac{8}{3}\} \quad \text{and } \forall \tau>0, \sum_{j\in\mathbb{Z}} \Psi(2^{-j}\tau)=1,$$

$$\text{supp}\chi \subset \{\tau \in \mathbb{R} : |\tau| \le \frac{4}{3}\} \quad \text{and } \forall \tau \in \mathbb{R}, \chi(\tau)+\sum_{j\in\mathbb{Z}} \Psi(2^{-j}\tau)=1.$$

Definition 1.2. *Let s and s' be two real numbers. Let $q \in (1,\infty)$ and $r \in [1,\infty)$, the anisotropic Besov space $\dot{B}^{s,s'}_{(r,\infty),q}$ denotes the space of homogeneous tempered distribution a such that:*

$$\|a\|_{\dot{B}^{s,s'}_{(r,\infty),q}} = \Bigg(\sum_{k,j=1}^{\infty} 2^{js+ks'} \|\Delta^h_j \Delta^v_k a\|_{L^\infty_v L^r_h}\Bigg)^{\frac{1}{q}}.$$

Lemma 1.1. *Let $\mathbf{B}_h$ (resp.$\mathbf{B}_v$) a ball of $\mathbb{R}^2_h$ (resp.$\mathbb{R}_V$), and $\mathcal{C}_h$ (resp.$\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (resp.$\mathbb{R}_v$). Let $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$. then there holds*

$$if\, \text{supp}\hat{a} \subset 2^k\mathbf{B}_h \Rightarrow \|\partial^\alpha_{x_h} a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k(|\alpha|+\frac{2}{p_2}-\frac{2}{p_1})} \|a\|_{L^{p_2}_h(L^{q_1}_v)},$$

$$if\, \mathrm{supp}\hat{a} \subset 2^l\ \mathbf{B}_v \Rightarrow \|\partial_{x_3}^{\beta} a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha|+\frac{2}{p_2}-\frac{2}{p_1})}\|a\|_{L_h^{p_1}(L_v^{q_2})},$$
$$if\, \mathrm{supp}\hat{a} \subset 2^k\ \mathcal{C}_h \Rightarrow \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^{\alpha} a\|_{L_h^{p_1}(L_v^{q_1})},$$
$$if\, \mathrm{supp}\hat{a} \subset 2^l\mathcal{C}_v \Rightarrow \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN}\|\partial_{x_3}^{N} a\|_{L_h^{p_1}(L_v^{q_1})}.$$

.

Lemma 1.2. *Let $\alpha > 0$, $a \in L^\infty((0,\infty);L^2(\mathbb{R}^3)) \cap L^2((0,\infty);\dot{H}^{\alpha,0}(\mathbb{R}^3))$, we have, for every $2 \le p \le \infty$,*

$$a \in L^p(\mathbb{R}^+;\dot{H}^{\frac{2\alpha}{p},0}(\mathbb{R}^3))$$

with

$$\|a\|_{L^p(\mathbb{R}^+;\dot{H}^{\frac{2\alpha}{p},0}(\mathbb{R}^3))} \le C\|a\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^3))}^{1-\frac{2}{p}}\|a\|_{L^2(\mathbb{R}^+;\dot{H}^{\alpha,0}(\mathbb{R}^3))}^{\frac{2}{p}}.$$

Lemma 1.3. *Let $(\alpha,q,r) \in \mathbb{R}^3$, such that $\alpha > \frac{1}{2}$, $2 \le r < \infty$, $2 < q < \infty$ and $\frac{2}{r}+\frac{2\alpha}{q} > \alpha$, $\beta = \frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2$, then, for $\forall T > 0$, the trilinear form:*

$$(a,b,c) \in \mathcal{L}^2 \times L^q(0,T;\dot{B}^{\beta,0}_{(r,\infty),q}) \mapsto \int_0^T \int_{\mathbb{R}^3} (a\cdot\nabla b)\cdot c dxdt$$

is continuous, and we have

$$\begin{aligned}\Big|\int_0^t \int_{\mathbb{R}^3}(a\cdot\nabla b)\cdot cdxdt\Big| \le\ & C \quad \Big(\|a\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{2}{q}}\|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{2}{q}}\|\Lambda_h^\alpha b\|_{L^2(\mathbb{R}^+;L^2)}\\ &+ \quad \|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}\|b\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{2}{q}}\|\Lambda_h^\alpha b\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{2}{q}}\\ &+ \quad \|a\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{1}{q}}\|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{1}{q}}\|\|b\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{1}{q}}\|\Lambda_h^\alpha b\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{1}{q}}\Big)\|c\|_{L^q(0,T;\dot{B}^{\beta,0}_{(r,\infty),q})}\end{aligned}$$

and

$$\Big|\int_0^t \int_{\mathbb{R}^3}(a\cdot\nabla a)\cdot cdxdt\Big| \le v_h\|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}^2 + C\int_0^t \|a\|_{L^2}^2\|c\|_{\dot{B}^{\beta,0}_{(r,\infty),q}}^q ds.$$

