# A new numerical method for nonlinear Volterra-Fredholm integro-differential equations ☆

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**Abstract**

Based on the good properties of reproducing kernel space, a new method combining the simplified reproducing kernel method (SRKM) and homotopy perturbation method (HPM) for solving the nonlinear Volterra-Fredholm integro-differential equations (V-FIDE) is proposed. The HPM can convert nonlinear problems into linear problems. And then using the SRKM to solve linear problems. The uniform convergence of the approximate solution is proved. Some numerical examples are prepared to illustrate the efficiency and rapidity of this method.

*Keywords:* Nonlinear Volterra-Fredholm integro-differential equations, Simplified reproducing kernel method, Homotopy perturbation method

## 1. Introduction

In this paper, we mainly discuss the nonlinear Volterra-Fredholm integro-differential equations:

$$\begin{cases} u'(x) + q(x)u(x) + \lambda_1 \int_0^x K_1(x,t)F(u(t))\,dt + \lambda_2 \int_0^1 K_2(x,t)G(u(t))\,dt = y(x), \\ u(a) = \alpha. \end{cases} \tag{1.1}$$

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The parameters $\lambda_1, \lambda_2$ are constant. $F(u(x))$ and $G(u(x))$ are constant coefficient polynomial with respect to $u(x)$. The V-FIDE has been widely used in physics, biological and engineering[1-3]. In order to obtain accurate numerical solutions more quickly, many methods to solve such problems have been proposed in recent years. The author of [1] introduced the Taylor polynomial method. E.Ba proposed the triangular functions method and the operational matrix with block-pulse functions in [2-3]. Hybrid Legendre polynomials and block-pulse functions approach were applied in [4]. In [5], the author discussed the Laplace discrete Adomian decomposition method over the integro-differential equation. J.Biazar and M.Eslami [12] presented He's homotopy perturbation method. F.S.Zulkarnain et al. [13] using the modified decomposition method obtained approximate solution of nonlinear Volterra-Fredholm integral equation. The numerical solvability of nonlinear V-FIDE and other related equations can be found in [6-8]. The authors in [14] formulated homotopy approximation technique for solving V-FIDE. In [15] the reproducing kernel method was applied to solve nonlinear equations. But the traditional reproducing kernel method is difficult to deal with the integral term, while the homotopy perturbation method can be effectively dealt with the integral term. Because the traditional reproducing kernel method needs orthogonalization, the calculation method is complex and time-consuming, our method avoids the Smith orthogonalization process, thus saving the calculation time and running memory. This article discusses the nonlinear V-FIDE by using SRKM and HPM in the reproducing kernel space, so that the equation can achieve higher accuracy.

In this paper, we described the homotopy perturbation theory in section 2. The reproducing kernel theory will be shown in sections 3 and 4. Some numerical examples are presented in section 5. In the end the conclusions are remarks.

## 2. Homotopy perturbation method

According to the theory of homotopy perturbation, we embed a small parameter $p(p \in [0,1])$ by constructing a homotopy map. The parameter p changes from 0 to 1, then the solution to the nonlinear equation $u(x)$ follows the homotopy path from the initial value problem to the original problem. And the solutions that satisfy the homotopy path can be expanded into a power series with respect to p.

To deal with the nonlinear part, we construct a homotopy for Eq.(1.1):

$$\begin{cases} u'(x) + q(x)u(x) + p\lambda_1 \int_0^x K_1(x,t)F(u(t))\,dt + p\lambda_2 \int_0^1 K_2(x,t)G(u(t))\,dt = y(x), \\ u(a) = \alpha. \end{cases} \tag{2.1}$$

when $p = 0$, the Eq.(2.1) is an initial value problem:

$$\begin{cases} u'(x) + q(x)u(x) = y(x), \\ u(a) = \alpha. \end{cases} \tag{2.2}$$

when $p = 1$, Eq.(2.1) is the original problem (1.1). According to the theory of homotopy perturbation, the solution of the operator equation that the homotopy path satisfies can be written as a power series about p:

