On the compressible viscous barotropic flows subject to large external potential forces in a half space with Navier's boundary conditions*

Changzhen Song† and Jianwen Zhang

School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China

Abstract. This paper is concerned with an initial and boundary value problem of the Navier-Stokes equations for compressible viscous barotropic flow subject to large external potential forces in a half space $\mathbb{R}^3_+$ with Navier's boundary conditions. The global well-posedness of strong solutions with large oscillations and vacuum is established, provided that the initial energy is suitably small and that the unique steady state is strictly away from vacuum. As a by-product, the stability of stationary solution is obtained.

Keywords. compressible Navier-Stokes equations; Navier's boundary conditions; half space; large external potential forces; global solutions

1 Introduction

The motion of compressible viscous barotropic flows occupying a domain $\Omega \subset \mathbb{R}^3$ is governed by the following Navier-Stokes equations:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \rho \nabla f, \tag{1.2}$$

where the unknown functions $\rho = \rho(x,t), u = (u^1, u^2, u^3)(x,t)$ and $P = P(\rho)$ are the density, velocity and pressure, respectively, and $f = f(x)$ is the external potential force (e.g., gravity). The pressure $P(\rho)$ is determined through the equation of state:

$$P(\rho) = A\rho^\gamma, \tag{1.3}$$

where $A > 0$ is the entropy constant and $\gamma > 1$ is the adiabatic exponent. The viscosity coefficients μ and λ satisfy the physical restrictions for Newtonian fluids:

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0, \tag{1.4}$$

which ensure that the Laḿe operator $\mathcal{L} \triangleq \mu\Delta + (\mu + \lambda)\nabla \operatorname{div}$ is a strongly elliptic operator.

In the past decades, the compressible Navier-Stokes equations (1.1)–(1.2) have been extensively studied by many people due to its physical importance and mathematical challenge. The local-in-time existence of strong/classical solutions was obtained in [25, 30, 32] and [3, 4, 5] for the non-vacuum case and the vacuum case, respectively. The first result about the global-in-time

*The work is supported by the National Natural Science Foundation of China (Grant No. 11671333)

†Corresponding author (E-mail: songczmath@163.com)

existence of smooth solutions is due to Matsumura-Nishida [23], when the initial perturbation around the non-vacuum equilibrium is sufficiently small in H^3. Later, Hoff [13, 14, 15] extended the Matsumura-Nishida's theorem [23] to weak solutions with discontinuous initial data. For the case of generally large data, the most important breakthrough is due to Lions [22] (see also [9, 11]), who proved the global existence of weak solutions with finite energy to the isentropic Navier-Stokes equations, when the adiabatic exponent γ is suitably large (i.e., $\gamma > 3/2$). However, the uniqueness of the Lions-Feireisl's type weak solutions is still completely unknown. If there is absent of external potential forces and the initial data satisfy some compatibility conditions as the one in [5], Huang-Li-Xin [18] established the global well-posedness of classical solutions of (1.1) and (1.2) with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support. Based on some ideas in [15], Duan [8] extended the Huang-Li-Xin's result (cf. [18]) to an initial-boundary value problem of (1.1) and (1.2) with Navier's type boundary conditions in the half space.

In reality, the large external force will significantly affect the dynamic motion of the flows. In the case when both the external force and the initial perturbation are sufficiently small, there have been many results about the global existence and the large-time behavior of smooth solutions, see, for example, [6, 7, 31, 33] and the references therein. However, if the external force could be arbitrarily large, then some seriously mathematical difficulties will arise. By virtue of the compactness technique in [22, 11] the authors [12, 28] showed that the density of weak solution converges to the steady density in L^γ as time goes to infinity, provided the adiabatic exponent $\gamma > 3/2$. For the case of large external potential force, if the initial perturbations are sufficiently small in $L^2 \cap L^\infty$ for density (non-vacuum) and in H^1 for velocity, Matsumura-Yamagata [24] obtained the convergence in L^p-norm with $2 < p \leq \infty$, when $\gamma > 1$ is close enough to 1 and the external potential forces decay suitably fast at infinity. This result was later improved by Li-Matsumura [19] by removing the smallness condition on $|\gamma - 1|$ and the far-field decay conditions of potential force, and then was extended to the vacuum case by the authors [20].

Let $\Omega = \mathbb{R}^3_+ \triangleq \{x \in \mathbb{R}^3 | x_3 > 0\}$ be the half space with the boundary $\partial\Omega \triangleq \{x \in \mathbb{R}^3 | x_3 = 0\}$. The present paper aims to study an initial-boundary value problem of (1.1)–(1.4) in $\mathbb{R}^3_+$ with the following initial and boundary conditions:

$$(\rho, \rho u)(x, 0) = (\rho_0(x), m_0(x)) \quad \text{for} \quad x \in \Omega, \tag{1.5}$$

$$\left(u^1, u^2, u^3\right)(x, t) = \beta\left(\partial_3 u^1, \partial_3 u^2, 0\right)(x, t) \quad \text{for} \quad x \in \partial\Omega,\ t > 0, \tag{1.6}$$

and the far-field behavior:

$$(\rho, u)(x, t) \to (\rho_\infty, 0) \quad \text{as} \quad |x| \to +\infty, \tag{1.7}$$

where $\rho_\infty > 0$ and $\beta > 0$ are given positive numbers. Such a kind of boundary conditions was proposed by Navier [26] and implies that the velocity on $\partial\Omega$ is proportional to the tangential component of the stress. The flat case of half space in the form (1.6) has been usually applied for incompressible flows, see, for example, [1, 2, 29] and the references cited therein.

The main purpose of this paper is to study the global well-posedness and asymptotic behavior of strong solutions of the problem (1.1)–(1.7). To formulate our main result precisely, we first consider the stationary problem:

$$\operatorname{div}(\rho_s u_s) = 0, \tag{1.8}$$

$$\rho_s u_s \cdot \nabla u_s + \nabla P(\rho_s) = \mu \Delta u_s + (\mu + \lambda) \nabla \operatorname{div} u_s + \rho_s \nabla f, \tag{1.9}$$

with the boundary and far-field conditions:

$$\left(u_s^1(x), u_s^2(x), u_s^3(x)\right) = \beta\left(\partial_3 u_s^1(x), \partial_3 u_s^2(x), 0\right) \quad \text{for} \quad x \in \partial\Omega, \tag{1.10}$$

$$(\rho_s, u_s)(x) \to (\rho_\infty, 0) \quad \text{as} \quad |x| \to +\infty, \tag{1.11}$$

Assume that $(\rho_s(x), u_s(x))$ with $\inf\limits_{x\in\Omega} \rho_s(x) > 0$ is a smooth solution of (1.8)–(1.11). Then, multiplying (1.9) by u_s in L^2 and integrating by parts, by (1.8) and (1.10) one easily infers that

$$\mu \int_\Omega |\nabla u_s|^2 \mathrm{d}x + (\mu + \lambda) \int_\Omega |\mathrm{div} u_s|^2 \mathrm{d}x + \mu \int_{\partial\Omega} \beta |u_s|^2 \mathrm{d}S = 0,$$

so that, $u_s(x) = 0$ for $x \in \Omega$. Thus, it follows from $(1.9)_2$ that $\rho_s = \rho_s(x)$ is determined by

$$\nabla P(\rho_s) = \rho_s \nabla f \Leftrightarrow \nabla\left(\int_{\rho_\infty}^{\rho_s} \frac{P'(s)}{s} \mathrm{d}s - f\right) = 0 \tag{1.12}$$

which, together with (1.3) and (1.11), yields

$$\rho_s(x) = \left(\rho_\infty^{\gamma-1} + \frac{\gamma-1}{A\gamma} f(x)\right)^{\frac{1}{\gamma-1}},$$

provided that $f \in H^2$ satisfies

$$\inf_{x\in\Omega} f(x) > -\frac{A\gamma}{\gamma-1}\rho_\infty^{\gamma-1}. \tag{1.13}$$

To summarize up, we have shown the following proposition.

Proposition 1.1 *Assume that $f \in H^2$ and (1.12) are satisfied. Then there exists a unique steady solution $(\rho_s(x), 0)$ to the problem (1.8)–(1.11) such that*

$$\rho_s - \rho_\infty \in H^2, \quad 0 < \underline{\rho} \le \inf_{x\in\bar\Omega} \rho_s(x) \le \sup_{x\in\bar\Omega} \rho_s(x) \le \bar\rho < \infty, \tag{1.14}$$

where $\underline{\rho}, \bar\rho$ are positive constants depending only on A, γ, ρ_∞, $\inf\limits_{x\in\bar\Omega} f(x)$, and $\sup\limits_{x\in\bar\Omega} f(x)$.

In order to measure the size of the initial data, we define

$$C_0 \triangleq \int \left(\frac{1}{2}\rho_0 |u_0|^2 + G(\rho_0)\right) \mathrm{d}x, \tag{1.15}$$

where $G(\cdot)$ is the potential energy density given by

$$G(\rho) \triangleq \int_{\rho_s}^{\rho}\int_{\rho_s}^{r} \frac{P'(\xi)}{\xi} \mathrm{d}\xi \mathrm{d}r = \rho \int_{\rho_s}^{\rho} \frac{P(s) - P(\rho_s)}{s^2} \mathrm{d}s. \tag{1.16}$$

It is easy to see from (1.13) that if $0 \le \rho \le 2\tilde\rho$, then there exists a positive constant $C(\underline{\rho}, \bar\rho, \tilde\rho)$, depending on $\underline{\rho}, \bar\rho$ and $\tilde\rho$, such that

$$C(\tilde\rho)^{-1}(\rho - \rho_s)^2 \le G(\rho) \le C(\tilde\rho)(\rho - \rho_s)^2. \tag{1.17}$$

The main result of this paper can be stated as follows.

Theorem 1.1 *Let Proposition 1.1 be in force. For any given positive numbers $\tilde\rho \ge \bar\rho + 1$ and M (not necessarily small), assume that*

$$0 \le \inf_{x\in\bar\Omega} \rho_0(x) \le \sup_{x\in\bar\Omega} \rho_0(x) \le \tilde\rho, \quad \rho_0 - \rho_\infty \in H^1 \cap W^{1,p}, \quad 3 < p < 6. \tag{1.18}$$

Moreover, assume that m_0/ρ_0 is well defined and satisfies

$$u_0 \triangleq \frac{m_0}{\rho_0} \quad \text{with} \quad \rho_0^{1/2} u_0 \in L^2, \quad \|\nabla u_0\|_{L^2}^2 \le M. \tag{1.19}$$

Then there exists a positive constant $\varepsilon > 0$, depending only on $\mu, \lambda, A, \gamma, \rho_\infty, \tilde\rho, M, \inf f(x)$ and $\|f\|_{H^2}$, such that if the initial energy satisfies

$$C_0 \le \varepsilon \tag{1.20}$$

then the problem (1.1)–(1.7) has a unique global strong solution (ρ, u) on $\mathbb{R}^3 \times (0,T]$ for any $0 < T < \infty$ such that

$$0 \le \inf_{x\in\mathbb{R}^3_+, t\ge 0} \rho(x,t) \le \sup_{x\in\mathbb{R}^3_+, t\ge 0} \rho(x,t) \le 2\tilde\rho, \tag{1.21}$$

and

$$\begin{cases} \rho - \rho_\infty \in C([0,T]; H^1 \cap W^{1,p}), \quad \rho u \in C([0,T]; L^2), \\ \sqrt{\rho} u \in L^\infty(0,T; L^2), \quad \nabla u \in L^\infty(0,T; L^2) \cap L^q(0,T; L^\infty), \\ (\nabla^2 u, \sqrt{\rho} u_t) \in L^2(0,T; L^2) \cap L^q(0,T; L^p), \\ \sqrt{t} \nabla u_t \in L^2(0,T; L^2), \quad \sqrt{t}(\nabla^2 u, \sqrt{\rho} u_t) \in L^\infty(0,T; L^2), \end{cases} \tag{1.22}$$

where $1 < q < \frac{4p}{5p-6}$. Moreover, the following large-time behavior holds,

$$\lim_{t\to\infty} \left(\|\rho(\cdot,t) - \rho_s\|_{L^p} + \|u(\cdot,t)\|_{L^p\cap L^\infty} \right) = 0, \quad \forall\, p \in (2, \infty). \tag{1.23}$$

Remark 1.1. Similarly to that in [20], we establish the global existence of strong solutions under the condition that $u_0 = m_0/\rho_0$ is well defined, which is much weaker than the one in [3, 4], and thus, improve the result in [8].

