Fractional Parabolic Equations with Generalized Mittag-Leffler Kernels

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Abstract

In this paper we apply the fractional integrals with arbitrary order depending on the fractional operators of Riemann type (ABR) and Caputo type (ABC) with kernels of Mittag Lefller in three parameters $E^{\gamma}_{\alpha,\mu}(\lambda, t)$ for solving the time fractional parabolic nonlinear equation. We utilize these operators with homotopy analysis method (HAM) for constructing the new scheme for generating the successive approximations. This procedure are used successfully on two examples for finding the solutions. The effectiveness and accuracy are verified by clarifying the convergence region in the h-curves as well as by calculating the residual error and the results were accurate. Depending on this results, this treatment can be used to find the approximate solutions to many fractional differential equations.

Keywords Mittag Lefller Kernel ; Homotopy analysis method , Time fractional parabolic nonlinear equation

MSC 2010: 34A08; 35A22; 41A30; 65N22.

1. Introduction

Differential equations play an important role in modeling real-world problems in biology, physics, engineering, and many areas in chemistry. Each of these applications can be modeled through differential equations [4, 5]. Recently, many applications in the field have been modeled through fractional differential equations: fluid mechanics ([1] - [3]), chemistry [6], biology [18], viscoelasticity (see [8] and [9]), engineering, finance, and physics [10, 11, 12], and so on. In addition,there are a lot of papers that have a new approach and their applied of fractional type operators with exponential or ML kernels [13]-[24].

Recently, many researchers have caught the attention of modeling many real-world problems using fractional operators Mittag-Leffler (ML) kernels. These kernels are actually not singular and has one parameter. These kernels actually useful and has an advantage in facilitating and simplifying the modeling and solution of many problems numerically. In a more recent time, Abdeljawad and Baleanu are used Mittag-Leffler kernel but with three parameters. They formulated the corresponding integral operators with arbitrary ML parameters and study their action on the Atangana-Baleanu fractional derivative in the Caputo sense (ABC) fractional operators. One of the most advantages of the used for the new Mittag-Leffler kernel is the existence of the solution for fractional differential equations. For example the solution of the fractional differential equation ${}_{0}^{ABC}D_t^{\alpha}y(t) = a$, $y(0) = \beta$ where a, β are constant does not exist for $0 < \alpha < 1$, whereas via the new definition the solution exists [26]. Further more general discussion to this issue by means of a necessary vanishing condition on the right hand side of the ABC initial value problem can be found in [27]. Very recently, the authors in [28] studied the relation between two models of fractional calculus which are defined using three-parameter Mittag-Leffler functions: the Prabhakar definition and a recently defined extension of the Atangana-Baleanu definition given in [25, 26].

In this paper we study approximate solutions using HAM for the time fractional parabolic equations. To the best of our knowledge, this is the first study of the time fractional parabolic equations by HAM based on the fractional integrals with arbitrary order with the kernels of Mittag Lefller with three parameters.

The present paper is organized as follows. The second is devoted to introduce the preliminaries and notations. The third is devoted to construct the scheme of HAM with new fractional integral. While the forth for applying the new scheme with two examples. Conclusions are presented in section five.

2. Preliminaries and notations

In this section, we introduce some definitions and properties for the generalized ABC fractional derivative which introduced by Abdeljawad and Baleanu in 2018 [25]and in [26], where Abdeljawad set up a study for more basics and properties of these operators and in [29] originated the discrete versions.

Definition 1. [25] *The generalized ABC fractional derivative with kernel $E^{\gamma}_{\alpha,\mu}(\lambda, t)$ where $0 < \alpha < 1$, $Re(\mu) > 0$,$\gamma \in \Re$, and $\lambda = \frac{-\alpha}{1-\alpha}$, is defined by*

$$({}^{\mathrm{ABC}}_{a}D^{\alpha,\mu,\gamma}f)(x) = \frac{M(\alpha)}{1-\alpha}\int_a^x E^{\gamma}_{\alpha,\mu}(\lambda, x-t)f'(t)dt$$

where

$$E^{\gamma}_{\alpha,\mu}(\lambda, t) = \sum_{k=0}^{\infty}\lambda^k(\gamma)_k\frac{t^{\alpha k+\mu-1}}{k!\Gamma(\alpha k+\mu)}$$

and $(\gamma)_k = \gamma(\gamma+1)\ldots(\gamma+k-1)$.