$$u(x,p) = u_0(x) + pu_1(x) + p^2u_2(x) + \ldots + p^nu_n(x) + \ldots$$

In this way, when $p \to 1$, the approximate solution of the nonlinear operator equation is obtained

$$u(x) = \lim_{p \to 1} u(x,p) = u_0(x) + u_1(x) + u_2(x) + \ldots$$

Let the k derivatives of $F$ and $G$ and set $p = 0$:

$$\sum_{n=0}^{\infty} p^n(u_n'(x)+q(x)u_n(x))+p\lambda_1 \int_0^x K_1(x,t)\sum_{k=0}^{\infty} A_kp^k dt+p\lambda_2 \int_0^1 K_2(x,t)\sum_{k=0}^{\infty} B_kp^k dt = y(x).$$

where

$$F(\sum_{n=0}^{\infty} p^nu_n(x)) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k}{dp^k}F(\sum_{n=0}^{\infty} p^nu_n(x)) \mid_{p=0}= \sum_{k=0}^{\infty} A_kp^k,$$

$$G(\sum_{n=0}^{\infty} p^nu_n(x)) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k}{dp^k}G(\sum_{n=0}^{\infty} p^nu_n(x)) \mid_{p=0}= \sum_{k=0}^{\infty} B_kp^k.$$

Comparing the coefficients of $p^i$ on both sides of this equation and set them equal, we can get when $k=0$

$$\begin{cases} u_0'(x) + q(x)u_0(x) = y(x), \\ u(a) = \alpha. \end{cases} \tag{2.3}$$

for $p^{k+1}$,

$$\begin{cases} u_k'(x) + q(x)u_k(x) = -\lambda_1 p \int_0^x K_1(x,t) \sum\limits_{k=0}^{\infty} A_k p^k dt - \lambda_2 p \int_0^1 K_2(x,t) \sum\limits_{k=0}^{\infty} B_k p^k dt, \\ u_k(a) = \alpha. \end{cases} \tag{2.4}$$

Adding the solution $u_k$ of Eq.(2.3) and (2.4), we obtain the true solutions to nonlinear equations

$$u_n(x) = u_0(x) + u_1(x) + u_2(x) + \dots$$

## 3. Reproducing kernel Hilbert space

We will solve the Eq.(1.1) with the support of the reproducing kernel space theory.

**Definition 3.1.** *([9]) Let $F \neq 0$. The reproducing kernel function $R : F \times F \to C$ of Hilbert space $H$ defined by*

$$(a)\ R(\cdot, x) \in H\ for\ all\ x \in F;$$
$$(b)\ \langle u, R(\cdot, x)\rangle = u(x)\ for\ all\ x \in F\ and\ all\ u \in H.$$

To solve Eq.(1.1), the reproducing kernel space $W_2^2[a,b]$ and $W_2^1[a,b]$ are introduced next.

**Definition 3.2.** *([9]) Reproducing kernel Hilbert space $W_2^2[a,b]$ is defined by*

$$\begin{aligned} W_2^2[a,b] =& \{u(x) \mid u'(x)\ is\ an\ absolutely\ continuous\ real\ value\ function\ in\ [a,b], \\ & u''(x) \in L^2[a,b]\}. \end{aligned}$$

*The inner product and norm in* $W_2^2[a,b]$ *are given as*

$$\langle u,v\rangle_{W_2^2} = \int_a^b (uv + 2u'v' + u''v'')dx, \quad u,v \in W_2^2[a,b],$$
$$\|u\|_{W_2^2} = \sqrt{\langle u,u\rangle_{W_2^2}}.$$

Similar to $W_2^2[a,b]$, we define inner production and norm of $W_2^1[a,b]$ as follows:

$$W_2^1[a,b] = \{u(x) \mid u(x)\ is\ an\ absolutely\ continuous\ real\ value\ function\ in[a,b], \\ u'(x) \in L^2[a,b]\}.$$

$$\langle u,v\rangle_{W_2^1} = u(a)v(a) + \int_a^b u'v'dx, \quad u,v \in W_2^1[a,b],$$
$$\|u\|_{W_2^1} = \sqrt{\langle u,u\rangle_{W_2^1}}.$$