2. Proof of the weak-strong uniqueness result

In this section we will prove Theorem 1.1. Let us recall the situation, we consider two divergence-free vector fields v_0 and u_0, such that

$$v_0 \in L^2(\mathbb{R}^3) \quad \text{and} \quad u_0 \in L^2 \cap \dot{B}^{\frac{2}{r}-2\alpha+2,0}_{(r,\infty),q}(\mathbb{R}^3).$$

We have chosen $2 \le r < \infty$,$2 < q < \infty$, with we associtate with v_0 and u_0 two weak solutions v and u such that $H(1)-H(3)$ in the space $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^3)) \cap L^p(\mathbb{R}^+;\dot{H}^{\frac{2\alpha}{p},0}(\mathbb{R}^3))$, with additionally, accoding proposition 1.1:

$$u \in L^q(0,T;\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}(\mathbb{R}^3)).$$

Proof. we have known conditions:

$$\begin{aligned}\|u\|_{L^2}^2 + 2\int_0^t \|\Lambda_h^\alpha u(s)\|_{L^2}^2 ds &\leq \|u_0\|_{L^2}^2,\\ \|v\|_{L^2}^2 + 2\int_0^t \|\Lambda_h^\alpha v(s)\|_{L^2}^2 ds &\leq \|v_0\|_{L^2}^2.\end{aligned}$$

If $w = u - v$, then

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda_h^\alpha w(s)\|_{L^2}^2 ds = \quad \|u\|_{L^2}^2 \quad & +2\int_0^t \|\Lambda_h^\alpha u(s)\|_{L^2}^2 ds + \|v\|_{L^2}^2\\ + \quad & 2\int_0^t \|\Lambda_h^\alpha v(s)\|_{L^2}^2 ds - 2(v(t), u(t)) - 4\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds,\end{aligned}$$

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda_h^\alpha w(s)\|_{L^2}^2 ds &\leq \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 - 2(v(t), u(t)) - 4\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds\\ &= \|v_0 - u_0\|_{L^2}^2 - 2(v_0, u_0) + 2(v(t), u(t)) - 4\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds.\end{aligned}$$

According to the Lemma(2.1), we have

$$(u(t), v(t)) + 2\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds = (v_0, u_0) + \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla w) \cdot u dx dt,$$

therefore,

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda_h^\alpha w(s)\|_{L^2}^2 ds \leq \|v_0 - u_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla w) \cdot u dx dt.$$

Thanks to the Lemma (1.3), we have

$$\Big| \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla w) \cdot u dx dt \Big| \leq v_h |\Lambda_h^\alpha w\|_{L^2(\mathbb{R}^+; L^2)}^2 + C\int_0^t \|w\|_{L^2}^2 \|u\|_{\dot{B}^{\beta,0}_{(r,\infty),q}}^q ds.$$

By the Gronwall's inequality, then

$$\|w(t)\|_{L^2}^2 \leq \|v_0 - u_0\|_{L^2}^2 \exp\Big(C\int_0^t \|u(s)\|^q_{\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2}_{(r,\infty),q}} ds \Big).$$

In particular, when $u_0 = v_0$,

$$u = v.$$

Theorem 1.1 is proved. $\square$

Let us prove the following result.