Remark 1.2. In [15], Hoff considered the case that $\beta = \beta(x_1, x_2)$ is a positive smooth function of x_1, x_2. Unfortunately, it will produce some additional lower-order terms, when the standard theory of elliptic system was applied to derive the L^p-estimate of the "effective viscous flux" and the vorticity. For example, as that in [15, (2.14)], one has to deal with the term $\|u\|_{L^2(0,T;L^2)}$, however, it seems difficult to derive the t-independent estimate of $\|u\|_{L^2(0,T;L^2)}$, and consequently, the norm of $\|\nabla u\|_{L^4(0,T;L^4)}$ cannot be well controlled as desired.

Theorem 1.1 will be proven in Section 3, based on the global a priori estimates established in Section 2. We now comment on the analysis of this paper. Indeed, the strategy for the proof of global existence is analogous to the one in [19, 20]. Roughly speaking, we first use the well-known Matsumura-Nishida's theorem (cf. [23]) to guarantee the local existence of classical solutions with strictly positive density, then extend the local classical solutions globally in time just under the condition that the initial energy is suitably small, and finally let the lower bound of the initial density go to zero. Since the scheme is standard, the main part of this paper is to derive some global a priori estimates which are independent of the lower bound of density. To do this, we will borrow some ideas from [8, 15, 18, 19, 20]. We shall make a full use of the mathematical structure of the stationary solutions to deal with the pressure and the external potential force by the deviation of the density ρ from the steady state ρ_s. Moreover, as it was mentioned in [15] that because ∇u may be discontinuous across the hypersurface of $\mathbb{R}^3_+$, it is difficult to show that $\nabla^2 u$ is locally integrable, and thus, one cannot expect to control $\|\nabla u\|_{L^4}$ by $\|\nabla u\|_{H^1}$ directly. Note that one has to deal with the term $\|\nabla u\|_{L^4(0,T;L^4)}$ induced the nonlinear terms in (1.1) after integrating by parts. To this end, similarly to that [19, 20], we introduce the following modified "effective viscous flux" $\tilde{F}$ and vorticity $\tilde{\omega}$

$$\tilde{F} \triangleq \rho_s^{-1}\left[(\lambda+2\mu)\mathrm{div}u - (P(\rho)-P(\rho_s))\right], \quad \tilde{\omega} \triangleq \rho_s^{-1}\nabla\times u. \tag{1.24}$$

However, unlike the Cauchy problem considered in [19, 20], the estimates of $\tilde{F}$ and $\tilde{\omega}$ become a bit more complicated, due to the boundary effects. Indeed, due to the Navier's type boundary conditions in (1.6), one can use the elliptic theory to obtain some desired estimates for the gradients of $\tilde{\omega}^1, \tilde{\omega}^2$ in a similar manner as that in [8, 15]. But, the gradient estimates of the third component $\tilde{\omega}^3$ needs more works (see Lemma 2.3). Based on some careful computations, we find that $\Delta_h\tilde{\omega}^3$ with $\Delta_h \triangleq \partial_{11}^2+\partial_{22}^2$ being Laplacian operator in horizonal direction equals to some terms involving the desired derivatives in either x_1 or x_2 direction (see (2.36)). This plays an important role when one applies the standard L^p-theory to the equation of $\Delta_h\tilde{\omega}^3$ and to derive the L^p-estimates of $\nabla_h\tilde{\omega}^3$ in dimension two. It is worth mentioning that once the estimates of $\tilde{\omega}$ is obtained, one also gets the estimates of $\tilde{F}$ (due to (2.27)). With these estimates at hand, one then can prove the upper bound of density by applying the Zlotnik inequlaity (see Lemma 2.2) in a similar manner as that in [18].

To prove the existence of strong solutions, we still need to estimates of the gradient of density, which relies strongly on the bound of $\|\nabla u\|_{L^\infty} \in L^1(0,T)$. As that in [16, 20], this will be achieved by using the fact that the t-weighted estimate in (2.49) implies $\|\rho\dot{u}\|_{L^p} \in L^q(0,T)$ for some $3 < p < 6$ and $q > 1$. This, combined with the BKM's type logarithmic estimate, immediately yields a desired estimate of $\|\nabla u\|_{L^\infty}$, and enables us to get that $\|\nabla\rho\|_{L^2\cap L^p} \in L^\infty(0,T)$ after solving a logarithmic inequality.

2 A priori estimates

This section is devoted to the derivations of the global a priori estimates of the solutions to the problem (1.1)–(1.7), and is split into two subsections which are concerned with the t-independent lower-order and the t-dependent gradient estimates for the existence of strong solutions, respectively. Throughout this section, we assume that (ρ, u) is a smooth solution of (1.1)–(1.7) defined on $\bar{\Omega}\times[0,T]$ with some $T \in (0,\infty)$.

2.1 t-independent lower-order estimates

The purpose of this subsection is mainly to derive some necessary lower-order estimates of the solutions. To do so, similarly to that in [8, 18, 20], we set

$$A_1(T) \triangleq \sup_{t\in[0,T]}\left(\sigma\|\nabla u\|_{L^2}^2\right) + \int_0^T \sigma\|\sqrt{\rho}\dot{u}\|_{L^2}^2\mathrm{d}t, \tag{2.1}$$

$$A_2(T) \triangleq \sup_{t\in[0,T]}\left(\sigma^3\|\sqrt{\rho}\dot{u}\|_{L^2}^2\right) + \int_0^T \sigma^3\|\nabla\dot{u}\|_{L^2}^2\mathrm{d}t, \tag{2.2}$$

and,

$$A_3(T) \triangleq \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2 \tag{2.3}$$

where

$$\sigma(t) \triangleq \min\{1,t\} \quad \text{and} \quad \dot{f} \triangleq f_t + u\cdot\nabla f.$$

We aim to prove the following key a priori estimates on $A_i(T)$ $(i=1,2,3)$ and the upper bound of the density.

Proposition 2.1 *Let the conditions of Theorem 1.1 hold. Then there exist two positive constants $\tilde{\varepsilon}$ and K, depending only on $\mu,\lambda,A,\gamma,\rho_\infty,\tilde{\rho},M,\beta(x)$ and $f(x)$, such that if (ρ,u) is a smooth solution of (1.1)–(1.7) on $\bar{\Omega}\times[0,T]$ satisfying*

$$\begin{cases} 0\le \rho(x,t)\le 2\tilde{\rho} \quad \text{for all} \quad (x,t)\in\bar{\Omega}\times[0,T], \\ A_1(T)+A_2(T)\le 2C_0^{1/2} \quad \text{and} \quad A_3(\sigma(T))\le 3K, \end{cases} \tag{2.4}$$

the following estimates hold:

$$\begin{cases} 0\le \rho(x,t)\le 7\tilde{\rho}/4 \quad \text{for all} \quad (x,t)\in\bar{\Omega}\times[0,T], \\ A_1(T)+A_2(T)\le C_0^{1/2} \quad \text{and} \quad A_3(\sigma(T))\le 2K, \end{cases} \tag{2.5}$$

provided

$$C_0\le\tilde{\varepsilon}. \tag{2.6}$$

Proof. Proposition 2.1 follows directly from Lemmas 2.4, 2.6 and 2.10 with K and $\tilde{\varepsilon}$ being the same ones chosen in Lemmas 2.4 and 2.10, respectively. □

For simplicity, throughout this subsection we denote by C or C_i $(i=1,2,\ldots)$ the generic positive constants which may depend on $\mu,\lambda,A,\gamma,\rho_\infty,\inf f(x),\|f\|_{H^2},\tilde{\rho}$, and M, but not on T. We also sometimes write $C(\alpha)$ to emphasize that C relies on α.

We begin with the following elementary energy estimate.

Lemma 2.1 *Let (ρ,u) be a smooth solution of (1.1)-(1.7) on $\bar{\Omega}\times[0,T]$. Then,*

$$\begin{aligned} &\sup_{0\le t\le T}\left(\frac{1}{2}\|\sqrt{\rho}u\|_{L^2}^2+\|G(\rho)\|_{L^1}\right)+\int_0^T\int_{\partial\Omega}\beta^{-1}|u|^2\mathrm{d}S\mathrm{d}t \\ &\quad+\int_0^T\left(\mu\|\nabla u\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}u\|_{L^2}^2\right)\mathrm{d}t\le C_0 \end{aligned} \tag{2.7}$$

Proof. Multiplying (1.2) by u in L^2 and integrating by parts, by virtue of (1.1), (1.6) and (1.12) we easily obtain (2.7). □

To estimate $A_1(T)$ and $A_2(T)$, we need the following preliminary estimates.

Lemma 2.2 *Let (ρ,u) with $0\le\rho(x,t)\le 2\tilde{\rho}$ be a smooth solution of (1.1)-(1.7) on $\bar{\Omega}\times[0,T]$. Then there is a positive constant $C(\tilde{\rho})$ such that*

$$\begin{aligned} A_1(T)\le{}& C(\tilde{\rho})C_0+C(\tilde{\rho})\int_0^T\sigma\|\nabla u\|_{L^3}^3\mathrm{d}t \\ &+C(\tilde{\rho})\int_0^T\int\sigma\left(|u|^2|\nabla u|+|u||\nabla u|^2\right)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} A_2(T)\le{}& C(\tilde{\rho})C_0+C(\tilde{\rho})A_1(T)+C(\tilde{\rho})\int_0^T\sigma^3\|\nabla u\|_{L^4}^4\mathrm{d}t \\ &+C(\tilde{\rho})\int_0^T\int\sigma^3\left(|u|^4+|\dot{u}||\nabla u||u|+|\dot{u}||\nabla u|^2\right)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{2.9}$$

Proof. In terms of (1.12), we rewrite (1.2) in the form:

$$\rho\dot{u}-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div}u+\nabla\left(P(\rho)-P\left(\rho_s\right)\right)=\left(\rho-\rho_s\right)\nabla f, \tag{2.10}$$

which, multiplied by $\sigma^m\dot{u}$ with $m\in\mathbb{Z}^+$ in L^2, yields

$$\begin{aligned}\sigma^m\int\rho|\dot{u}|^2\mathrm{d}x=&\mu\sigma^m\int\Delta u\cdot\dot{u}\mathrm{d}x+(\lambda+\mu)\sigma^m\int\dot{u}\cdot\nabla\mathrm{div}u\mathrm{d}x\\&-\sigma^m\int\dot{u}\cdot\nabla\left(P(\rho)-P(\rho_s)\right)\mathrm{d}x\\&+\sigma^m\int\left(\rho-\rho_s\right)\dot{u}\cdot\nabla f\mathrm{d}x\triangleq\sum_{i=1}^4 I_i.\end{aligned} \tag{2.11}$$