Similar to the proof in [10], we can prove that the space $W_2^2[a,b]$ and $W_2^1[a,b]$ are complete inner product spaces and reproducing kernel spaces with reproducing kernel functions $R_x(y)$ and $r_x(y)$

$$R_x(y) = xy + \frac{xy^2}{2} - \frac{y^3}{6}, \quad y \le x,$$

when $y > x, R_x(y) = R_y(x)$.

$$r_x(y) = \begin{cases} 1 - a + x, y > x, \\ 1 - a + y, y \le x. \end{cases}$$

## 4. Description of the method

In section 2, we describe the Homotopy perturbation method for nonlinear equation. To solve the Eq.(2.4), we introduce reproducing kernel method. Denote a liner operator $L : W_2^2[a,b] \to W_2^1[a,b]$, the Eq.(2.4) is equivalent to

$$\begin{cases} Lu(x) = f(x), \\ u_n(a) = \alpha,\ n = 0,1,2,\dots \end{cases} \tag{4.1}$$

where $Lu(x) = u_n'(x) + q(x)u_n(x)$. It's easy to prove that $L$ is a bounded linear operator. Let $\psi_i(x) = L^* r_s(t)(x_i),\ i = 1,2,\cdots,\ r_s(t)$ is the reproducing kernel, where $\{x_i\}$ is subset on $[a,b]$, and $L^*$ is the adjoint operator of $L$. We can obtain the following conclusions:

**Lemma 4.1.**

$$L^* r_s(t)(x_i) = LR_s(t)(x_i),\ i = 1, 2, \cdots$$

*$r_s(t)$ is the reproducing kernel for $W_2^2[a,b]$.*

*Proof.* According to the properties of the reproducing kernel function and the properties of the conjugate operator,

$$(L^* r_s)(t) = \langle L^* r_s, R_t \rangle_{W_2^2} = \langle r_s, LR_t \rangle_{W_2^1} = (LR_t)(s) = (LR_s)(t).$$

□

**Theorem 4.1.** *If $\{x_i\}_{i=1}^{\infty}$ is a set of mutually distinct dense points defined on $[a,b]$. Then $\{\psi_i(x)\}_{i=1}^{\infty}$ is linearly independent and a complete system of space $W_2^2[a,b]$.*

*Proof.* Assume

$$\sum_{i=1}^{n} c_i \psi_i(x) = 0,$$

because $L$ is invertible, and

$$\sum_{i=1}^{n} c_i \psi_i(x) = \sum_{i=1}^{n} c_i LR_{x_i}(x) = L(\sum_{i=1}^{n} c_i R_{x_i}(x)),$$

that is

$$\sum_{i=1}^{n} c_i R_{x_i}(x) = 0.$$

We derive that $c_i = 0 (i = 1, 2, \ldots, n)$, so $\{\psi_i(x)\}_{i=1}^{\infty}$ is linearly independent. In addition, for $u(x) \in W_2^2[a,b]$, if

$$\langle u(x), \psi_i \rangle_{W_2^2} = u(x_i) = 0, \quad i = 1, 2, \ldots$$

because of the density of $\{x_i\}_{i=1}^{\infty}$ and continuity of $u(x)$, we have $u(x) \equiv 0$. Therefore $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system. □

Let $\varphi_1(x) = R(x,a)$ and $S_{n+1} = span\{\psi_1(x), \psi_2(x), \ldots, \psi_n(x), \varphi_1(x)\}$. Define an orthogonal projection operator $P_n : W_2^2[a,b] \to W_2^1[a,b]$. We can obtain the follow conclusions:

**Theorem 4.2.** *If $u \in W_2^2[a,b]$ is the solution of Eq.(4.1), then $u_n = P_n u$ satisfies the following equations:*

$$\begin{aligned} &\langle u_n, \psi_i \rangle = f(x_i, u(x_i)), \quad i = 1, 2, \ldots, n, \\ &\langle u_n, \varphi_1 \rangle = \alpha. \end{aligned} \tag{4.2}$$

