# The existence of normalized solutions for $L^2$-critical quasilinear Schrödinger equations*

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**Abstract**

In this paper, we study the existence of critical points for the following functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{N}{4(N+1)}\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}},$$

constrained on $S_c = \{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 < +\infty,\ |u|_2 = c, c > 0\}$, where $N \geq 1$. The constraint problem is $L^2$-critical. We prove that the minimization problem $i_c = \inf\limits_{u \in S_c} I(u)$ has no minimizer for all $c > 0$. We also obtain a threshold value of $c$ separating the existence and nonexistence of critical points for $I(u)$ restricted to $S_c$.

**Keywords:** $L^2$-critical; Constrained minimization; Sharp existence; Quasilinear Schrödinger equations

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## 1 Introduction and main result

In the past years, the following quasilinear Schrödinger equation

$$i\partial_t \varphi + \Delta\varphi + \varphi\Delta(|\varphi|^2) + |\varphi|^{p-2}\varphi = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{1.1}$$

has attracted considerable attention, where $i$ denotes the imaginary unit and $\varphi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$, $p \in (2, 2 \cdot 2^*)$, $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$. Quasilinear Schrödinger equation (1.1) appears in various physical fields, such as in dissipative quantum mechanics, in plasma physics and in fluid mechanics, see more information in [7, 8, 18]. One usually searches for standing waves solutions of (1.1),

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i.e. solutions of the form $\varphi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter and $u : \mathbb{R}^N \to \mathbb{R}$ is a function to be founded, then (1.1) is reduced to be the following stationary equation

$$-\Delta u + u\Delta(|u|^2) - |u|^{p-2}u = \lambda u, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

We firstly consider the case where $\lambda$ is a fixed and assigned parameter. In such direction, the critical point theory is used to look for nontrivial solutions of the following functional

$$\Phi_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p$$

defined on the natural space

$$H := \left\{ u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 < +\infty \right\}.$$

However, nothing can be given a priori on the $L^2$-norm of the solutions. We say $u$ a weak solution of (1.2) if $u \in H$ and $\langle \Phi_p'(u), \phi\rangle = \lim\limits_{t\to 0^+} \frac{\Phi_p(u+t\phi)-\Phi_p(u)}{t} = 0$ for every direction $\phi \in C_0^\infty(\mathbb{R}^N)$. Different from semilinear equations, the quasilinear term $\int_{\mathbb{R}^N} |u|^2|\nabla u|^2$ in the functional $\Phi_p$ is not differentiable in $H$ when $N \geq 2$. This causes some mathematical difficulties which make the study of (1.2) particularly interesting. To overcome this difficulty, during the past ten years, researchers considered such quasilinear Schrödinger problems and a lot of existence and multiplicity results have been obtained by using minimizations, change of variables, Nehari method and perturbation method, see e.g. [1, 2, 4, 5, 12, 13, 14, 15, 16, 17, 19] and their references therein.

Recently, since the physicists are often interested in "normalized solutions", i.e. solutions with prescribed $L^2$-norm, it is interesting for us to study whether (1.2) has a normalized solution. For any fixed $c > 0$, a solution of (1.2) with $(\int_{\mathbb{R}^N} |u|^2)^{\frac{1}{2}} = c$ can be viewed as a critical point of the following functional

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \tag{1.3}$$

constrained on the $L^2$-spheres in $H$:

$$S_c = \{u \in H | \ |u|_2 = c, \ c > 0\},$$

where $|u|_2 := (\int_{\mathbb{R}^N} |u|^2)^{\frac{1}{2}}$. In this case, the parameter $\lambda$ is not fixed any longer but appears as an associated Lagrange multiplier. We call $(u_c, \lambda_c) \in S_c \times \mathbb{R}$ a couple of solution to (1.2) if $u_c$ is a critical point of $I_p(u)$ constrained on $S_c$ and $\lambda_c$ is the associated Lagrange parameter. To obtain the normalized solutions, there are some papers studying the following minimization problem