Remark 1. *We noted that if $\alpha = \mu = \gamma \to 1$, we obtained the standard first derivative $f'(x)$.*

Theorem 2.1. *The ABC fractional derivative for x^{β} where $\beta > 0, (\alpha+\mu) > 0$ given by*

$${}^{\mathrm{ABC}}_{0}D^{\alpha,\mu,\gamma}x^{\beta} = \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)\lambda^k(\gamma)_k x^{\alpha k+\mu+\beta-1}}{k!\Gamma(k\alpha+\beta+\mu)} = \frac{M(\alpha)\Gamma(\beta+1)}{1-\alpha}E^{\gamma}_{\alpha,\mu+\beta}(\lambda, x).$$

Proof: Using the definition of generalized ABC fractional derivative and the series of $E^{\gamma}_{\alpha,\mu}(\lambda, t)$ we have

$$\begin{aligned}
{}^{\mathrm{ABC}}_{0}D^{\alpha,\mu,\gamma}x^{\beta} &= \frac{M(\alpha)}{1-\alpha}\int_0^x\sum_{k=0}^{\infty}\lambda^k(\gamma)_k\frac{(x-t)^{\alpha k+\mu-1}}{k!\,\Gamma(\alpha k+\mu)}\beta\, t^{\beta-1}dt \\
&= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\int_0^x\lambda^k(\gamma)_k\frac{(x-t)^{\alpha k+\mu-1}}{k!\,\Gamma(\alpha k+\mu)}\beta\, t^{\beta-1}dt \\
&= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)\lambda^k(\gamma)_k x^{\alpha k+\mu+\beta-1}}{k!\,\Gamma(k\alpha+\beta+\mu)}. \\
&= \frac{M(\alpha)\Gamma(\beta+1)}{1-\alpha}E^{\gamma}_{\alpha,\mu+\beta}(\lambda, x).
\end{aligned}$$

Remark 2. *Upon Remark 1 and Theorem 2.1, we conclude that*

$$\lim_{\alpha\to 1}\frac{M(\alpha)\Gamma(\beta+1)}{1-\alpha}E_{\alpha,\beta+1}(\lambda,x)=\beta x^{\beta-1},\ \beta>0,x>0,\ \lambda=\frac{-\alpha}{1-\alpha}. \quad (1)$$

The analogous limit of (2) in the power law case (Caputo-fractional case) is

$$\lim_{\alpha\to 1}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}=\beta x^{\beta-1}. \quad (2)$$

Example: The generalized ABC fractional derivatives for $f(x)=x^3+2x^{1.5}$ for various order of α and μ with $\gamma=1$ are given in figure 1. It is clear that not only α but also μ can change the behavior of the derivative of function in the generalized ABC fractional derivative sense.

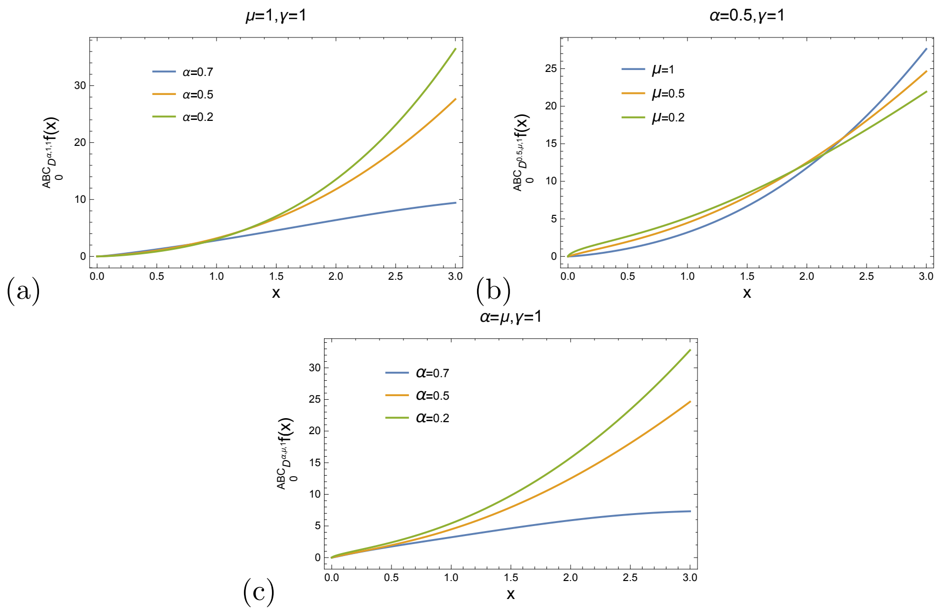

FIGURE 1. ${}_{0}^{\mathrm{ABC}}D^{\alpha,\mu,\gamma}\left(x^3+2x^{1.5}\right)$ for various values of α,μ with $\gamma=1$.

Definition 2. *Let f be a continuous function defined on an interval $[a,b]$ and assume $0<\alpha<1,\mu>0$. Then the left fractional integral of two parameters α and μ is defined by,*

$$({}_{a}^{\mathrm{ABC}}I^{\alpha,\mu}f)(x)=\frac{1-\alpha}{M(\alpha)}({}_{a}I^{1-\mu}f)(x)+\frac{\alpha}{M(\alpha)}({}_{a}I^{1-\mu+\alpha}f)(x)$$

where $({}_{a}I^{\beta}f)(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-s)^{\beta-1}f(s)ds$ is the Riemann-Liouville fractional integrals for $f(x)$ of order β.