In view of (1.12), we deduce after integrating by parts that

$$\begin{aligned}I_1=&-\frac{\mu}{2}\left(\sigma^m\|\nabla u\|_{L^2}^2\right)_t+\frac{\mu}{2}m\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2-\mu\sigma^m\int_{\partial\Omega}\partial_3u^j\dot{u}^j\mathrm{d}S\\&-\mu\sigma^m\int\left(\partial_ku^j\partial_ku^i\partial_iu^j-\frac{1}{2}|\nabla u|^2(\mathrm{div}u)\right)\mathrm{d}x\\\le&-\frac{\mu}{2}\left(\sigma^m\|\nabla u\|_{L^2}^2\right)_t+\frac{\mu}{2}m\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2-\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\sigma^m\beta^{-1}|u|^2\mathrm{d}S\\&-\frac{\mu}{2}m\sigma^{m-1}\sigma'\int_{\partial\Omega}\beta^{-1}|u|^2\mathrm{d}S-\mu\sigma^m\int_{\partial\Omega}\beta^{-1}u^ju^i\partial_iu^j\mathrm{d}S+C\sigma^m\|\nabla u\|_{L^3}^3.\end{aligned} \tag{2.12}$$

In view of the following identity

$$\int_{\partial\Omega}h(x)\mathrm{d}S=\int_{\Omega\cap\{0\le x_3\le 1\}}\left(h(x)+(x_3-1)\partial_3h(x)\right)\mathrm{d}x, \tag{2.13}$$

we infer from integrating by parts that

$$\left|\int_{\partial\Omega}\beta^{-1}|u|^2\mathrm{d}S\right|\le C\int\left(|u|^2+|\nabla u|^2\right)\mathrm{d}x$$

and

$$\left|\int_{\partial\Omega}\beta^{-1}u^ju^i\partial_iu^j\mathrm{d}S\right|\le C\int\left(|u||\nabla u|^2+|u|^2|\nabla u|\right)\mathrm{d}x,$$

since $i,j\in\{1,2\}$ due to the fact that $u^3=0$ on $\partial\Omega$. This, together with (2.12), gives

$$\begin{aligned}I_1\le&-\frac{\mu}{2}\left(\sigma^m\|\nabla u\|_{L^2}^2\right)_t-\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\sigma^m\beta^{-1}|u|^2\mathrm{d}S+C\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2\\&+C\sigma^{m-1}\sigma'\|u\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^3}^3+C\sigma^m\int\left(|u||\nabla u|^2+|u|^2|\nabla u|\right)\mathrm{d}x.\end{aligned} \tag{2.14}$$

In a similar manner,

$$I_2\le-\frac{\mu+\lambda}{2}\left(\sigma^m\|\mathrm{div}u\|_{L^2}^2\right)_t+\frac{\mu+\lambda}{2}m\sigma^{m-1}\sigma'\|\mathrm{div}u\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^3}^3. \tag{2.15}$$

It follows from (1.1) that

$$P(\rho)_t+\mathrm{div}(P(\rho)u)+\left(\rho P'(\rho)-P(\rho)\right)\operatorname{div}u=0, \tag{2.16}$$

and hence, using the fact that $0 \le \rho \le 2\tilde{\rho}$, we find

$$
\begin{aligned}
I_3 &= \sigma^m \int (\mathrm{div} u_t\,(P(\rho) - P(\rho_s)) - (u \cdot \nabla u) \cdot \nabla (P(\rho) - P(\rho_s)))\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int \sigma^m \mathrm{div} u\,(P(\rho) - P(\rho_s))\,\mathrm{d}x - m\sigma^{m-1}\sigma' \int \mathrm{div} u\,(P(\rho) - P(\rho_s))\,\mathrm{d}x \\
&\quad + \sigma^m \int \left(\left(\rho P'(\rho) - P(\rho)\right) (\mathrm{div} u)^2 + P(\rho)\partial_i u^j \partial_j u^i + u \cdot \nabla u \cdot \nabla P(\rho_s) \right) \mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t} \int \sigma^m \mathrm{div} u\,(P(\rho) - P(\rho_s))\,\mathrm{d}x + C(\tilde{\rho})\|\nabla u\|_{L^2}^2 + Cm^2\sigma^{2(m-1)}\sigma' C_0,
\end{aligned}
\tag{2.17}
$$

where we have also used (2.7), (1.14) and the following Gagliardo-Nirenberg's inequality:

$$
\|v\|_{L^p} \le C\|v\|_{L^2}^{\frac{6-p}{2p}} \|\nabla v\|_{L^2}^{\frac{3p-6}{2p}}, \quad \forall\, v \in H^1 \quad \text{and} \quad 2 \le p \le 6. \tag{2.18}
$$

Thanks to (1.1), one has $\rho_t + \mathrm{div}\,((\rho - \rho_s)\,u) + \mathrm{div}\,(\rho_s u) = 0$. Thus, we have by (1.14), (2.7) and (2.18) that

$$
\begin{aligned}
I_4 \le& \frac{\mathrm{d}}{\mathrm{d}t} \int \sigma^m (\rho - \rho_s)\,u \cdot \nabla f \mathrm{d}x + Cm\sigma^{m-1}\sigma' \|\rho - \rho_s\|_{L^2} \|u\|_{L^6} \|\nabla f\|_{L^3} \\
&+ C(\tilde{\rho})\sigma^m \int \left(|u||\nabla u||\nabla f| + |\nabla \rho_s||u|^2|\nabla f| + |\rho - \rho_s||u|^2|\nabla^2 f| \right) \mathrm{d}x \\
\le& \frac{\mathrm{d}}{\mathrm{d}t} \int \sigma^m (\rho - \rho_s)\,u \cdot \nabla f \mathrm{d}x + C(\tilde{\rho})\|\nabla u\|_{L^2}^2 + C(\tilde{\rho})m^2\sigma^{2(m-1)}\sigma' C_0.
\end{aligned}
\tag{2.19}
$$

Using (2.7), (2.18) and (1.14), we observe that

$$
\begin{aligned}
\int |u|^2 \mathrm{d}x &\le \underline{\rho}^{-1} \int \rho_s |u|^2 \mathrm{d}x \le C \int \left(\rho|u|^2 + |\rho - \rho_s||u|^2 \right) \mathrm{d}x \\
&\le CC_0 + \|\rho - \rho_s\|_{L^2} \|u\|_{L^4}^2 \le CC_0 + C_0^{1/2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{3/2},
\end{aligned}
$$

so that

$$
\|u\|_{L^2}^2 \le CC_0 + CC_0^{2/3} \|\nabla u\|_{L^2}^2. \tag{2.20}
$$

Thus, inserting (2.14), (2.15), (2.17) and (2.19) into (2.11), integrating it over $(0, t)$, and taking (2.17) and (2.20) into account, we immediately obtain (2.8) by choosing $m = 1$.

Next, operating $\partial_t + \mathrm{div}(u\cdot)$ to $(2.10)^j$, multiplying it by $\sigma^m \dot{u}^j$, and integrating by parts over Ω, we obtain after summing them up that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\sigma^m}{2} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \right) - \frac{m}{2} \sigma^{m-1}\sigma' \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \\
&= \mu\sigma^m \int \dot{u}^j \left[\Delta u_t^j + \mathrm{div}\left(u \Delta u^j \right) \right] \mathrm{d}x \\
&\quad + (\lambda + \mu)\sigma^m \int \dot{u}^j \left[\partial_t \partial_j \mathrm{div} u + \mathrm{div}\,(u \partial_j \mathrm{div} u) \right] \mathrm{d}x \\
&\quad - \sigma^m \int \dot{u}^j \left[(\partial_j P(\rho))_t + \partial_k \left(u^k \partial_j (P(\rho) - P(\rho_s)) \right) \right] \mathrm{d}x \\
&\quad + \sigma^m \int \dot{u}^j \left[\rho_t \partial_j f + \partial_k \left(u^k (\rho - \rho_s)\, \partial_j f \right) \right] \mathrm{d}x \triangleq \sum_{i=1}^{4} II_i
\end{aligned}
\tag{2.21}
$$

In terms of (1.6) and (2.13), we deduce after integrating by parts that

$$
\begin{aligned}
II_1 &= -\mu\sigma^m \int \left(|\nabla\dot{u}|^2 - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j + \partial_i\dot{u}^j\partial_k u^k\partial_i u^j - \partial_k\dot{u}^j\partial_i u^j\partial_i u^k\right)\mathrm{d}x \\
&\quad - \mu\sigma^m \int_{\partial\Omega} \beta^{-1}\left(|\dot{u}|^2 - \dot{u}^j u^k\partial_k u^j - \partial_k\dot{u}^j u^k u^j\right)\mathrm{d}S \\
&\le -\mu\sigma^m\|\nabla\dot{u}\|_{L^2}^2 + C\sigma^m\|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^4}^2 - \mu\sigma^m\int_{\partial\Omega}\beta^{-1}|\dot{u}|^2\mathrm{d}S \\
&\quad + C\sigma^m\int\left(|u||\nabla u||\dot{u}| + |u||\nabla u||\nabla\dot{u}| + |\nabla u|^2|\dot{u}|\right)\mathrm{d}x.
\end{aligned}
\tag{2.22}
$$

Similarly,

$$
II_2 \le -(\mu+\lambda)\sigma^m\|\mathrm{div}\dot{u}\|_{L^2}^2 + \frac{\mu}{8}\sigma^m\|\nabla\dot{u}\|_{L^2}^2 + C\sigma^m\|\nabla u\|_{L^4}^4 \tag{2.23}
$$

Based on (1.6), (1.14) and (2.16) and integration by parts, we obtain

$$
\begin{aligned}
II_3 &= -\sigma^m\int\left[\mathrm{div}\dot{u}\left(\rho P'(\rho) - P(\rho)\right)\mathrm{div}u + (\mathrm{div}\dot{u})\mathrm{div}(P(\rho_s)u)\right]\mathrm{d}x \\
&\quad - \sigma^m\int\partial_k\dot{u}^j\partial_j u^k\left(P(\rho) - P(\rho_s)\right)\mathrm{d}x \\
&\le \frac{\mu}{8}\sigma^m\|\nabla\dot{u}\|_{L^2}^2 + C(\tilde{\rho})\sigma^m\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{2.24}
$$

and analogously,

$$
\begin{aligned}
II_4 &\le C(\tilde{\rho})\left(\|\nabla\dot{u}\|_{L^2} + \|\rho\dot{u}\|_{L^3}\right)\|u\|_{L^6}\|\nabla f\|_{H^1} \\
&\le \frac{\mu}{8}\sigma^m\|\nabla\dot{u}\|_{L^2}^2 + C(\tilde{\rho})\sigma^m\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).
\end{aligned}
\tag{2.25}
$$

Thus, substituting (2.21)-(2.25) into (2.21) and integrating it over $(0,t)$, we arrive at the desired estimate of (2.9), choosing $m=3$ and using the Cauchy-Schwarz's inequality. $\square$

In order to estimate the right-hand side of (2.8) and (2.9), we introduce the following modified "effective viscous flux" $\tilde{F}$ and vorticity $\tilde{\omega}$:

$$
\begin{cases}
\tilde{F} \triangleq \rho_s^{-1}\left[(2\mu+\lambda)\mathrm{div}u - (P(\rho) - P(\rho_s))\right], \\
\tilde{\omega} \triangleq \rho_s^{-1}\nabla\times u = \rho_s^{-1}\left(\partial_2 u^3 - \partial_3 u^2, \partial_3 u^1 - \partial_1 u^3, \partial_1 u^2 - \partial_2 u^1\right)^\top,
\end{cases}
\tag{2.26}
$$

where $\tilde{\omega} = (\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3)^\top$. As observed in [19, 20], one easily deduces from (1.12) that

$$
\begin{aligned}
&\rho_s^{-1}(\nabla P(\rho) - \rho\nabla f) \\
&\quad = \rho_s^{-1}\left[\nabla\left(P(\rho) - P(\rho_s)\right) - \rho_s^{-1}(\rho-\rho_s)\nabla P(\rho_s)\right] \\
&\quad = \nabla\left[\rho_s^{-1}\left(P(\rho) - P(\rho_s)\right)\right] + \rho_s^{-2}\left[P(\rho) - P(\rho_s) - P'(\rho_s)(\rho-\rho_s)\right]\nabla\rho_s,
\end{aligned}
$$

which, combined with (1.2), gives

$$
\rho_s^{-1}\rho\dot{u} - \nabla\tilde{F} + \mu\nabla\times\tilde{\omega} = G_1 + G_2, \tag{2.27}
$$

where

$$
\begin{cases}
G_1 \triangleq -\left[(2\mu+\lambda)(\mathrm{div}u)\nabla\rho_s^{-1} - \mu\nabla\rho_s^{-1}\times(\nabla\times u)\right], \\
G_2 \triangleq \left[P(\rho) - P(\rho_s) - P'(\rho_s)(\rho-\rho_s)\right]\nabla\rho_s^{-1}.
\end{cases}
$$