$$i_{p,c} := \inf_{u \in S_c} I_p(u), \tag{1.4}$$

see [3, 9, 10]. It has been shown in [3, 10] that minimizers of $i_{p,c}$ are exactly critical points of $I_p|_{S_c}$. In [3], Colin, Jeanjean and Squassina proved that $p = \frac{4(N+1)}{N}$ is $L^2$-critical exponent for (1.4), namely, for all $c > 0$, $I_p(u)$ is bounded from below and coercive on $S_c$ if $p \in (2, \frac{4(N+1)}{N})$ and $i_{p,c} = -\infty$ if $p \in (\frac{4(N+1)}{N}, 2\cdot 2^*)$. When $p = \frac{4(N+1)}{N}$, Jeanjean and Luo showed in [9] that there exists $c_N \in (0, +\infty)$ such that $i_{\frac{4(N+1)}{N},c} = 0$ for $c \in (0, c_N)$ and $i_{\frac{4(N+1)}{N},c} = -\infty$ for all $c > c_N$. However, the accurate expression of $c_N$ and the accurate value of $i_{\frac{4(N+1)}{N},c_N}$ are unknown yet. Actually the method in [9] cannot do that. In this paper, by an alternative method we succeeded in obtaining a threshold value of $c$ to separate the existence and nonexistence of critical points for $I_{\frac{4(N+1)}{N}}(u)$ constrained on $S_c$.

For simplicity, we use $I(u)$ and $i_c$ to denote $I_{\frac{4(N+1)}{N}}(u)$ and $i_{\frac{4(N+1)}{N},c}$ respectively. Recall in (4.5) of [3] that there exists a positive constant $C$ depending only on $N$ such that for any $u \in H$, $\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \leq C \left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{2}{N}} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2$. Set

$$A := \inf_{u \in H\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{2}{N}} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2}{\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}} \geq \frac{1}{C} > 0. \tag{1.5}$$

Then our main result is as follows:

**Theorem 1.1.** *For $p = \frac{4(N+1)}{N}$ and $N \geq 1$, let $c_* = \left(\frac{4(N+1)}{N} A\right)^{\frac{N}{4}}$. Then*

(1) $i_c = \begin{cases} 0, & 0 < c \leq c_*, \\ -\infty & c > c_*. \end{cases}$

(2) *$i_c$ has no minimizer for all $c > 0$.*

(3) *$I(u)$ has no critical point on the constraint $S_c$ for all $0 < c \leq c_*$.*

Since it has been proved that problem (1.2) has at least one nontrivial solution when $p = \frac{4(N+1)}{N}$ ( see e.g. [5, 14]), it is reasonable to conjecture that $I(u)$ has at least one critical point constrained on $S_c$ for some $c > c_*$. In this paper, we did so. To the best of our knowledge, there is no paper on this respect. To state our main result, we set

$$N_c := \left\{ u \in S_c | \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 < \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \right\}, \tag{1.6}$$

then it follows from Theorem 1.1 (1) that $N_c \neq \varnothing$ for each $c > c_*$. Define

$$M_c := \{u \in N_c | \ G(u) = 0\},$$

where

$$G(u) := \int_{\mathbb{R}^N} |\nabla u|^2 + (N+2) \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{N(N+2)}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}.$$

Then we have the following result:

**Theorem 1.2.** *Assume that $N \leq 3$, $p = \frac{4(N+1)}{N}$ and $c > c_*$, where $c_*$ is given in Theorem 1.1. Then there exists a couple of solution $(u_c, \lambda_c) \in M_c \times \mathbb{R}_-$ satisfying the following equation*

$$-\Delta u + u\Delta(|u|^2) - |u|^{\frac{2N+4}{N}} u = \lambda_c u, \qquad x \in \mathbb{R}^N \tag{1.7}$$

