Multiple solutions for a modified quasilinear Schrödinger elliptic equation with a non-square diffusion term

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Abstract. In this paper, we establish the results of multiple solutions for a class of modified nonlinear Schrödinger equation involving the p-Laplacian. The main tools used for analysis is the critical points theorems by Ricceri and the dual approach.

Keywords. Quasilinear Schrödinger equation; Dual approach; Multiple solutions; Non-square diffusion term.

1 Introduction

Let Ω be a nonempty bounded open set of the real Euclidean space $\mathbb{R}^N (N \geq 2)$ with C^1-boundary $\partial\Omega$, consider the multiple solutions for the following quasilinear Schrödinger elliptic equation with the p-Laplacian and non-square diffusion term

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-1}u - \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), N < p \leq 2\alpha$, $\lambda \geq 0$ is a parameter, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The equation (1.1) involves a quasilinear and nonconvex diffusion term $\Delta_p(|u|^{2\alpha})\,|u|^{2\alpha-2}u$, so in the literature it is referred as so-called modified nonlinear Schrödinger equation. For the case $p = 2$, the solution of (1.1) is related to standing wave solutions of the following quasilinear Schrödinger equation

$$iz_t + \triangle z - \omega(x)z + \kappa\Delta(h(|z|^2))h'(|z|^2)z + g(x,z) = 0, \ x \in \mathbb{R}^n, \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $\omega : \mathbb{R}^n \to \mathbb{R}$ is a given potential, h and g are real functions, κ is a real constant. Putting $z(t,x) = e^{-i\beta t}u(x)$ in (1.2), where $\beta \in \mathbb{R}$ and $u(x) > 0$ is a real function, then the quasilinear equation (1.2) reduces to the following modified ellipic form

$$-\Delta u + V(x)u - \kappa\Delta(h(|u|^2))h'(|u|^2)u = f(x,u), \ x \in \mathbb{R}^n. \tag{1.3}$$

If $h(s) = s$, then (1.3) turns into a superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = f(x,u), \ x \in \mathbb{R}^n. \tag{1.4}$$

Kurihara [1] used this equation to model the time evolution of the condensate wave function in superfluid film. Moreover, if $h(s) = (1+s)^{\frac{1}{2}}$, the equation (1.3) is transformed to the following elliptic form

$$-\Delta u + V(x)u - \kappa\Delta\big[(1+u^2)^{\frac{1}{2}}\big]\frac{u}{(1+u^2)^{\frac{1}{2}}} = f(x,u), \ x \in \mathbb{R}^n, \tag{1.5}$$

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which is a model of the self-channeling of a high-power ultrashort laser in matter [2, 3].

Many mathematical methods such as dual approach [4–9], iterative techniques [10–15], fixed point theorem [16–19], variational methods [20–27] have been employed to solve the various differential equations. In particalr, by using a constrained minimization argument, Poppenberg et al [26] established the existence of positive ground state solution for quasilinear Schrödinger equation (1.4). Colin and Jeanjean [4], João Marcos and Severo [27] studied the existence of positive solutions for (1.4) by the change of variables. The Nehari method and the symmetric mountain pass lemma were also used to establish the existence of solutions in [28–30]. In [31], Liu et al. studied the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda|u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $\lambda \geq 0, 4\alpha < p+1 < \frac{4\alpha N}{N-2},\ \alpha \geq \frac{1}{2}$, $V \in C(\mathbb{R}^N)$ and

$(\widetilde{\mathbf{V}})$ There exists $V_0 > 0$ such that $V(x) \geq V_0$ in $\mathbb{R}^N$. Moreover, $V(x) \to \infty$ as $|x| \to \infty$, or more generally, for every $M > 0$, $meas(x \in \mathbb{R}^N : V(x) \leq M) < \infty$, in which "meas" denotes the Lebesgue measure in $\mathbb{R}^N$.

The condition $(\widetilde{\mathbf{V}})$ is an essential assumption which guarantees that the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \leq s < \frac{2N}{N-2}$, where

$$E = \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty \right\}$$

is a subspace of $W^{1,2}(\mathbb{R}^N)$ with the norm

$$||u||_E = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right)^{\frac{1}{2}}.$$

Clearly, the assumptions $(\widetilde{\mathbf{V}})$ fails to hold for a general continuous and bounded function. Thus if the potential $V(x)$ fails to satisfy $(\widetilde{\mathbf{V}})$, whether the multiple solutions of problem (1.6) still exist or not? In order to answer this question, in this paper, we investigate the more general modified nonlinear Schrödinger equation (1.1) and get a positive answer, i.e., if the potential $V(x)$ is a general continuous and bounded function, then there exist the multiple solutions to the quasilinear Schrödinger elliptic equation with the p-Laplacian and non-square diffusion term (1.1) under suitable growth conditions.

The rest of this paper is organized as follows. In Section 2, with help of a change of variables, we set up the variational framework for problem (1.1) and give some lemmas of the functional associated with problem (1.1). In Section 3 and Section 4, by using Riccer's critical point theorem, we give the proof of main results.

