The existence of nontrivial solutions for a critically coupled Schrödinger system in a bounded domain of $\mathbb{R}^3$*

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Abstract

In this paper, we consider the following coupled Schrödinger system with doubly critical exponents, which can be seen as a counterpart of the Brezis-Nirenberg problem

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1 u^5+\beta u^2v^3, & x\in\Omega,\\ -\Delta v+\lambda_2 v=\mu_2 v^5+\beta v^2u^3, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega,\end{cases}$$

where Ω is a ball in $\mathbb{R}^3$, $-\lambda_1(\Omega)<\lambda_1,\lambda_2<-\frac{1}{4}\lambda_1(\Omega)$, $\mu_1,\mu_2>0$ and $\beta>0$. Here $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω. We show that the problem has at least one nontrivial solution for all $\beta>0$.

Keywords: Coupled Brezis-Nirenberg problem; Nontrivial solutions; Critical exponents; Variational methods.

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1 Introduction and main results

In this paper, we study the following critically coupled perturbed Brezis-Nirenberg system

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1 u^5+\beta u^2v^3, & x\in\Omega,\\ -\Delta v+\lambda_2 v=\mu_2 v^5+\beta v^2u^3, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^3$, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$. Here $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

In recent years, there have been a lot of researches on the following coupled system of the time-dependent nonlinear Schrödinger equations

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 - \Delta\Phi_1 = \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, & x \in \Omega,\ t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 - \Delta\Phi_2 = \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, & x \in \Omega,\ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1,2, & \\ \Phi_j(x,t) = 0, \quad j = 1,2, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \leq 3)$ is the whole domain $\mathbb{R}^N$ or a smooth bounded domain. i is the imaginary unit, $\mu_1, \mu_2 > 0$ and a coupling constant $\beta \neq 0$. System (1.2) arises from many branches of physics, including the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in two different hyperfine states and an application of nonlinear topics to birefringent optical fibers, see more details in [1, 14, 21] and references theiren.

To obtain solitary wave solutions of system (1.2), we set $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$, then (1.2) turns to be the following elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta v u^2, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

For a coupled system, we are interested in the existence of a nontrivial solution (u, v), i.e. (u, v) satisfying the system with both $u \not\equiv 0$ and $v \not\equiv 0$. However, the system problem may have solutions of the form $(u, 0)$ or $(0, v)$ with $u, v \not\equiv 0$, which we call semi-trivial solutions and may cause some difficulties. When $N \leq 3$, system (1.3) is a system problem of subcritical growth. It was first studied by Lin and Wei in [17], who showed that (1.3) has a nontrivial solution when $\Omega = \mathbb{R}^N$ and $0 < \beta < \sqrt{\mu_1\mu_2}$. After that, the existence and multiplicity results have been extensively studied, see e.g. [2, 3, 4, 6, 7, 13, 18, 19, 20, 26, 27, 28] and the references therein.

Recently, there have been some papers studying critical system problems related to (1.3) in which the nonlinearity and coupling terms are of Sobolev critical growth, i.e.

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1|u|^{2^*-2}u + \beta|u|^{\frac{2^*}{2}-2}u|v|^{\frac{2^*}{2}}, & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2|v|^{2^*-2}v + \beta|v|^{\frac{2^*}{2}-2}v|u|^{\frac{2^*}{2}}, & x \in \Omega, \\ u, v > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $2^* = \frac{2N}{N-2}$, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$. Our problem (1.1) is a special case of (1.4) with $N = 3$.

When $\beta = 0$, (1.4) turns out to be the well known Brezis-Nirenberg single equation

$$-\Delta u + \lambda_i u = \mu_i|u|^{2^*-2}u, \qquad u \in H_0^1(\Omega), \quad i = 1,2, \tag{1.5}$$

which has been widely investigated in the past years, see e.g. [5, 8]. It is proved in [5] that (1.5) has a positive least energy solution u_{μ_i} if $-\lambda_1(\Omega) < \lambda_i < 0$ for $N \geq 4$ or $-\lambda_1(\Omega) < \lambda_i < -\lambda_*(\Omega)$ (this set may be vacuous) for $N = 3$, where $\lambda_*(\Omega) = \frac{\pi^2}{4R_0^2}$ with $R_0 = \sup\{R| \ B_R(x) \subset \Omega\}$, (see Theorems 1.1 and 1.2' in [5]). In particular, for Ω being a ball in $\mathbb{R}^3$, $\lambda_*(\Omega) = \frac{1}{4}\lambda_1(\Omega)$.

When $\beta \neq 0$, one easily sees that $(u_{\mu_1}, 0)$ and $(0, u_{\mu_2})$ are both semitrivial solutions of problem (1.4). We are interested in the existence of nontrivial solutions. There are some papers on this respect in the literature, see e.g. [9, 10, 11, 12, 15, 22, 23, 24, 25, 16, 30, 31] and the references therein. Chen and Zou studied the case where $N = 4$ in [11]. They showed that for a special case where $\lambda_1 = \lambda_2 = \lambda$, (1.4) has a positive least energy solution of the form $(\sqrt{k}w, \sqrt{l}w)$ if $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, where w is a positive least energy solution of $-\Delta w + \lambda w = w^3$ in $H_0^1(\Omega)$ and $k, l > 0$ is the unique solution of the linear system $\mu_1 k + \beta l = 1, \beta k + \mu_2 l = 1$. For the general case where $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, by Ekeland's variational principle and the mountain pass theorem, they showed that there exist $0 < \beta_1 \leq \min\{\mu_1, \mu_2\}$, $\beta_2 \geq \max\{\mu_1, \mu_2\}$ such that (1.4) has a positive least energy solution for all $\beta \in (-\infty, 0) \cup (0, \beta_1) \cup (\beta_2, +\infty)$. (1.4) does not have a nontrivial nonnegative solution if $\min\{\mu_1, \mu_2\} \leq \beta \leq \max\{\mu_1, \mu_2\}$ and $\mu_1 \neq \mu_2$. But it is unknown whether the least energy solution exists or not if $\beta \in [\beta_1, \beta_2]$ (see Remark 1.3 in [12]). Recently, by introducing a suitable submanifold, the author in [31] fills the narrow gap of the range of $\beta > 0$ for the existence of positive solutions given in [11] and proved that (1.4) has a positive solution for $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$. When $N = 4$, A. Pistoia and H. Tavares studied the existence of spiked solutions for (1.4) with $\beta > 0$ small or $\beta < 0$ (see [25]).

In [12], Chen and Zou studied the higher dimensional case $N \geq 5$. By using an essential fact that $2^* < 4$ and the mountain pass lemma, they proved that if $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, then (1.4) has a positive least energy solution for any $\beta \neq 0$. When $\lambda_1 = \lambda_2 = \lambda$, they showed that (1.4) has a least energy solution $(\sqrt{k_0}w, \sqrt{l_0}w)$ if $\beta \geq \frac{2}{N-2}\max\{\mu_1, \mu_2\}$, where w is a positive least energy solution of $-\Delta w + \lambda w = |w|^{2^*-2}w$ in $H_0^1(\Omega)$ and (k_0, l_0) satisfies the following system

$$\mu_1 k^{\frac{2^*}{2}-1} + \beta k^{\frac{2^*}{4}-1} l^{\frac{2^*}{4}} = 1, \qquad \beta k^{\frac{2^*}{4}} l^{\frac{2^*}{4}-1} + \mu_2 l^{\frac{2^*}{2}-1} = 1, \quad k, l > 0 \tag{1.6}$$

and $k_0 = \min\{k| \ (k, l) \text{ is a solution of } (1.6)\}$; By an alternative method, Ye and Peng in [30] prove that when $\lambda_1 = \lambda_2$, (1.4) has a least energy solution of the form $(\sqrt{k}w, \sqrt{l}w)$ for all $\beta > 0$.

Chen and Lin in [9] proved the asymptotic behavior of least energy solutions of (1.4) when $N \geq 4$. The existence of signchanging solutions for (1.4) with $\beta < 0$ has been studied by Chen, Lin and Zou in [10] ($N \geq 6$) and by Peng, Peng and Wang in [22] ($N = 5$).

When $N = 3$, problem (1.4) becomes exactly (1.1) (if $u, v > 0$). In [16], Kim proved that when $0 < \mu_1 \leq \mu_2$, there exists sufficiently large β_3 such that for $\beta > \beta_3$, (1.1) has a nontrivial least energy solution if $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$

and $\dfrac{\lambda_1+\lambda_2\theta^2}{1+\theta^2} < -\lambda_*(\Omega)$, where $\theta = \theta(\beta) \to 1$ as $\beta \to +\infty$ and satisfies that $\beta\theta^3 - \mu_2\theta^4 - \beta\theta + \mu_1 = 0$. Kim also proved that (1.1) has a nontrivial solution for small $|\beta|$ if $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < -\lambda_*(\Omega)$. In [30], Ye and Peng showed that when $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < -\lambda_*(\Omega)$, (1.1) has a positive least energy solution for $\beta > \max\{\frac{\sqrt{2}(19\mu_2-\mu_1)}{8}, \frac{\sqrt{2}(19\mu_1-\mu_2)}{8}, \frac{9}{\sqrt{2}\min\{B_{\mu_1}^2, B_{\mu_2}^2\}}\}$, where B_{μ_1}, B_{μ_2} respectively denote the least energy of the positive least energy solution to (1.5). For $\lambda_1 = \lambda_2$, it is proved in [30] that (1.1) has at least one positive solution for all $\beta > 0$, and has a positive least energy solution if $\beta > \max\{\frac{\sqrt{2}(19\mu_2-\mu_1)}{8}, \frac{\sqrt{2}(19\mu_1-\mu_2)}{8}\}$. Guo and Zou showed in [15] that (1.1) has a positive least energy solution when $0 < \beta \leq 2\min\{\mu_1, \mu_2\}$. However, as far as we know, whether problem (1.1) has a nontrivial solution for all $\beta \neq 0$ is unknown yet. It is quite interesting to find out what the optimal ranges of β for the existence of nontrivial solutions might be. The present paper is devoted to this aspect and partially answers this question.

When $\Omega = \Omega_\varepsilon \subset \mathbb{R}^N (N = 3, 4)$ with small shrinking holes as the parameter $\varepsilon \to 0$, A. Pistoia, N. Soave and H. Tavares studied the existence of positive solutions for (1.4) (see [23, 24]).

In this paper, without loss of generality, we may assume that Ω is a ball in $\mathbb{R}^3$. Define $H := H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm defined as $\|(u,v)\|_H = (\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2)^{\frac{1}{2}}$. Weak solutions of (1.1) correspond to critical points of the following functional $I : H \to \mathbb{R}$:

$$I(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) - \frac{1}{6}\int_\Omega (\mu_1|u|^6 + \mu_2|v|^6 + 2\beta|u|^3|v|^3),$$

for any $(u,v) \in H$. We call (u,v) a positive solution of (1.1) if (u,v) is a nontrivial solution and $u, v > 0$ a.e. in Ω. We call (u,v) a positive least energy solution of (1.1) if (u,v) is a positive solution and

$$I(u,v) = \inf\{I(\varphi,\psi)|\ (\varphi,\psi) \text{ is a nontrivial solution of (1.1)}\}.$$

Motivated by [11, 12], we define

$$\begin{aligned} M = \Big\{ \quad & (u,v) \in H|\ u \not\equiv 0, v \not\equiv 0, \int_\Omega (|\nabla u|^2 + \lambda_1 u^2) = \int_\Omega \mu_1|u|^6 + \int_\Omega \beta|u|^3|v|^3, \\ & \int_\Omega (|\nabla v|^2 + \lambda_2 v^2) = \int_\Omega \mu_2|v|^6 + \int_\Omega \beta|u|^3|v|^3 \Big\}. \end{aligned}$$

Then M contains all nontrivial solutions of (1.1). Take $\varphi, \psi \in C_0^\infty(\Omega)\backslash\{0\}$ with $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) = \varnothing$, then there exist $t_1, t_2 > 0$ such that $(t_1\varphi, t_2\psi) \in M$, so $M \neq \varnothing$. Set

$$B := \inf_{(u,v)\in M} I(u,v) = \inf_{(u,v)\in M} \frac{1}{3}\int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2), \tag{1.7}$$

then $B > 0$.