*with $I(u_c) = \inf\limits_{u \in M_c} I(u)$.*

To prove Theorem 1.2, since $i_c = -\infty$ for $c > c_*$, the minimization problem constrained on $S_c$ does not work. We try to construct a submanifold of $S_c$, on which $I(u)$ admits a minimizer. As $\int_{\mathbb{R}^N} |u|^2|\nabla u|^2$ and $\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}$ behave at the same way under $L^2$-preserving scaling of $u$, it may occur that $I(u^t) > 0$ and $I(u^t)$ is strictly increasing with respect to $t$ on $(0, +\infty)$ for some $u \in S_c$, where $u^t(x) = t^{\frac{N}{2}} u(tx)$. Then usual arguments which allowed us to benefit from the Pohozaev-Nehari constraint $\{u \in S_c | G(u) = 0\}$ cannot be applied here. We need to exclude the interference of the functions satisfying that $\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 \geq \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}$, which is the reason why the set $N_c$ is introduced. We can show that for each $u \in N_c$, there exists a unique $t(u) > 0$ such that $G(u^{t(u)}) = 0$ and $I(u^{t(u)}) = \max\limits_{t>0} I(u^t)$. Then $M_c$ can be viewed as the suitable submanifold. To prove Theorem 1.2, we consider the following minimization problem

$$m_c = \inf_{u \in M_c} I(u)$$

and prove that $m_c$ is attained. There are two difficulties. First, it is not easy to prove that $M_c$ is a natural constraint of $I|_{S_c}$, i.e. minimizers of $m_c$ are critical points of $I(u)$ constrained on $S_c$ since there may be two Lagrange multipliers. We overcome this difficulty by using the Pohozaev identity and the well-known Gagliardo-Nirenberg inequality, which requires more careful analysis. Second, it is difficult to show that $M_c$ is weakly closed due to a possible lack of compactness for the minimizing sequences. In our case it seems impossible to reduce the problem to the classical vanishing-dichotomy-compactness scenario and to use the concentration-compactness principle since we search for solutions constrained on $S_c$. To overcome this difficulty, we construct a Schwartz symmetric minimizing sequence of $m_c$ and prove the strict monotonicity of the function $c \mapsto m_c$ to avoid possible vanishing and dichotomy of the sequence. In the proof of the essential strictly monotonicity of $m_c$, we use the scaling arguments in which $\frac{4(N+1)}{N} < 2^*$ and $N \leq 3$ is required.

**Remark 1.3.** *When $N \geq 4$, the $L^2$-critical exponent $\frac{4(N+1)}{N} > 2^*$. It seems impossible to show the strict monotonicity of $m_c$ (see details in Remark 2.10 below), which makes that our method cannot be used to deal with the case where $N \geq 4$. However, we conjecture that the conclusion of Theorem 1.2 also holds for $N \geq 4$.*

We also concern the behavior of the solutions $u_c$ and $\lambda_c$ obtained in Theorem 1.2 upon the value of $c > 0$.

**Proposition 1.4.** *For any $c > c_*$, let $(u_c, \lambda_c)$ be the couple of solution obtained in Theorem 1.2. Then*

*(1)* $\begin{cases} |\nabla u_c|_2 \to +\infty, \quad m_c \to +\infty, \\ \lambda_c \to -\infty \end{cases}$ *as* $c \to (c_*)^+$.

*(2)* $\begin{cases} |\nabla u_c|_2 \to 0, \quad m_c \to 0, \\ \lambda_c \to 0 \end{cases}$ *as* $c \to +\infty$.

We finally obtain a supplementary result in the special case where $p = \frac{2N+4}{N}$. In [9], Jeanjean and Luo conjecture that $i_{\frac{2N+4}{N},c}$ has a minimizer for some $c > 0$. We succeeded in proving this conjecture.

Recall in [6, 11, 20] the well-known Gagliardo-Nirenberg inequality with the best constant: Let $p \in [2, 2^*)$ if $N \geq 3$ and $p \geq 2$ if $N = 1, 2$, then

$$|u|_p^p \leq \frac{p}{2|Q_p|_2^{p-2}} |u|_2^{p-\frac{N(p-2)}{2}} |\nabla u|_2^{\frac{N(p-2)}{2}}, \qquad \forall u \in H^1(\mathbb{R}^N), \tag{1.8}$$

with equality only for $u = Q_p$, where up to translations, $Q_p$ is the unique ground state solution of

$$-\frac{N(p-2)}{4}\Delta Q + \left(1 + \frac{p-2}{4}(2-N)\right) Q = |Q|^{p-2}Q, \quad x \in \mathbb{R}^N. \tag{1.9}$$

Moreover, when $p = \frac{2N+4}{N}$, it is proved in [6, 11] that $Q_{\frac{2N+4}{N}}$ is monotonically decreasing away from the origin and

$$Q_{\frac{2N+4}{N}}(x),\ |\nabla Q_{\frac{2N+4}{N}}(x)| = O(|x|^{-\frac{1}{2}} e^{-|x|}) \quad \text{as } |x| \to +\infty. \tag{1.10}$$

Then we have the following existence result.