It follows from Proposition 1.1 that

$$\begin{cases} \|G_1\|_{L^2} \le C\|\nabla u\|_{L^3}\|\nabla \rho_s\|_{L^6} \le C\|\nabla u\|_{L^3}, \\ \|G_2\|_{L^2} \le C(\tilde\rho)\|\nabla \rho_s\|_{L^6}\|\rho-\rho_s\|_{L^6}^2 \le C(\tilde\rho)\|\rho-\rho_s\|_{L^6}^2, \end{cases} \tag{2.28}$$

where we have used the fact $0 \le \rho \le 2\tilde\rho$ to get that

$$\left|P(\rho)-P(\rho_s)-P'(\rho_s)(\rho-\rho_s)\right| \le C(\tilde\rho)|\rho-\rho_s|^2.$$

Operating div and curl to both sides of (2.27), we get

$$\begin{cases} \Delta \tilde F = \operatorname{div}\left(\rho_s^{-1}\rho\dot u - G_1 - G_2\right), \\ \mu\Delta\tilde\omega = \mu\nabla\left((\nabla\times u)\cdot\nabla\rho_s^{-1}\right) + \nabla\times\left(\rho_s^{-1}\rho\dot u - G_1 - G_2\right), \end{cases} \tag{2.29}$$

and moreover, by (1.6) we find

$$\begin{cases} \tilde\omega^1 = \rho_s^{-1}\left(\partial_2 u^3 - \partial_3 u^2\right) = -\rho_s^{-1}\partial_3 u^2 = -(\beta\rho_s(x))^{-1}u^2, \quad x\in\partial\Omega, \\ \tilde\omega^2 = \rho_s^{-1}\left(\partial_3 u^1 - \partial_1 u^3\right) = \rho_s^{-1}\partial_3 u^1 = (\beta\rho_s(x))^{-1}u^1, \quad x\in\partial\Omega. \end{cases} \tag{2.30}$$

Based on the standard elliptic theory, we infer from (2.28)–(2.30) that

Lemma 2.3 *Let (ρ, u) with $0 \le \rho(x,t) \le 2\tilde\rho$ be a smooth solution of* (1.1)-(1.7) *on $\bar\Omega\times[0,T]$. Then, for $\tilde F$ and $\tilde\omega$ be the ones defined in* (2.26)*, one has*

$$\|\nabla\tilde F\|_{L^2} + \|\nabla\tilde\omega\|_{L^2} \le C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho-\rho_s\|_{L^6}^2\right). \tag{2.31}$$

Proof. Let

$$H^1 \triangleq \tilde\omega^1 + (\beta\rho_s(x))^{-1}u^2, \quad H^2 \triangleq \tilde\omega^2 - (\beta\rho_s(x))^{-1}u^1. \tag{2.32}$$

It is clear that $H^i = 0$ $(i=1,2)$ on $\partial\Omega$. Using (2.18), (2.28) and Proposition 1.1, we infer from $(2.29)_2$ that

$$\begin{aligned} \|\nabla H^i\|_{L^2} &\le C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \|\nabla\rho_s\|_{L^3}\|u\|_{L^6} + \|\nabla u\|_{L^2} + \|G_1\|_{L^2} + \|G_2\|_{L^2}\right) \\ &\le C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho-\rho_s\|_{L^6}^2\right), \end{aligned}$$

and hence, by (2.32) we easily get that

$$\|\nabla\tilde\omega^i\|_{L^2} \le C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho-\rho_s\|_{L^6}^2\right), \quad i=1,2. \tag{2.33}$$

To estimate $\|\nabla\tilde\omega^3\|_{L^2}$, we first observe from direct calculations that

$$\begin{aligned} \partial_3\tilde\omega^3 &= \rho_s^{-1}\partial_1\left(\partial_3 u^2 - \partial_2 u^3\right) - \rho_s^{-1}\partial_2\left(\partial_3 u^1 - \partial_1 u^3\right) + \partial_3\rho_s^{-1}\left(\partial_1 u^2 - \partial_2 u^1\right) \\ &= -\partial_1\tilde\omega^1 - \partial_2\tilde\omega^2 + \nabla\rho_s^{-1}\cdot(\nabla\times u), \end{aligned} \tag{2.34}$$

which, combined with (2.33) and Proposition 1.1, results in

$$\begin{aligned} \|\partial_3\tilde\omega^3\|_{L^2} &\le C\left(\|\nabla\tilde\omega^1\|_{L^2} + \|\nabla\tilde\omega^2\|_{L^2} + \|\nabla\rho_s\|_{L^6}\|\nabla\times u\|_{L^3}\right) \\ &\le C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho-\rho_s\|_{L^6}^2\right). \end{aligned} \tag{2.35}$$

Moreover, substituting (2.34) into the third equation of $(2.29)_2$, we obtain

$$\begin{aligned} \mu\left(\partial_{11}^2 + \partial_{22}^2\right)\tilde\omega^3 &= \partial_1\left(\rho_s^{-1}\rho\dot u^2 - G_1^2 - G_2^2\right) - \partial_2\left(\rho_s^{-1}\rho\dot u^1 - G_1^2 - G_2^1\right) \\ &\quad + \mu\partial_3\left((\nabla\times u)\cdot\nabla\rho_s^{-1}\right) - \mu\partial_{33}^2\tilde\omega^3 \\ &= \partial_1\left(\rho_s^{-1}\rho\dot u^2 - G_1^2 - G_2^2\right) - \partial_2\left(\rho_s^{-1}\rho\dot u^1 - G_1^2 - G_2^1\right) \\ &\quad + \mu\partial_{13}^2\tilde\omega^1 + \mu\partial_{23}^2\tilde\omega^2, \end{aligned} \tag{2.36}$$

where f^i denotes the i-th component of $f \in \mathbb{R}^3$. So, it follows from (2.36) that for fixed $x_3 \in \mathbb{R}_+$,

$$\|\partial_i \tilde{\omega}^3(x_3)\|_{L^2(\mathbb{R}^2)}^2 \le C(\tilde{\rho}) \left(\|\sqrt{\rho}\dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \|G_1\|_{L^2(\mathbb{R}^2)}^2 + \|G_2\|_{L^2(\mathbb{R}^2)}^2 \right) + C \left(\|\nabla \tilde{\omega}^1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \tilde{\omega}^2\|_{L^2(\mathbb{R}^2)}^2 \right),$$

which, integrated with respect to x_3 over $\mathbb{R}_+$ and combined with (2.28) and (2.33), yields

$$\|\partial_i \tilde{\omega}^3\|_{L^2} \le C(\tilde{\rho}) \left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^6}^2 \right), \quad i = 1, 2. \tag{2.37}$$

Thus, collecting (2.33), (2.35) and (2.37) together gives

$$\|\nabla \tilde{\omega}^3\|_{L^2} \le C(\tilde{\rho}) \left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^6}^2 \right),$$

which, combined with (2.27) and (2.28), also proves the desired estimate of $\|\nabla \tilde{F}\|_{L^2}$. The proof of (2.31) is therefore complete. $\square$

The following lemma is concerned with the short-time estimate of $A_3(\sigma(T))$.

Lemma 2.4 *Let (ρ, u) with $0 \le \rho(x,t) \le 2\tilde{\rho}$ be a smooth solution of (1.1)–(1.7) on $\bar{\Omega} \times [0,T]$. Then there exist positive constants K and $\varepsilon_0 > 0$, depending on $\tilde{\rho}$ and M, such that*

$$A_3(\sigma(T)) + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \mathrm{d}t \le 2K, \tag{2.38}$$

provided $A_3(\sigma(T)) \le 3K$ and $C_0 \le \varepsilon_0$.

Proof. Indeed, choosing $m = 0$ in (2.11) and integrating it over $(0, \sigma(T))$, we deduce from (2.14), (2.15), (2.17) and (2.18) that

$$\begin{aligned} & A_3(\sigma(T)) + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \mathrm{d}t \\ & \le C(\tilde{\rho})(C_0 + M) + C(\tilde{\rho}) \int_0^{\sigma(T)} \int \left(|\nabla u|^3 + |u||\nabla u|^2 + |u|^2 |\nabla u| \right) \mathrm{d}x\mathrm{d}t \\ & \le C(\tilde{\rho})(C_0 + M) + C(\tilde{\rho}) \int_0^{\sigma(T)} \left(\|\nabla u\|_{L^3}^3 + \|u\|_{L^3}^3 \right) \mathrm{d}t \end{aligned} \tag{2.39}$$

where we have also used (2.7), (2.20), Proposition 1.1 and the Cauchy-Schwarz's inequality.

In order to deal with $\|\nabla u\|_{L^3}^3$, we first infer from the identity $\nabla \times (\nabla \times u) = \nabla \mathrm{div} u - \Delta u$ and (2.26) that

$$\begin{aligned} (2\mu + \lambda)\Delta u &= (2\mu + \lambda)\nabla \mathrm{div} u - (2\mu + \lambda)\nabla \times (\nabla \times u) \\ &= \nabla(\rho_s \tilde{F}) + \nabla(P(\rho) - P(\rho_s)) - (2\mu + \lambda)\nabla \times (\rho_s \tilde{\omega}), \end{aligned}$$

subject to the boundary conditions $\partial_3 u^i = \beta^{-1} u^i$ with $i = 1, 2$, and $u^3 = 0$ for $x \in \partial\Omega$. Thus, it follows from the standard elliptic theory that for any $p > 1$

$$\|\nabla u\|_{L^p} \le C(p) \left(\|\tilde{F}\|_{L^p} + \|\tilde{\omega}\|_{L^p} + \|P(\rho) - P(\rho_s)\|_{L^p} + \|u\|_{L^p} \right). \tag{2.40}$$

Thus, using (2.7), (2.28), (2.31) and (2.40) with $p = 6$, we have

$$\begin{aligned} \|\nabla u\|_{L^3}^3 &\le C\|\nabla u\|_{L^2}^{3/2} \left(\|\nabla \tilde{F}\|_{L^2}^{3/2} + \|\nabla \tilde{\omega}\|_{L^2}^{3/2} + \|P(\rho) - P(\rho_s)\|_{L^6}^{3/2} + \|\nabla u\|_{L^2}^{3/2} \right) \\ &\le C\|\nabla u\|_{L^2}^{3/2} \left(C_0^{1/4} + \|\sqrt{\rho}\dot{u}\|_{L^2}^{3/2} + \|\nabla u\|_{L^3}^{3/2} + \|\nabla u\|_{L^2}^{3/2} \right) \\ &\le \frac{1}{2}\|\nabla u\|_{L^3}^3 + C\|\nabla u\|_{L^2}^{3/2} \|\sqrt{\rho}\dot{u}\|_{L^2}^{3/2} + C\|\nabla u\|_{L^2}^{3/2} \left(C_0^{1/4} + \|\nabla u\|_{L^2}^{3/2} \right), \end{aligned} \tag{2.41}$$

which, inserted into (2.39) and combined with the Cacuhy-Schwarz's inequality, yields

$$
\begin{aligned}
& A_3(\sigma(T)) + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \mathrm{d}t \\
& \le C(\tilde{\rho}, M) + C(\tilde{\rho}) C_0 [A_3(\sigma(T))]^{1/2} + C(\tilde{\rho}) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^6 \mathrm{d}t \\
& \le K + C(\tilde{\rho}) C_0 [A_3(\sigma(T))]^2,
\end{aligned}
$$

with $K \triangleq C(\tilde{\rho}, M)$. Here, we have also used (2.7) and (2.20) to get that

$$
\int_0^{\sigma(T)} \|u\|_{L^3}^3 \mathrm{d}t \le C + C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^3 \mathrm{d}t \le C + C C_0 [A_3(\sigma(T))]^{1/2}.
$$

So, if $A_3(\sigma(T)) \le 3K$ and $C_0 \le \varepsilon_0 \triangleq \min\{1, (9K)^{-1}\}$, then (2.38) follows. $\square$

Based on Lemma 2.3, we have the following important estimates, which will be used to deal with $A_1(T)$ and $A_2(T)$.