We first consider a special case $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < -\frac{1}{4}\lambda_1(\Omega)$. Recall that it has been showed in [5] and [32] that

$$-\Delta w + \lambda w = w^5, \qquad w \in H_0^1(\Omega) \tag{1.8}$$

has a unique positive least energy solution w with its energy

$$B_1 := \frac{1}{3}\int_\Omega (|\nabla w|^2 + \lambda w^2) = \frac{1}{3}\int_\Omega |w|^6 < \frac{1}{3}S^{\frac{3}{2}}, \tag{1.9}$$

where S is the sharp constant of $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, i.e. $S = \inf\limits_{u\in D^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3}|\nabla u|^2}{(\int_{\mathbb{R}^3}|u|^6)^{\frac{1}{3}}}$. Here $D^{1,2}(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) | |\nabla u| \in L^2(\mathbb{R}^3)\}$. Moreover, it is showed in [29] that

$$(3B_1)^{\frac{2}{3}} = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^2 + \lambda u^2)}{\left(\int_\Omega |u|^6\right)^{\frac{1}{3}}}. \tag{1.10}$$

For any $\mu_1, \mu_2, \beta > 0$, the following system

$$\begin{cases} \mu_1 t^4 + \beta t s^3 = 1, \\ \mu_2 s^4 + \beta s t^3 = 1, \\ t, s > 0 \end{cases} \tag{1.11}$$

has a unique solution (t_0, s_0) such that

$$t_0^2 + s_0^2 = \min\{t^2 + s^2 |\ (t, s) \text{ is a solution of (1.11)}\}$$

(see Lemma 2.9 below). Then our main results are as follows.

Theorem 1.1. *Suppose that $\mu_1, \mu_2, \beta > 0$ and $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < -\frac{1}{4}\lambda_1(\Omega)$ for Ω being a ball in $\mathbb{R}^3$.*

(1) If $\mu_1 = \mu_2$, then the following two conclusions hold.

(a_1) For $\beta \le 2\mu_1$, (1.1) has a unique positive solution of the form (tw, sw);

(a_2) For $\beta > 2\mu_1$, (1.1) has three positive solutions of the form (tw, sw);

Moreover, $(t_0 w, s_0 w)$ is the unique least energy solution of (1.1) with $B = (t_0^2 + s_0^2)B_1$ for all $\beta > 0$.

(2) If $\mu_1 \ne \mu_2$, then there exist $\beta^* > 2\max\{\mu_1, \mu_2\}$ such that the following three conclusions hold.

(b_1) For $\beta < \beta^*$, (1.1) has a unique positive solution of the form (tw, sw);

(b_2) For $\beta = \beta^*$, (1.1) has exactly two positive solutions of the form (tw, sw);

(b_3) For $\beta > \beta^*$, (1.1) has three positive solutions of the form (tw, sw);

Moreover, there exists $2\min\{\mu_1, \mu_2\} < \beta_* < \beta^*$ such that $(t_0 w, s_0 w)$ is the unique least energy solution of (1.1) with $B = (t_0^2 + s_0^2)B_1$ for all $\beta \in (0, \beta_*] \cup [\beta^*, +\infty)$.

Theorem 1.2. *Suppose that $\mu_1, \mu_2 > 0$ and $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < -\frac{1}{4}\lambda_1(\Omega)$ for Ω being a ball in $\mathbb{R}^3$, then (1.1) has a nontrivial solution for all $\beta > 0$. Moreover, if $0 < \beta < 2\sqrt{\mu_1\mu_2}$, then (1.1) has a positive least energy solution (u, v) with $I(u, v) = B$.*

Remark 1.3. *For arbitrary domain $\Omega \subset \mathbb{R}^3$, by the same proof, Theorems 1.2 and 1.1 hold for $-\lambda_1(\Omega) < \lambda_i < -\lambda_*(\Omega)$ if the set $(-\lambda_1(\Omega), -\lambda_*(\Omega)) \neq \varnothing$.*

We give the main idea in the proof of the main theorems. To prove Theorem 1.1, the key point is to search for positive solutions for the system (1.11), which can be viewed as a special case of (1.6) with $N = 3$. However, the existence of solutions to (1.11) is totally different from (1.6) with $N \geq 4$. It has been proved in [11, 30] that (1.6) has a unique positive solution when $N \geq 4$. But for (1.11), by delicate calculations, we show that if $\mu_1 = \mu_2$, then

$$\#\{(t,s)|(t,s) \text{ is a solution of } (1.11)\} = \begin{cases} 1, & \text{if } 0 < \beta \leq 2\mu_1, \\ 3, & \text{if } \beta > 2\mu_1 \end{cases}$$

and if $\mu_1 \neq \mu_2$, then

$$\#\{(t,s)|(t,s) \text{ is a solution of } (1.11)\} = \begin{cases} 1, & \text{if } 0 < \beta < \beta^*, \\ 2, & \text{if } \beta = \beta^*, \\ 3, & \text{if } \beta > \beta^*, \end{cases}$$

here we denote $\#A$ to be the number of elements in a set A. We succeed in proving Theorem 1.1 by getting a solution (t_0, s_0) of (1.11) satisfying $t_0^2 + s_0^2 = \min\{t^2 + s^2|\ (t,s) \text{ is a solution of } (1.11)\}$ and proving the following inequality

$$\begin{cases} \mu_1 t^4 + \beta t s^3 \geq 1, \\ \mu_2 s^4 + \beta s t^3 \geq 1, \\ t, s > 0, \end{cases} \quad \Longrightarrow \quad t^2 + s^2 \geq t_0^2 + s_0^2 \tag{1.12}$$

holds for all $\beta > 0$ if $\mu_1 = \mu_2$ or $\beta \in (0, \beta_*] \cup [\beta^*, +\infty)$ if $\mu_1 \neq \mu_2$.

To prove Theorem 1.2, since (1.1) is a doubly Sobolev critical system, the functional I does not satisfy $(PS)_c$ condition at every level c. We have to deal with two bad possibilities that $(PS)_c$ sequences of I weakly converge to semetrivial solutions or weakly converge to the trivial solution. This difficulty is usually overcome by pulling the energy level down below some critical energy level. We recall that for problem (1.4) with $N \geq 4$, the critical energy level is obtained in [11, 12] by using the important fact that $2^* \leq 4$ and that the system (1.6) has a unique solution. In [16, 30], by requiring that $\beta > 0$ is large enough, they show that for the mountain pass level c, the corresponding function satisfies the $(PS)_c$ condition. However, since we deal with all $\beta > 0$ in (1.1), the methods in [16, 30] does not work here. In (1.1), $2^* = 6 > 4$. The system (1.11), which is a special case of (1.6) with $N = 3$, may have multiple solutions. Then the methods in [11, 12] to get the critical energy level cannot be directly applied to (1.1), To overcome this difficulty, we first prove that for any $\beta > 0$ and any minimizing sequence $\{(u_n, v_n)\} \subset M$ of B, there exist a constant $C_0 > 0$ such that $\int_\Omega |\nabla u_n|^6, \int_\Omega |v_n|^6 \geq C_0$ and that for $0 < \beta < 2\sqrt{\mu_1\mu_2}$, each minimizer of B is a nontrivial critical point of I, which give us a clue that when $\beta > 0$ is small, we could obtain least energy solutions of (1.1) by searching for minimizers of B. Using the uniqueness of positive solutions to (1.11), we show that for $\beta \leq 2\mu_1$ if $\mu_1 = \mu_2$ or $\beta < \beta^*$ if $\mu_1 \neq \mu_2$, $B < \min\{B_{\mu_1} + B_{\mu_2}, \frac{1}{3}(t_0^2 + s_0^2)S^{\frac{3}{2}}\}$. Therefore by using the constrained minimization method, we succeed in proving that each minimizing sequence of B may weakly converge to a positive least energy solution of (1.1) for $\beta < 2\sqrt{\mu_1\mu_2}$.

For $\beta \geq 2\sqrt{\mu_1\mu_2}$, the critical energy level cannot be similarly obtained since (1.11) may have two or three solutions. To do so, without loss of generality, we may assume that $\mu_1 \leq \mu_2$. We introduce a suitable submanifold of the associated Nehari manifold

$$\begin{aligned}\mathcal{M} \ &= \Big\{(u,v) \in H\backslash\{(0,0)\}| \int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) = \int_\Omega (\mu_1|u|^6 + \mu_2|v|^6 \\ &\quad + 2\beta|u|^3|v|^3),\ 2\mu_1 \int_\Omega |u|^6 \leq \beta\tau_0^3 \int_\Omega |u|^3|v|^3,\ \tau_*^3 \int_\Omega |v|^6 \leq \int_\Omega |u|^3|v|^3\Big\}.\end{aligned}$$

where $\tau_0 = \frac{t_0}{s_0}$ and $\tau_* \in (0, \tau_0)$ is chosen to satisfy that $\frac{\tau_*^2+1}{(\mu_1\tau_*^4+\beta\tau_*)^{\frac{1}{2}}} \geq t_0^2 + s_0^2$ (see (3.7) below). Indeed, the subset $\mathcal{M}$ excludes the bad possibility of semitrivial solutions in the Nehari manifold. The definition of $\mathcal{M}$ requires us to learn more about the solutions of (1.11) and the inequality (1.12). Then the constrained minimization method is carried out on $\mathcal{M}$, i.e. $m_\beta := \inf\limits_{(u,v)\in\mathcal{M}} I(u,v) > 0$, We prove that $m_\beta < \frac{1}{3}(t_0^2 + s_0^2)S^{\frac{3}{2}}$ and each minimizing sequence of m_β weakly converges to a nontrivial critical point of I. Then the theorem is proved.

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^3$, $L^p(\Omega)$ $(1 \leq p < +\infty)$ is the usual Lebesgue space with the standard norm $|u|_p$. We use "$\to$, $\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. $B_r(x) := \{y \in \mathbb{R}^3|\ |x-y| < r\}$. C will denote a positive constant unless specified. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In § 2, we prove Theorem 1.1. In § 3, we will prove Theorem 1.2.

2 Proof of Theorem 1.1

In this section, we consider the case where $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < -\frac{1}{4}\lambda_1(\Omega)$. First we give some preliminary results, which are useful to prove the theorem.

Denote

$$\overline{\mu} := \max\{\mu_1, \mu_2\}, \qquad \underline{\mu} := \min\{\mu_1, \mu_2\}.$$

We consider the following function $\phi_\beta : (0, +\infty) \to \mathbb{R}$ defined as

$$\phi_\beta(\tau) = \overline{\mu} + \beta\tau^3 - \beta\tau - \underline{\mu}\tau^4, \quad \forall\, \tau > 0. \tag{2.1}$$

Note that

$$\lim_{\tau\to 0^+} \phi_\beta(\tau) = \overline{\mu} > 0, \quad \phi_\beta(1) = \overline{\mu} - \underline{\mu} \geq 0 \quad \text{and} \quad \lim_{\tau\to+\infty} \phi_\beta(\tau) = -\infty, \tag{2.2}$$

then $\phi_\beta(\tau)$ has at least one positive zero point, i.e. $\{\tau > 0 |\ \phi_\beta(\tau) = 0\} \neq \varnothing$.

Lemma 2.1. *Let $\beta > 0$, the following system*

$$\begin{cases} \mu_1 t^4 + \beta t s^3 = 1, \\ \mu_2 s^4 + \beta s t^3 = 1, \\ t, s > 0 \end{cases} \tag{2.3}$$

has at least one solution.

Proof. Without loss of generality, we may assume that $\mu_1 \leq \mu_2$. Suppose that $\tau > 0$ is a zero point of ϕ_β, i.e. $\mu_2 + \beta\tau^3 = \beta\tau + \mu_1\tau^4$, then

$$\mu_1[\tau(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}]^4 + \beta\tau(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}[(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}]^3 = 1$$

and

$$\mu_2[(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}]^4 + \beta(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}[\tau(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}]^3 = 1,$$

i.e. $(\tau(\mu_1\tau + \beta\tau^4)^{-\frac{1}{4}}, (\mu_1\tau + \beta\tau^4)^{-\frac{1}{4}})$ is a solution of (2.3). Then the lemma is proved. □

For $\beta > 0$, set

$$T_\beta := \{(t, s) |\ (t, s) \text{ is a solution of system (2.3)}\}, \tag{2.4}$$

then Lemma 2.1 shows that $T_\beta \neq \varnothing$.

Lemma 2.2. *Let $\beta > 0$.*

(1) *If $(t, s) \in T_\beta$, then $\phi_\beta(\frac{t}{s}) = 0$ for $\mu_1 \leq \mu_2$ and $\phi_\beta(\frac{s}{t}) = 0$ for $\mu_2 < \mu_1$.*

(2) *If $\phi_\beta(\tau) = 0$, then* $\begin{cases} (\tau(\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}, (\mu_1\tau^4 + \beta\tau)^{-\frac{1}{4}}) \in T_\beta, & \text{if } \mu_1 \leq \mu_2, \\ ((\mu_2\tau^4 + \beta\tau)^{-\frac{1}{4}}, \tau(\mu_2\tau^4 + \beta\tau)^{-\frac{1}{4}}) \in T_\beta, & \text{if } \mu_2 < \mu_1. \end{cases}$

(3)
$$\#T_\beta = \#\{\tau > 0 |\ \phi_\beta(\tau) = 0\}.$$

Proof. (1) If $(t, s) \in T_\beta$, then $\mu_1(\frac{t}{s})^4 + \beta\frac{t}{s} = \frac{1}{s^4} = \mu_2 + \beta(\frac{t}{s})^3$ and $\mu_1 + \beta(\frac{s}{t})^3 = \frac{1}{t^4} = \mu_2(\frac{s}{t})^4 + \beta\frac{s}{t}$. So (1) holds.