**Theorem 1.5.** *For $p = \frac{2N+4}{N}$ and $N \geq 1$, let $c^* = |Q_{\frac{2N+4}{N}}|_2$, then*

(1) $i_{\frac{2N+4}{N},c} = 0$ *for all* $0 < c \leq c^*$ *and* $i_{\frac{2N+4}{N},c} < 0$ *for all* $c > c^*$.

(2) $i_{\frac{2N+4}{N},c}$ *has a minimizer if and only if* $c > c^*$.

(3) $I_{\frac{2N+4}{N}}(u)$ *has no critical point on the constraint* $S_c$ *for all* $0 < c \leq c^*$.

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^N$. $L^p := L^p(\mathbb{R}^N)$ $(1 \leq p \leq +\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively. $C$ will denote a positive constant unless specified. We use " $:=$ " to denote definitions. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In § 2, we prove Theorems 1.1 and 1.2. In § 3, we prove Proposition 1.4. In § 4, we prove Theorem 1.5.

## 2 Proof of Theorems 1.1 and 1.2

In this section, we first prove Theorem 1.1. By (1.5), we have

$$\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \le \frac{1}{A}|u|_2^{\frac{4}{N}} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2, \qquad \forall u \in H. \tag{2.1}$$

In particular, for any $c > 0$ and any $u \in S_c$, since $c_* = \left(\frac{4(N+1)}{N}A\right)^{\frac{N}{4}}$, we have

$$\frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \le \left(\frac{c}{c_*}\right)^{\frac{4}{N}} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2. \tag{2.2}$$

**Lemma 2.1.** $i_c = \begin{cases} 0, & 0 < c \le c_*, \\ -\infty & c > c_*. \end{cases}$

*Proof.* (1) For any $0 < c \le c_*$ and any $u \in S_c$, by (2.2) we have $\frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \le \int_{\mathbb{R}^N} |u|^2|\nabla u|^2$, then

$$I(u) \ge \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 > 0, \tag{2.3}$$

which shows that $i_c \ge 0$ by the arbitrary of $u$.

On the other hand, for any $t > 0$, set $u^t(x) := t^{\frac{N}{2}} u(tx)$, then

$$I(u^t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + t^{N+2} \left[ \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} - \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \right] \to 0 \quad \text{as } t \to 0^+,$$

hence $i_c \le 0$. So $i_c = 0$ for each $0 < c \le c_*$.

(2) For any $c > c_* = \left(\frac{4(N+1)}{N}A\right)^{\frac{N}{4}}$, then $A < \frac{N}{4(N+1)} c^{\frac{4}{N}}$. By the definition of $A$ there exists $u \in H\backslash\{0\}$ such that $\frac{|u|_2^{\frac{4}{N}} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2}{\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}} < \frac{N}{4(N+1)} c^{\frac{4}{N}}$. Set $v := \frac{c}{|u|_2} u$, then $v \in S_c$ and

$$\begin{aligned} \int_{\mathbb{R}^N} |v|^2|\nabla v|^2 = \left(\frac{c}{|u|_2}\right)^4 \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 &< \frac{N}{4(N+1)} \left(\frac{c}{|u|_2}\right)^{4+\frac{4}{N}} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \\ &= \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |v|^{\frac{4(N+1)}{N}}. \end{aligned} \tag{2.4}$$

Hence for any $t > 0$,

$$I(v^t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - t^{N+2} \left[ \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |v|^{\frac{4(N+1)}{N}} - \int_{\mathbb{R}^N} |v|^2|\nabla v|^2 \right] \to -\infty$$

as $t \to +\infty$, which implies that $i_c = -\infty$ for any $c > c_*$. □

By the proof of Lemma 2.1, we have the following result.