*Proof.* On account of the self-conjugation of the operator $P_n$ and the properties of the reproducing kernel, it can be obtained that

$$\begin{aligned} \langle P_n u, \psi_i \rangle_{W_2^2} = \langle u, \psi_i \rangle_{W_2^2} = \langle u, L^* r_{x_i} \rangle_{W_2^2} = \langle Lu, r_{x_i} \rangle_{W_2^1} = Lu(x_i) \\ = f(x_i, u(x_i)), \quad i = 1, 2, \ldots, n. \end{aligned}$$

$$\langle P_n u, \varphi_1 \rangle_{W_2^2} = \langle u, P_n \varphi_1 \rangle_{W_2^2} = \langle u, \varphi_1 \rangle_{W_2^2} = \langle u, R_a \rangle_{W_2^2} = u(a) = \alpha.$$

□

**Theorem 4.3.** *$u_n(x)$ is the approximate solution of eq.(4.1), and $u_n(x)$ uniformly converges to $u(x)$ on $[a,b]$.*

*Proof.* Obviously, $\|u_n - u\| \to 0$ holds as $n \to \infty$ in $W_2^2[a,b]$. So that $u_n(x)$ is the approximate solution of Eq.(4.1). Besides,

$$| u_n(x) - u(x) | = | \langle u_n - u, R_x(y) \rangle_{W_2^2} | \leq \|u_n - u\|_{W_2^2} \|R_x(y)\|_{W_2^2}.$$

Note that $R_x(y)$ is continuous on $[a,b]$, thus

$$| u_n(x) - u(x) | \leq M \|u_n - u\| \to 0.$$

we obtain $u_n(x)$ converges uniformly to $u(x)$ on $[a,b]$. □

Consequently, $u_n \in S_{N+1}$ is the approximation solution of Eq.(4.1) can be expressed in the form

$$u_n(x) = b_1 \varphi_1 + \sum_{j=1}^{n} a_j \psi_j(x). \tag{4.3}$$

Then substituting Eq.(4.3) into Eq.(4.2), we can obtain the coefficients of $\varphi_1$ and each $\psi_j(x)$

$$\begin{aligned} & b_1 \langle \varphi_1, \varphi_1 \rangle + \sum_{j=1}^{n} a_j \langle \psi_j(x), \varphi_1 \rangle = \alpha, \\ & b_1 \langle \varphi_1, \psi_i \rangle + \sum_{j=1}^{n} a_j \langle \psi_j(x), \psi_i \rangle = f(x_i, u(x_i)). \quad i = 1, 2, \dots, n. \end{aligned} \tag{4.4}$$

Let

$$G = \begin{pmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \psi_1, \varphi_1 \rangle & \langle \psi_2, \varphi_1 \rangle & \dots & \langle \psi_n, \varphi_1 \rangle \\ \langle \varphi_1, \psi_1 \rangle & \langle \psi_1, \psi_1 \rangle & \langle \psi_1, \psi_1 \rangle & \dots & \langle \psi_n, \psi_1 \rangle \\ \langle \varphi_1, \psi_2 \rangle & \langle \psi_1, \psi_2 \rangle & \langle \psi_2, \psi_2 \rangle & \dots & \langle \psi_n, \psi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \varphi_1, \psi_n \rangle & \langle \psi_1, \psi_n \rangle & \langle \psi_2, \psi_n \rangle & \dots & \langle \psi_n, \psi_n \rangle \end{pmatrix}$$

$$f = \begin{pmatrix} \alpha \\ f(x_1, u(x_1)) \\ f(x_2, u(x_2)) \\ \vdots \\ f(x_n, u(x_n)) \end{pmatrix}$$

Thus, we have $(b_1, a_1, a_2, \dots, a_n)^T = G^{-1} f$ as required.

## 5. Numerical examples

To demonstrate the feasibility of the present method, some numerical examples are given to illustrate its effectiveness.