(2) is a direct consequence of Lemma 2.1. (3) follows from (1)(2). □

Lemma 2.3. *Let $\beta > 0$, then T_β is a finite set and $1 \leq \#T_\beta \leq 3$.*

Proof. This result follows from Lemma 2.2 and a fact that $1 \leq \#\{\tau > 0 | \ \phi_\beta(\tau) = 0\} \leq 3$. $\square$

In order to prove Theorem 1.1, we need to look for positive zero points of ϕ_β. We first consider a special case where $\mu_1 = \mu_2$.

Lemma 2.4. *Suppose that $\beta > 0$ and $\mu_1 = \mu_2$.*

(1) *If $\beta \leq 2\mu_1$, then $\tau_1 = 1$ is the unique positive zero point of ϕ_β and $\phi_\beta(\tau)(\tau - 1) < 0$ for each $\tau \neq 1$.*

(2) *If $\beta > 2\mu_1$, then ϕ_β has three positive zero points $\tau_1 = \frac{\beta-\sqrt{\beta^2-4\mu_1^2}}{2\mu_1}$, $\tau_2 = 1$ and $\tau_3 = \frac{\beta+\sqrt{\beta^2-4\mu_1^2}}{2\mu_1}$. Moreover, $\phi_\beta(\tau)(\tau - \tau_1)(\tau - 1)(\tau - \tau_3) < 0$ for each $\tau \neq \tau_1, 1, \tau_3$.*

Proof. Suppose that $\mu_1 = \mu_2$, then $\phi_\beta(\tau)$ can be rewritten as

$$\phi_\beta(\tau) = (\tau^2 - 1)(-\mu_1\tau^2 + \beta\tau - \mu_1).$$

Hence $\phi_\beta(\tau) = 0 \Leftrightarrow \tau = 1$ or $-\mu_1\tau^2 + \beta\tau - \mu_1 = 0$.

If $\beta < 2\mu_1$, then $-\mu_1\tau^2 + \beta\tau - \mu_1 < 0$ for all $\tau \in \mathbb{R}$; If $\beta = 2\mu_1$, then $-\mu_1\tau^2 + \beta\tau - \mu_1 = -\mu_1(\tau - 1)^2$. So $\phi_\beta(\tau) = 0 \Leftrightarrow \tau = 1$ if $\beta \leq 2\mu_1$.

If $\beta > 2\mu_1$, then $-\mu_1\tau^2 + \beta\tau - \mu_1 = 0 \Leftrightarrow \tau = \frac{\beta\pm\sqrt{\beta^2-4\mu_1^2}}{2\mu_1}$. So the lemma is proved. $\square$

For the general case where $\mu_1, \mu_2 > 0$, the situation is more complicated. By direct calculations, we have

$$\phi'_\beta(\tau) = 3\beta\tau^2 - \beta - 4\underline{\mu}\tau^3 \qquad \text{and} \qquad \phi''_\beta(\tau) = 6\beta\tau - 12\underline{\mu}\tau^2, \qquad \forall\, \tau > 0.$$

Then $\phi''_\beta(\tau) = 0 \Leftrightarrow \tau = \frac{\beta}{2\underline{\mu}}$. Moreover, $\phi''_\beta(\tau) > 0$ if $\tau \in (0, \frac{\beta}{2\underline{\mu}})$ and $\phi''_\beta(\tau) < 0$ if $\tau > \frac{\beta}{2\underline{\mu}}$. Thus

$$\phi'_\beta(\tau) < \phi'_\beta(\frac{\beta}{2\underline{\mu}}) = \frac{\beta^3}{4\underline{\mu}^2} - \beta, \quad \forall\, \tau \neq \frac{\beta}{2\underline{\mu}}. \tag{2.5}$$

We have to discuss the following cases:

(i) If $0 < \beta \leq 2\underline{\mu}$, then $\phi'_\beta(\frac{\beta}{2\underline{\mu}}) \leq 0$. (2.6)

(ii) If $\beta > 2\underline{\mu}$, then $\phi'_\beta(\frac{\beta}{2\underline{\mu}}) > \phi'_\beta(1) > 0$. Since $\lim_{\tau\to 0^+} \phi'_\beta(\tau) = -\beta$ and $\lim_{\tau\to+\infty} \phi'_\beta(\tau) = -\infty$, there exist $0 < a_\beta < 1 < \frac{\beta}{2\underline{\mu}} < b_\beta$ depending on β such that

$$\phi'_\beta(a_\beta) = \phi'_\beta(b_\beta) = 0 \text{ and } \phi'_\beta(\tau)(\tau - a_\beta)(\tau - b_\beta) < 0 \text{ for all } \tau \neq a_\beta, b_\beta. \tag{2.7}$$

Moreover, by $\phi'_\beta(a_\beta) = 0$ and $a_\beta < 1$ we have

$$\begin{aligned}\phi'_\beta(\frac{1}{a_\beta}) = \frac{3\beta}{a_\beta^2} - \beta - \frac{4\underline{\mu}}{a_\beta^3} \quad &= \beta\left(\frac{1}{a_\beta^4} - 1\right) + \frac{4\underline{\mu}}{a_\beta}\left(1 - \frac{1}{a_\beta^2}\right) \\ &= \left(\frac{1}{a_\beta^2} - 1\right)\left(\frac{\beta}{a_\beta^2} + \beta - \frac{4\underline{\mu}}{a_\beta}\right) > 0.\end{aligned}$$

Then $1 < \frac{1}{a_\beta} < b_\beta$.

Lemma 2.5. *Suppose that $\beta > 0$ and $\mu_1 \neq \mu_2$, then there exists $\beta^* > 2\overline{\mu}$ such that the following three cases hold:*

(1) *If $0 < \beta < \beta^*$, then ϕ_β has a unique positive zero point τ_1 and $\phi_\beta(\tau)(\tau - \tau_1) < 0$ for each $\tau \neq \tau_1$. Moreover, $\tau_1 > 1$ if $0 < \beta \leq 2\underline{\mu}$; $\tau_1 > b_\beta$ if $2\underline{\mu} < \beta < \beta^*$.*

(2) *If $\beta = \beta^*$, then ϕ_{β^*} has exactly two positive zero points τ_1, τ_2. Moreover, $\min\{\tau_1, \tau_2\} = a_{\beta^*}$, $\max\{\tau_1, \tau_2\} > b_{\beta^*}$ and $\phi_{\beta^*}(\tau)(\tau - \max\{\tau_1, \tau_2\}) < 0$ for each $\tau \neq \tau_1, \tau_2$.*

(3) *If $\beta > \beta^*$, then ϕ_β has three positive zero points τ_1, τ_2, τ_3. Let*

$$\tau_{mid} := \sum_{i=1}^{3} \tau_i - \min_{1\leq i\leq 3}\{\tau_i\} - \max_{1\leq i\leq 3}\{\tau_i\}, \tag{2.8}$$

then $\min_{1\leq i\leq 3}\{\tau_i\} < a_\beta < \tau_{mid} < 1 < b_\beta < \max_{1\leq i\leq 3}\{\tau_i\}$. Moreover, $\phi_\beta(\tau)(\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3) < 0$ for each $\tau \neq \tau_1, \tau_2, \tau_3$.

Proof. By (2.5) and (2.6), we easily see that when $0 < \beta \leq 2\underline{\mu}$, $\phi_\beta(\tau)$ is strictly decreasing on $(0, +\infty)$. Then we conclude from (2.2) that there exists a unique $\tau_1 > 0$ such that $\phi_\beta(\tau_1) = 0$ and $\phi_\beta(\tau) > 0$ for $0 < \tau < \tau_1$, $\phi_\beta(\tau) < 0$ for $\tau > \tau_1$. Moreover, by $\phi_\beta(1) > 0$, we have $\tau_1 > 1$.

For $\beta > 2\underline{\mu}$, by (2.7) we see that $\phi_\beta(\tau)$ is strictly decreasing on $(0, a_\beta]$, $[b_\beta, +\infty)$ and strictly increasing on $[a_\beta, b_\beta]$. Then $\phi_\beta(b_\beta) > \phi_\beta(1) > 0$. So by (2.2), we see that

$$\min_{\tau\in(0,b_\beta]} \phi_\beta(\tau) = \phi_\beta(a_\beta) \quad \text{and} \quad \phi_\beta \text{ has a unique zero point on } [b_\beta, +\infty). \tag{2.9}$$

We consider the following function

$$h_\beta(\tau) := 4\phi_\beta(\tau) - \tau\phi'_\beta(\tau) = 4\overline{\mu} + \beta\tau^3 - 3\beta\tau, \quad \tau > 0. \tag{2.10}$$

Then it follows from (2.7) that

$$4\phi_\beta(a_\beta) = h_\beta(a_\beta) \quad \text{and} \quad 4\phi_\beta(b_\beta) = h_\beta(b_\beta). \tag{2.11}$$

Note that $h'_\beta(\tau) = 3\beta(\tau^2 - 1) = 0, \tau > 0 \Leftrightarrow \tau = 1$ and $h'_\beta(\tau)(\tau - 1) > 0$, $\forall \tau \neq 1$. Then

$$\min_{\tau>0} h_\beta(\tau) = h_\beta(1) = 4\overline{\mu} - 2\beta.$$

If $2\underline{\mu} < \beta \leq 2\overline{\mu}$, then $h_\beta(\tau) > h_\beta(1) \geq 0$ for all $\tau \neq 1$. By (2.11) we have $\phi_\beta(a_\beta) > 0$, which and (2.9) imply that ϕ_β has a unique positive zero point τ_1 with $\tau_1 > b_\beta$ and $\phi_\beta(\tau) > 0$ for $0 < \tau < \tau_1$, $\phi_\beta(\tau) < 0$ for $\tau > \tau_1$.

If $\beta > 2\overline{\mu}$, then $h_\beta(1) < 0$. There exist $0 < c_\beta < 1 < d_\beta$ such that $h_\beta(c_\beta) = h_\beta(d_\beta) = 0$ and $h_\beta(\tau)(\tau - c_\beta)(\tau - d_\beta) > 0$, $\forall \tau \neq c_\beta, d_\beta$. Moreover,

$$h_\beta(\tau) = 4(\overline{\mu} - \underline{\mu}) - \tau^3 \phi_\beta'(\frac{1}{\tau}). \tag{2.12}$$

then $h_\beta(\frac{1}{a_\beta}) > 0$ and $h_\beta(\frac{1}{b_\beta}) > 0$. Thus we have $\frac{1}{b_\beta} < c_\beta < d_\beta < \frac{1}{a_\beta} < b_\beta$.

We rewrite $\phi_\beta'(a_\beta) = 0$ and $h_\beta(c_\beta) = 0$ as

$$\frac{3a_\beta^2 - 1}{a_\beta^3} = \frac{4\underline{\mu}}{\beta}, \qquad 3c_\beta - c_\beta^3 = \frac{4\overline{\mu}}{\beta}. \tag{2.13}$$

Then $\frac{\sqrt{3}}{3} < a_\beta < 1$ and $\lim_{\beta \to +\infty} c_\beta = 0$. There exists β sufficiently large such that $c_\beta < \frac{\sqrt{3}}{3} < a_\beta < d_\beta$, hence $h_\beta(a_\beta) < 0$. Since ϕ_β is continuous with respect to β and $h_{2\overline{\mu}}(a_{2\overline{\mu}}) > 0$, there exists $\beta^* > 2\overline{\mu}$ such that $h_{\beta^*}(a_{\beta^*}) = 0$, i.e. $\phi_{\beta^*}(a_{\beta^*}) = 0$ and $a_{\beta^*} = c_{\beta^*}$. Then ϕ_{β^*} has exactly two positive zero points, denoted by τ_1, τ_2 with $\min\{\tau_1, \tau_2\} = a_{\beta^*}$ and $\max\{\tau_1, \tau_2\} > b_{\beta^*}$.

By $a_{\beta^*} < 1 < b_{\beta^*}$ and (2.9), we have $\phi_{\beta^*}(\tau) \geq 0$ for all $0 < \tau < 1$. For each $2\overline{\mu} < \beta < \beta^*$, we see that $\beta(\tau^3 - \tau) > \beta^*(\tau^3 - \tau)$ for $0 < \tau < 1$. Then

$$\phi_\beta(\tau) = \overline{\mu} - \underline{\mu}\tau^4 + \beta(\tau^3 - \tau) > \overline{\mu} - \underline{\mu}\tau^4 + \beta^*(\tau^3 - \tau) = \phi_{\beta^*}(\tau) \geq 0, \quad \forall\, 0 < \tau < 1,$$

which implies that $\phi_\beta(a_\beta) > 0$. Then we see from (2.9) that ϕ_β has a unique positive zero point τ_1 with $\tau_1 > b_\beta$ and $\phi_\beta(\tau)(\tau - \tau_1) < 0$ for each $\tau \neq \tau_1$.