**Corollary 2.2.**
$$\begin{cases} N_c = \varnothing, & 0 < c \le c_*, \\ N_c \ne \varnothing, & c > c_*, \end{cases} \tag{2.5}$$
*where $N_c$ is defined as in* (1.6)*. Moreover, for any $0 < c \le c_*$ and any $u \in S_c$, $I(u) > 0$.*

**Lemma 2.3.** *$i_c$ has no minimizer for all $c > 0$.*

*Proof.* The Lemma follows directly from Lemma 2.1 and Corollary 2.2. $\square$

**Lemma 2.4.** *$I(u)$ has no critical point constrained on $S_c$ for each $c \in (0, c_*]$.*

*Proof.* By contradiction, we just suppose that there exists some $c \in (0, c_*]$ and some $u_c \in S_c$ such that $(I|_{S_c})'(u_c) = 0$, then there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $I'(u_c) - \lambda_c u_c = 0$. Hence by Lemma 3.1 in [3], we see that $u_c$ satisfies the following Pohozaev identity:

$$(N-2)\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_c|^2 + \int_{\mathbb{R}^N} |u_c|^2|\nabla u_c|^2\right) - \frac{N}{2}\lambda_c \int_{\mathbb{R}^N} |u_c|^2 - \frac{N^2}{4(N+1)}\int_{\mathbb{R}^N} |u_c|^{\frac{4(N+1)}{N}} = 0,$$

so
$$\int_{\mathbb{R}^N} |\nabla u_c|^2 + (N+2)\int_{\mathbb{R}^N} |u_c|^2|\nabla u_c|^2 = \frac{N(N+2)}{4(N+1)}\int_{\mathbb{R}^N} |u_c|^{\frac{4(N+1)}{N}},$$
which implies that $u_c \in N_c$. It is a contradiction with Corollary 2.2. Then the lemma is proved. $\square$

**Proof of Theorem 1.1**

*Proof.* Theorem 1.1 follows from Lemmas 2.1-2.4. $\square$

Next we deal with the existence of normalized solutions for $I(u)$ restricted to $S_c$ when $c > c_*$ and $N \le 3$. Motivated by Lemma 2.4 and Corollary 2.2, we try to search for normalized solutions constrained on $N_c$.

**Lemma 2.5.** *For any $u \in N_c$, there exists a unique $\tilde{t} > 0$ such that $I(u^{\tilde{t}}) = \max\limits_{t>0} I(u^t)$ and $G(u^{\tilde{t}}) = 0$, where $u^t(x) = t^{\frac{N}{2}} u(tx)$ and*

$$G(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + (N+2)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{N(N+2)}{4(N+1)}\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}}. \tag{2.6}$$

*Proof.* For any $u \in N_c$, we consider the following path $\gamma : (0, +\infty) \to \mathbb{R}$ defined as

$$\gamma(t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - t^{N+2} \left[ \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} - \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \right],$$

i.e. $\gamma(t) = I(u^t)$. Then by an elementary analysis, we see that $\gamma$ has a unique positive critical point $\tilde{t}$ corresponding to its maximum, i.e. $\gamma'(\tilde{t}) = 0$ and $\gamma(\tilde{t}) = \max_{t>0} \gamma(t)$. Hence $I(u^{\tilde{t}}) = \max_{t>0} I(u^t)$ and

$$\tilde{t}^2 \int_{\mathbb{R}^N} |\nabla u|^2 + (N+2)\tilde{t}^{N+2} \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 - \frac{N(N+2)}{4(N+1)} \tilde{t}^{N+2} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} = 0.$$

So $G(u^{\tilde{t}}) = 0$. □

For any $c > c_*$, we define a manifold as follows:

$$M_c = \{u \in N_c |\ G(u) = 0\},$$

then Lemma 2.5 shows that $M_c \neq \varnothing$.