Lemma 2.5 *Let (ρ, u) with $0 \le \rho(x,t) \le 2\tilde{\rho}$ be a smooth solution of (1.1)–(1.7) on $\bar{\Omega} \times [0,T]$. Then there exists a positive constant $\varepsilon_1 > 0$, depending on $\tilde{\rho}$, such that*

$$
\int_0^T \sigma^3 \left(\|\nabla u\|_{L^4}^4 + \|\rho - \rho_s\|_{L^4}^4 + \|\tilde{F}\|_{L^4}^4 + \|\tilde{\omega}\|_{L^4}^4 \right) dt \le C C_0. \tag{2.42}
$$

provided $A_1(T) + A_2(T) \le 2C_0^{1/2}$ and $C_0 \le \varepsilon_1$.

Proof. In terms of $\tilde{F}$ in (2.26), we can rewrite (1.1) as

$$
(\rho - \rho_s)_t + \frac{\rho_s}{2\mu + \lambda} (P(\rho) - P(\rho_s)) = -\mathrm{div}\,(u(\rho - \rho_s)) - u \cdot \nabla \rho_s - \frac{\rho_s^2 \tilde{F}}{2\mu + \lambda},
$$

which, multiplied by $4(\rho - \rho_s)^3$ and integrated by parts over Ω, gives

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \int (\rho - \rho_s)^4 \mathrm{d}x + \frac{4}{2\mu + \lambda} \int \rho_s (P(\rho) - P(\rho_s)) (\rho - \rho_s)^3 \mathrm{d}x \\
& \le C \int \left((\rho - \rho_s)^4 |\nabla u| + |\rho - \rho_s|^3 |u| |\nabla \rho_s| + |\rho - \rho_s|^3 |\tilde{F}| \right) \mathrm{d}x \\
& \le \delta \|\rho - \rho_s\|_{L^4}^4 + C(\delta, \tilde{\rho}) \left(\|\nabla u\|_{L^2}^2 + \|\tilde{F}\|_{L^4}^4 \right), \quad \delta > 0,
\end{aligned}
$$

where we have used (1.14) and (2.18). Noting that

$$
\rho_s (P(\rho) - P(\rho_s)) (\rho - \rho_s)^3 \ge C(\rho - \rho_s)^4,
$$

and choosing $\delta > 0$ suitably small, by (2.7) we deduce

$$
\int_0^T \sigma^3 \|\rho - \rho_s\|_{L^4}^4 dt \le C C_0 + C \int_0^T \sigma^3 \|\tilde{F}\|_{L^4}^4 \mathrm{d}t. \tag{2.43}
$$

Since $A_1(T) \le 2C_0^{1/2}$, by (2.7), (2.20) and (2.28) we have

$$
\int_0^T \sigma \|u\|_{L^4}^4 \mathrm{d}t \le C \sup_{0 \le t \le T} (\sigma \|u\|_{L^2} \|\nabla u\|_{L^2}) \int_0^T \|\nabla u\|_{L^2}^2 \mathrm{d}t \le C C_0. \tag{2.44}
$$

It is easy to see that

$$\|\tilde{F}\|_{L^2}+\|\tilde{\omega}\|_{L^2}\le C\left(\|\nabla u\|_{L^2}+C_0^{1/2}\right),$$

and hence, using (2.7), (2.28), (2.31) and (2.44), we find

$$\begin{aligned}
&\int_0^T\sigma^3\left(\|\tilde{F}\|_{L^4}^4+\|\tilde{\omega}\|_{L^4}^4\right)\mathrm{d}t\\
&\quad\le C\int_0^T\sigma^3\left(\|\tilde{F}\|_{L^2}\|\nabla\tilde{F}\|_{L^2}^3+\|\tilde{\omega}\|_{L^2}\|\nabla\tilde{\omega}\|_{L^2}^3\right)\mathrm{d}t\\
&\quad\le C\int_0^T\sigma^3\left(\|\nabla u\|_{L^2}+C_0^{1/2}\right)\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^3+\|\nabla u\|_{L^3}^3+\|\nabla u\|_{L^2}^3+\|\rho-\rho_s\|_{L^6}^6\right)\mathrm{d}t\\
&\quad\le C\int_0^T\left(\sigma^{1/2}\|\nabla u\|_{L^2}+C_0^{1/2}\right)\left(\sigma^{3/2}\|\sqrt{\rho}\dot{u}\|_{L^2}\right)\sigma\|\sqrt{\rho}\dot{u}\|_{L^2}^2\mathrm{d}t\\
&\qquad+C\int_0^T\left(\sigma^{1/2}\|\nabla u\|_{L^2}+C_0^{1/2}\right)\|\nabla u\|_{L^2}\left(\sigma^{3/2}\|\nabla u\|_{L^4}^2\right)\mathrm{d}t\\
&\qquad+C\int_0^T\left[\left(\sigma\|\nabla u\|_{L^2}^2\right)\|\nabla u\|_{L^2}^2+C_0^{1/2}\|\nabla u\|_{L^2}\left(\sigma^{3/2}\|\rho-\rho_s\|_{L^4}^2\right)\right]\mathrm{d}t\\
&\qquad+CC_0^{1/2}\int_0^T\left[\left(\sigma^{1/2}\|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}^2+\|\rho-\rho_s\|_{L^4}^4\right]\mathrm{d}t\\
&\quad\le CC_0+CC_0^{1/2}\int\sigma^3\left(\|\nabla u\|_{L^4}^4+\|\rho-\rho_s\|_{L^4}^4\right)\mathrm{d}t,
\end{aligned}\tag{2.45}$$

where we have also used the fact that $A_1(T)+A_2(T)\le 2C_0^{1/2}$ and the following inequality:

$$\|\rho-\rho_s\|_{L^6}^6\le C\|\rho-\rho_s\|_{L^4}^4\le C\|\rho-\rho_s\|_{L^3}^3\le CC_0^{1/2}\|\rho-\rho_s\|_{L^4}^2.$$

Now, by choosing $p=4$ in (2.40), we conclude from (2.43)–(2.45) that

$$\begin{aligned}
&\int_0^T\sigma^3\left(\|\nabla u\|_{L^4}^4+\|\tilde{F}\|_{L^4}^4+\|\tilde{\omega}\|_{L^4}^4+\|\rho-\rho_s\|_{L^4}^4\right)\mathrm{d}t\\
&\quad\le CC_0+CC_0^{1/2}\int\sigma^3\left(\|\nabla u\|_{L^4}^4+\|\rho-\rho_s\|_{L^4}^4\right)\mathrm{d}t,
\end{aligned}$$

and thus, if C_0 is chosen to be small enough such that $C_0\le\varepsilon_1\triangleq\min\{\varepsilon_0,(2C)^{-2}\}$, then one immediately obtains (2.42). □

We are now in a position of closing the estimates of $A_1(T)$ and $A_2(T)$.

Lemma 2.6 *Assume that* (ρ,u) *is a smooth solution of* (1.1)–(1.7) *on* $\bar{\Omega}\times[0,T]$*, satisfying* (2.4) *with* $K>0$ *being the same one determined in* (2.38)*. Then there exists a positive constant* $\varepsilon_2>0$*, depending on* $\tilde{\rho}$*, such that*

$$A_1(T)+A_2(T)\le C_0^{1/2},\tag{2.46}$$

provided $C_0\le\varepsilon_2$*.*

Proof. By virtue of (2.4), (2.7), (2.42) and (2.44), we infer from Lemma 2.2 that

$$\begin{aligned}
A_1(T)+A_2(T) \le\, & C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^T \sigma\|\nabla u\|_{L^3}^3 + \int_0^T\int \sigma|\nabla u|^2|u|\mathrm{d}x\mathrm{d}t\\
& + C(\tilde\rho)\int_0^T\int \sigma^3\left(|u||\nabla u||\dot u| + |\nabla u|^2|\dot u|\right)\mathrm{d}x\mathrm{d}t\\
\le\, & C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^T \sigma\left(\|\nabla u\|_{L^3}^3 + \|\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}\right)\mathrm{d}t\\
& + C(\tilde\rho)\int_0^T \sigma^3\left(\|u\|_{L^3} + \|\nabla u\|_{L^3}\right)\|\nabla u\|_{L^2}\|\dot u\|_{L^6}\mathrm{d}t\\
\le\, & C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^T \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t + \frac12 A_2(T),
\end{aligned}$$

and hence,

$$A_1(T)+A_2(T) \le C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^T \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t. \tag{2.47}$$

Here, we have also used (2.4), (2.18) and (2.20) to obgtain

$$\sigma\|u\|_{L^3}^2 \le C\sigma\|\nabla u\|_{L^2}\left(C_0^{1/2} + CC_0^{1/3}\|\nabla u\|_{L^2}\right) \le CC_0^{1/2}, \quad \forall\, t\in[0,T].$$

Clearly, it remains to deal with $\|\nabla u\|_{L^3}$. To do so, we first infer from (2.7) and (2.42) that

$$\int_{\sigma(T)}^T \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t \le \int_{\sigma(T)}^T\left(\|\nabla u\|_{L^4}^4 + \|\nabla u\|_{L^2}^2\right)\mathrm{d}t \le C(\tilde\rho)C_0,$$

and thus, using (2.4), (2.7), (2.38) and (2.41), we find

$$\begin{aligned}
\int_0^T \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t \le\, & \int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t + \int_{\sigma(T)}^T \sigma\|\nabla u\|_{L^3}^3\mathrm{d}t\\
\le\, & C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^2}^{3/2}\left(C_0^{1/4} + \|\nabla u\|_{L^2}^{3/2} + \|\sqrt\rho\dot u\|_{L^2}^{3/2}\right)\mathrm{d}t\\
\le\, & C(\tilde\rho)C_0 + C(\tilde\rho)\int_0^{\sigma(T)}\left(\sigma\|\nabla u\|_{L^2}^2\right)^{1/4}\|\nabla u\|_{L^2}\left(\sigma\|\sqrt\rho\dot u\|_{L^2}^2\right)^{3/4}\mathrm{d}t\\
\le\, & C(\tilde\rho,M)C_0 + C(\tilde\rho,M)C_0^{1/8}\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^{1/2}\left(\sigma\|\sqrt\rho\dot u\|_{L^2}^2\right)^{3/4}\mathrm{d}t\\
\le\, & C(\tilde\rho,M)C_0^{3/4},
\end{aligned} \tag{2.48}$$

which, inserted into (2.47), yields

$$A_1(T)+A_2(T) \le C(\tilde\rho,M)C_0^{3/4} \le C_0^{1/2},$$

provided $C_0 \le \varepsilon_2 \triangleq \min\{\varepsilon_1, (C(\tilde\rho,M))^{-4}\}$. This finishes the proof of (2.46). □

To derive the uniform upper bound of density, we need the following refined estimate.