**Example 5.1** Consider the nonlinear integro-differential equation:

$$u'(x) + u(x) + \frac{1}{2} \int_0^x x u^2(t) dt - \frac{1}{4} \int_0^1 t u^3(t) dt = y(x).$$

where $y(x) = 2x + x^2 + \frac{1}{10} x^6 - \frac{1}{32}$, with the initial condition $u(0) = 0$, and the exact solution $u(x) = x^2$. Comparison of the numerical results and absolute error listed in Table 1, our method is more accurate than the method in [4] as $n = 64$. The red line is exact solution and the blue points are the approximate solution in Figure 1.

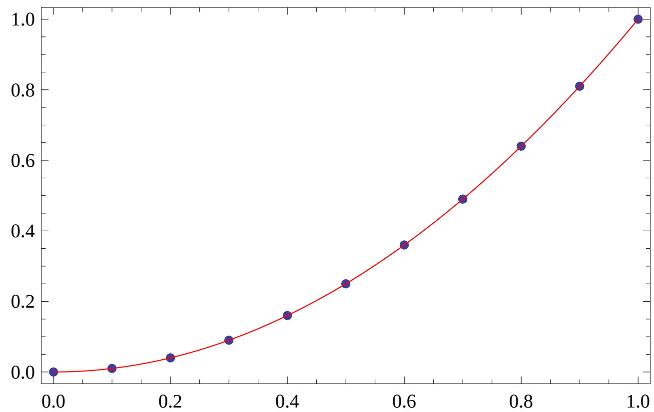


Figure 1: Approximate solution of Example 5.1

Table 1: Numerical result and absolute error for Example 5.1

| $x$ | Exact solution | Presented method | In[4] | Absolute error |
|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.00000 |
| 0.1 | 0.010000 | 0.010002 | 0.010917 | 1.6697E-6 |
| 0.2 | 0.040000 | 0.040003 | 0.041703 | 3.3887E-6 |
| 0.3 | 0.090000 | 0.090005 | 0.092364 | 5.0243E-6 |
| 0.4 | 0.160000 | 0.160006 | 0.162911 | 6.7948E-6 |
| 0.5 | 0.250000 | 0.250008 | 0.253371 | 8.5438E-6 |
| 0.6 | 0.360000 | 0.360010 | 0.364244 | 1.0344E-5 |
| 0.7 | 0.490000 | 0.490012 | 0.493830 | 1.2289E-5 |
| 0.8 | 0.640000 | 0.640014 | 0.642375 | 1.4131E-5 |
| 0.9 | 0.810000 | 0.810016 | 0.810337 | 1.6191E-5 |
| 1.0 | 1.000000 | 1.000018 | 0.998506 | 1.8166E-5 |

**Example 5.2** For the following nonlinear integro-differential equation:

$$u'(x) + u(x) - 2\int_0^x sin(x)u^2(t)dt = cosx + (1-x)sinx + cosxsin^2x.$$

with the initial condition $u(0) = 0$, and the exact solution $u(x) = sinx$. Table 2 illustrates the approximate solution and absolute error, comparison of the numerical results and absolute error listed in Table 2, our method is more accurate than the method in [4] as $n = 64$. Figure 2 describes the image of exact solution and approximate solution by our method.