Lemma 2.1. *Under the assumption of theorem* 1.1, $\forall t \leq T$, *the following equality holds*

$$(u(t), v(t)) + 2\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds = (v_0, u_0) + \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla w) \cdot u dx dt.$$

Proof. Take two smooth sequences $\{u_n\}$ and $\{v_n\}$ such that $\mathrm{div} u_n = 0$ and $\mathrm{div} v_n = 0$, then

$$\begin{aligned} \lim_{n\to\infty} v_n &= v \quad \text{in} \quad L^2(\mathbb{R}^+, \dot{H}^{\alpha,0}), \\ \lim_{n\to\infty} u_n &= u \quad \text{in} \quad L^2(\mathbb{R}^+, \dot{H}^{\alpha,0}) \cap L^q([0,T]; \dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}. \end{aligned} \tag{2.1}$$

Take the scalar product with v_n and u_n of the generalized anisotropic Navier-stokes equations on u and u respectively yields, after integration in time and integration by parts in the space variables

$$\begin{aligned} \int_0^t \Big((\partial_s u, v_n) + (\Lambda_h^\alpha u, \Lambda_h^\alpha v_n) + (u \cdot \nabla u, v_n) \Big)(s) ds &= 0, \\ \int_0^t \Big((\partial_s v, u_n) + (\Lambda_h^\alpha v, \Lambda_h^\alpha u_n) + (v \cdot \nabla v, u_n) \Big)(s) ds &= 0. \end{aligned}$$

Because $\{\Lambda_h^\alpha u_n\}$ and $\{\Lambda_h^\alpha v_n\}$ converge in $L^2(\mathbb{R}^+, L^2)$,

$$\lim_{n\to\infty} \int_0^t \Big((\Lambda_h^\alpha u, \Lambda_h^\alpha v_n) + (\Lambda_h^\alpha v, \Lambda_h^\alpha u_n) \Big)(s) ds = 2\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v) ds.$$

Lemma 1.3 implies that

$$\lim_{n\to\infty} \int_0^t (v \cdot \nabla v, u_n)(s) ds = \int_0^t (v \cdot \nabla v, u)(s) ds.$$

Similarly, since $\mathrm{div} u = 0$,

$$\lim_{n\to\infty} \int_0^t (u \cdot \nabla u, v_n)(s) ds = \int_0^t (u \cdot \nabla v_n, u)(s) ds,$$

$$\lim_{n\to\infty} \int_0^t (u \cdot \nabla u, v_n)(s) ds = \int_0^t (u \cdot \nabla u, v)(s) ds.$$

However $\partial_s v = -(-\Delta_h)^\alpha v - \mathbb{P}(v \cdot \nabla v)$ in $\mathcal{D}'(\mathbb{R}^3)$ where $\mathbb{P}$ stands for the Projector Operator, so

$$\begin{aligned} \lim_{n\to\infty} \int_0^t (\partial_s v, u_n)(s) ds &= -\lim_{n\to\infty} \int_0^t \Big((\Lambda_h^\alpha v, \Lambda_h^\alpha u_n) + (v \cdot \nabla v, u_n) \Big)(s) ds \\ &= -\int_0^t \Big((\Lambda_h^\alpha v, \Lambda_h^\alpha u) + (v \cdot \nabla v, u) \Big)(s) ds \\ &= \int_0^t (\partial_s v, u)(s) ds. \end{aligned}$$

Similarly,

$$\lim_{n\to\infty}\int_0^t (\partial_s u, v_n)(s)ds = \int_0^t (\partial_s u, v)(s)ds.$$

Then, we have the limit

$$\begin{aligned}\lim_{n\to\infty} \quad \Big\{ & \int_0^t \Big((\partial_s u, v_n) + (\Lambda_h^\alpha u, \Lambda_h^\alpha v_n) + (u\cdot\nabla u, v_n)\Big)(s)ds \\ + & \int_0^t \Big((\partial_s v, u_n) + (\Lambda_h^\alpha v, \Lambda_h^\alpha u_n) + (v\cdot\nabla v, u_n)\Big)(s)ds\Big\}\end{aligned}$$

is

$$\begin{aligned}\int_0^t & \Big((\partial_s u, v) + (\partial_s v, u) + 2(\Lambda_h^\alpha u, \Lambda_h^\alpha v) + (u\cdot\nabla u, v) + (v\cdot\nabla v, u)\Big)(s)ds \\ &= \int_0^t \Big(\frac{d}{dt}(u,v) + 2(\Lambda_h^\alpha u, \Lambda_h^\alpha v) + (u\cdot\nabla u, v) + (v\cdot\nabla v, u)\Big)(s)ds \\ &= 0.\end{aligned}$$

So

$$(u(t), v(t)) + 2\int_0^t (\Lambda_h^\alpha u, \Lambda_h^\alpha v)(s)ds = (u_0, v_0) + \int_0^t (w\cdot\nabla u, w)(s)ds.$$

$\square$

Lemma 2.1 is proved.