For any $\beta > \beta^*$, similarly, we see that $\phi_\beta(a_{\beta^*}) < \phi_{\beta^*}(a_{\beta^*}) = 0$, which and (2.7)(2.9) show that ϕ_β has three positive zero points, denoted by τ_1, τ_2, τ_3 with $\min_{1 \leq i \leq 3}\{\tau_i\} < a_\beta < \tau_{mid} < 1 < b_\beta < \max_{1 \leq i \leq 3}\{\tau_i\}$ and $\phi_\beta(\tau)(\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3) < 0$ for each $\tau \neq \tau_1, \tau_2, \tau_3$. $\square$

Remark 2.6. *(1) The positive zero points τ_i obtained in Lemma 2.5 are indeed dependent on β, i.e. $\tau_i = \tau_i(\beta)$.*

(2) When $\mu_1 = \mu_2$, by (2.12) we have $h_\beta(\tau) = -\tau^3 \phi_\beta'(\frac{1}{\tau})$. For $\beta > 2\overline{\mu} = 2\mu_1$, we have $c_\beta = \frac{1}{b_\beta}$ and $d_\beta = \frac{1}{a_\beta}$. Then $c_\beta < a_\beta < d_\beta$. So $4\phi_\beta(a_\beta) = h_\beta(a_\beta) < 0$, which and (2.7) show that $\phi_\beta(\tau)$ has three positive zero points for $\mu_1 = \mu_2$ and $\beta > 2\mu_1$.

Lemma 2.7. *When $\beta = \beta^*$, then $a_{\beta^*} = \left(\frac{5(\beta^*)^2 - 8\mu_1\mu_2 - 4\sqrt{(4\mu_1\mu_2 - (\beta^*)^2)(\mu_1\mu_2 - (\beta^*)^2)}}{3(\beta^*)^2}\right)^{\frac{1}{2}}$.*

Proof. When $\beta = \beta^*$, by $\phi_{\beta^*}(a_{\beta^*}) = \frac{1}{4}h_{\beta^*}(a_{\beta^*}) = 0$ and (2.13), we see that

$$\frac{3a_{\beta^*}^2 - 1}{a_{\beta^*}^3}(3a_{\beta^*} - a_{\beta^*}^3) = \frac{16\mu_1\mu_2}{(\beta^*)^2},$$

i.e. a_{β^*} is a positive solution of the equation $3a_{\beta^*}^4 + [\frac{16\mu_1\mu_2}{(\beta^*)^2} - 10]a_{\beta^*}^2 + 3 = 0$. By $\beta^* > 2\overline{\mu}$, then $(\beta^*)^2 > 4\mu_1\mu_2$, which implies that $\frac{16\mu_1\mu_2}{(\beta^*)^2} - 10 < -6$. So by $0 < a_{\beta^*} < 1$ we have $a_{\beta^*}^2 = \frac{10-16\frac{\mu_1\mu_2}{(\beta^*)^2} - \sqrt{\left[16\frac{\mu_1\mu_2}{(\beta^*)^2}-10\right]^2-36}}{6}$. The lemma is proved. $\square$

Based on Lemmas 2.2-2.5 we have the following lemma.

Lemma 2.8. *(1) If $\mu_1 = \mu_2$, then* $\#T_\beta = \begin{cases} 1, & \text{if } 0 < \beta \le 2\mu_1, \\ 3, & \text{if } \beta > 2\mu_1. \end{cases}$.

(2) If $\mu_1 \ne \mu_2$, then $\#T_\beta = \begin{cases} 1, & \text{if } 0 < \beta < \beta^*, \\ 2, & \text{if } \beta = \beta^*, \\ 3, & \text{if } \beta > \beta^*. \end{cases}$

By Lemmas 2.2-2.8, we see that

$$T_\beta = \begin{cases} \left\{(\tau_i(\mu_1\tau_i^4 + \beta\tau_i)^{-\frac{1}{4}}, (\mu_1\tau_i^4 + \beta\tau_i)^{-\frac{1}{4}})\right\}_{i=1}^{\#T_\beta}, & \text{if } \mu_1 \le \mu_2, \\ \left\{((\mu_2\tau_i^4 + \beta\tau_i)^{-\frac{1}{4}}, \tau_i(\mu_2\tau_i^4 + \beta\tau_i)^{-\frac{1}{4}})\right\}_{i=1}^{\#T_\beta}, & \text{if } \mu_2 < \mu_1, \end{cases} \tag{2.14}$$

where τ_i is defined in Lemmas 2.4 and 2.5.

Set

$$m_0^2 := \min\left\{\frac{\tau_i^2 + 1}{(\underline{\mu}\tau_i^4 + \beta\tau_i)^{\frac{1}{2}}} \mid 1 \le i \le \#T_\beta\right\} \tag{2.15}$$

and

$$P_\beta := \left\{\tau_i \mid \frac{\tau_i^2 + 1}{(\underline{\mu}\tau_i^4 + \beta\tau_i)^{\frac{1}{2}}} = m_0^2,\ 1 \le i \le \#T_\beta\right\}. \tag{2.16}$$

Lemma 2.9. *Let $\beta > 0$. Then $P_\beta = \{\tau_0\}$ with $\tau_0 = 1$ if $\mu_1 = \mu_2$ and*

$$\tau_0 = \begin{cases} \tau_1, & \text{if } \beta < \beta^*, \\ a_{\beta^*}, & \text{if } \beta = \beta^*, \\ \tau_{mid}, & \text{if } \beta > \beta^*, \end{cases} \qquad \text{if } \mu_1 \ne \mu_2,$$

where a_β and τ_1, τ_{mid} are given in (2.7) and Lemma 2.5.

Proof. For $\beta \le 2\mu_1$ if $\mu_1 = \mu_2$ or $\beta < \beta^*$ if $\mu_1 \ne \mu_2$, then obviously, by Lemmas 2.4 and 2.5, we see that $P_\beta = \{\tau_1\}$.

For $\beta > 2\mu_1$ if $\mu_1 = \mu_2$ or $\beta \ge \beta^*$ if $\mu_1 \ne \mu_2$, to prove the lemma, we consider the following function introduced in [30]:

$$F_\beta(\tau) = \frac{1 + \tau^2}{(\overline{\mu} + \underline{\mu}\tau^6 + 2\beta\tau^3)^{\frac{1}{3}}}, \qquad \tau > 0.$$

Note that $F'_\beta(\tau) = \frac{2\tau}{(\overline{\mu}+\underline{\mu}\tau^6+2\beta\tau^3)^{\frac{4}{3}}}\phi_\beta(\tau)$, where $\phi_\beta(\tau)$ is defined in (2.1). Moreover, by $\phi_\beta(\tau_i) = 0$, we have $[F_\beta(\tau_i)]^{\frac{3}{2}} = \frac{1+\tau_i^2}{(\underline{\mu}\tau_i^4+\beta\tau_i)^{\frac{1}{2}}}$. It is enough to consider $F_\beta(\tau_i), 1 \le i \le \#T_\beta$.

If $\mu_1 \ne \mu_2$ and $\beta = \beta^*$, then we conclude from Lemma 2.5 that $F'_{\beta^*}(\tau)(\tau - \max\{\tau_1,\tau_2\}) < 0$ for all $\tau \ne \tau_1, \tau_2$. Hence $F_{\beta^*}(\tau)$ is strictly increasing on $(0, \max\{\tau_1,\tau_2\}]$ and strictly decreasing on $[\max\{\tau_1,\tau_2\}, +\infty)$. So $F_{\beta^*}(\min\{\tau_1,\tau_2\}) < F_{\beta^*}(\max\{\tau_1,\tau_2\})$.

If $\mu_1 \ne \mu_2$ and $\beta > \beta^*$, then similarly, $F_\beta(\tau)$ is strictly increasing on $(0, \min\limits_{1\le i\le 3}\{\tau_i\}]$, $[\tau_{mid}, \max\limits_{1\le i\le 3}\{\tau_i\}]$ and strictly decreasing on $[\min\limits_{1\le i\le 3}\{\tau_i\}, \tau_{mid}]$, $[\max\limits_{1\le i\le 3}\{\tau_i\}, +\infty)$. Therefore, $F_\beta(\tau_{mid}) < F_\beta(\min\limits_{1\le i\le 3}\{\tau_i\}), F_\beta(\max\limits_{1\le i\le 3}\{\tau_i\})$. Similarly, if $\mu_1 = \mu_2$ and $\beta > 2\mu_1$, we can show that $F_\beta(1) < F_\beta(\tau_1), F_\beta(\tau_3)$. □

Lemma 2.10. *Suppose that (t,s) satisfies*

$$\begin{cases} \mu_1 t^4 + \beta t s^3 \ge 1, \\ \mu_2 s^4 + \beta s t^3 \ge 1, \\ t, s > 0. \end{cases} \tag{2.17}$$

(1) If $\mu_1 = \mu_2$, then $t^2 + s^2 \ge m_0^2$ for all $\beta > 0$.

(2) If $\mu_1 \ne \mu_2$, then there exists $\beta_* \in (2\underline{\mu}, \beta^*)$ such that $t^2 + s^2 \ge m_0^2$ for all $\beta \in (0, \beta_*] \cup [\beta^*, +\infty)$.

Proof. By (2.17), we have

$$(t^2+s^2)^2 \ge g_\beta(\tau) := \frac{(\tau^2+1)^2}{\underline{\mu}\tau^4 + \beta\tau}, \qquad (t^2+s^2)^2 \ge f_\beta(\tau) := \frac{(\tau^2+1)^2}{\overline{\mu} + \beta\tau^3}, \tag{2.18}$$

where $\tau = \frac{t}{s}$ if $\mu_1 \le \mu_2$ and $\tau = \frac{s}{t}$ if $\mu_2 > \mu_1$. Then $(t^2+s^2)^2 \ge \max\{g_\beta(\tau), f_\beta(\tau), \tau > 0\}$. It is enough to show that $\max\{g_\beta(\tau), f_\beta(\tau), \tau > 0\} \ge m_0^4$.

Note that

$$g_\beta(\tau) = f_\beta(\tau) \Longleftrightarrow \phi_\beta(\tau) = 0 \Longleftrightarrow \tau \in \{\tau_i\}_{i=1}^{\#T_\beta}, \tag{2.19}$$

where τ_i is defined in Lemmas 2.4 and 2.5. Moreover,

$$(g_\beta(\tau) - f_\beta(\tau))\mathrm{sgn}(\phi_\beta(\tau)) > 0, \quad \forall\, \tau \notin \{\tau_i\}_{i=1}^{\#T_\beta}. \tag{2.20}$$

By direct calculations, we have

$$\lim_{\tau\to 0^+} g_\beta(\tau) = +\infty, \quad \lim_{\tau\to+\infty} g_\beta(\tau) = \underline{\mu}^{-1} \quad \text{and} \quad g'_\beta(\tau) = \frac{\tau^2+1}{(\underline{\mu}\tau^4+\beta\tau)^2}\phi'_\beta(\tau).$$

The function $g_\beta(\tau)$ has the same monotonicity as $\phi_\beta(\tau)$, i.e. if $\beta \le 2\underline{\mu}$, then $g_\beta(\tau)$ is strictly decreasing on $(0, +\infty)$; if $\beta > 2\underline{\mu}$, then $g_\beta(\tau)$ is strictly decreasing on $(0, a_\beta]$, $[b_\beta, +\infty)$ and strictly increasing on $[a_\beta, b_\beta]$.

Similarly, we have

$$\lim_{\tau\to 0^+} f_\beta(\tau) = \overline{\mu}^{-1}, \quad \lim_{\tau\to+\infty} f_\beta(\tau) = +\infty \qquad \text{and} \qquad f'_\beta(\tau) = \frac{\tau(\tau^2+1)}{(\overline{\mu}+\beta\tau^3)^2} h_\beta(\tau),$$

where $h_\beta(\tau)$ is defined in (2.10). By the same proof as in Lemma 2.5, then $f_\beta(\tau)$ is strictly increasing on $(0,+\infty)$ if $\beta \le 2\overline{\mu}$; $f_\beta(\tau)$ is strictly increasing on $(0, c_\beta]$, $[d_\beta, +\infty)$ and strictly decreasing on $[c_\beta, d_\beta]$ if $\beta > 2\overline{\mu}$. Moreover, $b_\beta > d_\beta$ if $\beta > 2\overline{\mu}$. For $\beta \ge \beta^*$, by $4\phi_\beta(a_\beta) = h_\beta(a_\beta) \le 0$, we see that $a_\beta \ge c_\beta$.