Remark 3. [26] *If $\gamma=1$, then $({}_{a}^{\mathrm{ABC}}I^{\alpha,\mu}\,{}_{a}^{\mathrm{ABC}}D^{\alpha,\mu,\gamma}f)(x)=f(x)-f(a)$.*

More general properties and formulations under the existence of the third parameter $\gamma \neq 1$ can be found in [26]

3. Solution by homotopy analysis method

In this section we consider the parabolic fractional nonlinear differential equation in the sense of generalized ABC, which can be written in the form

$$ {}_{0}^{\mathrm{ABC}}D_{t}^{\alpha,\mu,\gamma}u(x,t) = N[u(x,t)] \tag{3} $$

subject to the initial condition $u(x,0) = f(x)$. According to the homotopy analysis method framework, we write the unknown function $u(x,t)$ in the form of Taylor series

$$ \phi(u(x,t);q) = u_0(x,t) + \sum_{i=1}^{\infty} u_n(x,t)q^i \tag{4} $$

where $q \in [0,1]$ is the embedding parameter. Now we construct the homotopy map

$$ (1-q)L[\phi(u(x,t);q)-u_0(x,t)] = hq({}_{0}^{\mathrm{ABC}}D_{t}^{\alpha,\mu,\gamma}\phi(u(x,t);q)-N[\phi(u(x,t);q)]) \tag{5} $$

where L is called the linear operator, $\hbar$ is the convergent control parameter.

Differentiate Eq. (5) n-times with respect to q and set $q = 0$, then divide by $n!$, we get

$$ L[u_n - \chi_n u_{n-1}] = \hbar({}_{0}^{\mathrm{ABC}}D_{t}^{\alpha,\mu,\gamma}u_{n-1} + R_n[\overrightarrow{u}_{n-1}]) $$

where

$$ R_n[\overrightarrow{u}_{n-1}] = \frac{1}{n!}\frac{\partial^n(qN[\phi(x,t;q)])}{\partial q^n}\Big|_{q=0}, $$

$\chi_n = \begin{cases} 0, & n \leq 1, \\ 1, & n > 1, \end{cases}$ and $\overrightarrow{u}_{n-1} = \{u_0, u_1, \ldots, u_{n-1}\}$.

The initial condition becomes

$$ \phi(u(x,0;q)) = u_0(x,0) + \sum_{i=0}^{\infty} u_i(x,0)q^i = f(x), $$

so that $u_i(x,0) = 0$ for $i = 1,2,3,\ldots$ and $u_0(x,0) = f(x)$.

By take the linear operator as $L = {}_{0}^{\mathrm{ABC}}D_{t}^{\alpha,\mu,\gamma}$. For sake of simplicity, we consider $\gamma = 1$. Thus our fractional derivative depends on α and μ.

$$ {}_{0}^{\mathrm{ABC}}D^{\alpha,\mu,\gamma}[u_n(x,t) - \chi_n u_{n-1}(x,t)] \;=\; \hbar\left[{}_{0}^{\mathrm{ABC}}D^{\alpha,\mu,\gamma}u_{n-1}(x,t) + R_n[\overrightarrow{u}_{n-1}]\right]. $$

Now, apply ${}_{0}^{\mathrm{ABC}}I^{\alpha,\mu}$ on the above equation, with the help of Remark 5, we have

$$ u_n \quad = (\chi_n + h)u_{n-1}(x,t) - (\chi_n + \hbar)u_{n-1}(x,0) + \hbar\,{}_{0}^{\mathrm{ABC}}I^{\alpha,\mu}\left[R_n[\overrightarrow{u}_{n-1}]\right] \tag{6} $$

We then end up with the series solution given by

$$ u(x,t) = u_0(x,t) + \sum_{i=1}^{\infty} u_i(x,t). \tag{7} $$

Finally we note that, we should have an initial guess $u_0(x,t)$ which satisfy the initial condition. For that, we simply assume $u_0(x,t) = f(x)$.

4. Applications

In this section, we apply HAM to the fractional parabolic nonlinear equation in the generalized ABC sense.

Example 1. *Consider the nonlinear parabolic PDE in the generalized ABC fractional derivative sense.*

$$ {}^{\mathrm{ABC}}_{\ 0}D_t^{\alpha,\mu,\gamma} u(x,t) = u_{xx}(x,t) + au(x,t) - bu^2(x,t). \tag{8}$$

subject to the initial condition $u(x,0) = \frac{1}{\left(\sqrt{\frac{b}{a}}+e^{\frac{\sqrt{a}x}{\sqrt{6}}}\right)^2}$.

When $\alpha = \mu = 1$ the equation has the exact solution [30]

$$u(x,t) = \frac{1}{\left(\sqrt{\frac{b}{a}} + e^{\frac{\sqrt{a}x}{\sqrt{6}} - \frac{5at}{6}}\right)^2}. \tag{9}$$