Note that $\frac{4(N+1)}{N} < 2^*$ for $N = 1, 2, 3$. Recall by the Gagliardo-Nirenberg inequality (1.8) that when $N = 1, 2, 3$, there exists a positive constant $C$ depending only on $N$ such that

$$\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \leq C |\nabla u|_2^{N+2} |u|_2^{\frac{-N^2+2N+4}{N}}, \tag{2.7}$$

where we note that

$$\frac{-N^2 + 2N + 4}{N} > 0 \quad \text{for} \quad N \leq 3. \tag{2.8}$$

**Lemma 2.6.** *For any $c > c_*$,*

*(1) $I(u)$ is bounded from below and coercive on $M_c$.*
*(2) There exists a constant $C_0 > 0$ such that $\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \geq C_0$ for all $u \in M_c$.*
*(3) There exists a constant $C_1 > 0$ such that $I(u) \geq C_1$ for all $u \in M_c$.*

*Proof.* For any $u \in M_c$, $G(u) = 0$ and

$$I(u) = I(u) - \frac{1}{N+2} G(u) = \frac{N}{2(N+2)} \int_{\mathbb{R}^N} |\nabla u|^2 \geq 0. \tag{2.9}$$

Then $I$ is bounded from below and coercive on $M_c$. Moreover, by $G(u) = 0$ and (2.7), we see that there exists $C > 0$ depending only on $N$ and $c$ such that

$$\left( \frac{1}{C} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \right)^{\frac{2}{N+2}} \leq |\nabla u|_2^2 \leq \frac{N(N+2)}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \leq \frac{N(N+2)}{4(N+1)} C |\nabla u|_2^{N+2},$$

then

$$\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \geq \left( \frac{4(N+1)}{N(N+2) C^{\frac{2}{N+2}}} \right)^{\frac{N+2}{N}} := C_0$$

and $|\nabla u|_2 \geq \left(\frac{4(N+1)}{N(N+2)C}\right)^{\frac{1}{N}}$, which and (2.9) show that

$$I(u) \geq \frac{N}{2(N+2)} \left(\frac{4(N+1)}{N(N+2)}C\right)^{\frac{2}{N}} := C_1$$

for all $u \in M_c$. □

For any $c > c_*$, set

$$m_c := \inf_{u \in M_c} I(u), \tag{2.10}$$

we see from Lemma 2.6 that $m_c > 0$.

To prove Theorem 1.2, we need the following essential lemmas.

**Lemma 2.7.** *The function $c \mapsto m_c$ is strictly decreasing on $(c_*, +\infty)$.*

*Proof.* For any $c_1, c_2 \in (c_*, +\infty)$ satisfying that $c_1 < c_2$, it is enough to prove that $m_{c_2} < m_{c_1}$.

By the definition of $m_{c_1}$ and Lemma 2.5, there exists $u_n \in M_{c_1}$ such that $I(u_n) \leq m_{c_1} + \frac{1}{n}$ and $I(u_n) = \max_{t>0} I(u_n^t)$.

**Case 1**: $N = 2, 3$.

Set $v_n(x) := (\frac{c_1}{c_2})^{\frac{N}{2}-1} u_n(\frac{c_1}{c_2}x)$, then $|v_n|_2 = c_2$ and $|\nabla v_n|_2 = |\nabla u_n|_2$. Moreover,

$$\int_{\mathbb{R}^N} |v_n|^2 |\nabla v_n|^2 = \left(\frac{c_1}{c_2}\right)^{N-2} \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \leq \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2, \tag{2.11}$$

and by (2.8),

$$\int_{\mathbb{R}^N} |v_n|^{\frac{4(N+1)}{N}} = \left(\frac{c_2}{c_1}\right)^{\frac{-N^2+2N+4}{N}} \int_{\mathbb{R}^N} |u_n|^{\frac{4(N+1)}{N}} > \int_{\mathbb{R}^N} |u_n|^{\frac{4(N+1)}{N}}, \tag{2.12}$$

i.e. $v_n \in N_{c_2}$ since $u_n \in M_{c_1}$. Then by Lemma 2.5 there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $v_n^{t_n} \in M_{c_2}$ and $I(v_n^{t_n}) = \max_{t>0} I(v_n^t)$.