So by Lemmas 2.4-2.9 and (2.20), we have the following four conclusions.

(1) For $\beta \le 2\underline{\mu}$, then

$$\begin{aligned}\max\{g_\beta(\tau), f_\beta(\tau), \tau>0\} &= \begin{cases} g_\beta(\tau), & \text{if } 0<\tau\le\tau_1 \\ f_\beta(\tau), & \text{if } \tau \ge \tau_1 \end{cases} \\ &\ge g_\beta(\tau_1) = m_0^4.\end{aligned}$$

(2) For $\mu_1 \ne \mu_2$ and $\beta = \beta^*$, then

$$\begin{aligned}\max\{g_{\beta^*}(\tau), f_{\beta^*}(\tau), \tau>0\} &= \begin{cases} g_{\beta^*}(\tau), & \text{if } \tau \in (0, \max\{\tau_1,\tau_2\}] \\ f_{\beta^*}(\tau), & \text{if } \tau \ge \max\{\tau_1,\tau_2\} \end{cases} \\ &\ge g_{\beta^*}(\min\{\tau_1,\tau_2\}) = g_{\beta^*}(a_{\beta^*}) = m_0^4.\end{aligned}$$

(3) For $\beta > 2\underline{\mu}$ if $\mu_1 = \mu_2$ or $\beta > \beta^*$ if $\mu_1 \ne \mu_2$, then

$$\begin{aligned}\max\{g_\beta(\tau), f_\beta(\tau), \tau>0\} &= \begin{cases} g_\beta(\tau), & \text{if } \tau \in (0, \min\limits_{1\le i\le 3}\{\tau_i\}] \cup [\tau_{mid}, \max\limits_{1\le i\le 3}\{\tau_i\}] \\ f_\beta(\tau), & \text{if } \tau \in [\min\limits_{1\le i\le 3}\{\tau_i\}, \tau_{mid}] \cup [\max\limits_{1\le i\le 3}\{\tau_i\}, +\infty) \end{cases} \\ &\ge g_\beta(\tau_{mid}) = m_0^4.\end{aligned}$$

(4) For $\mu_1 \ne \mu_2$ and $2\underline{\mu} < \beta < \beta^*$, then we also have

$$\begin{aligned}\max\{g_\beta(\tau), f_\beta(\tau), \tau>0\} &= \begin{cases} g_\beta(\tau), & \text{if } 0<\tau\le\tau_1 \\ f_\beta(\tau), & \text{if } \tau \ge \tau_1 \end{cases} \\ &\ge \min\{g_\beta(a_\beta), g_\beta(\tau_1)\}.\end{aligned}$$

We should consider the value of $g_\beta(a_\beta)$ and $g_\beta(\tau_1)$. We claim that there exists $2\underline{\mu} < \beta_* < \beta^*$ such that

$$(g_\beta(a_\beta) - g_\beta(\tau_1))(\beta - \beta_*) \le 0 \qquad \text{for all } 2\underline{\mu} < \beta < \beta^*, \tag{2.21}$$

where the equality $g_\beta(a_\beta) = g_\beta(\tau_1)$ holds only if $\beta = \beta_*$. Indeed, we consider the following two functions $H, G : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$H(\beta, a) = 3\beta a^2 - \beta - 4\underline{\mu} a^3, \qquad G(\beta,\tau) = \overline{\mu} + \beta\tau^3 - \beta\tau - \underline{\mu}\tau^4.$$

For each $2\underline{\mu} < \beta < \beta^*$, by (2.7) and Lemma 2.5, we see that $H(\beta, a_\beta) = 0$ and $H_a(\beta, a_\beta) = \phi''_\beta(a_\beta) > 0$; $G(\beta, \tau_1) = 0$ and $G_\tau(\beta, \tau_1) = \phi'_\beta(\tau_1) < 0$. By the implicit function theorem, there exist two functions $a(\beta), \tau(\beta) \in C^1(2\underline{\mu}, \beta^*)$ satisfying that

$$a(\beta) = a_\beta, \quad H(\beta, a(\beta)) = 0 \text{ and } a'(\beta) = -\frac{H_\beta(\beta, a)}{H_a(\beta, a)}$$

and $\tau(\beta) = \tau_1$, $G(\beta, \tau(\beta)) = 0$ and $\tau'(\beta) = -\frac{G_\beta(\beta,\tau)}{G_\tau(\beta,\tau)}$. In particular,

$$a'(\beta) = -\frac{3a_\beta^2 - 1}{\phi''_\beta(a_\beta)} < 0, \qquad \tau'(\beta) = -\frac{\tau_1^3 - \tau_1}{\phi'_\beta(\tau_1)} > 0.$$

Then $g_\beta(a_\beta) = g_\beta(a(\beta))$ and $g_\beta(\tau_1) = g_\beta(\tau(\beta))$. Thus by $\phi'_\beta(a_\beta) = 0$ and $\phi'_\beta(\tau_1) < 0$ we see that

$$\frac{dg_\beta(a(\beta))}{d\beta} = -\frac{a_\beta(a_\beta^2+1)^2}{(\beta a_\beta + \underline{\mu} a_\beta^4)^2} = -\frac{16}{9\beta^2 a_\beta}, \qquad \frac{dg_\beta(\tau(\beta))}{d\beta} = -\frac{2\tau_1(\tau_1^2+1)}{(\beta + \underline{\mu}\tau_1^3)^2} > -\frac{16}{9\beta^2\tau_1^2}.$$

Note that $0 < a_\beta < 1 < \tau_1$, we have $|\frac{dg_\beta(\tau(\beta))}{d\beta}| < |\frac{dg_\beta(a(\beta))}{d\beta}|$, i.e. $g_\beta(a(\beta))$, $g_\beta(\tau(\beta))$ are both strictly decreasing with respect to β and $g_\beta(\tau(\beta))$ decreases more slowly than $g_\beta(a(\beta))$. We see from (1) that $g_{2\underline{\mu}}(\tau) > g_{2\underline{\mu}}(\tau_1)$ for all $0 < \tau < \tau_1$ and from (2) that $g_{\beta^*}(a_{\beta^*}) < g_{\beta^*}(\tau_1)$. So by the continuity of the functions $g_\beta(a_\beta)$ and $g_\beta(\tau_1)$ with respect to β, we conclude that (2.21) holds. Then $\max\{g_\beta(\tau), f_\beta(\tau)\} \geq g_\beta(\tau_1) = m_0^4$ for all $0 < \beta \leq \beta_*$.

Therefore, we conclude from (2.18) and (1)-(4) that the lemma is proved. □

The following lemma is a consequence of the proof of Lemma 2.10.

Lemma 2.11. *Suppose that $\mu_1 \neq \mu_2$. Then*

$$\#\left\{\tau > 0 | \sqrt{g_\beta(\tau)} = m_0^2, 2\underline{\mu} < \beta < \beta^*\right\} = \begin{cases} 1, & 2\underline{\mu} < \beta < \beta_*, \\ 2, & \beta = \beta_*, \\ 3, & \beta_* < \beta < \beta^*. \end{cases}$$

Proof of Theorem 1.1

Proof. When $\lambda_1 = \lambda_2$, for any $\beta > 0$, problem (1.1) has at least one positive solution of the form $(t_i w, s_i w)$, where $(t_i, s_i) \in T_\beta, 1 \leq i \leq \#T_\beta$ and w is defined in (1.8).

Let

$$(t_0, s_0) := \begin{cases} (\tau_0(\mu_1\tau_0^4 + \beta\tau_0)^{-\frac{1}{4}}, (\mu_1\tau_0^4 + \beta\tau_0)^{-\frac{1}{4}}), & \text{if } \mu_1 \leq \mu_2, \\ ((\mu_2\tau_0^4 + \beta\tau_0)^{-\frac{1}{4}}, \tau_0(\mu_2\tau_0^4 + \beta\tau_0)^{-\frac{1}{4}}), & \text{if } \mu_2 < \mu_1, \end{cases} \tag{2.22}$$

where τ_0 is given in Lemma 2.9. We see that $(t_0 w, s_0 w)$ is a positive solution of (1.1) and

$$B \leq I(t_0 w, s_0 w) = \frac{1}{3}(t_0^2 + s_0^2)\int_\Omega (|\nabla w|^2 + \lambda w^2) = (t_0^2 + s_0^2)B_1 = m_0^2 B_1,$$

where B_1 is defined in (1.9).

For any $(u,v)\in M$, by the Hölder inequality and (1.10) we have

$$(3B_1)^{\frac{2}{3}}\Big[\Big(\int_\Omega |u|^6\Big)^{\frac{1}{3}}+\Big(\int_\Omega |v|^6\Big)^{\frac{1}{3}}\Big]\le \int_\Omega(|\nabla u|^2+\lambda u^2+|\nabla v|^2+\lambda v^2)=3I(u,v),\quad (2.23)$$

$$(3B_1)^{\frac{2}{3}}\Big(\int_\Omega |u|^6\Big)^{\frac{1}{3}}\le \int_\Omega(|\nabla u|^2+\lambda u^2)\le \mu_1\int_\Omega |u|^6+\beta\Big(\int_\Omega |u|^6\Big)^{\frac{1}{2}}\Big(\int_\Omega |v|^6\Big)^{\frac{1}{2}},\quad (2.24)$$

$$(3B_1)^{\frac{2}{3}}\Big(\int_\Omega |v|^6\Big)^{\frac{1}{3}}\le \int_\Omega(|\nabla v|^2+\lambda v^2)\le \mu_2\int_\Omega |v|^6+\beta\Big(\int_\Omega |v|^6\Big)^{\frac{1}{2}}\Big(\int_\Omega |u|^6\Big)^{\frac{1}{2}}.\quad (2.25)$$

Set

$$t:=\Big(\frac{\int_\Omega |u|^6}{3B_1}\Big)^{\frac{1}{6}}>0,\qquad s:=\Big(\frac{\int_\Omega |v|^6}{3B_1}\Big)^{\frac{1}{6}}>0,$$

then by (2.24) and (2.25) we have

$$1\le \mu_1 t^4+\beta t s^3,\qquad 1\le \mu_2 s^4+\beta s t^3.\quad (2.26)$$

By Lemma 2.10 we see that $t^2+s^2\ge m_0^2$ when $\beta>0$ if $\mu_1=\mu_2$ or when $\beta\in(0,\beta_*]\cup[\beta^*,+\infty)$ if $\mu_1\neq\mu_2$. We conclude from (2.23) that $B=m_0^2B_1=I(t_0w,s_0w)$. So (t_0w,s_0w) is a positive least energy solution of (1.1).

Next we prove that for $\beta>0$ if $\mu_1=\mu_2$ or for $\beta\in(0,\beta_*]\cup[\beta^*,+\infty)$ if $\mu_1\neq\mu_2$, (t_0w,s_0w) is the unique least energy solution of (1.1). Let (u,v) be any a nontrivial least energy solution of (1.1), then $(u,v)\in M$ and $I(u,v)=B$. Similarly by (2.23)-(2.26), we see that

$$\Big(\frac{\int_\Omega |u|^6}{3B_1}\Big)^{\frac{1}{3}}+\Big(\frac{\int_\Omega |v|^6}{3B_1}\Big)^{\frac{1}{3}}=m_0^2$$

and $(3B_1)^{\frac{2}{3}}=\frac{\int_\Omega(|\nabla u|^2+\lambda u^2)}{(\int_\Omega |u|^6)^{\frac{1}{3}}}=\frac{\int_\Omega(|\nabla v|^2+\lambda v^2)}{(\int_\Omega |v|^6)^{\frac{1}{3}}}$. So $u,v\in H_0^1(\Omega)\backslash\{0\}$ is a minimizer of $(3B_1)^{\frac{2}{3}}$ defined in (1.10). Then there exist two Lagrange multipliers $L_1=(\frac{3B_1}{\int_\Omega |u|^6})^{\frac{2}{3}}>0, L_2=(\frac{3B_1}{\int_\Omega |v|^6})^{\frac{2}{3}}>0$ such that

$$-\Delta u+\lambda u-L_1u^5=0,\qquad -\Delta v+\lambda v-L_2v^5=0.$$

Set $(\hat u,\hat v):=(L_1^{\frac{1}{4}}u, L_2^{\frac{1}{4}}v)$, then $\hat u,\hat v$ are both nontrivial solution of $-\Delta u+\lambda u=u^5$ in $H_0^1(\Omega)$ with $\frac{1}{3}\int_\Omega(|\nabla\hat u|^2+\lambda\hat u^2)=\frac{1}{3}\int_\Omega(|\nabla\hat v|^2+\lambda\hat v^2)=B_1$. Thus $\hat u=\hat v=w$. So we see that $(u,v)=(L_1^{-\frac{1}{4}}w, L_2^{-\frac{1}{4}}w)$. We conclude from $(u,v)\in M$ that $(L_1^{-\frac{1}{4}}, L_2^{-\frac{1}{4}})$ is a solution of (2.3) and $L_1^{-\frac{1}{2}}+L_2^{-\frac{1}{2}}=m_0^2$. It follows from Lemmas 2.4-2.9 that $(L_1^{-\frac{1}{4}}, L_2^{-\frac{1}{4}})=(t_0,s_0)$. Therefore we have $(u,v)=(t_0w,s_0w)$. Then the theorem is proved.