According to the framework of HAM, we choose the initial guess $u_0(x,t) = f(x) = \frac{1}{\left(\sqrt{\frac{b}{a}}+e^{\frac{\sqrt{a}x}{\sqrt{6}}}\right)^2}$, which then gives the following set of infinite linear fractional differential equations:

$$\begin{aligned} {}^{\mathrm{ABC}}_{\ 0}D^{\alpha,\mu,\gamma}[u_n(x,t) - \chi_n u_{n-1}(x,t)] \ &= \ \hbar \Big[{}^{\mathrm{ABC}}_{\ 0}D^{\alpha,\mu,\gamma} u_{n-1}(x,t) - (u_{n-1})_{xx}(x,t) \\ &\quad -au_{n-1}(x,t) + b\sum_{j=0}^{n-1} u_j(x,t)u_{n-1-j}(x,t)\Big], \end{aligned}$$

for $n = 1, 2, \ldots$. Then, applying ${}^{\mathrm{ABC}}_{\ 0}I^{\alpha,\mu}$ on the above equation, one obtains

$$\begin{aligned} u_n \quad &= (\chi_n + h)u_{n-1}(x,t) - (\chi_n + \hbar)u_{n-1}(x,0) + \\ &\hbar\, {}^{\mathrm{ABC}}_{\ 0}I^{\alpha,\mu}\left[-(u_{n-1}(x,t))_{xx} - au_{n-1}(x,t) + b\sum_{j=0}^{n-1} u_j(x,t)u_{n-1-j}(x,t)\right]. \end{aligned} \tag{10}$$

The first two terms are given by

$$u_1(x,t) = -\frac{\hbar t^{1-\mu}\left((1-\alpha)\Gamma(\alpha-\mu+2) + \alpha\Gamma(2-\mu)t^{\alpha}\right)\left(f(x)(a-bf(x)) + f''(x)\right)}{\Gamma(2-\mu)\Gamma(\alpha-\mu+2)},$$

$$u_2(x,t) = \hbar t^{1-2\mu}\left(g_1 - g_2 + g_3 - g_4 + g_5\right),$$

where

$$g_1 = \frac{\alpha^2\hbar t^{2\alpha+1}\left(a^2 f(x) + 2f''(x)(a-2bf(x)) - 3abf(x)^2 + 2b^2 f(x)^3 - 2bf'(x)^2 + f^{(4)}(x)\right)}{\Gamma(2\alpha-2\mu+3)},$$

$$g_2 = \frac{2(\alpha-1)\alpha\hbar t^{\alpha+1}\left(a^2 f(x) + 2f''(x)(a-2bf(x)) - 3abf(x)^2 + 2b^2 f(x)^3 - 2bf'(x)^2 + f^{(4)}(x)\right)}{\Gamma(\alpha-2\mu+3)},$$

$$g_3 = \frac{(\alpha-1)^2\hbar t\left(a^2 f(x) + 2f''(x)(a-2bf(x)) - 3abf(x)^2 + 2b^2 f(x)^3 - 2bf'(x)^2 + f^{(4)}(x)\right)}{\Gamma(3-2\mu)},$$

$$g_4 = \frac{\alpha(\hbar+1)t^{\alpha+\mu}\left(f(x)(a-bf(x)) + f''(x)\right)}{\Gamma(\alpha-\mu+2)},$$

$$g_5 = \frac{(\alpha-1)(\hbar+1)t^{\mu}\left(f(x)(a-bf(x)) + f''(x)\right)}{\Gamma(2-\mu)}.$$

By the same manner, we find $u_3, u_4, \ldots u_N$ and the M-th order of approximate solution is defined by

$$u(x,t) =\simeq u_0(x,t) + \sum_{i=1}^{M} u_i(x,t).$$

The approximate solution depends on the value of $\hbar$, so firstly, we fixed $\mu = \alpha = 1$ and we find the optimal value of $\hbar$ by using least square method. For that, we define the residual error

$$Res(x,t) = {}_0^{\mathrm{ABC}} D_t^{\alpha,\mu,\gamma} u(x,t) - u_x x(x,t) - au(x,t) + bu^2(x,t),$$

and consider the function

$$\zeta = \frac{1}{(N_1+1)(N_2+1)} \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} Res^2(x_i,t_j)$$

where $x_i = \frac{iL}{N1+1}, t_j = \frac{jK}{N_2+1}$, K is the endpoint of time and L is the endpoint of space along x.

It is clear that, the solution depends of the fractional parameters α and μ. For simply, we assume $\alpha = \mu$ and vary α from 0 to 1. The $\hbar-$ curve for several values of α with $a = 0.02, b = 0.03$ plotted in figure 2, the horizontal line that is parallel to the x-axis is representing the convergent region. We can also minimize the $\zeta(\hbar)$ to find the optimal value of $\hbar$ as given in figure 3. The solution $u(x,t)$ for different values of α are given in figure 4. Moreover, the residual error for the solution with various values of α are plotted in figure 5. It is worthily to mention that, if $\mu = 1$ and vary $0 < \alpha < 1$ (which is the standard ABC derivative) the solution does not satisfy the initial condition.

Example 2. *Consider the nonlinear parabolic equation with* $0 < \alpha \leq 1$

$${}_0^{\mathrm{ABC}} D_t^{\alpha,\mu,\gamma} u(x,t) = u_{xx}(x,t) - \rho u(x,t) \tag{11}$$

subject to the initial condition $u(x,0) = \sqrt{\frac{2}{\rho}}\frac{2x}{x^2+1}$.