2 Dual approach

Let $E = W^{1,p}(\Omega)$ $(p \geq 1)$ be the Sobolev spaces with the norm

$$||u|| = \left(\int_{\Omega} (|\nabla u|^p) + V(x)|u|^p dx \right)^{\frac{1}{p}}.$$

We focus on the existence of nontrivial weak solutions of problem (1.1). A function u is called a weak solution of the problem (1.1) if $u \in W_0^{1,p}(\Omega)$ and for any $\varphi \in C_0^\infty(\Omega)$, one has

$$\begin{aligned} &\int_{\Omega} \left[(1 + (2\alpha)^{p-1}|u|^{(2\alpha-1)p}|\nabla u|^{p-2}\nabla u \nabla\varphi + (2\alpha)^{p-1}(2\alpha-1)|u|^{p(2\alpha-1)-2}u|\nabla u|^p\varphi \right] dx \\ &\quad = -\int_{\Omega} V(x)|u|^{p-2}u\varphi dx + \lambda \int_{\Omega} F(x,u)dx, \end{aligned}$$

where $F(x,u)=\int_0^u f(t,\xi)d\xi$. But we notice that the natural functional of problem (1.1)

$$I(u)=\frac{1}{p}\int_\Omega\left[(1+(2\alpha)^{p-1}|u|^{(2\alpha-1)p}|\nabla u|^p\right]dx+\frac{1}{p}\int_\Omega V(x)|u|^p dx-\lambda\int_\Omega F(x,u)dx$$

may not be well defined and not Gâteaux differentiable in the corresponding Sobolev space E.

Thus inspired by [32], we define a function h by

$$\begin{aligned} &h'(t)=\frac{1}{\sqrt[p]{1+(2\alpha)^{p-1}|h(t)|^{p(2\alpha-1)}}}, \quad t\geq 0,\\ &h(0)=0, \quad h(-t)=-h(t), \quad t\leq 0. \end{aligned} \tag{2.1}$$

Let $u=h(v)$, then

$$J(v)=I(h(v))=\frac{1}{p}\int_\Omega|\nabla v|^p+V(x)|h(v)|^p dx-\lambda\int_\Omega F(x,h(v))dx.$$

Moreover the corresponding energy functional $J(v)$ is well defined on $W^{1,p}(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$, if $v\in W^{1,p}(\Omega)$ is a critical point of the functional J, i.e, for any $\varphi\in W^{1,p}(\Omega)$,

$$\begin{aligned}\langle J'(v),\varphi\rangle=&\frac{1}{p}\int_\Omega|\nabla v|^{p-2}\nabla v\nabla\varphi+V(x)|h(v)|^{p-2}h(v)h'(v)\varphi dx\\ &-\lambda\int_\Omega f(x,h(v))h'(v)\varphi dx,\end{aligned}$$

then v is a weak solution of the equation

$$-\Delta_p v=-V(x)|h(v)|^{p-2}h(v)h'(v)+\lambda f(x,h(v))h'(v), \quad x\in\Omega. \tag{2.2}$$

Thus, from (2.1) and (2.2), it is easy to know that $u=h(v)$ is a weak solution of the problem (1.1). In the result, it is sufficient to consider the existence of solutions of (2.2) in $W^{1,p}(\Omega)$.

The following lemma can be found in [28]:

Lemma 1. *The function $h(t)$ enjoys the following properties:*

(h_1) *$h\in C^2$ is uniquely defined, odd, increasing and and invertible in $\mathbb{R}$.*

(h_2) $0<h'(t)\leq 1, \forall t\in\mathbb{R}$.

(h_3) $|h(t)|\leq|t|, \forall t\in\mathbb{R}$.

(h_4) $\lim_{t\to 0}\frac{h(t)}{t}=1$.

(h_5) $|h(t)|\leq(2\alpha)^{\frac{1}{2p\alpha}}|t|^{\frac{1}{2\alpha}}, \forall t\in\mathbb{R}$.

(h_6) $\frac{h(t)}{2}\leq\alpha th'(t)\leq\alpha h(t), \forall t\geq 0, \quad \alpha h(t)\leq\alpha th'(t)|\leq\frac{h(t)}{2}, \forall t\leq 0$.

(h_7) *there exists $a\in(0,(2\alpha)^{\frac{1}{2p\alpha}}]$ such that $h(t)t^{-\frac{1}{2\alpha}}\to a$ as $t\to+\infty$.*

(h_8) *there exists $b_0>0$ such that*

$$|h(t)|\geq\begin{cases} b_0|t|, & \text{if } |t|\leq 1,\\ b_0|t|^{\frac{1}{2\alpha}}, & \text{if } |t|\geq 1,\end{cases}$$

(h_9) *for each $\tau>0$, there exists $\chi(\tau)=m$ if $\tau=m$ and $\chi(\tau)=m+1$ if $\tau\in(m,m+1)$, $m\in N$ such that*

$$|h(\tau t)|\leq\chi(\tau)|h(t)|, \quad \forall t\in\ \mathbb{R}.$$

(h_{10}) $\frac{1}{2}h^2(t)\leq\alpha th'(t)h(t)\leq\alpha h^2(t)$ *for all $t\in\mathbb{R}$.*

Notice that $p > N$, $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact. Thus there exists a positive constant $c > 0$ such that

$$||u||_\infty \le c||u||, \quad \forall u \in W^{1,p}(\Omega), \tag{2.3}$$

where $||u||_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$.