Lemma 2.7 *Assume that* (ρ,u) *is a smooth solution of* (1.1)–(1.7) *on* $\bar\Omega\times[0,T]$*, satisfying* (2.4) *with* $K>0$ *being the same one determined in* (2.38)*. Then,*

$$\sup_{0\le t\le T}\left(\sigma\|\sqrt\rho\dot u\|_{L^2}^2\right) + \int_0^T \sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t \le C(\tilde\rho,M), \tag{2.49}$$

provided $C_0 \le \varepsilon_2$.

Proof. First, it is easily deduced from (2.38) and (2.46) that

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2+\int_0^T\|\sqrt{\rho}\dot u\|_{L^2}^2\mathrm{d}t\le C(\tilde\rho,M). \tag{2.50}$$

Next, choosing $m=1$ in (2.21), integrating it over $(0,T)$, using (2.7), (2.18), (2.44) and (2.50), we deduce from (2.22)–(2.25) that

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\right)+\int_0^T\sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho)\int_0^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t+C(\tilde\rho)\int_0^T\int\sigma\left(|u||\nabla u||\dot u|+|\nabla u|^2|\dot u|\right)\mathrm{d}x\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho)\int_0^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t+C(\tilde\rho)\int_0^T\sigma\left(\|u\|_{L^3}+\|\nabla u\|_{L^3}\right)\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho,M)\int_0^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t+\frac12\int_0^T\sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t,
\end{aligned}$$

where we have used the fact that $\sigma(t)\|u(t)\|_{L^3}\le C$ for any $0\le t\le T$ (see (2.47)). Thus,

$$\sup_{0\le t\le T}\left(\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\right)+\int_0^T\sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t\le C(\tilde\rho,M)+C(\tilde\rho,M)\int_0^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t. \tag{2.51}$$

Using (2.7), (2.18), (2.31), (2.40) with $p=6$, (2.42), (2.48) and (2.50), we have

$$\begin{aligned}
\int_0^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t&\le\int_0^{\sigma(T)}\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t+\int_{\sigma(T)}^T\sigma\|\nabla u\|_{L^4}^4\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho)\int_0^{\sigma(T)}\sigma\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho)\int_0^{\sigma(T)}\sigma\|\nabla u\|_{L^2}\|\sqrt{\rho}\dot u\|_{L^2}^3\mathrm{d}t\\
&\le C(\tilde\rho,M)+C(\tilde\rho,M)\sup_{0\le t\le T}\left(\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\right)^{1/2},
\end{aligned}$$

which, inserted into (2.51) and combined with the Cauchy-Schwarz's inequality, finishes the proof of (2.49). □

To be continued, as that in [8], we introduce the standard "effective viscous flux" F and vorticity ω as follows:

$$F\triangleq(2\mu+\lambda)\mathrm{div}u-(P(\rho)-P(\rho_s)),\quad \omega\triangleq\nabla\times u, \tag{2.52}$$

which, together with $(1.1)_2$ and (1.12), yields

$$\nabla F-\mu\nabla\times\omega=\rho\dot u-(\rho-\rho_s)\nabla f \tag{2.53}$$

and

$$\begin{cases}\Delta F=\mathrm{div}(\rho\dot u-(\rho-\rho_s)\nabla f),\\ \mu\Delta\omega=\nabla\times(\rho\dot u-(\rho-\rho_s)\nabla f).\end{cases} \tag{2.54}$$

Moreover, it is clear that

$$\begin{cases}\omega^1=\partial_2u^3-\partial_3u^2=-\beta^{-1}u^2,\quad x\in\partial\Omega,\\ \omega^2=\partial_3u^1-\partial_1u^3=\beta^{-1}u^1,\quad x\in\partial\Omega.\end{cases} \tag{2.55}$$

Thus, similarly to the derivation of Lemma 2.3, by virtue of the elliptic theory we infer from (2.52)–(2.55) that

Lemma 2.8 *Let (ρ, u) with $0 \le \rho(x,t) \le 2\tilde{\rho}$ be a smooth solution of* (1.1)-(1.7) *on $\bar{\Omega} \times [0,T]$. Then, for F and ω be the ones defined in* (2.52)*, one has*

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \le C(\tilde{\rho}) \left(\|\sqrt{\rho}\dot{u}\|_{L^p} + \|(\rho - \rho_s)\nabla f\|_{L^p} + \|u\|_{L^p} \right). \tag{2.56}$$

To be continued, we recall the following Zlotnik's inequality, which is useful for the proof of the upper bound of density.

Lemma 2.9 ([34]) *Assume that $y \in W^{1,1}(0,T)$ solves the ODE system:*

$$y' = g(y) + b'(t) \quad on \quad [0,T], \quad y(0) = y_0,$$

where $b \in W^{1,1}(0,T)$ and $g \in C(\mathbb{R})$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.57}$$

for all $0 \le t_1 \le t_2 \le T$ with some positive constants N_0 and N_1, then one has

$$y(t) \le \max\{y_0, \xi^*\} + N_0 < +\infty \quad on \quad [0,T], \tag{2.58}$$

where $\xi^* \in \mathbb{R}$ is a constant such that

$$g(\xi) \le -N_1 \quad for \quad \xi \ge \xi^*. \tag{2.59}$$

With the help of Lemmas 2.8 and 2.9, we are now ready to derive the t-independent upper bound of density.

Lemma 2.10 *Assume that (ρ, u) is a smooth solution of* (1.1)–(1.7) *on $\bar{\Omega} \times [0,T]$, satisfying* (2.4) *with $K > 0$ being the same one as in* (2.38)*. Then there exists a positive constant ε, depending on $\tilde{\rho}$ and M, such that*

$$\sup_{0 \le t \le T} \|\rho(t)\|_{L^\infty} \le \frac{7\tilde{\rho}}{4}, \tag{2.60}$$

provided $C_0 \le \tilde{\varepsilon}$.

Proof. Due to (2.52), one has

$$\mathrm{div} u = (2\mu + \lambda)^{-1} \left(\rho F + P(\rho) - P(\rho_s) \right),$$

which, together with (1.1), implies

$$D_t \rho = g(\rho) + b'(t),$$

where $D_t \rho \triangleq \rho_t + u \cdot \nabla \rho$ denotes the material derivative,

$$g(\rho) \triangleq -\frac{A\rho}{2\mu + \lambda} \left(\rho^\gamma - \rho_s^\gamma \right), \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho F \mathrm{d}t$$

To apply the Zlotnik's inequality (cf. Lemma 2.9), we need to deal with $b(t)$. To do so, we first observe from (2.18), the Sobolev embedding inequality and Lemma 2.8 that

$$
\begin{aligned}
\|F\|_{L^\infty} &\le C\left(\|F\|_{L^4}+\|\nabla F\|_{L^4}\right) \le C\|F\|_{L^2}^{1/4}\|\nabla F\|_{L^2}^{3/4}+C\|\nabla F\|_{L^4}\\
&\le C(\tilde\rho)\left(\|\nabla u\|_{L^2}+\|\rho-\rho_s\|_{L^2}\right)^{1/4}\left(\|\sqrt{\rho}\dot u\|_{L^2}+\|(\rho-\rho_s)\nabla f\|_{L^2}+\|u\|_{L^2}\right)^{3/4}\\
&\quad +C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^4}+\|(\rho-\rho_s)\nabla f\|_{L^4}+\|u\|_{L^4}\right)\\
&\le C(\tilde\rho)\left(\|\nabla u\|_{L^2}^{1/4}+C_0^{1/8}\right)\left(\|\sqrt{\rho}\dot u\|_{L^2}^{3/4}+1\right)\\
&\quad +C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2}^{1/4}\|\nabla\dot u\|_{L^2}^{3/4}+\|\rho-\rho_s\|_{L^{12}}\|\nabla f\|_{L^6}+\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4}\right)\\
&\le C(\tilde\rho)\left(\|\nabla u\|_{L^2}^{1/4}+C_0^{1/8}\right)\left(\|\sqrt{\rho}\dot u\|_{L^2}^{3/4}+1\right)\\
&\quad +C(\tilde\rho)\left(\|\sqrt{\rho}\dot u\|_{L^2}^{1/4}\|\nabla\dot u\|_{L^2}^{3/4}+C_0^{1/12}+\|\nabla u\|_{L^2}^{3/4}\right),
\end{aligned}
\tag{2.61}
$$

where we have also used (2.7), (2.20) and (2.50). So, using (2.4), (2.7), (2.49) and (2.50), we have from (2.61) that for $0\le t_1<t_2\le\sigma(T)\le 1$,

$$
\begin{aligned}
|b(t_2)-b(t_1)| &\le C(\tilde\rho)\int_0^{\sigma(T)}\|F\|_{L^\infty}\mathrm{d}t\\
&\le C(\tilde\rho)C_0^{1/12}+C(\tilde\rho)\left(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2\mathrm{d}t\right)^{1/8}\left(\int_0^{\sigma(T)}\|\sqrt{\rho}\dot u\|_{L^2}^2\mathrm{d}t\right)^{3/8}\\
&\quad +C(\tilde\rho)\int_0^{\sigma(T)}\left(\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\right)^{1/8}\left(\sigma\|\nabla\dot u\|_{L^2}^2\right)^{3/8}\sigma^{-1/2}\mathrm{d}t\\
&\le C(\tilde\rho)C_0^{1/12}+C(\tilde\rho)\sup_{0\le t\le\sigma(T)}\left(\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\right)^{1/16}\left(\int_0^{\sigma(T)}\sigma\|\sqrt{\rho}\dot u\|_{L^2}^2\mathrm{d}t\right)^{1/16}\\
&\qquad\qquad\times\left(\int_0^{\sigma(T)}\sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t\right)^{3/8}\left(\int_0^{\sigma(T)}\sigma^{-8/9}\mathrm{d}t\right)^{9/16}\\
&\le C(\tilde\rho)C_0^{1/12}+C(\tilde\rho)C_0^{1/32}\le C(\tilde\rho)C_0^{1/32}
\end{aligned}
$$

Thus, for any $t\in[0,\sigma(T)]$, one can choose N_0,N_1 in (2.57) and ξ^* in (2.59) as follows:

$$N_0=C(\tilde\rho)C_0^{1/32},\quad N_1=0,\quad \xi^*=\bar\rho.$$

Noting that $\underline{\rho}\le\rho_s\le\bar\rho$ (see Proposition 1.1) and

$$g(\xi)\le-\frac{A\xi}{\lambda+2\mu}\left(\xi^\gamma-\bar\rho^\gamma\right)\le -N_1=0,\quad \forall\,\xi\ge\xi^*=\bar\rho,$$

we infers from (2.58) that (keeping in mind that $0\le\rho_0\le\tilde\rho$ and $\tilde\rho\ge\bar\rho+1$)

$$\sup_{0\le t\le\sigma(T)}\|\rho(t)\|_{L^\infty}\le\max\{\tilde\rho,\bar\rho\}+N_0\le\tilde\rho+C(\tilde\rho)C_0^{1/32}\le\frac{3}{2}\tilde\rho, \tag{2.62}$$

provided C_0 is chosen to be such that

$$C_0\le\min\{\varepsilon_2,\varepsilon_3\}\quad\text{with}\quad \varepsilon_3\triangleq\left(\frac{\tilde\rho}{2C(\tilde\rho)}\right)^{32}.$$

For any $\sigma(T) \le t_1 < t_2 \le T$, by (2.4) and (2.7)we have from (2.61) and the Cauchy-Schwarz's inequality that

$$
\begin{aligned}
|b(t_2) - b(t_1)| &\le \left(C(\tilde\rho) C_0^{1/12} + \frac{A}{2(2\mu+\lambda)} \right) (t_2 - t_1) \\
&\quad + C(\tilde\rho) \int_{\sigma(T)}^T \left(\sigma \|\sqrt{\rho}\dot u\|_{L^2}^2 + \sigma^3 \|\nabla \dot u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \mathrm{d}t \\
&\le \frac{A}{2\mu+\lambda}(t_2 - t_1) + C(\tilde\rho) C_0^{1/2},
\end{aligned} \tag{2.63}
$$

where in the last inequality we have chosen C_0 to be such that

$$
C_0 \le \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\} \quad \text{with} \quad \varepsilon_4 \triangleq \left(\frac{A}{2C(\tilde\rho)(2\mu+\lambda)} \right)^{12}.
$$