3. Proof of the Lemma 1.3

The aim of this section is to prove Lemma 1.3. Lemma 1.3 was stated in the introduction and used several times in the proof of the theorem. Let us recall that we consider a trilinear form

$$(a,b,c) \in \mathcal{L}^2 \times L^q(0,T;\dot{B}^{\beta,0}_{(r,\infty),q}) \mapsto \int_0^T\int_{\mathbb{R}^3}(a\cdot\nabla b)\cdot c dxdt,$$

for which we hope to prove the continuity as well as the estimate

$$\Big|\int_0^t\int_{\mathbb{R}^3}(a\cdot\nabla a)\cdot cdxdt\Big| \le v_h\|\Lambda_h^\alpha a\|^2_{L^2(\mathbb{R}^+;L^2)} + C\int_0^t \|a\|^2_{L^2}\|c\|^q_{\dot{B}^{\beta,0}_{(r,\infty),q}}ds.$$

Now we start to the proof of Lemma1.3.

Proof. We are going to use the paraproduct algorithm introduction by Bony, we have

$$\int_0^t\int_{\mathbb{R}^3}(a\cdot\nabla b)\cdot cdxdt = \sum_{j\in\mathbb{Z}}\int_0^T\int_{\mathbb{R}^3}\Delta_j^h\Big((a\cdot\nabla b)\cdot c\Big)dxdt.$$

We are going to split the term $\sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^3} \Delta_j^h\Big((a\cdot\nabla b)\cdot c\Big)dxdt$ in the following way

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^3} \Delta_j^h\Big((a\cdot\nabla b)\cdot c\Big)dxdt\\
=\ & \sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^3} \Delta_j^h\Big((a_h\cdot\nabla_h b)\cdot c\Big)dxdt + \sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^3} \Delta_j^h\Big((a_3\cdot\partial_3 b)\cdot c\Big)dxdt\\
=\ & I_j^h + I_j^v.
\end{aligned}
$$

We shall split the horizontal term I_j^h, using the h0mogeneous Bony's decomposition, we have

$$
I_j^h = I_1^{h,j} + I_2^{h,j} + R^{h,j},
$$

where

$$
\begin{aligned}
I_1^{h,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} \Big(S_{k-1}^h a_h \Delta_k^h \nabla_h b\Big)\Delta_j^h c dxds,\\
I_2^{h,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} \Big(S_{k-1}^h \nabla_h b \Delta_k^h a_h\Big)\Delta_j^h c dxds,\\
R^{h,j} &= \sum_{k<j-N_0,|k-k'|\le 1} \int_0^t \int_{\mathbb{R}^3} \Big(\Delta_{k'}^h \nabla_h b \Delta_k^h a_h\Big)\Delta_j^h c dxds.
\end{aligned}
$$

<u>Estimate of $I_1^{h,j}$</u>
we use Hölder inequality to infer

$$
|I_1^{h,j}| \le \sum_{|j-k|<N_0} \int_0^t \|S_{k-1}^h a_h\|_{L_v^2 L_h^{\overline{r}}} \|\Delta_k^h \nabla_h b\|_{L^2} \|\Delta_j^h c\|_{L_v^\infty L_h^r} ds,
$$

where

$$
\frac{1}{r} + \frac{1}{\overline{r}} = \frac{1}{2}.
$$

As a result of Lemma 1.1

$$
\begin{aligned}
\|\Delta_k^h \nabla_h b\|_{L^2} &\le 2^{2k} \|\Delta_k^h b\|_{L^2} 2^{\alpha k} 2^{-\alpha k},\\
\|S_{k-1}^h a_h\|_{L_v^2 L_h^{\overline{r}}} &\le \sum_{k'<k} 2^{2(\frac{1}{2}-\frac{1}{\overline{r}})} \|\Delta_{k'}^h a_h\|_{L^2} 2^{\frac{2\alpha k'}{q}} 2^{-\frac{2\alpha k'}{q}}.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
|I_1^{h,j}| \le\ & \sum_{k'<k} \int_0^t \|\Delta_{k'}^h a_h\|_{L^2} 2^{\frac{2\alpha k'}{q}} \|\Delta_k^h b\|_{L^2} 2^{\alpha k} 2^{k'(1-\frac{2}{\overline{r}}-\frac{2\alpha}{q})}\\
& 2^{-j(\frac{2}{r}+\frac{2\alpha}{q}-\alpha)} 2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)} \|\Delta_j^h c\|_{L_v^\infty L_h^r} ds.
\end{aligned}
$$