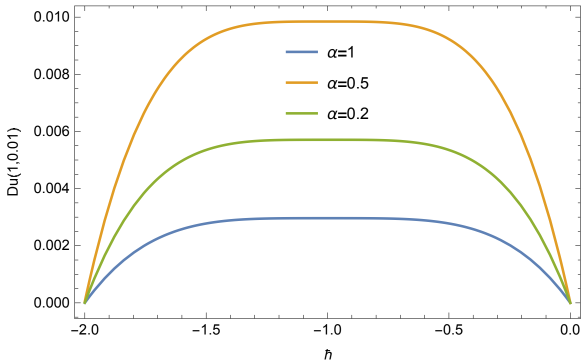

FIGURE 2. $\hbar-$ curve for example 1 with $\mu = \alpha = 1, 0.5, 0.2$. Using 4-order of approximation.

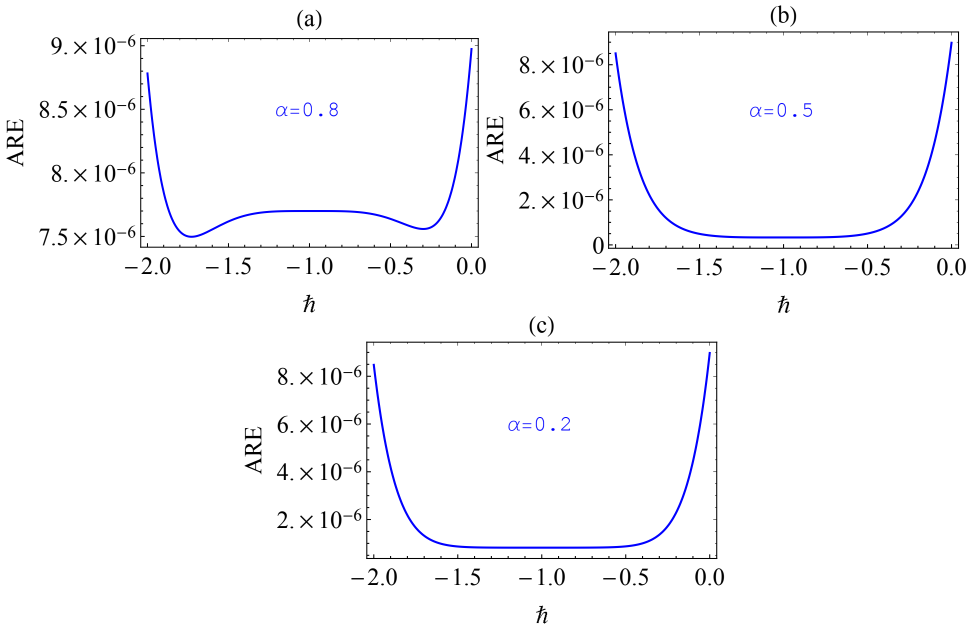

FIGURE 3. Average residual error with $\hbar$ for $\alpha = 0.8, 0.5, 0.2$.

The exact solution when $\alpha = \mu = 1$ is $u(x,t) = \sqrt{\frac{2}{\rho}}\frac{2x}{x^2+6t+1}$. Similar to the first example, we choose the initial guess $u_0(x,t) = f(x) = \sqrt{\frac{2}{\rho}}\frac{2x}{x^2+1}$. The $n-$ order can be written as

$$\begin{aligned} u_n \quad &= (\chi_n + h)u_{n-1}(x,t) - (\chi_n + \hbar)u_{n-1}(x,0) + \\ &\hbar\, {}_0^{\mathrm{ABC}}I^{\alpha,\mu}\left[-(u_{n-1}(x,t))_{xx} + \rho\sum_{j=0}^{n-1}\sum_{k=0}^{j} u_k(x,t)u_{k-j}(x,t)u_{n-1-j}(x,t)\right]. \end{aligned} \tag{12}$$