Thus, for any $t \in [\sigma(T), T]$, we can choose N_0 and N_1 in (2.57) and ξ^* in (2.59) as follows:

$$
N_0 = C(\tilde\rho) C_0^{1/2}, \quad N_1 = \frac{A}{2\mu+\lambda}, \quad \xi^* = \bar\rho + 1.
$$

It is easily seen that for any $\xi \ge \xi^*$,

$$
g(\xi) \le -\frac{A\xi}{2\mu+\lambda}(\xi^\gamma - \bar\rho^\gamma) \le -N_1 = -\frac{A}{2\mu+\lambda},
$$

so that, it follows from (2.58), (2.62) and (2.63) that

$$
\sup_{\sigma(T) \le t \le T} \|\rho(t)\|_{L^\infty} \le \max\left\{ \frac{3}{2}\tilde\rho, \bar\rho + 1 \right\} + N_0 \le \frac{3}{2}\tilde\rho + C(\tilde\rho) C_0^{1/2} \le \frac{7}{4}\tilde\rho, \tag{2.64}
$$

provided the initial energy C_0 satisfies

$$
C_0 \le \tilde\varepsilon \triangleq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} \quad \text{with} \quad \varepsilon_5 \triangleq \left(\frac{\tilde\rho}{4C(\tilde\rho)} \right)^2.
$$

Therefore, collecting (2.62) and (2.64) together finishes the proof fo Lemma 2.10. □

2.2 L^p-estimates of the gradient of denstiy

This subsection concerns the necessary estimates for the existence of strong solutions. To do this, let the conditions of Theorem 1.1 be in force. We always assume that (2.4) holds and the initial energy C_0 satisfies (2.6). For simplicity, we denote by C the various positive constants which may depend on

$$
\mu, \lambda, A, \gamma, \beta, \rho_\infty, \inf f(x), \tilde\rho, M, \|f\|_{H^2}, \|\rho_0 - \rho_\infty\|_{H^1 \cap W^{1,p}}, \text{ and } T.
$$

We aim to prove the following proposition, which is mainly concerned with the L^p-estimates of the gradient of density.

Proposition 2.2 *Let (ρ, u) be a smooth solution of (1.1)–(1.7) on $\bar\Omega \times [0, T]$, satisfying (2.4) and (2.6). Then there exists a positive constant $C(T)$, depending on T, such that*

$$
\sup_{0 \le t \le T} \|\nabla\rho\|_{L^2 \cap L^p} + \int_0^T \left(\|\nabla^2 u\|_{L^p}^q + \|\nabla u\|_{L^\infty}^q \right) \mathrm{d}t \le C(T). \tag{2.65}
$$

where

$$
3 < p < 6, \quad 1 < q < \frac{4p}{5p-6}. \tag{2.66}
$$

Lemma 2.11 *There exists a positive constant $C(T)$, depending on T, such that*

$$\int_0^T \left(\|\sqrt{\rho}\dot{u}\|_{L^p}^q + \|\mathrm{div}u\|_{L^\infty}^q + \|\mathrm{curl}u\|_{L^\infty}^q\right) \mathrm{d}t \le C(T) \tag{2.67}$$

where p, q are the same ones in (2.66).

Proof. Thanks to (2.18) and (2.49), we see that

$$\begin{aligned}
\int_0^T \|\sqrt{\rho}\dot{u}\|_{L^p}^q \mathrm{d}t &\le C \int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{q(6-p)}{2p}} \|\dot{u}\|_{L^6}^{\frac{q(3p-6)}{2p}} \mathrm{d}t \\
&\le C \sup_{0\le t\le T} \left(t\|\sqrt{\rho}\dot{u}\|_{L^2}^2\right)^{\frac{q(6-p)}{4p}} \int_0^T t^{-\frac{q}{2}} \left(\sigma(t)\|\nabla\dot{u}\|_{L^2}^2\right)^{\frac{q(3p-6)}{4p}} \mathrm{d}t \\
&\le C \left(\int_0^T t^{-\frac{2pq}{4p-3pq+6q}} \mathrm{d}t\right)^{\frac{4p-3pq+6q}{4p}} \left(\int_0^T t\|\nabla\dot{u}\|_{L^2}^2 \mathrm{d}t\right)^{\frac{q(3p-6)}{4p}} \\
&\le C(T),
\end{aligned} \tag{2.68}$$

since $1 < q < \frac{4p}{5p-6}$ implies that $0 < \frac{2pq}{4p-3pq+6q} < 1$. As a result, it follows from (2.7), (2.50), (2.52), (2.56) and the Sobolev's embedding inequality that

$$\begin{aligned}
\|\mathrm{div}u\|_{L^\infty} + \|\mathrm{curl}u\|_{L^\infty} &\le C\left(1 + \|F\|_{L^\infty} + \|\omega\|_{L^\infty}\right) \\
&\le C\left(1 + \|\nabla u\|_{L^p} + \|\nabla F\|_{L^p} + \|\nabla\omega\|_{L^p}\right) \\
&\le C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^p}\right) \in L^q(0,T).
\end{aligned}$$

This finishes the proof of Lemma 2.11. □

To be continued, we recall the following logarithm estimate for the Lamé system, which will be used to estimate $\|\nabla u\|_{L^\infty}$.

Lemma 2.12 ([8, 17]) *Assume that μ, λ satisfy* (1.4), *and that $v = v(x)$ is a solution of the Lamé system:*

$$-\mu\Delta v - (\mu+\lambda)\nabla\mathrm{div}v = \mathrm{div}g$$

with the boundary conditions (1.6), *where $g = (g_{ij})_{3\times3}$ satisfies $g \in L^2 \cap W^{1,r}$ with $3 < r < \infty$. Then there exists a constant $C > 0$, depending on r, such that*

$$\|\nabla v\|_{L^\infty} \le C\left(1 + \ln(e + \|\nabla g\|_{L^r})\|g\|_{L^\infty} + \|g\|_{L^2}\right). \tag{2.69}$$

We are now in a position of estimating the L^p-norm of the gradient of density.

Proof of Proposition 2.2. First, it is easily derived from (1.1) that for any $2 \le p \le 6$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p} &\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p} \\
&\le C\left(1 + \|\nabla u\|_{L^\infty}\right)\|\nabla\rho\|_{L^p} + C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^p}\right),
\end{aligned} \tag{2.70}$$

where we have also used the theory of elliptic system to get that

$$\|\nabla^2 u\|_{L^p} \le C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^p} + \|\nabla\rho\|_{L^p}\right), \quad 2 \le p \le 6. \tag{2.71}$$

To estimate $\|\nabla u\|_{L^\infty}$, we decompose u into two parts: v and w, where

$$\mu\Delta v + (\mu+\lambda)\mathrm{div}v = \nabla(P(\rho) - P(\rho_s))$$

and $w = u - v$ satisfies

$$\mu \Delta w + (\mu + \lambda) \mathrm{div} w = \rho \dot{u} + (\rho - \rho_s) \nabla f$$

with the Navier's type boundary conditions (1.6) for v and w on $\partial\Omega$. Then it follows from Lemma 2.11 that for $p \in (3, \infty)$,

$$\begin{aligned} \|\nabla v\|_{L^\infty} &\le C\left(1 + \ln(e + \|\nabla(\rho - \rho_s)\|_{L^p})\|\rho - \rho_s\|_{L^\infty} + \|\rho - \rho_s\|_{L^2}\right) \\ &\le C\left(1 + \ln(e + \|\nabla \rho\|_{L^p})\right). \end{aligned} \tag{2.72}$$

In view of the theory of elliptic system, one has

$$\|\nabla^2 w\|_{L^p} \le C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^p}\right),$$

which, together with the Sobolev's embedding inequality, gives

$$\|\nabla w\|_{L^\infty} \le C\left(\|\nabla w\|_{L^p} + \|\nabla^2 w\|_{L^p}\right) \le C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^p}\right), \tag{2.73}$$

where we have used the fact that

$$\|\nabla w\|_{L^2} \le C\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \le C\left(\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}\right) \le C.$$

Substituting (2.72) and (2.73) into (2.70), we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \rho\|_{L^p} \le{}& C\left(1 + \ln(e + \|\nabla \rho\|_{L^p})\right)\|\nabla \rho\|_{L^p} \\ &+ C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^p}\right)\left(1 + \|\nabla \rho\|_{L^p}\right), \end{aligned}$$

and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(e + \|\nabla \rho\|_{L^p}\right) \le C\left(1 + \ln(e + \|\nabla \rho\|_{L^p})\right) + C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^p}\right),$$

which, combined with (2.67), shows that $\|\nabla \rho\|_{L^p}$ is bounded for any $3 < p < 6$. As an immediate result, one also infers from (2.71) that $\|\nabla^2 u\|_{L^p} \in L^q(0, T)$, and thus, $\|\nabla u\|_{L^\infty} \in L^1(0, T)$ due to the Sobolev embedding's inequality. This, together with (2.70) with $p = 2$, yields that $\|\nabla \rho\|_{L^2}$ is bounded. The proof of Proposition 2.2 is therefore complete. □

3 Proof of Theorem 1.1

With the help of the Propositions 2.1 and 2.2, we can now prove Theorem 1.1. Indeed, we can first construct approximate solutions with positive density by applying the local existence theorem due to Matsumura-Nishida [23], then combine Proposition 2.1 with the bootstrap arguments to extend the local approximate solutions globally in time under the smallness condition of initial energy (i.e., (2.6)), and finally pass to the limit based on the global uniform a priori estimates in Propositions 2.1 and 2.2. Since the proofs are standard and analogous to that in [19, 20], we omit here for simplicity. Note that the large-time behavior stated in (1.23) is an immediate result of the t-independent estimates given by Proposition 2.1.

Next, we prove the uniqueness of the solutions, which is an immediate consequence of the following more general result.

Theorem 3.1 *Let $(\rho, u) \in V$ be a weak solution of (1.1)–(1.7) on $\Omega \times [0,T]$, where*

$$V \triangleq \left\{ (\rho,u) \middle| \begin{array}{l} \rho - \rho_\infty \in C([0,T]; L^2 \cap L^\infty), \ \rho u \in C([0,T]; L^2), \\ \sqrt{\rho} u \in L^\infty(0,T;L^2), \ \nabla u \in L^2(0,T;L^2) \end{array} \right\}. \tag{3.1}$$

Assume that $(\tilde\rho, \tilde u) \in V$ is another solution of (1.1)–(1.7) on $\mathbb{R}^3 \times [0,T]$, which enjoys the same data f and (ρ_0, m_0) with u_0 being well defined by $m_0 = \rho_0 u_0$ as that of (ρ, u) and possesses the additional regularities:

$$\nabla\tilde\rho \in L^\infty(0,T;L^3), \quad \nabla\tilde u \in L^1(0,T;L^\infty), \quad \sqrt{t}(\tilde u_t + \tilde u \cdot \nabla \tilde u) \in L^2(0,T;L^3). \tag{3.2}$$

Then, it holds that $(\rho,u) = (\tilde\rho, \tilde u)$ almost everywhere on $\mathbb{R}^3 \times [0,T]$.