We apply Lemma 1.2 to imply that

$$|I_1^{h,j}| \le \|a\|^{\frac{2}{q}}_{L^\infty(\mathbb{R}^+;L^2)} \|\Lambda_h^\alpha a\|^{1-\frac{2}{q}}_{L^2(\mathbb{R}^+;L^2)} \|\Lambda_h^\alpha b\|_{L^\infty(\mathbb{R}^+;L^2)} \|c\|_{L^q(0,T;\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q})}$$

and

$$\frac{1}{q}+\frac{1}{\overline{q}}=\frac{1}{2}.$$

Furthermore, in the case when $a=b$ we can also get

$$\begin{aligned} |I_1^{h,j}| &\le \int_0^t \|a\|^{\frac{2}{q}}_{L^2} \|\Lambda_h^\alpha a\|^{1-\frac{2}{q}}_{L^2} \|c\|_{\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}} ds \\ &\le v_h \|\Lambda_h^\alpha a\|^2_{L^2} + C\int_0^t \|a(s)\|^2_{L^2} \|c(s)\|^q_{\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}} ds. \end{aligned}$$

<u>Estimate of $I_2^{h,j}$</u>

By Hölder inequality, then

$$|I_2^{h,j}| \le \sum_{|j-k|<N_0} \int_0^t \|\Delta_k^h a_h\|_{L^2} \|S_{k-1}^h \nabla_h b\|_{L_v^2 L_h^{\overline{r}}} \|\Delta_j^h c\|_{L_v^\infty L_h^r} ds$$

with

$$\frac{1}{r}+\frac{1}{\overline{r}}=\frac{1}{2}.$$

We also have used Lemma 1.1 that

$$\begin{aligned} \|S_{k-1}^h \nabla_h b\|_{L_v^2 L_h^{\overline{r}}} &\le \sum_{k'<k} 2^{2(\frac{1}{2}-\frac{1}{\overline{r}})} \|\Delta_{k'}^h b\|_{L^2} 2^{\frac{2\alpha k'}{\overline{q}}} 2^{-\frac{2\alpha k'}{\overline{q}}} \\ &= \sum_{k'<k} \|\Delta_{k'}^h b\|_{L^2} 2^{\frac{2\alpha k'}{\overline{q}}} 2^{k'(1-\frac{2}{r}-\frac{2\alpha}{\overline{q}}+2)}, \end{aligned}$$

$$\begin{aligned} |I_2^{h,j}| \le \sum_{k'<k} \int_0^t & \|\Delta_k^h a_h\|_{L^2} 2^{\alpha k} \|\Delta_{k'}^h b\|_{L^2} 2^{2\alpha k'} 2^{k'(1-\frac{2}{\overline{r}}-\frac{2\alpha}{\overline{q}}+2)} \\ & 2^{-j(\frac{2}{r}+\frac{2\alpha}{q}-\alpha+2)} 2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)} \|\Delta_j^h c\|_{L_v^\infty L_h^r} ds, \end{aligned}$$

and similarly for b, then we get

$$|I_2^{h,j}| \le \|b\|^{\frac{2}{q}}_{L^\infty(\mathbb{R}^+;L^2)} \|\Lambda_h^\alpha b\|^{1-\frac{2}{q}}_{L^2(\mathbb{R}^+;L^2)} \|\Lambda_h^\alpha a\|_{L^\infty(\mathbb{R}^+;L^2)} \|c\|_{L^q(0,T;\dot{B}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q})}$$

with

$$\frac{1}{q}+\frac{1}{\overline{q}}=\frac{1}{2},$$

$$\frac{2}{r}+\frac{2\alpha}{q}-\alpha+2=1-\frac{2}{\overline{r}}-\frac{2\alpha}{\overline{q}}+2.$$

Furthermore, in the case when $a=b$ we can also get

$$\begin{aligned}|I_2^{h,j}| &\leq \int_0^t v_h\|a\|_{L^2}^{\frac{2}{q}}\|\Lambda_h^\alpha a\|_{L^2}^{1-\frac{2}{q}}\|c\|_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}}ds\\ &\leq v_h\|\Lambda_h^\alpha a\|_{L^2}^2+C\int_0^t\|a(s)\|_{L^2}^2\|c(s)\|^q_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}}ds.\end{aligned}$$