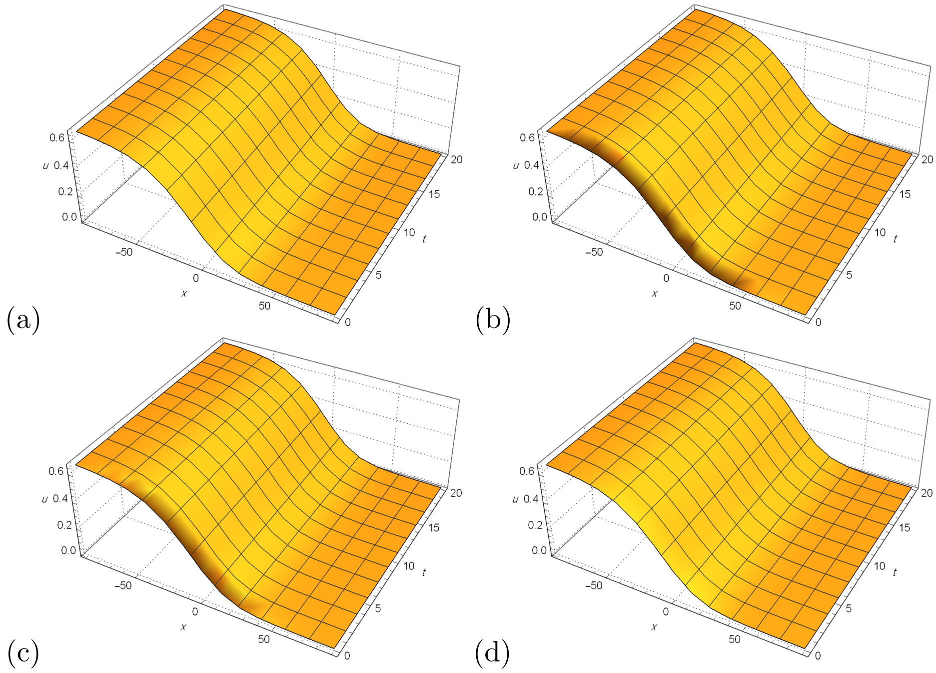

FIGURE 4. $u(x,t)$ for example 1 with different value of α where (a),(b),(c),(d) for $\alpha = 1, 0.8, 0.5, 0.2$ respectively.

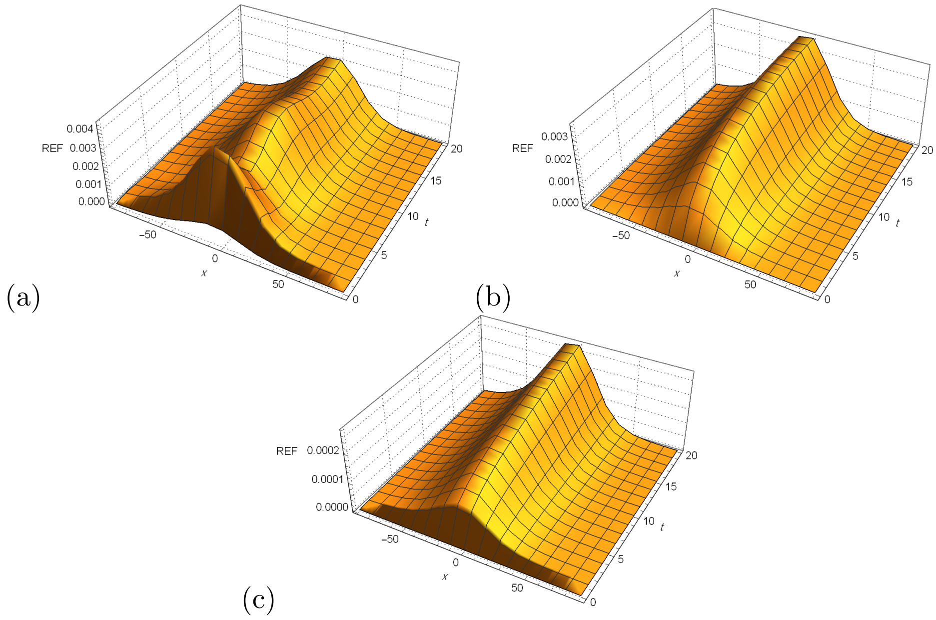

FIGURE 5. Residual error function for example 1 with different value of α where (a),(b),(c) for $\alpha = 0.8, 0.5, 0.2$ respectively.

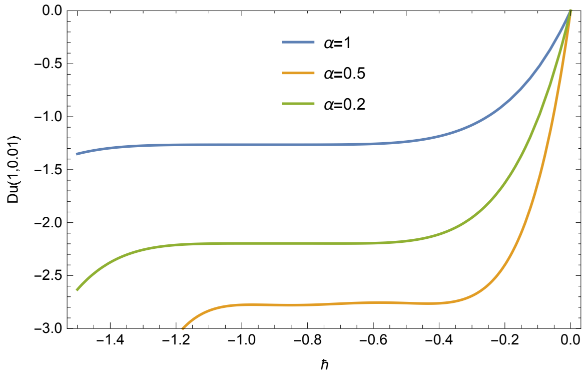

FIGURE 6. $\hbar-$ curve for example 2 with different values of α.

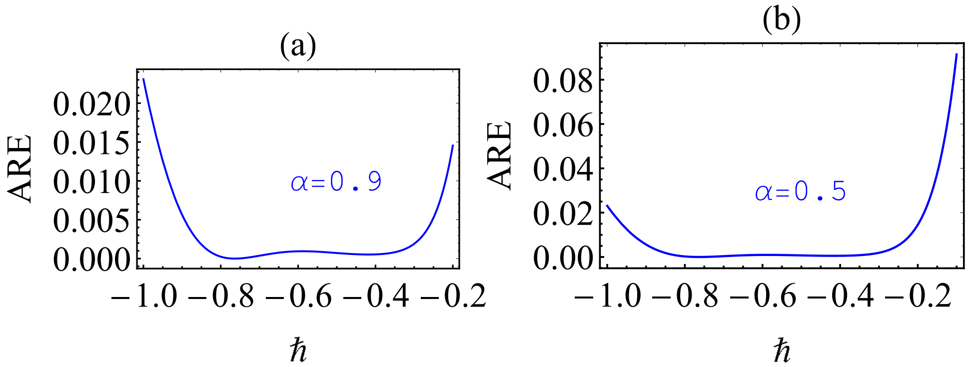

FIGURE 7. Average residual error with $\hbar$ for $\alpha = 0.9, 0.5$.