Different from [29–31], the following assumption on potential is adopted in this paper:

(V) $V \in C(\Omega)$ and there exist two constants $V_0, V_1 > 0$ such that

$$V_0 \le V(x) \le V_1, \quad x \in \Omega.$$

Now define two functionals $\Phi, \Psi : E \to \mathbb{R}$ as follows:

$$\begin{aligned}\Phi(v) &= \frac{1}{p}\int_\Omega (|\nabla v|^p + V(x)|h(v)|^p)\, dx,\\ \Psi(v) &= -\int_\Omega F(x, h(v))dx.\end{aligned}$$

For any $v, w \in E$, we have $\Phi, \Psi \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle \Phi'(v), w\rangle &= \int_\Omega \left(|\nabla v|^{p-2}\nabla v\nabla w + V(x)|h(v)|^{p-2}h(v)h'(v)w\right) dx,\\ \langle \Psi'(v), w\rangle &= -\int_\Omega f(x, h(v))h'(v)wdx.\end{aligned}$$

Lemma 2. *For fixed $r > 0$ with $\Phi(v) \le r, v \in E$, then there exists a constant $\varrho > 0$ independent of r such that*

$$\Phi(v) \ge \varrho||v||^p. \tag{2.4}$$

Proof. Let $v \ne 0$, otherwise, the conclusion holds. In the following, we argue by contradiction to prove (2.4).

Suppose that there exists a sequence $\{v_n\} \subset E$ satisfying $v_n \ne 0$ for all $n \in N$ such that

$$\int_\Omega \frac{|\nabla v_n|^p}{||v_n||^p}dx + \int_\Omega \frac{V(x)|h(v_n)|^p}{||v_n||^p}dx \to 0, \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Set $w_n = \frac{v_n}{||v_n||}$, then $||w_n|| = 1$. Noticing that the compactness of embedding $E \hookrightarrow L^s$ for $s \in [1, +\infty)$, up to a subsequence, we have $w_n(x) \rightharpoonup w(x)$ in E, $w_n(x) \to w(x)$ in $L^s(\Omega)$ for $s \in [1, +\infty)$ and $w_n(x) \to w(x)$ a.e on Ω. It follows from (2.5) that

$$\int_\Omega |\nabla w_n|^p dx \to 0, \quad \int_\Omega \frac{V(x)|h(v_n)|^p}{||v_n||^p}dx \to 0, \quad \int_\Omega V(x)w_n^p dx \to 1. \tag{2.6}$$

We assert that for any $\varepsilon > 0$, there exists a constant $\tau > 0$ independent of n such that $meas(B_n) \le \varepsilon$, where $meas(\cdot)$ denotes the standard Lebesgue measure and $B_n = \{x \in \Omega : |v_n| \ge \tau\}$.

In fact, if not, there exists $\varepsilon_0 > 0$ such that $meas(A_n) \ge \varepsilon_0$, where $A_n = \{x \in \Omega : |v_n| \ge n\}$. By (h_8) and the Fatou Lemma, we get

$$\begin{aligned}\int_\Omega (|\nabla v_n|^p + V(x)|h(v_n)|^p)\, dx &\ge \int_\Omega V(x)|h(v_n)|^p dx\\ &\ge \int_\Omega V_0 b_0 |v_n|^{\frac{p}{2\alpha}} dx \ge V_0 b_0 n^{\frac{p}{2\alpha}}\varepsilon_0 \to +\infty, \quad \text{as} \quad n \to \infty.\end{aligned} \tag{2.7}$$

The above fact contradicts with the boundedness of $\Phi(\{v_n\})$. Therefore the above conclusion is valid.

Next it follows from the Hölder inequality and the Sobolev embedding theorem that there exists ε small enough such that

$$\begin{aligned}\int_{B_n} V(x)w_n^2dx &\leq [V_1 meas(B_n)]^{\frac{p-2}{p}}||w_n||_p^2 \leq [V_1 meas(B_n)]^{\frac{p-2}{p}}||w_n||_p^2\\ &\leq C_1\varepsilon^{\frac{p-2}{p}} \leq \frac{1}{4}, \quad \forall n,\end{aligned} \tag{2.8}$$

where C_1 is a constant which is independent of ε.