<u>Estimate of $R^{h,j}$</u>

Using the Hölder inequality, we infer

$$|R^{h,j}|\leq\sum_{k>j-N_0,|k-k'|\leq 1}\int_0^t\|\Delta_k^h a_h\|_{L^2}\|\Delta_{k'}^h\nabla_h b\|_{L^2}\|\Delta_j^h c\|_{L^\infty}ds.$$

Using Lemma 1.1, we get

$$\begin{aligned}\|\Delta_j^h c\|_{L^\infty} &\leq 2^{\frac{2j}{r}}\|\Delta_j^h c\|_{L_v^\infty L_h^r},\\ \|\Delta_{k'}^h\nabla_h b\|_{L^2} &\leq 2^{2k'}\|\Delta_{k'}^h b\|_{L^2}2^{\frac{\alpha k'}{q'}}2^{-\frac{\alpha k'}{q'}},\end{aligned}$$

$$\begin{aligned}|R^{h,j}| \leq \sum_{k>j-N_0}&\int_0^t\|\Delta_k^h a_h\|_{L^2}2^{\frac{\alpha k}{q'}}\|\Delta_{k'}^h b\|_{L^2}2^{\frac{\alpha k'}{q'}}2^{k(2-\frac{2\alpha k'}{q'})}\\ &2^{-j(-2\alpha+\frac{2\alpha}{q}+2)}2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)}\|\Delta_j^h c\|_{L_v^\infty L_h^r}.\end{aligned}$$

Lemma 1.2 implies that

$$|R^{h,j}|\leq\|a\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{1}{q}}\|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{1}{q}}\|b\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{1}{q}}\|\Lambda_h^\alpha b\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{1}{q}}\|c\|_{L^q(0,T;\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0})}$$

and

$$\frac{1}{q}+\frac{1}{q'}=1.$$

Furthermore, in the case when $a=b$ we can also get

$$\begin{aligned}|R^{v,j}| &\leq \int_0^t v_h\|a\|_{L^2}^{\frac{2}{q}}\|\Lambda_h^\alpha a\|_{L^2}^{1-\frac{2}{q}}\|c\|_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}}ds\\ &\leq v_h\|\Lambda_h^\alpha a\|_{L^2}^2+C\int_0^t\|a(s)\|_{L^2}^2\|c(s)\|^q_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}}ds.\end{aligned}$$

We shall split the horizontal term I_j^v, using the homogeneous Bony's decomposition, we have

$$I_j^{v,j}=I_1^{v,j}+I_2^{v,j}+R^{v,j},$$

where

$$\begin{aligned}
I_1^{v,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} S_{k-1}^h a_3 \Delta_k^h \partial_3 b \Delta_j^h c dx ds \\
&= -I_1^{h,j} - I_{1'}^{v,j}, \\
I_2^{v,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} S_{k-1}^h \partial_3 b \Delta_k^h a_3 \Delta_j^h c dx ds \\
&= -I_2^{h,j} - I_{2'}^{v,j}, \\
R^{v,j} &= \sum_{k<j-N_0, |k-k'|\leq 1} \int_0^t \int_{\mathbb{R}^3} \Delta_{k'}^h \partial_3 b \Delta_k^h a_3 \Delta_j^h c dx ds \\
&= -R^{h,j} - {R'}^{v,j},
\end{aligned} \tag{3.2}$$

with

$$\begin{aligned}
I_{1'}^{v,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} S_{k-1}^h a_h \Delta_k^h b \Delta_j^h \nabla_h c dx ds, \\
I_{2'}^{v,j} &= \sum_{|j-k|<N_0} \int_0^t \int_{\mathbb{R}^3} S_{k-1}^h b \Delta_k^h a_h \Delta_j^h \nabla_h c dx ds, \\
{R'}^{v,j} &= \sum_{k<j-N_0, |k-k'|\leq 1} \int_0^t \int_{\mathbb{R}^3} \Delta_{k'}^h b \Delta_k^h a_h \Delta_j^h \nabla_h c dx ds.
\end{aligned}$$