The first term is

$$u_1(x,t) = \hbar t^{1-\mu}\left(\rho f(x)^3 - f''(x)\right)\left(\frac{1-\alpha}{\Gamma(2-\mu)} + \frac{\alpha t^{\alpha}}{\Gamma(\alpha-\mu+2)}\right).$$

Using this manner, we find the other $M-$terms. Now, fixed $\rho = 10, \gamma = 1, \mu = \alpha$ and we vary α from 0 to 1. The $\hbar-$cureve is plotted in figure 6. The average residual error for $\alpha = 0.9$ and 0.5 are presented in figure 7. The exact solution for $\alpha = \mu = 1$ and the solution for different values of α are given in figure 8. Finally, we noted that the solution for $\mu = 1 = \gamma$ is the standard ABC fractional sense, but this solution does not satisfying the initial condition.

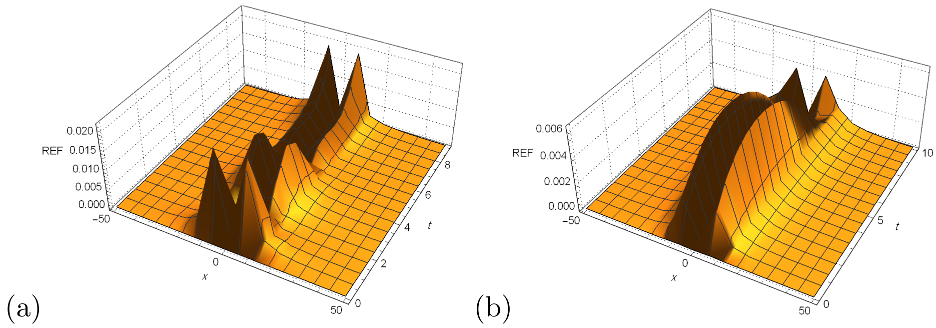

FIGURE 8. Residual error function for example 2 with different value of α where (a),(b) for $\alpha = 0.9, 0.5$ respectively.

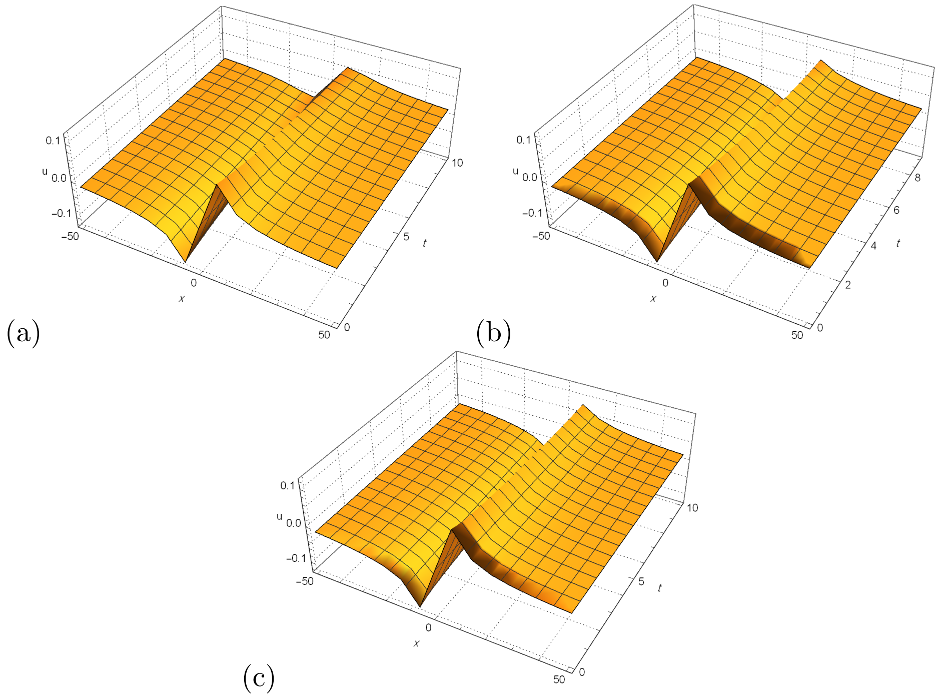

FIGURE 9. $u(x,t)$ for example 2 with different value of α where (a),(b),(c) for $\alpha = 1, 0.9, 0.5$ respectively.

5. Conclusion

In this paper, we haveapplied the Homotopy Analysis Method using the fractional integrals with arbitrary order depending on the fractional operators of Riemann type (ABR) and Caputo type (ABC) with kernels of Mittag Lefller in three parameters. We investigated the approximate solutions of the time fractional order parabolic equation. The accuracy of the approximate solutions was verified by comparing the proximate solutions with exact solutions at the case of classical parabolic equation. While in the case of fractional parabolic we satisfied the accuracy of the approximate solutions by computing the average residual error. In all cases the order of the errors are very small and a good agreement found.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[2] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1993.

[3] A. K. Alomari, V. S. Erturk, S. Momani, A. Alsaedi, An approximate solution method for the fractional version of a singular BVP occurring in the electrohydrodynamic flow in a circular cylindrical conduit. The European Physical Journal Plus, (2019), 134.4: 158.