□

3 Proof of Theorem 1.2

In this section, we consider the general case where $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < -\frac{1}{4}\lambda_1(\Omega)$. The Aubin-Talenti instanton [1] defined as

$$U(x) = \frac{3^{\frac{1}{4}}}{(1+|x|^2)^{\frac{1}{2}}}$$

satisfies the equation $-\Delta u = u^5$ in $\mathbb{R}^3$ and $\int_{\mathbb{R}^3} |\nabla U|^2 = \int_{\mathbb{R}^3} |U|^6 = S^{\frac{3}{2}}$.

As recalled in section 1, $-\Delta u + \lambda_i u = \mu_i u^5$ in $H_0^1(\Omega)$ has a unique positive least energy solution $u_{\mu_i} \in C^2(\Omega) \cap C(\overline{\Omega})$ with its energy

$$B_{\mu_i} := \frac{1}{3}\int_\Omega (|\nabla u_{\mu_i}|^2 + \lambda_i u_{\mu_i}^2) = \frac{1}{3}\mu_i \int_\Omega |u_{\mu_i}|^6 < \frac{1}{3}\mu_i^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.1}$$

We have

$$\int_\Omega (|\nabla u|^2 + \lambda_i u^2) \geq (3B_{\mu_i})^{\frac{2}{3}} \Big(\mu_i \int_\Omega |u|^6\Big)^{\frac{1}{3}}, \quad \forall\, u \in H_0^1(\Omega). \tag{3.2}$$

Lemma 3.1. *Suppose that $\beta > 0$, then there exists $C_0 > 0$ such that $\int_\Omega |u|^6, \int_\Omega |v|^6 \geq C_0$ for any $(u,v) \in M$ with $I(u,v) \leq \frac{1}{3} m_0^2 S^{\frac{3}{2}}$.*

Proof. For any $(u,v) \in M$, by (3.2) and the Hölder inequality we have

$$(3\mu_1^{\frac{1}{2}} B_{\mu_1})^{\frac{2}{3}} \Big(\int_\Omega |u|^6\Big)^{\frac{1}{3}} \leq \int_\Omega (|\nabla u|^2 + \lambda_1 u^2) \leq \mu_1 \int_\Omega |u|^6 + \beta \Big(\int_\Omega |u|^6\Big)^{\frac{1}{2}} \Big(\int_\Omega |v|^6\Big)^{\frac{1}{2}},$$

$$(3\mu_2^{\frac{1}{2}} B_{\mu_2})^{\frac{2}{3}} \Big(\int_\Omega |v|^6\Big)^{\frac{1}{3}} \leq \int_\Omega (|\nabla v|^2 + \lambda_2 v^2) \leq \mu_2 \int_\Omega |v|^6 + \beta \Big(\int_\Omega |u|^6\Big)^{\frac{1}{2}} \Big(\int_\Omega |v|^6\Big)^{\frac{1}{2}}.$$

By contradiction, we suppose that there exists a sequence $\{(u_n, v_n)\} \subset M$ with $I(u_n, v_n) \leq \frac{1}{3} m_0^2 S^{\frac{3}{2}}$ satisfying that $\int_\Omega |u_n|^6 \to 0$ as $n \to +\infty$. Since $\int_\Omega (|\nabla u_n|^2 + \lambda_1 u_n^2 + |\nabla v_n|^2 + \lambda_2 v_n^2) = 3I(u_n, v_n) \leq m_0^2 S^{\frac{3}{2}}$ and $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, $\{(u_n, v_n)\}$ is uniformly bounded in H. Hence

$$(3\mu_1^{\frac{1}{2}} B_{\mu_1})^{\frac{2}{3}} \leq \mu_1 \left(\int_\Omega |u_n|^6\right)^{\frac{2}{3}} + \beta \left(\int_\Omega |u_n|^6\right)^{\frac{1}{6}} \left(\int_\Omega |v_n|^6\right)^{\frac{1}{2}} \to 0 \ \text{ as } n \to +\infty,$$

which is impossible. Similarly, $\int_\Omega |v_n|^6 \to 0$ is also impossible. So there exists a constant $C_0 > 0$ such that $\int_\Omega |u|^6, \int_\Omega |v|^6 \geq C_0$. Then the lemma is proved. □

Lemma 3.2. *Suppose that $\beta > 0$, for any $u, v \in H_0^1(\Omega) \backslash \{0\}$, there exist $t, s > 0$ such that $(tu, sv) \in M$.*

Proof. It is enough to prove that there exist $t, s > 0$ such that

$$\begin{cases} \int_\Omega(|\nabla u|^2+\lambda_1 u^2) - t^4\mu_1\int_\Omega |u|^6 - ts^3\beta\int_\Omega |u|^3|v|^3 = 0, \\ \int_\Omega(|\nabla v|^2+\lambda_2 v^2) - s^4\mu_2\int_\Omega |v|^6 - st^3\beta\int_\Omega |u|^3|v|^3 = 0. \end{cases} \tag{3.3}$$

By $\beta > 0$ and the first equation in (3.3), we see that

$$s = f_1(t) := \Big(\frac{\int_\Omega(|\nabla u|^2+\lambda_1 u^2) - t^4\mu_1\int_\Omega |u|^6}{t\beta\int_\Omega |u|^3|v|^3}\Big)^{\frac{1}{3}}, \quad 0 < t < T,$$

where $T = (\frac{\int_\Omega(|\nabla u|^2+\lambda_1 u^2)}{\mu_1\int_\Omega |u|^6})^{\frac{1}{4}}$. Moreover, $\lim\limits_{t\to 0^+} f_1(t) = +\infty$ and $\lim\limits_{t\to T^-} f_1(t) = 0$. Then to prove (3.3) is equivalent to show that

$$f_2(t) := \int_\Omega(|\nabla v|^2+\lambda_2 v^2) - [f_1(t)]^4\mu_2\int_\Omega |v|^6 - f_1(t)t^3\beta\int_\Omega |u|^3|v|^3 = 0, \quad 0 < t < T$$

has a solution. Note that $\lim\limits_{t\to 0^+} f_2(t) = -\infty$ and

$$\lim_{t\to T^-} f_2(t) = \int_\Omega(|\nabla v|^2+\lambda_2 v^2) \geq (\lambda_1(\Omega)+\lambda_2)\int_\Omega v^2 > 0$$

since $\lambda_2 > -\lambda_1(\Omega)$, then there exists $0 < t < T$ such that $f_2(t) = 0$. Set $s = f_1(t) > 0$, then $(tu, sv) \in M$. □

Lemma 3.3. *Suppose that $0 < \beta \leq 2\mu_1$ if $\mu_1 = \mu_2$ or $0 < \beta < \beta^*$ if $\mu_1 \neq \mu_2$, then $B < \min\Big\{B_{\mu_1} + B_{\mu_2}, \frac{1}{3}m_0^2 S^{\frac{3}{2}}\Big\}$, where B, B_{μ_i}, m_0^2 are respectively given as in* (1.7),(3.1) *and* (2.15).

Proof. For simplicity, we take $\Omega = B_1(0)$. Then $\lambda_1(\Omega) = \pi^2$. Set $\varphi(x) = \cos(\frac{\pi|x|}{2})$ and

$$w_\varepsilon(x) = \varphi(x)\varepsilon^{-\frac{1}{2}}U(\frac{x}{\varepsilon}) \tag{3.4}$$

for any $\varepsilon > 0$. Then we obtain that (see e.g. Lemma 1.3 in [5]),

$$\int_{B_1(0)} |\nabla w_\varepsilon|^2 = S^{\frac{3}{2}} + \frac{\sqrt{3}}{2}\pi^3\varepsilon + O(\varepsilon^2), \qquad \int_{B_1(0)} |w_\varepsilon|^6 = S^{\frac{3}{2}} + O(\varepsilon^2), \tag{3.5}$$

$$\int_{B_1(0)} |w_\varepsilon|^2 = 2\sqrt{3}\pi\varepsilon + O(\varepsilon^2). \tag{3.6}$$

For $0 < \beta \leq 2\mu_1$ if $\mu_1 = \mu_2$ or $0 < \beta < \beta^*$ if $\mu_1 \neq \mu_2$, then by Lemmas 2.4-2.9, then $T_\beta = \{(t_0, s_0)\}$. Set $C_1 := 2\sqrt{3}\pi\min\{-\lambda_1 - \frac{\pi^2}{4}, -\lambda_2 - \frac{\pi^2}{4}\} > 0$ since $\lambda_1, \lambda_2 < -\frac{\pi^2}{4}$. We consider the maximal point of the following function

$$f(t,s) = \frac{t^2+s^2}{2}(S^{\frac{3}{2}} - C_1\varepsilon + O(\varepsilon^2)) - \frac{\mu_1 t^6 + \mu_2 s^6 + 2\beta t^3 s^3}{6}(S^{\frac{3}{2}} + O(\varepsilon^2)),\ t, s > 0.$$

By $f_t(t,s)=f_s(t,s)=0$, we have

$$\begin{cases} \frac{S^{\frac{3}{2}}-C_1\varepsilon+O(\varepsilon^2)}{S^{\frac{3}{2}}+O(\varepsilon^2)} = \mu_1 t^4+\beta t s^3, \\ \frac{S^{\frac{3}{2}}-C_1\varepsilon+O(\varepsilon^2)}{S^{\frac{3}{2}}+O(\varepsilon^2)} = \mu_2 s^4+\beta s t^3, \end{cases}$$

then $(t,s)=\left(\frac{S^{\frac{3}{2}}-C_1\varepsilon+O(\varepsilon^2)}{S^{\frac{3}{2}}+O(\varepsilon^2)}\right)^{\frac{1}{4}}(t_0,s_0)$.

By Lemma 3.2, there exist $t_\varepsilon, s_\varepsilon>0$ such that $(t_\varepsilon w_\varepsilon, s_\varepsilon w_\varepsilon)\in M$. So we see from (3.4)-(3.6) that

$$\begin{aligned} B \le I(t_\varepsilon w_\varepsilon, s_\varepsilon w_\varepsilon) \le \max_{t,s>0} I(tw_\varepsilon, sw_\varepsilon) &\le \max_{t,s>0} f(t,s) \\ &= \frac{1}{3}(t_0^2+s_0^2)\frac{[S^{\frac{3}{2}}-C_1\varepsilon+O(\varepsilon^2)]^{\frac{3}{2}}}{[S^{\frac{3}{2}}+O(\varepsilon^2)]^{\frac{1}{2}}} \\ &< \frac{1}{3}m_0^2 S^{\frac{3}{2}}, \end{aligned}$$

for ε small enough. For any ball in $\mathbb{R}^3$, applying the above argument, we also get $B<\frac{1}{3}m_0^2S^{\frac{3}{2}}$.