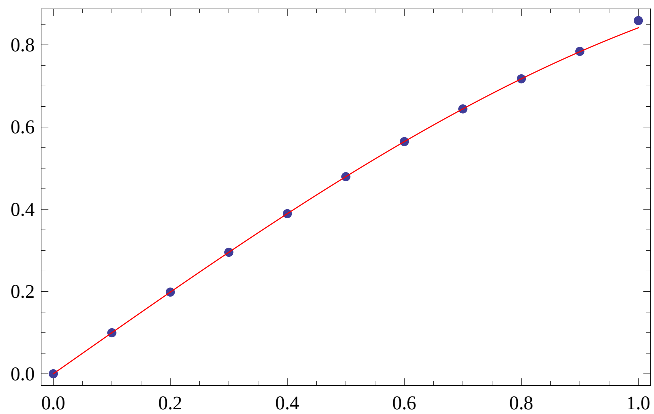


Figure 2: Approximate solution of Example 5.2

Table 2: Numerical result and absolute error for Example 5.2

| $x$ | Exact solution | Presented method | Absolute error in[4] | Absolute error |
|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000032 | 4.11116E-13 |
| 0.1 | 0.099833 | 0.099832 | 0.099825 | 1.52134E-6 |
| 0.2 | 0.198669 | 0.198666 | 0.198678 | 3.09097E-6 |
| 0.3 | 0.295520 | 0.295516 | 0.295603 | 4.56079E-6 |
| 0.4 | 0.389418 | 0.389412 | 0.389605 | 6.13961E-6 |
| 0.5 | 0.479425 | 0.479418 | 0.479398 | 7.68377E-6 |
| 0.6 | 0.564642 | 0.564633 | 0.563598 | 9.28797E-6 |
| 0.7 | 0.644217 | 0.644207 | 0.642606 | 1.10732E-5 |
| 0.8 | 0.717356 | 0.717343 | 0.715049 | 1.28998E-5 |
| 0.9 | 0.783326 | 0.783312 | 0.779882 | 1.51106E-5 |
| 1.0 | 0.841470 | 0.841453 | 0.837683 | 1.75819E-5 |

**Example 5.3** Consider the nonlinear Volterra-Fredholm integro-differential equation

$$u'(x) = \frac{1}{5}x^5 - \int_0^x (u^2(t) - 2)dt, u(0) = 0.$$

with the exact solution given by $u(x) = x^2$. Comparison of the numerical results and absolute error listed in Table 3, our method is more accurate than the method in [5] as $n = 64$. The exact solution and approximate solution by our method are shown in Figure 3.

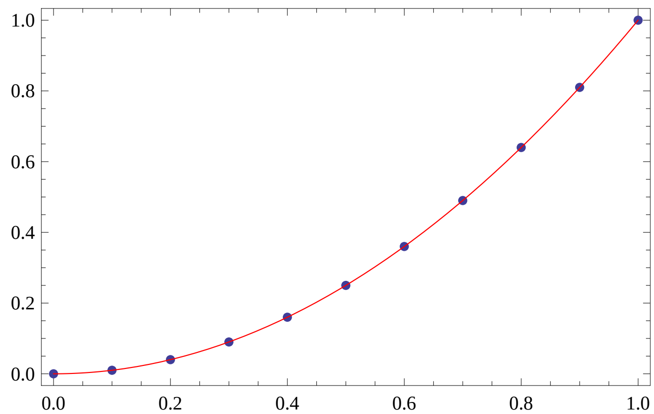


Figure 3: Approximate solution of Example 5.3

Table 3: Numerical result and absolute error for Example 5.3

| $x$ | Exact solution | Presented method | In[5] | Absolute error in[5] | Absolute error |
|---|---|---|---|---|---|
| 0.0 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 0.2 | 0.04000000 | 0.040000000 | 0.03999986 | 1.41640E-7 | 5.77316E-15 |
| 0.4 | 0.16000000 | 0.160000000 | 0.15999094 | 9.05930E-6 | 3.98628E-11 |
| 0.6 | 0.36000000 | 0.360000011 | 0.35989712 | 1.02879E-4 | 1.13421E-8 |
| 0.8 | 0.64000000 | 0.640000628 | 0.63942742 | 5.72582E-4 | 6.28234E-7 |
| 1.0 | 1.00000000 | 1.000014077 | 0.99787295 | 2.12705E-3 | 1.40770E-5 |

## 5. Conclusions

In this work, the simplified reproducing kernel method and homotopy perturbation method were applied successfully for solving the nonlinear V-FIDE. We got the uniformly approximate solution. Besides, compared with the method of Hybrid Legendre polynomials[4], Laplace discrete adomian decomposition method[5], the convergence speed and accuracy of solution were better. Numerical experiments confirm the new algorithm is efficient and stable.

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