On the other hand, noticing that if $|v_n(x)| \leq \tau$, then

$$\frac{|v_n(x)|}{\tau} \leq 1,$$

by (h_8), we have

$$\left|h\left(\frac{|v_n(x)|}{\tau}\right)\right| \geq b_0\frac{|v_n(x)|}{\tau}. \tag{2.9}$$

Thus it follows from (h_9) of Lemma 1, (2.9) and (2.5) that

$$\begin{aligned}\int_{\Omega\setminus B_n} V(x)w_n^pdx &= \int_{\Omega\setminus B_n} V(x)\frac{|v_n|^p}{||v_n||^p}dx = \tau^p\int_{\Omega\setminus B_n} V(x)\frac{|\frac{v_n}{\tau}|^p}{||v_n||^p}dx\\ &\leq \left(\frac{\tau}{b_0}\right)^p\int_{\Omega\setminus B_n} V(x)\frac{|h\left(\frac{v_n}{\tau}\right)|^p}{||v_n||^p}dx\\ &\leq \chi\left(\frac{1}{\tau}\right)\left(\frac{\tau}{b_0}\right)^p\int_{\Omega\setminus B_n} V(x)\frac{|h\left(v_n\right)|^p}{||v_n||^p}dx\\ &\to 0, \quad \text{as} \quad n\to\infty.\end{aligned} \tag{2.10}$$

Combining (2.8) and (2.10), one has

$$\int_\Omega V(x)w_n^pdx = \int_{B_n} V(x)w_n^pdx + \int_{\Omega\setminus B_n} V(x)w_n^pdx \leq \frac{1}{4} + o(1),$$

which implies that $1 \leq \frac{1}{4}$, a contradiction. So the proof is completed. □

Lemma 3. *Assume that $V(x)$ satisfies* $(\mathbf{V})$, *then Φ' is coercive, hemicontinuous and uniformly monotone.*

Proof. Firstly, by (h_4) and (h_7) of Lemma 2.1, we have

$$\lim_{t\to 0}\frac{|h(t)|^p}{t^p} = 1, \qquad \lim_{t\to\infty}\frac{|h(t)|^p}{t^{\frac{p}{2\alpha}}} = a^p,$$

which implies that for any sufficiently small $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|h(t)|^p \geq (1-\epsilon)t^p - C_\epsilon t^{\frac{p}{2\alpha}}, \ t\in(0,+\infty). \tag{2.11}$$

On the other hand, for any $v \in E$ with $||v|| > 1$, (h_{10}) of Lemma 1 and (2.11) yield

$$\begin{aligned}\frac{\langle\Phi'(v),v\rangle}{||v||} &= \frac{\int_\Omega\left(|\nabla v|^p + V(x)|h(v)|^{p-2}h(v)h'(v)v\right)dx}{||v||}\\ &\geq \frac{\int_\Omega\left(|\nabla v|^p + V(x)|h(v)|^p\right)dx}{2\alpha||v||}\\ &\geq \frac{\int_\Omega\left(|\nabla v|^p + V(x)[(1-\epsilon)v^p - C_\epsilon v^{\frac{p}{2\alpha}}]\right)dx}{2\alpha||v||}.\end{aligned} \tag{2.12}$$

Notice that $E \hookrightarrow L^s$ for $s \in [p, p^*)$ is continuous, then for any $v \in E$ with $||v|| > 1$, choose sufficiently small ε such that

$$\begin{aligned} \int_\Omega \left(|\nabla v|^p + V(x)[(1-\epsilon)v^p - C_\epsilon v^{\frac{p}{2\alpha}}] \right) dx &\geq \frac{1}{2}||v||^p - C_\varepsilon V_1 \int_\Omega v^{\frac{p}{2\alpha}} dx \\ &\geq \frac{1}{2}||v||^p - C_\varepsilon V_1 |\Omega|^{1-\frac{1}{2\alpha}} ||v||_{L^p}^{\frac{p}{2\alpha}} \geq \frac{1}{2}||v||^p - \widetilde{C_\varepsilon}||v||^{\frac{p}{2\alpha}}. \end{aligned} \tag{2.13}$$

It follows from $N < p \leq 2\alpha, N \geq 2$, (2.12) and (2.13) that

$$\lim_{||v|| \to \infty} \frac{\langle \Phi'(v), v \rangle}{||v||} = \infty, \tag{2.14}$$

which implies that Φ' is coercive. The fact that Φ' is hemicontinuous can be verified using standard arguments. In addition, with the help of Theorem 26 (A) in [**?**] as well as, $J(v) = I(h(v))$ and the inequality

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p|\xi - \eta|^p, \; p \geq 2, c_p > 0, \; \forall \xi, \eta \in \mathbb{R}^N,$$

we know that Φ' exists and is continuous. □

3 The existence of three solutions

In this section, we show the existence of three solutions of (1.1), the main tool used for analysis is the Riccer's critical point theorem [33, 34], which is given below for reader's convenience.

Lemma 4. *Let E be a separable and reflexive real Banach space, $\Phi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on E^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i)

$$\lim_{||u|| \to +\infty} (\Phi(z) + \lambda\Psi(z)) = +\infty$$

for all $\lambda \in (0, +\infty)$,

Further, assume that there are $r > 0, z_0, z_1 \in E$ such that

(ii) $\Phi(z_0) < r < \Phi(z_1)$,

(iii)

$$\inf_{u \in \Phi^{-1}((-\infty, r))} \Psi(z) > \frac{(\Phi(z_1) - r)\Psi(z_0) + (r - \Phi(z_0))\Psi(z_1)}{\Phi(z_1) - \Psi(z_0)}.$$

Then, there exist an open interval $\Lambda \subset (0, \infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$, the equation

$$\Phi'(z) + \lambda\Psi'(z) = 0$$

has at least three solutions in E, whose norms are less than ρ.