Remark 3.1. Obviously, the solutions (ρ, u) obtained in Theorem 1.1 belong to the set V and satisfy (3.2). Indeed, by virtue of (1.22) and the Hölder inequality, one gets that

$$\nabla\rho \in L^\infty(0,T;L^3), \quad \nabla u \in L^1(0,T;L^\infty),$$

and

$$\begin{aligned} \int_0^T t\|u_t + u\cdot\nabla u\|_{L^3}^2 \mathrm{d}t &\le \int_0^T t \left(\|u_t\|_{L^2}\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \right) \mathrm{d}t \\ &\le C(T) \int_0^T \left(\|\sqrt{\rho} u_t\|_{L^2}^2 + t\|\nabla u_t\|_{L^2}^2 + t\|\nabla^2 u\|_{L^2}^2 \right) \mathrm{d}t \\ &\le C, \end{aligned}$$

where we have used the following Poincaré type inequality:

$$\|v\|_{L^2}^2 \le C\left(\|\sqrt{\rho} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right), \quad \forall\, v \in H^1, \tag{3.3}$$

due to the Cauchy-Schwarz's inequality and the fact that

$$\begin{aligned} \int |v|^2 \mathrm{d}x \le \underline{\rho}^{-1} \int \rho_s |v|^2 \mathrm{d}x &\le C \int \left(\rho|v|^2 + |\rho - \rho_s||v|^2 \right) \mathrm{d}x \\ &\le C\left(\|\sqrt{\rho} v\|_{L^2}^2 + \|\rho - \rho_s\|_{L^3} \|v\|_{L^2}^{1/2} \|\nabla v\|_{L^2}^{3/2} \right). \end{aligned}$$

Proof of Theorem 3.1. Let (ρ,u) and $(\tilde\rho,\tilde u)$ be the solutions of the problem (1.1)–(1.7) as the ones given in Theorem 3.1. Define $R \triangleq \rho - \tilde\rho$ and $U \triangleq u - \tilde u$. Then it is easy to check that the pair of functions (R, U) satisfies

$$R_t + \rho \mathrm{div} U + R \mathrm{div} \tilde u + U \cdot \nabla R + \tilde u \cdot \nabla R + U \cdot \nabla \tilde \rho = 0, \tag{3.4}$$

and

$$\begin{aligned} &\rho U_t + \rho u \cdot \nabla U + \nabla(P(\rho) - P(\tilde\rho)) - \mu\Delta U - (\mu+\lambda)\nabla \mathrm{div} U \\ &\quad = -\rho U \cdot \nabla \tilde u + R\nabla f - R(\tilde u_t + \tilde u \cdot \nabla \tilde u). \end{aligned} \tag{3.5}$$

with zero initial conditions and the Navier's type boundary conditions:

$$(U^1, U^2, U^3) = \beta(\partial_3 U^1, \partial_3 U^2, 0)(x,t), \quad x \in \partial\Omega, \ t > 0 \tag{3.6}$$

Since $\|R\|_{L^\infty} \le \|\rho\|_{L^\infty} + \|\tilde\rho\|_{L^\infty}$, by (3.6) we obtain after multiplying (3.4) by R in L^2 and integrating by parts that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|R\|_{L^2}^2 &\le C\|\rho\|_{L^\infty}\|\nabla U\|_{L^2}\|R\|_{L^2} + C\|\nabla\tilde u\|_{L^\infty}\|R\|_{L^2}^2 \\
&\quad + C\|R\|_{L^\infty}\|\nabla U\|_{L^2}\|R\|_{L^2} + C\|U\|_{L^6}\|\nabla\tilde\rho\|_{L^3}\|R\|_{L^2} \\
&\le C\left(\|\rho\|_{L^\infty} + \|\tilde\rho\|_{L^\infty} + \|\nabla\tilde\rho\|_{L^3}\right)\|\nabla U\|_{L^2}\|R\|_{L^2} + C\|\nabla\tilde u\|_{L^\infty}\|R\|_{L^2}^2 \\
&\le C\|\nabla U\|_{L^2}\|R\|_{L^2} + C\|\nabla\tilde u\|_{L^\infty}\|R\|_{L^2}^2,
\end{aligned}
$$

which, combined with the fact that $\nabla\tilde u \in L^1(0,T;L^\infty)$, gives

$$
\|R(t)\|_{L^2} \le C\int_0^t \|\nabla U\|_{L^2}\mathrm{d}s \le C\sqrt{t}\left(\int_0^t \|\nabla U\|_{L^2}^2\mathrm{d}s\right)^{1/2}. \tag{3.7}
$$

Next, multiplying (3.5) by U in L^2 and integrating by parts, we have from (3.6) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}U\|_{L^2}^2 + \mu\|\nabla U\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}U\|_{L^2}^2 + \mu\int_{\partial\Omega}\beta^{-1}|U|^2\mathrm{d}S \\
&\le C\|R\|_{L^2}\|\nabla U\|_{L^2} + C\|\nabla\tilde u\|_{L^\infty}\|\sqrt{\rho}U\|_{L^2}^2 + C\|R\|_{L^2}\|\nabla f\|_{L^3}\|U\|_{L^6} \\
&\quad + C\|\tilde u_t + \tilde u\cdot\nabla\tilde u\|_{L^3}\|R\|_{L^2}\|U\|_{L^6} \\
&\le \frac{\mu}{2}\|\nabla U\|_{L^2}^2 + C\|\nabla\tilde u\|_{L^\infty}\|\sqrt{\rho}U\|_{L^2}^2 + C\left(1 + \|\tilde u_t + \tilde u\cdot\nabla\tilde u\|_{L^3}^2\right)\|R\|_{L^2}^2,
\end{aligned}
$$

and hence, by virtue of (3.7) we find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 + \mu\int_{\partial\Omega}\beta^{-1}|U|^2\mathrm{d}S \\
&\le C\left(1 + \|\nabla\tilde u\|_{L^\infty} + t\|\tilde u_t + \tilde u\cdot\nabla\tilde u\|_{L^3}^2\right)\left(\|\sqrt{\rho}U\|_{L^2}^2 + \int_0^t \|\nabla U\|_{L^2}^2\mathrm{d}s\right),
\end{aligned} \tag{3.8}
$$

where it follows from (3.2) that

$$
\left(1 + \|\nabla\tilde u\|_{L^\infty} + t\|\tilde u_t + \tilde u\cdot\nabla\tilde u\|_{L^3}^2\right) \in L^1(0,T).
$$

In view of the continuity of $(\rho,\tilde\rho)$ and $(\rho u, \tilde\rho\tilde u)$, we have

$$
\begin{aligned}
\|\sqrt{\rho}U(t)\|_{L^2}^2 &= \int \rho(u-\tilde u)\cdot(u-\tilde u)\mathrm{d}x \\
&= \int (\rho u - \tilde\rho\tilde u)\cdot(u-\tilde u)\mathrm{d}x + \int(\rho-\tilde\rho)\tilde u\cdot(u-\tilde u)\mathrm{d}x \\
&\le C\left(\|\rho u - \tilde\rho\tilde u\|_{L^2} + \|\rho-\tilde\rho\|_{L^3}\|\tilde u\|_{L^6}\right)\left(\|u\|_{L^2} + \|\tilde u\|_{L^2}\right) \\
&\le C\left(\|\rho u - \tilde\rho\tilde u\|_{L^2} + \|\rho-\tilde\rho\|_{L^3}\right) \to 0 \quad \text{as} \quad t\to 0,
\end{aligned}
$$

which, combined with (3.7), (3.8) and the Gronwall's inequality, shows that $R=0$ and $U=0$ a.e. This finishes the proof of Theorem 3.1. □

References

[1] T. Arbogast, H. L. Lehr, Homogenization of a Darcy-Stokes system modeling vuggy porous media, Comput. Geosci. 10 (3), 291-302 (2006).

[2] R. E. Caflisch, J. H. Rubinstein, Lecture on the mathematical theory of multiphase flow, notes, New York University, New York, (1985).

[3] Y. Cho, H. J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. (9) 83 (2), 243-275 (2004).

[4] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations 228 (2), 377-411 (2006).

[5] Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, Manuscripta Math. 120 (1), 91-129 (2006).

[6] R. J. Duan, H. X. Liu, U. Ukai, T. Yang, Optimal L^p-L^q convergence rates for the compressible Navier-Stokes equations with potential force. J. Differential Equations. 238, 220-233 (2007).

[7] R. J. Duan, S. Ukai, T. Yang, H. J. Zhao, Optimal convergence rates for the compressible Navier-Stokes equations with potential forces. Math. Models Meth. Appl. Sci. 17, 737-758 (2007).

[8] Q. Duan, Global well-posedness of classical solutions to the compressible Navier-Stokes equations in a half-space. J. Differential Equations, 253 167-202 (2012).

[9] E. Feireisl, Dynamics of Viscous Compressible Fluids. Oxford Science Publication, Oxford, 2004.

[10] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid. Indiana Univ. Math. J. 53, 1707-1740 (2004)

[11] E. Feireisl, A. Novotny, H. Petzeltová: On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. 3(4), 358-392 (2001).

[12] E. Feireisl, H. Petzeltová, Large-time behaviour of solutions to the Navier-Stokes equations of compressible flow. Arch. Rational Mech. Anal. 150(1), 77-96 (1999).

[13] D. Hoff, Global solutions of the Navie-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Diff. Eqs. **120** , 215-254 (1995)

[14] D. Hoff, Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Rational Mech. Anal. **132**, 1-14 (1995)

[15] D. Hoff, Compressible flow in a half-space with Navier boundary conditions, J. Math. Fluid Mech. 7 (3), 315-338 (2005).

[16] X. D. Huang, On local strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum. arXiv:1905.11809

[17] X. D. Huang, J. Li, Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and magnetohydrodynamic flows. Commun. Math. Phys., 324(1), 147-171, (2013).

[18] X. D. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible NavierC Stokes equations. Commun. Pure Appl. Math. 65(4), 549-585 (2012)

[19] J. Li, A. Matsumura, On the Navier-Stokes equations for three-dimensional compressible barotropic flow subject to large external potential forces with discontinuous initial data. J. Math. Pures Appl. 95(5), 495-512 (2011).

[20] J. Li, J. W. Zhang, J. N. Zhao, On the global motion of viscous compressible barotropic flows subject to large external potential forces and vacuum. SIAM J. Math. Anal. 47(2), 1121-1153 (2015).

[21] P. L. Lions, Mathematical Topics in Fluid Mechanics, Vol. 1. Incompressible Models. Oxford Science, Oxford, 1996.

[22] P. L. Lions, Mathematical Topics in Fluid Mechanics, Vol. 2. Compressible Models. Oxford University Press, New York, 1998.

[23] A, Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20, 67-104 (1980).

[24] A. Matsumura, N. Yamagata, Global weak solutions of the Navier-Stokes equations for multidimensional compressible flow subject to large external potential forces. Osaka J. Math. 38(2), 399-418 (2001).

[25] J. Nash, Problème de Cauchy pour les équations differentielles d'un fluide général. Bull. Soc. Math. France. 90, 487-497 (1962).

[26] N. Navier, Sur les Lois du Mouvement des Fluides, Mémoires de L'Académie Royale des Sciences. Paris, Tome VI, 1823

[27] L. Nirenberg: On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa (3), 13, 115-162 (1959).

[28] A. Novotny, I. Straskraba, Stabilization of weak solutions to compressible Navier-Stokes equations. J. Math. Kyoto Univ. 40(2), 217-245 (2000).

[29] P.G. Saffman, On the boundary condition at the surface of a porous medium, Stud. Appl. Math. 50, 93-101 (1971).

[30] J. Serrin, On the uniqueness of compressible fluid motion. Arch. Rational. Mech. Anal. 3, 271-288 (1959).

[31] Y. Shibata, K. Tanaka, On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance. J. Math. Soc. Japan 55(3), 797-826 (2003).

[32] A. Tani, On the first initial-boundary value problem of compressible viscous fluid motion. Publ. Res. Inst. Math. Sci. 13, 193-253 (1977).

[33] A. Valli, Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method. Ann. Scuola. Norm. Super. Pisa CI. Sci. IV 10, 607-647 (1983).

[34] A. A. Zlotnik, Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Diff. Eqs. **36**, 701-716 (2000)