By Lemma 3.2, there exist $\bar{t},\bar{s}>0$ such that $(\bar{t}u_{\mu_1}, \bar{s}u_{\mu_2})\in M$. Then

$$\begin{aligned} B \le I(\bar{t}u_{\mu_1}, \bar{s}u_{\mu_2}) &< \max_{t>0}\left\{\frac{t^2}{2}\int_\Omega (|\nabla u_{\mu_1}|^2+\lambda_1|u_{\mu_1}|^2) - \frac{t^6}{6}\mu_1\int_\Omega |u_{\mu_1}|^6\right\} \\ &\quad + \max_{s>0}\left\{\frac{s^2}{2}\int_\Omega (|\nabla u_{\mu_2}|^2+\lambda_2|u_{\mu_2}|^2) - \frac{s^6}{6}\mu_2\int_\Omega |u_{\mu_2}|^6\right\} \\ &< B_{\mu_1}+B_{\mu_2}. \end{aligned}$$

□

Lemma 3.4. *Let $0<\beta<2\sqrt{\mu_1\mu_2}$. If $(u,v)\in M$ is a minimizer of B, then (u,v) is a nontrivial critical point of I.*

Proof. Let

$$\begin{aligned} G_1(u,v) &= \int_\Omega (|\nabla u|^2+\lambda_1 u^2) - \int_\Omega (\mu_1|u|^6+\beta|u|^3|v|^3), \\ G_2(u,v) &= \int_\Omega (|\nabla v|^2+\lambda_2 v^2) - \int_\Omega (\mu_2|v|^6+\beta|u|^3|v|^3). \end{aligned}$$

Since $(u,v)\in M$ is a minimizer of B, there exist $L_1,L_2\in\mathbb{R}$ such that $\langle I'(u,v)+L_1G_1'(u,v)+L_2G_2'(u,v),(\varphi,\phi)\rangle=0$ for any $(\varphi,\phi)\in H$. Taking $(\varphi,\phi)=(u,0)$ and $(\varphi,\phi)=(0,v)$ respectively, we have the following linear system

$$\begin{cases} \int_\Omega (4\mu_1|u|^6+\beta|u|^3|v|^3)L_1 + 3\beta\int_\Omega |u|^3|v|^3 L_2 = 0, \\ 3\beta\int_\Omega |u|^3|v|^3 L_1 + \int_\Omega (4\mu_2|v|^6+\beta|u|^3|v|^3)L_2 = 0. \end{cases}$$

By $0<\beta<2\sqrt{\mu_1\mu_2}$, the Hölder inequality and the Cauchy inequality, we see that

$$\beta\int_\Omega |u|^3|v|^3 \le 2\Big(\mu_1\int_\Omega |u|^6\Big)^{\frac12}\Big(\mu_2\int_\Omega |v|^6\Big)^{\frac12} \le \mu_1\int_\Omega |u|^6+\mu_2\int_\Omega |v|^6,$$

which and Lemma 3.1 imply that

$$\begin{aligned}
&\int_\Omega(4\mu_1|u|^6+\beta|u|^3|v|^3)\int_\Omega(4\mu_2|v|^6+\beta|u|^3|v|^3)-9\Big(\beta\int_\Omega |u|^3|v|^3\Big)^2\\
&\ge 16\mu_1\mu_2\int_\Omega |u|^6\int_\Omega |v|^6-4\Big(\beta\int_\Omega |u|^3|v|^3\Big)^2\\
&=4(4\mu_1\mu_2-\beta^2)\int_\Omega |u|^6\int_\Omega |v|^6\\
&\ge 4(4\mu_1\mu_2-\beta^2)C_0^2>0.
\end{aligned}$$

So we see that $L_1=L_2=0$ and $I'(u,v)=0$. $\square$

Next we consider the case where $\beta\ge 2\sqrt{\mu_1\mu_2}$. We consider the following set defined in Lemma 2.11

$$X_\beta:=\Big\{\tau>0|\ \sqrt{g_\beta(\tau)}=\frac{\tau^2+1}{(\underline{\mu}\tau^4+\beta\tau)^{\frac12}}=m_0^2\Big\},$$

where m_0^2 is defined in (2.15). Then $\tau_0\in X_\beta$ for each $\beta>0$, where τ_0 is defined as in Lemma 2.9. Note that $2\underline{\mu}\le 2\sqrt{\mu_1\mu_2}\le 2\overline{\mu}$, by Lemma 2.11 and the proof of Lemma 2.10 we see that for $\beta_*<\beta<\beta^*$, $X_\beta=\{\widetilde{\tau}_1,\widetilde{\tau}_2,\tau_0\}$ with $0<\widetilde{\tau}_1<a_\beta<\widetilde{\tau}_2<b_\beta<\tau_0$. For $\beta\ge 2\sqrt{\mu_1\mu_2}$, we define

$$\tau_*:=\begin{cases}\widetilde{\tau}_2, & \max\{2\sqrt{\mu_1\mu_2},\beta_*\}<\beta<\beta^* \text{ or } \beta=2\sqrt{\mu_1\mu_2} \text{ if } \beta_*<2\sqrt{\mu_1\mu_2},\\ \text{any element in } (0,\tau_0), & \text{otherwise.}\end{cases} \tag{3.7}$$

Then we obtain the following modified version of Lemma 2.10, which will be crucial to deal with the case where $\beta\ge 2\sqrt{\mu_1\mu_2}$.

Lemma 3.5. *Let $\beta\ge 2\sqrt{\mu_1\mu_2}$. Suppose that (t,s) satisfies*

$$\begin{cases}\mu_1t^4+\beta ts^3\ge 1,\\ \mu_2s^4+\beta st^3\ge 1,\\ \frac{t}{s}\ge\tau_* \ \ \textit{if } \mu_1\le\mu_2 \ \ \textit{or} \ \ \frac{s}{t}\ge\tau_* \ \ \textit{if } \mu_1>\mu_2,\\ t,s>0,\end{cases}$$

then $t^2+s^2\ge m_0^2$.

Proof. By Lemmas 2.10 and 2.11, it is enough to prove the case where $\beta_* < \beta < \beta^*$. Using the definition of τ_*, similarly to the proof of Lemma 2.10, by the monotonicity of the functions $g_\beta(\tau)$ and $f_\beta(\tau)$ we see that

$$\begin{aligned}(t^2+s^2)^2 \geq \max\{g_\beta(\tau), f_\beta(\tau), \tau \geq \tau_*\} &= \begin{cases} g_\beta(\tau), & \text{if } \tau_* \leq \tau \leq \tau_0, \\ f_\beta(\tau), & \text{if } \tau \geq \tau_0, \end{cases} \\ &\geq g_\beta(\tau_0) = m_0^4, \end{aligned}$$

□

To prove the theorem, without loss of generality, we may assume that $\mu_1 \leq \mu_2$ in what follows. We consider the following manifold

$$\begin{aligned}\mathcal{M} &= \Big\{(u,v) \in H\backslash\{(0,0)\} | \int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) = \int_\Omega (\mu_1|u|^6 + \mu_2|v|^6 \\ &\quad + 2\beta|u|^3|v|^3),\, 2\mu_1 \int_\Omega |u|^6 \leq \beta\tau_0^3 \int_\Omega |u|^3|v|^3,\, \tau_*^3 \int_\Omega |v|^6 \leq \int_\Omega |u|^3|v|^3 \Big\}.\end{aligned}$$

For any $u \in H_0^1(\Omega)\backslash\{0\}$, then there exists $t > 0$ such that $(tt_0u, ts_0u) \in \mathcal{M}$, where (t_0, s_0) is defined in (2.22). So $\mathcal{M} \neq \varnothing$. Set

$$m_\beta := \inf_{(u,v)\in\mathcal{M}} I(u,v) = \inf_{(u,v)\in\mathcal{M}} \frac{1}{3}\int_\Omega (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2),$$

then $m_\beta > 0$.

Lemma 3.6. *Suppose that $\beta \geq 2\sqrt{\mu_1\mu_2}$. Then $m_\beta < \frac{1}{3}m_0^2 S^{\frac{3}{2}}$.*

Proof. We first consider $\Omega = B_1(0)$. For any $\varepsilon > 0$, let $w_\varepsilon(x)$ and (t_0, s_0) be defined in (3.4) and (2.22). Then there exists

$$t_\varepsilon = \Big(\frac{m_0^2 \int_\Omega |\nabla w_\varepsilon|^2 + (t_0^2\lambda_1 + s_0^2\lambda_2)\int_\Omega w_\varepsilon^2}{m_0^2 \int_\Omega w_\varepsilon^6}\Big)^{\frac{1}{4}}$$

such that $(t_\varepsilon t_0 w_\varepsilon, t_\varepsilon s_0 w_\varepsilon) \in \mathcal{M}$. So by (3.5) and (3.6), we have

$$\begin{aligned} m_\beta &\leq I(t_\varepsilon t_0 w_\varepsilon, t_\varepsilon s_0 w_\varepsilon) \\ &= \frac{1}{3}t_\varepsilon^2 \left[m_0^2 \int_\Omega |\nabla w_\varepsilon|^2 + (t_0^2\lambda_1 + s_0^2\lambda_2)\int_\Omega w_\varepsilon^2\right] \\ &= \frac{1}{3}\frac{\Big(m_0^2 S^{\frac{3}{2}} + [\frac{\pi^2}{4}m_0^2 + (\lambda_1 t_0^2 + \lambda_2 s_0^2)]2\sqrt{3}\pi\varepsilon + O(\varepsilon^2)\Big)^{\frac{3}{2}}}{\Big[m_0^2(S^{\frac{3}{2}} + O(\varepsilon^2))\Big]^{\frac{1}{2}}} \\ &< \frac{1}{3}m_0^2 S^{\frac{3}{2}} \end{aligned}$$

for $\varepsilon > 0$ small enough, where we have used a fact that $\lambda_1 t_0^2 + \lambda_2 s_0^2 < -\frac{\pi^2}{4} m_0^2$ since $\lambda_1, \lambda_2 < -\frac{1}{4}\lambda_1(B_1(0)) = -\frac{\pi^2}{4}$. So $m_\beta < \frac{1}{3} m_0^2 S^{\frac{3}{2}}$. We can similarly prove that the lemma holds for any ball in $\mathbb{R}^3$. □

Lemma 3.7. *Suppose that $\beta \geq 2\sqrt{\mu_1\mu_2}$, then there exists a bounded $(PS)_{m_\beta}$ sequence $\{(u_n, v_n)\} \subset \mathcal{M}$ for I.*

Proof. By the Ekeland variational principle, there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{M}$ satisfying that

$$m_\beta \leq I(u_n, v_n) \leq m_\beta + \frac{1}{n},$$

$$I(u, v) \geq I(u_n, v_n) - \frac{1}{n}\|(u_n, v_n) - (u, v)\|_H, \quad \forall\ (u, v) \in \mathcal{M}. \tag{3.8}$$

We easily see that $\{(u_n, v_n)\}$ is uniformly bounded in H. For any $\varphi, \phi \in H_0^1(\Omega)$, we consider a function $F_n : \mathbb{R}^2 \to \mathbb{R}$ as

$$F_n(t, s) = \langle I'(u_n + t\varphi + su_n, v_n + t\phi + sv_n), (u_n + t\varphi + su_n, v_n + t\phi + sv_n)\rangle.$$

Then $F_n(0, 0) = 0$ and $\frac{\partial F_n(0,0)}{\partial s} = -4\int_\Omega(\mu_1|u_n|^6 + \mu_2|v_n|^6 + 2\beta|u_n|^3|v_n|^3) \neq 0$. By the Implicit Function theorem, there exists $\{\delta_n\} \subset \mathbb{R}_+$ and a function $s_n(t) \in C^1(-\delta_n, \delta_n)$ such that

$$s_n(0) = 0, \quad F_n(t, s_n(t)) = 0,\ \forall\ t \in (-\delta_n, \delta_n) \quad \text{and} \quad s_n'(0) = -\frac{\frac{\partial F_n(0,0)}{\partial t}}{\frac{\partial F_n(0,0)}{\partial s}}.$$

Then $s_n'(0)$ is bounded since $\{(u_n, v_n)\}$ is uniformly bounded. Moreover, $(u_n + t\varphi + s_n(t)u_n, v_n + t\phi + s_n(t)v_n) \to (u_n, v_n)$ in H as $t \to 0$. Then there exists $\varepsilon_n \in (0, \delta_n)$ small such that $(u_n + t\varphi + s_n(t)u_n, v_n + t\phi + s_n(t)v_n) \in \mathcal{M}$ for all $t \in (-\varepsilon_n, \varepsilon_n)$. Denote

$$\varphi_{n,t} := u_n + t\varphi + s_n(t)u_n, \qquad \phi_{n,t} := v_n + t\phi + s_n(t)v_n,$$

then $(\varphi_{n,t}, \phi_{n,t}) \in \mathcal{M}$ for $\forall\ t \in (-\varepsilon_n, \varepsilon_n)$. It follows from (3.8) that

$$I(\varphi_{n,t}, \phi_{n,t}) - I(u_n, v_n) \geq -\frac{1}{n}\|(t\varphi + s_n(t)u_n, t\phi + s_n(t)v_n)\|_H. \tag{3.9}$$

By $(u_n, v_n) \in \mathcal{M}$ and the Taylor Expansion we have

$$\begin{aligned} I(\varphi_{n,t}, \phi_{n,t}) - I(u_n, v_n) &= \langle I'(u_n, v_n), (t\varphi + s_n(t)u_n, t\phi + s_n(t)v_n)\rangle + r(n, t) \\ &= t\langle I'(u_n, v_n), (\varphi, \phi)\rangle + r(n, t), \end{aligned} \tag{3.10}$$

where $r(n, t) = o(\|(t\varphi + s_n(t)u_n, t\phi + s_n(t)v_n)\|_H)$ as $t \to 0$. We see that

$$\limsup_{t\to 0} \|(\varphi + \frac{s_n(t)}{t}u_n, \phi + \frac{s_n(t)}{t}v_n)\|_H \leq C, \tag{3.11}$$

where C is independence of n. Hence $r(n,t) = o(t)$. By (3.8)-(3.11) and letting $t \to 0$, we have

$$|\langle I'(u_n, v_n), (\varphi, \phi)\rangle| \leq \frac{C}{n},$$

where C is independence of n. Hence $I'(u_n, v_n) \to 0$, i.e. $\{(u_n, v_n)\}$ is a bounded $(PS)_{m_\beta}$ sequence for I. $\square$