Before stating our main results, we firstly denote two constants,

$$k = c \left(\frac{V_0|\Omega|}{p\varrho} \right)^{\frac{1}{p}}, \quad \mu = \frac{V_0}{V_1|\Omega|},$$

where c, V_0, V_1 and ϱ are defined by (2.3), (**V**) and Lemma 1, $|\Omega|$ is the Lebesgue measure of Ω. And then some assumptions on $F(x, s)$ to be used are also list below:

$(\mathbf{F}_1)$ there exist a function $a(x) \in L^1(\Omega)$ and $0 < \sigma < p$ such that

$$F(x,s) \le a(x)(1 + |s|^{\sigma})$$

for all $(x,s) \in \Omega \times \mathbb{R}$.

$(\mathbf{F}_2)$ $F(x,0) = 0$ for any $x \in \Omega$.

$(\mathbf{F}_3)$ there exists $t_0 \in \mathbb{R}$ with $|t_0| > 1$ such that

$$\sup_{(x,|z|)\in\Omega\times[0,k]} F(x,z) < \mu \frac{\int_\Omega F(x,t_0)dx}{|t_0|^p}.$$

Now we state our main result here.

Theorem 3.1. *Suppose* $(\mathbf{V})$ *and* $(\mathbf{F_1})$-$(\mathbf{F_3})$ *hold. Then there exist an open interval* $\Lambda \subset (0,\infty)$ *and a positive real number* $\rho > 0$ *such that for any* $\lambda \in \Lambda$*, the quasilinear elliptic equation (1.1) has at least three weak solutions whose norms are less than* ρ*.*

Proof. By the definitions of Φ and Ψ, we know that Ψ' is compact and Φ is weakly lower semi-continuous. Further from Lemma 3, we know that $(\Phi')^{-1}$ is well defined and continuous. Now we show that the hypotheses of Lemma 4 are fulfilled.

It follows from $(\mathbf{F_1})$,(2.2) and (2.12)-(2.13) that, for any $\lambda \ge 0$,

$$\Phi(z) + \lambda\Psi(z) \ge \frac{1}{2p}||z||^p - \frac{\widetilde{C_\varepsilon}}{2}||z||^{\frac{p}{2\alpha}} - \lambda c||a||_{L^1}||z||^{\sigma} - \lambda||a||_{L^1},\ \forall z \in\ E.$$

Since $0 < \sigma < p \le 2\alpha$, we have

$$\lim_{||v||\to\infty}(\Phi(z) + \lambda\Psi(z)) = \infty$$

and (i) is verified.

Now let $z_0 = 0, z_1 = s_0 = h^{-1}(t_0),\ |t_0| > 1$, then $|t_0| = |h(s_0)|$. We denote $r = \frac{1}{p}V_0|\Omega|$, then

$$\Phi(z_1) = \frac{1}{p}\int_\Omega V(x)|h(s_0)|^p dx \ge \frac{1}{p}V_0|\Omega||h(s_0)|^p > \frac{1}{p}V_0|\Omega| = r > 0 = \Phi(z_0).$$

Thus, (ii) of Lemma 4 is satisfied.

On the other hand, from $(\mathbf{F_2})$ and $(\mathbf{F_3})$, we get $\int_\Omega F(x,t_0)dx \ge 0$ and

$$\begin{aligned}
&-\frac{(\Phi(z_1) - r)\Psi(z_0) + (r - \Phi(z_0))\Psi(z_1)}{\Phi(z_1) - \Psi(z_0)}\\
&= -r\frac{\Psi(z_1)}{\Phi(z_1)}\\
&= \frac{pr\int_\Omega F(x,h(s_0))dx}{\int_\Omega V(x)|h(s_0)|^p dx}\\
&= \frac{pr\int_\Omega F(x,t_0)dx}{\int_\Omega V(x)|t_0|^p dx}\\
&\ge \frac{pr\int_\Omega F(x,t_0)dx}{|\Omega|V_1|t_0|^p}\\
&= \frac{V_0}{V_1}\frac{\int_\Omega F(x,t_0)dx}{|t_0|^p}.
\end{aligned} \tag{3.1}$$