<u>Estimate of $I_{1'}^{v,j}$</u>

$$|I_{1'}^{v,j}| \leq \sum_{|j-k|<N_0} \int_0^t \|S_{k-1}^h a_h\|_{L_v^2 L_h^{\overline{r}}} \|\Delta_k^h b\|_{L^2} \|\Delta_j^h \nabla_h c\|_{L_v^\infty L_h^r} ds,$$

where

$$\frac{1}{r} + \frac{1}{\overline{r}} = \frac{1}{2}.$$

Using Lemma 1.1 we have

$$\begin{aligned}
\|\Delta_j^h \nabla_h c\|_{L_v^\infty L_h^2} &\leq 2^{2j} \|\Delta_j^h c\|_{L_v^\infty L_h^2}, \\
\|S_{k-1}^h a_h\|_{L_v^2 L_h^{\overline{r}}} &\leq \sum_{k'<k} 2^{2k'(\frac{1}{2}-\frac{1}{\overline{r}})} \|\Delta_{k'}^h a_h\|_{L^2} 2^{\frac{2\alpha k'}{q}} 2^{-\frac{2\alpha k'}{q}}.
\end{aligned}$$

Hence

$$
\begin{aligned}
|I_{1'}^{v,j}| &\le \sum_{k'<k}\int_0^t \|\Delta_{k'}^h a_h\|_{L^2} 2^{\frac{2\alpha k'}{\bar q}} \|\Delta_k^h b\|_{L^2} 2^{\alpha k} 2^{k'(1-\frac{2}{\bar r}-\frac{2\alpha}{\bar q})} \\
&\quad 2^{-j(\frac{2}{r}+\frac{2\alpha}{q}-\alpha)} 2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)} \|\Delta_j^h c\|_{L_v^\infty L_h^r} ds.
\end{aligned}
$$

By Lemma 1.2 we obtain

$$
|I_{1'}^{v,j}| \le \|a\|_{L^\infty(\mathbb{R}^+;L^2)}^{\frac{2}{q}} \|\Lambda_h^\alpha a\|_{L^2(\mathbb{R}^+;L^2)}^{1-\frac{2}{q}} \|\Lambda_h^\alpha b\|_{L^\infty(\mathbb{R}^+;L^2)} \|c\|_{L^q(0,T;\dot B_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0})}
$$

and

$$
\frac{1}{q}+\frac{1}{\bar q}=\frac{1}{2},
$$

$$
\frac{2}{r}+\frac{2\alpha}{q}-\alpha = 1-\frac{2}{\bar r}-\frac{2\alpha}{\bar q}.
$$

Furthermore, in the case when $a=b$ we can also get

$$
\begin{aligned}
|I_{1'}^{v,j}| &\le \int_0^t \|a\|_{L^2}^{\frac{2}{q}} \|\Lambda_h^\alpha a\|_{L^2}^{1-\frac{2}{q}} \|c\|_{\dot B_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}} ds \\
&\le v_h \|\Lambda_h^\alpha a\|_{L^2}^2 + C\int_0^t \|a(s)\|_{L^2}^2 \|c(s)\|_{\dot B_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}}^q ds.
\end{aligned}
$$

<u>Estimate of $I_{2'}^{v,j}$</u>

By Hölder inequality, then

$$
|I_{2'}^{v,j}| \le \sum_{|j-k|<N_0} \int_0^t \|\Delta_k^h a_h\|_{L^2} \|S_{k-1}^h b\|_{L_v^2 L_h^{\bar r}} \|\Delta_j^h \nabla_h c\|_{L_v^\infty L_h^r} ds
$$

with

$$
\frac{1}{r}+\frac{1}{\bar r}=\frac{1}{2}.
$$

We also have used Lemma 1.1 that

$$
\begin{aligned}
\|S_{k-1}^h b\|_{L_v^2 L_h^{\bar r}} &\le \sum_{k'<k} 2^{2(\frac{1}{2}-\frac{1}{\bar r})} \|\Delta_{k'}^h b\|_{L^2} 2^{\frac{2\alpha k'}{\bar q}} 2^{-\frac{2\alpha k'}{\bar q}} \\
&= \sum_{k\prime<k} \|\Delta_{k'}^h b\|_{L^2} 2^{\frac{2\alpha k'}{\bar q}} 2^{k'(1-\frac{2}{r}-\frac{2\alpha}{\bar q})}, \\
\|\Delta_j^h \nabla_h c\|_{L_v^\infty L_h^2} &\le 2^{2j} \|\Delta_j^h c\|_{L_v^\infty L_h^2},
\end{aligned}
$$

$$
\begin{aligned}
|I^{v,j}_{2'}| \;\; &\leq \;\; \sum_{k'<k}\int_0^t \|\Delta^h_k a_h\|_{L^2} 2^{\alpha k}\|\Delta^h_{k'} b\|_{L^2} 2^{2\alpha k'} 2^{k'(1-\frac{2}{\bar r}-\frac{2\alpha}{\bar q})} \\
& \qquad 2^{-j(\frac{2}{r}+\frac{2\alpha}{q}-\alpha)} 2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)}\|\Delta^h_j c\|_{L^\infty_v L^r_h} ds.
\end{aligned}
$$