[4] D. S. Jones, M. J. Plank and B. D. Sleeman, *Differential Equations and Mathematical Biology*, Second Edition, Series on Mathematical Biology, Chapman and Hall (CRC Press), Boca Raton, Florida, 2009.

[5] J. M. Amigó and M. Small, Mathematical methods in medicine: neuroscience, cardiology and pathology, *Philos. Trans. A Math. Phys. Engrg. Sci.* **375** (2017), Article ID 20170016, 1–9.

[6] K. M. Saad, Comparing the Caputo, Caputo-Fabrizio and Atangana-Baleanu derivative with fractional order: Fractional cubic isothermal auto-catalytic chemical system, *European Phys. J. Plus* **133** (2018), Article ID 94, 1–12.

[7] M. A. Khan, Z. Hammouch and D. Baleanu, Modeling the dynamics of hepatitis E via the Caputo-Fabrizio derivative, *Math. Model. Natur. Phenom.* **14** (2019), Article ID 311, 1–19.

[8] M. Caputo, *Elasticita e Dissipazioni*, Zanichelli, Bologna, 1969.

[9] I. Podlubny, *Fractional Differential Equations*: *An Introduction to Fractional Derivatives*, *Fractional Differential Equations*, *to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, Vol. **198**, Academic Press, San Diego, London, and Toronto, 1999.

[10] R. Hilfer (Editor), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000.

[11] C.-P. Li and W.-H. Deng, Remarks on fractional derivatives, *Appl. Math. Comput.* **187** (2007), 777–784.

[12] M. Popolizio and R. Garrappa, On the use of matrix functions for fractional partial differential equations, *Math. Comput. Simul.* **81** (2011), 1045–1056.

[13] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1** (2015), 73–85.

[14] I. Koca and A. Atangana, Solutions of Cattaneo-Hristov model of elastic heat diffusion with Caputo-Fabrizio and Atangana-Baleanu fractional derivatives, *Thermal Sci.* **21** (2017), 2299–2305.

[15] J. Losada and J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1** (2015), 87–92.

[16] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Sci.* **20** (2016), 763–769.

[17] J. Singh, D. Kumar, Z. Hammouch, A. Atangana. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. Applied Mathematics and Computation 2018; 316: 504-515.

[18] M. A. Khan, Z. Hammouch,D. Baleanu. Modeling the dynamics of hepatitis E via the CaputoFabrizio derivative. Mathematical Modelling of Natural Phenomena 2019; 14: DOI:10.1051/mmnp/2018074.

[19] Morales-Delgado VF Gmez-Aguilar JF Saad KM Escobar Jimnez RF, Mathematical Methods in the Applied Sciences Application of the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives to mathematical model of cancer chemotherapy effect 42(4), 2019, 1167-1193

[20] A. A. Alderremy, Khaled M.Saad, Praveen Agarwal, Shaban Aly Shilpi Jain Certain new models of the multi space-fractional Gardner equation, Physica A: Statistical Mechanics and its Applications 545, 2020, 123806

[21] K. M. Saad, A. Atangana, D. Baleanu, New fractional derivatives with non-singular kernel applied to the Burgers equation, Chaos: An Interdisciplinary Journal of Non-linear Science, 2018, 28(6), 063109

[22] K. M. Saad, D. Baleanu, A. Atangana, New fractional derivatives applied to the Kortewegde Vries and Kortewegde VriesBurgers equations, Computational and Applied Mathematics, 2018, 37 (4), 5203-5216.

[23] Khaled M. Saad, D. Baleanu, Sinan Deniz, On a new modified fractional analysis of Nagumo equation, International Journal of Biomathematics, 2019, 12(3), 1950034.

[24] S. Aljhani, M. S. Noorani, A. K. Alomari, Numerical Solution of Fractional-Order HIV Model Using Homotopy Method. Discrete Dynamics in Nature and Society, 2020, 2020.

[25] T. Abdeljawad, D. Baleanu, On fractional derivatives with generalized Mittag-Leffler kernels. Advances in Difference Equations, 2018, 2018.1: 468.

[26] T. Abdeljawad, Fractional operators with generalized Mittag-Leffler kernels and their iterated differintegrals. Chaos: An Interdisciplinary Journal of Nonlinear Science, 2019, 29.2: 023102.

[27] F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative, Chaos, Solitons and Fractals, Volume 117, December 2018, Pages 16-20.

[28] A. Fernandez, T. Abdeljawad, and D. Baleanu, Relations between fractional models with three-parameter Mittag-Leffler kernels, Advances in Difference Equations, (2020) 2020:186 .

[29] T. Abdeljawad, Fractional difference operators with discrete generalized MittagLeffler kernels, Chaos, Solitons and Fractals 126, pp. 315-324 (2019).

[30] Zaid Odibat, Dumitru Baleanu, A linearization-based approach of homotopy analysis method for non-linear time-fractional parabolic PDEs. Mathematical Methods in the Applied Sciences, 2019, 42.18: 7222-7232.