Next we focus our attention on the case when $v \in E$ with $\Phi(v) \le r$. By (2.2) and (2.3), we have

$$r \ge \Phi(v) \ge \varrho ||v||^p \ge \varrho \left(\frac{||v||_\infty}{c} \right)^p, \tag{3.2}$$

which implies that $|v(x)| \le c \left(\frac{r}{\varrho}\right)^{\frac{1}{p}} = c \left(\frac{V_0|\Omega|}{p\varrho}\right)^{\frac{1}{p}} = k, \quad \forall\, x \in \Omega$. The above inequality and (h_3) of Lemma 2.1 yield

$$\begin{aligned} -\inf_{v \in \Phi^{-1}((-\infty,r])} \Psi(v) &= \sup_{v \in \Phi^{-1}((-\infty,r])} -\Psi(v) \\ &\le \int_\Omega \sup_{|v| \in [0,k]} F(x, h(v)) dx \\ &\le |\Omega| \sup_{(x,|v|) \in \Omega \times [0,k]} F(x, h(v)) \\ &\le |\Omega| \sup_{(x,|h(v)|) \in \Omega \times [0,k]} F(x, h(v)) \\ &= |\Omega| \sup_{(x,|z|) \in \Omega \times [0,k]} F(x, z). \end{aligned} \tag{3.3}$$

From (3.1),(3.3) and $(\mathbf{F_3})$, it is easy to get that the (iii) of Lemma 4 holds.

Thus all the hypotheses of Lemma 4 are satisfied, and hence according to Lemma 4, there exist an open interval $\Lambda \subset (0, \infty)$ and a positive real number $\rho > 0$ such that for any $\lambda \in \Lambda$, the quasilinear elliptic equation (1.1) has at least three weak solutions whose norms are less than ρ.

Theorem 3.2. *Assume* $(\mathbf{V})$ *and* $(\mathbf{F_1})$-$(\mathbf{F_2})$ *and the following condition hold:*

- *(F_3^*) there exists a constant $M > 0$ such that $F(x, z) \le 0, (x, |z|) \in \Omega \times [0, M]$ and $\lim_{|z| \to \infty} F(x, z) > 0$ for $x \in \Omega$ uniformly holds.*

Then there exist an open interval $\Lambda \subset (0, \infty)$ and a positive real number $\rho > 0$ such that for any $\lambda \in \Lambda$, the quasilinear elliptic equation (1.1) has at least three weak solutions whose norms are less than ρ.

Proof. By $(\mathbf{F_1})$, similar as the proof of Theorem 3.1, it is easy to know that the hypotheses (i) of Lemma 4 holds. Thus we only need to verify the hypotheses (ii) and (iii). In fact, it follows from $(\mathbf{F_3^*})$ that, for any $x \in \Omega$, there exists a sufficiently large

$$|t_0| > \max\left\{1, \frac{M}{c}\left(\frac{p\varrho}{V_0|\Omega|}\right)^{\frac{1}{p}}\right\}$$

such that $F(x, t_0) > 0$. We take $z_0 = 0, z_1 = s_0 = h^{-1}(t_0)$, then $1 < |t_0| = |h(s_0)|$. Denote $r = \varrho\left(\frac{M}{c}\right)^p$, we have

$$\begin{aligned} \Phi(z_1) = \frac{1}{p}\int_\Omega V(x)|h(s_0)|^p dx &\ge \frac{1}{p} V_0 |\Omega| |t_0|^p \\ > \varrho \left(\frac{M}{c}\right)^p &= r > 0 = \Phi(z_0). \end{aligned}$$

Thus, (ii) of Lemma 4 is satisfied.

On the other hand, from $(\mathbf{F_2})$ and $(\mathbf{F_3^*})$, we have

$$\begin{aligned}
-\frac{(\Phi(z_1)-r)\Psi(z_0)+(r-\Phi(z_0))\Psi(z_1)}{\Phi(z_1)-\Psi(z_0)} \\
&= -r\frac{\Psi(z_1)}{\Phi(z_1)} \\
&= \frac{pr\int_\Omega F(x,h(s_0))dx}{\int_\Omega V(x)|h(s_0)|^p dx} \\
&= \frac{pr\int_\Omega F(x,t_0)dx}{\int_\Omega V(x)|t_0|^p dx} \\
&\geq \frac{p\varrho\left(\frac{M}{c}\right)^p\int_\Omega F(x,t_0)dx}{|\Omega|V_1|t_0|^p} \\
&> 0.
\end{aligned} \tag{3.4}$$

Moreover, for $\Phi(v)\leq r,\ v\in E$, by (2.3) and Lemma 2, we have

$$\begin{aligned}
|v(x)| \leq ||v||_\infty \leq c||v|| \leq c\left(\frac{\Phi(v)}{\varrho}\right)^{\frac{1}{p}} \\
\leq c\left(\frac{r}{\varrho}\right)^{\frac{1}{p}} = M, \quad \forall\ x\in\Omega.
\end{aligned}$$

The above inequality and (h_3) of Lemma 1 show that

$$\begin{aligned}
-\inf_{v\in\Phi^{-1}((-\infty,M])}\Psi(v) = \sup_{v\in\Phi^{-1}((-\infty,r])} -\Psi(v) \\
&\leq \int_\Omega \sup_{|v|\in[0,M]} F(x,h(v))dx \\
&\leq |\Omega| \sup_{(x,|v|)\in\Omega\times[0,M]} F(x,h(v)) \\
&\leq |\Omega| \sup_{(x,|h(v)|)\in\Omega\times[0,M]} F(x,h(v)) \\
&= |\Omega| \sup_{(x,|z|)\in\Omega\times[0,M]} F(x,z) \\
&\leq 0.
\end{aligned} \tag{3.5}$$

(3.3) and (3.5) show that the (iii) of Lemma 4 holds.