So, we get

$$
|I^{v,j}_{2'}| \leq \|b\|^{\frac{2}{q}}_{L^\infty(\mathbb{R}^+;L^2)}\|\Lambda^\alpha_h b\|^{1-\frac{2}{q}}_{L^2(\mathbb{R}^+;L^2)}\|\Lambda^\alpha_h a\|_{L^\infty(\mathbb{R}^+;L^2)}\|c\|_{L^q(0,T;\dot B^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q})}
$$

with

$$
\frac{1}{q}+\frac{1}{\bar q}=\frac{1}{2},
$$

$$
\frac{2}{r}+\frac{2\alpha}{q}-\alpha = 1-\frac{2}{\bar r}-\frac{2\alpha}{\bar q}.
$$

Furthermore, in the case when $a=b$ we can also get

$$
\begin{aligned}
|I^{v,j}_{2'}| \;\; &\leq \;\; \int_0^t \|a\|^{\frac{2}{q}}_{L^2}\|\Lambda^\alpha_h a\|^{1-\frac{2}{q}}_{L^2}\|c\|_{\dot B^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}} ds \\
&\leq \;\; v_h\|\Lambda^\alpha_h a\|^2_{L^2} + C\int_0^t \|a(s)\|^2_{L^2}\|c(s)\|^q_{\dot B^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q}} ds.
\end{aligned}
$$

<u>Estimate of $R^{'v,j}$</u>

Using the Hölder inequality, we infer

$$
|R^{'v,j}| \leq \sum_{k>j-N_0,|k-k'|\leq 1}\int_0^t \|\Delta^h_k a_h\|_{L^2}\|\Delta^h_{k'} b\|_{L^2}\|\Delta^h_j \nabla_h c\|_{L^\infty} ds.
$$

By Lemma 1.1, we get

$$
\|\Delta^h_j \nabla_h c\|_{L^\infty} \leq 2^{j(\frac{2}{r}+2)}\|\Delta^h_j c\|_{L^\infty_v L^r_h}.
$$

$$
\begin{aligned}
|R^{'v,j}| \;\; &\leq \;\; \sum_{k>j-N_0}\int_0^t \|\Delta^h_k a_h\|_{L^2} 2^{\frac{\alpha k}{q'}}\|\Delta^h_{k'} b\|_{L^2} 2^{\frac{\alpha k'}{q'}} 2^{k(-\frac{2\alpha k'}{q'})} \\
& \qquad 2^{-j(-2\alpha+\frac{2\alpha}{q})} 2^{j(\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2)}\|\Delta^h_j c\|_{L^\infty_v L^r_h}.
\end{aligned}
$$

Lemma 1.2 implies that

$$
|R^{'v,j}| \leq \|a\|^{\frac{1}{q}}_{L^\infty(\mathbb{R}^+;L^2)}\|\Lambda^\alpha_h a\|^{1-\frac{1}{q}}_{L^2(\mathbb{R}^+;L^2)}\|b\|^{\frac{1}{q}}_{L^\infty(\mathbb{R}^+;L^2)}\|\Lambda^\alpha_h b\|^{1-\frac{1}{q}}_{L^2(\mathbb{R}^+;L^2)}\|c\|_{L^q(0,T;\dot B^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}_{(r,\infty),q})}
$$

and

$$
\frac{1}{q}+\frac{1}{q'}=1.
$$

Furthermore, in the case when $a=b$ we can also get

$$
\begin{aligned}
|R'^{v,j}| &\leq \int_0^t \|a\|_{L^2}^{\frac{2}{q}} \|\Lambda_h^\alpha a\|_{L^2}^{1-\frac{2}{q}} \|c\|_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}} ds \\
&\leq v_h \|\Lambda_h^\alpha a\|_{L^2}^2 + C\int_0^t \|a(s)\|_{L^2}^2 \|c(s)\|^q_{\dot{B}_{(r,\infty),q}^{\frac{2}{r}+\frac{2\alpha}{q}-2\alpha+2,0}} ds.
\end{aligned}
$$

□

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