Lemma 3.8. *([29], Lemma 1.32) Let Ω be an open subset of $\mathbb{R}^N$ and let $\{u_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$. If $\{u_n\}$ is bounded in $L^p(\Omega)$ and $u_n \to u$ a.e. on Ω, then*

$$\lim_{n\to+\infty} (|u_n|_p^p - |u_n - u|_p^p) = |u|_p^p.$$

Lemma 3.9. *([12], Lemma 3.3) Let $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ as $n \to +\infty$ and $1 < p < +\infty$, then passing to a subsequence, there holds*

$$\lim_{n\to+\infty} \int_\Omega (|u_n|^p|v_n|^p - |u_n - u|^p|v_n - v|^p - |u|^p|v|^p) = 0.$$

Proof of Theorem 1.2

Proof. (1) For $0 < \beta < 2\sqrt{\mu_1\mu_2}$, let $\{(u_n, v_n)\} \subset M$ be a minimizing sequence for B, i.e. $I(u_n, v_n) \to B$ as $n \to +\infty$, then $\{(u_n, v_n)\}$ is uniformly bounded in H. By Lemma 3.1, we see that

$$\int_\Omega |u_n|^6, \int_\Omega |v_n|^6 \geq C_0, \tag{3.12}$$

where C_0 is a positive constant given in Lemma 3.1. Up to a subsequence, we may assume that $(u, v) \in H$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in H. By the Sobolev embedding inequality, we have

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{in } L^6(\Omega), \\ u_n^3 \rightharpoonup u^3, \quad v_n^3 \rightharpoonup v^3 \quad \text{in } L^2(\Omega), \\ u_n^5 \rightharpoonup u^5, \quad v_n^5 \rightharpoonup v^5 \quad \text{in } L^{\frac{6}{5}}(\Omega), \\ u_n \to u, \quad v_n \to v \quad \text{in } L^2(\Omega). \end{cases} \tag{3.13}$$

So $I'(u,v) = 0$. Let $w_n := u_n - u$, $\sigma_n := v_n - v$ and

$$b_1 := \lim_{n\to+\infty} \Big(\int_\Omega |w_n|^6\Big)^{\frac{1}{6}}, \qquad b_2 := \lim_{n\to+\infty} \Big(\int_\Omega |\sigma_n|^6\Big)^{\frac{1}{6}}. \tag{3.14}$$

By $(u_n, v_n) \in M$, the Sobolev embedding inequality and Lemma 3.8 we have

$$
\begin{aligned}
B &= \lim_{n\to+\infty} I(u_n, v_n) \\
&= \frac{1}{3}\lim_{n\to+\infty}\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2 + \lambda_1|u_n|^2 + \lambda_2|v_n|^2) \\
&= \frac{1}{3}\lim_{n\to+\infty}\int_\Omega (|\nabla w_n|^2 + |\nabla \sigma_n|^2) + \frac{1}{3}\int_\Omega (|\nabla u|^2 + \lambda_1|u|^2 + |\nabla v|^2 + \lambda_2|v|^2) \\
&\geq \frac{1}{3}S(b_1^2 + b_2^2) + I(u, v).
\end{aligned} \tag{3.15}
$$

We claim that

$$\text{both } u \neq 0 \text{ and } v \neq 0 \Longleftrightarrow b_1 = b_2 = 0.$$

Indeed, if $u \neq 0, v \neq 0$, then (u, v) is a nontrivial solution of (1.1), i.e. $(u, v) \in M$. Hence we have $I(u, v) \geq B$, which and (3.15) show that $b_1 = b_2 = 0$. On the other hand, if $b_1 = b_2 = 0$, then $(u_n, v_n) \to (u, v)$ in H. We conclude from (3.12) that $u \neq 0, v \neq 0$.

It is enough to show that $b_1 = b_2 = 0$. By contradiction, we just suppose that $b_1 = b_2 = 0$ does not hold, then $u = 0$ or $v = 0$ holds. By $(u_n, v_n) \in M$, Lemmas 3.8-3.9 and the Hölder inequality, we have

$$S\Big(\int_\Omega |w_n|^6\Big)^{\frac{1}{3}} \leq \int_\Omega |\nabla w_n|^2 \leq \mu_1 \int_\Omega |w_n|^6 + \beta\Big(\int_\Omega |w_n|^6\Big)^{\frac{1}{2}}\Big(\int_\Omega |\sigma_n|^6\Big)^{\frac{1}{2}} + o_n(1), \tag{3.16}$$

$$S\Big(\int_\Omega |\sigma_n|^6\Big)^{\frac{1}{3}} \leq \int_\Omega |\nabla \sigma_n|^2 \leq \mu_2 \int_\Omega |\sigma_n|^6 + \beta\Big(\int_\Omega |w_n|^6\Big)^{\frac{1}{2}}\Big(\int_\Omega |\sigma_n|^6\Big)^{\frac{1}{2}} + o_n(1), \tag{3.17}$$

where $o_n(1) \to 0$ as $n \to +\infty$.

If $b_1, b_2 > 0$, then

$$\begin{cases} \mu_1(\frac{b_1}{S^{\frac{1}{4}}})^4 + \beta(\frac{b_1}{S^{\frac{1}{4}}})(\frac{b_2}{S^{\frac{1}{4}}})^3 \geq 1, \\ \mu_2(\frac{b_2}{S^{\frac{1}{4}}})^4 + \beta(\frac{b_2}{S^{\frac{1}{4}}})(\frac{b_1}{S^{\frac{1}{4}}})^3 \geq 1. \end{cases} \tag{3.18}$$

By $0 < \beta < 2\sqrt{\mu_1\mu_2}$ and Lemma 2.10, we have $b_1^2 + b_2^2 \geq m_0^2 S^{\frac{1}{2}}$. We conclude from (3.15) that $B \geq \frac{1}{3}S(b_1^2 + b_2^2) \geq \frac{1}{3}m_0^2 S^{\frac{3}{2}}$, which contradicts Lemma 3.3.

If $b_1 = 0$ and $b_2 > 0$, then $u_n \to u$ in $H_0^1(\Omega)$. By (3.12), we see that u is a nontrivial solution of $-\Delta u + \lambda_1 u = \mu_1 u^5$ in $H_0^1(\Omega)$. Hence $v = 0$ and $I(u, 0) \geq B_{\mu_1}$. We conclude from (3.17) that $b_2^2 \geq \mu_2^{-\frac{1}{2}} S^{\frac{1}{2}}$. Then

$$B \geq \frac{1}{3}\int_\Omega (|\nabla u|^2 + \lambda_1 u^2) + \frac{1}{3}Sb_2^2 \geq B_{\mu_1} + \frac{1}{3}b_2^2 S = B_{\mu_1} + \frac{1}{3}\mu_2^{-\frac{1}{2}} S^{\frac{3}{2}},$$

which contradicts Lemma 3.3 and (3.1). Similarly, we can get a contradiction if $b_1 > 0$ and $b_2 = 0$.

Therefore, $b_1 = b_2 = 0$. Then $u \neq 0, v \neq 0$ and $I(u,v) = B$. Since the functional I and the manifold M are symmetric, we may assume that $u, v \geq 0$ in Ω. By the strong maximum principle, we see that $u, v > 0$ in Ω. So (u,v) is a positive least energy solution of (1.1).

(2) Let $\beta \geq 2\sqrt{\mu_1\mu_2}$. By Lemma 3.7, there exists a bounded $(PS)_{m_\beta}$ sequence $\{(u_n, v_n)\} \subset \mathcal{M}$ for I, then we may assume that $(u,v) \in H$ such that $(u_n, v_n) \rightharpoonup (u,v)$ in H and (3.13) holds. So $I'(u,v) = 0$. We first show that $\lim\limits_{n\to+\infty} \int_\Omega |u_n|^6 > 0$ and $\lim\limits_{n\to+\infty} \int_\Omega |v_n|^6 > 0$. By contradiction, if we assume that $\int_\Omega |u_n|^6 \to 0$, then by the Hölder inequality, we have $\int_\Omega |u_n|^3|v_n|^3 \to 0$ as $n \to +\infty$. By $(u_n, v_n) \in \mathcal{M}$, we see that $\int_\Omega |v_n|^6 \to 0$. Then $m_\beta = \lim\limits_{n\to+\infty} \frac{1}{3}\int_\Omega (\mu_1|u_n|^6 + \mu_2|v_n|^6 + 2\beta|u_n|^3|v_n|^3) = 0$, which contradicts $m_\beta > 0$. So $\lim\limits_{n\to+\infty} \int_\Omega |u_n|^6 > 0$. Similarly, we can prove that $\lim\limits_{n\to+\infty} \int_\Omega |v_n|^6 > 0$. Let w_n, σ_n, b_1, b_2 be defined as in (3.14). By the boundedness of $\{(u_n, v_n)\}$ and $I'(u_n, v_n) \to 0$ we see that

$$\begin{cases} \int_\Omega |\nabla w_n|^2 = \int_\Omega (\mu_1|w_n|^6 + \beta|w_n|^3|\sigma_n|^3) + \beta\int_\Omega |u|^3|v|^3 + o_n(1), \\ \int_\Omega |\nabla \sigma_n|^2 = \int_\Omega (\mu_2|\sigma_n|^6 + \beta|w_n|^3|\sigma_n|^3) + \beta\int_\Omega |u|^3|v|^3 + o_n(1). \end{cases}$$

Similarly to (3.15), we also have

$$\begin{aligned} m_\beta &= \frac{1}{3}\lim_{n\to+\infty}\int_\Omega (|\nabla w_n|^2 + |\nabla \sigma_n|^2) + \frac{1}{3}\int_\Omega (|\nabla u|^2 + \lambda_1|u|^2 + |\nabla v|^2 + \lambda_2|v|^2) \\ &\geq \frac{1}{3}S(b_1^2 + b_2^2) + I(u,v). \end{aligned}$$

If $u = 0$, then $b_1 > 0$. By $(u_n, v_n) \in \mathcal{M}$ and the Hölder inequality, we see that

$$\frac{2\mu_1}{\beta\tau_0^3}\int_\Omega |u_n|^6 \leq \int_\Omega |u_n|^3|v_n|^3 = \int_\Omega |u_n|^3|\sigma_n|^3 + o_n(1) \leq \left(\int_\Omega |u_n|^6\right)^{\frac{1}{2}}\left(\int_\Omega |\sigma_n|^6\right)^{\frac{1}{2}} + o_n(1),$$

which implies that $\frac{b_2}{b_1} \geq (\frac{2\mu_1}{\beta\tau_0^3})^{\frac{1}{3}} > 0$, i.e. $b_2 > 0$. Moreover, by Lemma 3.8, we have $\tau_*^3\int_\Omega |\sigma_n|^6 + o_n(1) = \tau_*^3\int_\Omega |v_n|^6 \leq \int_\Omega |u_n|^3|v_n|^3 = \int_\Omega |u_n|^3|\sigma_n|^3 + o_n(1) \leq \left(\int_\Omega |u_n|^6\right)^{\frac{1}{2}}\left(\int_\Omega |\sigma_n|^6\right)^{\frac{1}{2}} + o_n(1)$, which implies that $\frac{\int_\Omega |u_n|^6 + o_n(1)}{\int_\Omega |\sigma_n|^6} \geq \tau_*^6$. Similarly to the proof of (3.16)-(3.18), we get that

$$\begin{cases} \mu_1(\frac{b_1}{S^{\frac{1}{4}}})^4 + \beta(\frac{b_1}{S^{\frac{1}{4}}})(\frac{b_2}{S^{\frac{1}{4}}})^3 \geq 1, \\ \mu_2(\frac{b_2}{S^{\frac{1}{4}}})^4 + \beta(\frac{b_2}{S^{\frac{1}{4}}})(\frac{b_1}{S^{\frac{1}{4}}})^3 \geq 1, \\ \frac{b_1}{b_2} \geq \tau_*. \end{cases} \tag{3.19}$$

By Lemma 3.5 we have $b_1^2 + b_2^2 \geq m_0^2 S^{\frac{1}{2}}$. So $m_\beta \geq \frac{1}{3}S(b_1^2 + b_2^2) \geq \frac{1}{3}m_0^2 S^{\frac{3}{2}}$, which contradicts Lemma 3.6. So $u \neq 0$.

Similarly, we can show that $v \neq 0$. Therefore we have $u \neq 0$ and $v \neq 0$. Then (u,v) is a nontrivial solution of (1.1). □

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