According to Lemma 4, the conclusion of Theorem 3.2 also holds. □

4 The existence of infinitely many solutions

In this section, we use an infinitely many critical points theorem to obtain the multiple solutions result of the problem (1.1).

Let E be a reflexive real Banach space, $\Phi : E \to \mathbb{R}$ be a (strongly) continuous, coercive sequentially weakly lower semi- continuous and Gâteaux differentiable functional, $\Psi : E \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and Gâteaux differentiable functional.

For all $r > \inf_E \Phi$, let

$$\varphi(r) = \inf_{z\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{z\in\Phi^{-1}((-\infty,r))} \Psi(z)\right) - \Psi(z)}{r - \Phi(z)},$$

and

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \quad \delta = \liminf_{r\to(\inf_E \Phi)^+} \varphi(r).$$

Lemma 5. *[35] Suppose E, Φ, Ψ satisfy the above assumptions, then the following conclusions hold:*

(a) If $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either the functional $\Phi - \lambda\Psi$ has a global minimum, or there exists a sequence $\{z_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(z_n) = +\infty$.

(b) If $\delta < +\infty$ then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either there exists a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or there exists a sequence $\{z_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(z_n) = \inf_E \Phi$, which weakly converges to a global minimum of Φ.

Suppose $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is continuous and denote

$$l = \liminf_{\kappa\to+\infty} \frac{\int_\Omega \max_{|t|\le\kappa} F(x,t)dx}{\kappa^p}, \quad L = \limsup_{\kappa\to+\infty} \frac{\int_\Omega F(x,\kappa)dx}{\kappa^p}.$$

We state the result of the multiple solutions as follows:

Theorem 4.1. *Assume that and*

$$\frac{l}{L} < \frac{p\varrho}{c^p V_1|\Omega|}$$

hold. Then for any

$$\lambda \in (\frac{V_1|\Omega|}{pL}, \frac{\varrho}{c^p l})$$

the quasilinear elliptic equation (1.1) has an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

Proof. Firstly, for any $v \in E$, define

$$\Phi(v) = \frac{1}{p}\int_\Omega (|\nabla v|^p + V(x)|h(v)|^p)\, dx, \quad \Psi(v) = \int_\Omega F(x, h(v))dx.$$

Then $\Phi : E \to \mathbb{R}$ is a continuous, coercive sequentially weakly lower semi-continuous and Gâteaux differentiable functional, $\Psi : E \to \mathbb{R}$ is a sequentially weakly upper semicontinuous and Gâteaux differentiable functional.

Take $\lambda \in (\frac{V_1|\Omega|}{pL}, \frac{1}{pc^p l})$, and let $\{\kappa_n\}$ be a real sequence satisfying $\lim_{n\to\infty} \kappa_n = \infty$, and so we have

$$l = \liminf_{n\to+\infty} \frac{\int_\Omega \max_{|t|\le\kappa_n} F(x,t)dx}{\kappa_n^p}. \tag{4.1}$$

Letting $r_n = \varrho \left(\frac{\kappa_n}{c}\right)^p$, $n \in \mathbb{N}$ and considering $\Phi(z) < r_n$. According to (2.3) and Lemma 2, we have

$$|z(x)| \le ||z||_\infty \le c||z|| \le c\left(\frac{\Phi(z)}{\varrho}\right)^{\frac{1}{p}} \le c\left(\frac{r_n}{\varrho}\right)^{\frac{1}{p}} = \kappa_n, \quad \forall\, x \in \Omega. \tag{4.2}$$

Consequently, from (4.1) and (h_3) of Lemma 1, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{z\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{z\in\Phi^{-1}((-\infty,r_n))}\Psi(z)\right)-\Psi(z)}{r_n-\Phi(z)} \\
&= \inf_{\Phi(z)<r_n} \frac{\left(\sup_{\Phi(z)<r_n}\Psi(z)\right)-\Psi(z)}{r_n-\Phi(z)} \\
&\le \frac{\sup_{\Phi(z)<r_n}\int_\Omega F(x,h(z))dx}{r_n} \\
&\le \frac{\int_\Omega \max_{|z|\le\kappa_n} F(x,h(z))dx}{r_n} \\
&\le \frac{\int_\Omega \max_{|h(z)|\le\kappa_n} F(x,h(z))dx}{r_n} \\
&= \frac{\int_\Omega \max_{|t|\le\kappa_n} F(x,t)dx}{r_n} \\
&= \frac{c^p\int_\Omega \max_{|t|\le\kappa_n} F(x,t)dx}{\varrho\kappa_n^p}, \quad n\in\mathbb{N},
\end{aligned} \tag{4.3}$$

which implies that

$$\gamma = \liminf_{r\to+\infty}\varphi(r) \le \frac{c^p l}{\varrho} < +\infty.$$

Now we show that the functional $\Phi-\lambda\Psi$ is unbounded from below. To do this, we take a real sequence $\{e_n\}$ such that $\lim_{n\to\infty} e_n = +\infty$. Notice (h_8) of Lemma 1, we have $h(e_n)\ge b_0 e_n^{\frac{1}{2\alpha}} \to\infty,\ n\to\infty$, and then

$$L = \lim_{n\to+\infty}\int_\Omega \frac{F(x,e_n)}{e_n^p}dx = \lim_{n\to+\infty}\int_\Omega \frac{F(x,h(e_n))}{h(e_n)^p}dx. \tag{4.4}$$

Let $w_n(x)=e_n,\ n\in\mathbb{N}, x\in\Omega$, then we have

$$\Phi(w_n) = \frac{1}{p}\int_\Omega V(x)|h(w_n))|^p dx \le \frac{V_1|\Omega|}{p}h^p(e_n),$$

and

$$\Phi(w_n)-\lambda\Psi(w_n) \le \frac{V_1|\Omega|}{p}h^p(e_n) - \lambda\int_\Omega F(x,h(e_n))dx.$$

We divide L into two cases to prove that $\Phi-\lambda\Psi$ is unbounded from below.

Case 1: If $L<+\infty$, choose $0<\epsilon<L-\frac{V_1|\Omega|}{\lambda p}$, then by (4.4), there exists $N_0>0$ such that for any $n>N_0$, we have

$$\int_\Omega F(x,h(e_n))dx > (L-\epsilon)h(e_n)^p.$$

Thus,

$$\begin{aligned}
\Phi(w_n)-\lambda\Psi(w_n) &\le \frac{V_1|\Omega|}{p}h^p(e_n) - \lambda\int_\Omega F(x,h(e_n))dx \\
&\le \frac{V_1|\Omega|}{p}h^p(e_n) - \lambda(L-\epsilon)h^p(e_n) \\
&= h^p(e_n)\left(\frac{V_1|\Omega|}{p}-\lambda(L-\epsilon)\right).
\end{aligned}$$

It follows from the choice of ϵ that $\frac{V_1|\Omega|}{p}-\lambda(L-\epsilon)<0$, and then, one gets $\lim_{n\to\infty}(\Phi(w_n)-\lambda\Psi(w_n)) = -\infty$.

Case 2: If $L = +\infty$, we can choose sufficiently large $M_0 > \frac{V_1|}{\lambda p}\Omega|$, and from (4.4), there exists $N_{M_0} > 0$ such that for any $n > N_{M_0}$, we have

$$\int_\Omega F(x, h(e_n))dx > M_0 h^p(e_n).$$

Consequently,

$$\begin{aligned}\Phi(w_n) - \lambda\Psi(w_n) &\le \frac{V_1|\Omega|}{p} h^p(e_n) - \lambda \int_\Omega F(x, |h(e_n)|)dx \\ &\le \frac{V_1|\Omega|}{p} h^p(e_n) - \lambda M_0 h^p(e_n) \\ &= h^p(e_n)\left(\frac{V_1|\Omega|}{p} - \lambda M_0\right).\end{aligned}$$

It follows from the choice of M_0 that

$$\lim_{n\to\infty} (\Phi(w_n) - \lambda\Psi(w_n)) = -\infty.$$

The above facts show that the functional $\Phi - \lambda\Psi$ is unbounded from below. According to (a) of Lemma 5, the functional $\Phi - \lambda\Psi$ admits a sequence $\{v_n\}$ of critical points, that is, $\{h(v_n)\}$ are exactly the weak solutions of the quasilinear elliptic equation (1.1). □

It follows from Theorem 4.1 that we have the following corollary:

Corollary 1. *Assume* $(\mathbf{V})$ *holds, and* $l < +\infty,\ L = +\infty$. *Then for any*

$$\lambda \in (0, \frac{\varrho}{c^p \widetilde{l}})$$

the quasilinear elliptic equation (1.1) has an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

Denote

$$\widetilde{l} = \liminf_{\kappa\to 0^+} \frac{\int_\Omega \max_{|t|\le\kappa} F(x,t)dx}{\kappa^p}, \quad \widetilde{L} = \limsup_{\kappa\to 0^+} \frac{\int_\Omega F(x,\kappa)dx}{\kappa^p},$$

then with help of (h_3)-(h_4) of Lemma 1 and arguing as in the proof of Theorem 4.1, we easily obtain the following results:

Theorem 4.2. *Assume* $(\mathbf{V})$ *holds, and* $c^p V_1|\Omega|\widetilde{l} < \widetilde{L}p\varrho$. *Then for any*

$$\lambda \in (\frac{V_1|\Omega|}{p\widetilde{L}}, \frac{\varrho}{c^p\widetilde{l}}),$$

the quasilinear elliptic equation (1.1) has an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

Theorem 4.3. *Assume* $(\mathbf{V})$ *holds, and* $\widetilde{l} < +\infty,\ \widetilde{L} = +\infty$. *Then for any*

$$\lambda \in (0, \frac{\varrho}{c^p\widetilde{l}}),$$

the quasilinear elliptic equation (1.1) has an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

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