Furthermore, there exists $C > 0$ independent of $n$ such that $t_n \geq C$ for all $n$. Indeed, we just assume that $t_n \to 0$ as $n \to +\infty$. By the definition of $\{v_n\}$, we see that $\{v_n\}$ is uniformly bounded in $H$. Then we conclude from Lemma 2.6 (3) that $0 < m_{c_2} \leq \lim_{n \to +\infty} I(v_n^{t_n}) \to 0$, which is impossible. So by Lemma 2.6 (2) and (2.11) we have

$$\begin{aligned} m_{c_2} \leq I(v_n^{t_n}) \; & \leq I(u_n^{t_n}) - \left((\frac{c_2}{c_1})^{\frac{-N^2+2N+4}{N}} - 1\right) t_n^{N+2} \int_{\mathbb{R}^N} |u_n|^{\frac{4(N+1)}{N}} \\ & < \max_{t>0} I(u_n^t) - \left((\frac{c_2}{c_1})^{\frac{-N^2+2N+4}{N}} - 1\right) t_n^{N+2} C_0 \\ & \leq I(u_n) - \left((\frac{c_2}{c_1})^{\frac{-N^2+2N+4}{N}} - 1\right) t_n^{N+2} C_0 \\ & \leq m_{c_1} + \frac{1}{n} - \left((\frac{c_2}{c_1})^{\frac{-N^2+2N+4}{N}} - 1\right) C^{N+2} C_0, \end{aligned} \tag{2.13}$$

where $C_0$ is a positive constant independent of $n$ given in Lemma 2.6. Hence it follows that $m_{c_2} < m_{c_1}$.

**Case 1**: $N = 1$.

Set $v_n(x) := u_n((\frac{c_1}{c_2})^2 x)$, then $|v_n|_2 = c_2$ and $|\nabla v_n|_2 = (\frac{c_1}{c_2})^2 |\nabla u_n|_2$. Similarly to (2.11) and (2.12), we have

$$\int_{\mathbb{R}} |v_n|^2 |\nabla v_n|^2 = \left(\frac{c_1}{c_2}\right)^2 \int_{\mathbb{R}} |u_n|^2 |\nabla u_n|^2 < \int_{\mathbb{R}} |u_n|^2 |\nabla u_n|^2,$$

and

$$\int_{\mathbb{R}} |v_n|^{\frac{4(N+1)}{N}} = \left(\frac{c_2}{c_1}\right)^2 \int_{\mathbb{R}} |u_n|^{\frac{4(N+1)}{N}} > \int_{\mathbb{R}} |u_n|^{\frac{4(N+1)}{N}},$$

Then $v_n \in N_{c_2}$. By the same process as in (2.13), we see that

$$m_{c_2} \leq m_{c_1} + \frac{1}{n} - \left((\frac{c_2}{c_1})^2 - 1\right) C^{N+2} C_0,$$

so it follows that $m_{c_2} < m_{c_1}$. Then we complete the proof of the lemma. $\square$

**Lemma 2.8.** *For any $c > c_*$, each minimizer of $m_c$ is a critical point of $I(u)$ constrained on $S_c$.*

*Proof.* We note that $M_c = \{u \in S_c |\ G(u) = 0\}$. then $m_c = \inf_{\{u \in S_c |\ G(u)=0\}} I(u)$.

Let $\tilde{m}_c := \inf_{\{u \in S_c |\ G(u)=0\}} I(u)$. Suppose that $u \in M_c$ is a minimizer of $m_c$, then $u$ is also a minimizer of $\tilde{m}_c$. Hence by standard arguments, there exist $\lambda, \mu \in \mathbb{R}$ such that $I'(u) - \lambda u - \mu G'(u) = 0$, i.e. $u$ satisfies the following equation

$$-(1-2\mu)\Delta u + [1-(N+2)\mu] u \Delta(|u|^2) - [1-(N+2)\mu] |u|^{2+\frac{4}{N}} u = \lambda u. \tag{2.14}$$

It is enough to prove that $\lambda \neq 0$ and $\mu = 0$.

By contradiction, we just suppose that $\mu \neq 0$. By (2.14), we know that $u$ satisfies the following Pohozaev identity

$$\begin{aligned}(N-2)\left[\frac{1-2\mu}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + [1-(N+2)\mu]\int_{\mathbb{R}^N} |u|^2 |\nabla u|^2\right] - \frac{N}{2}\lambda \int_{\mathbb{R}^N} |u|^2 \\ - \frac{N^2}{4(N+1)}[1-(N+2)\mu]\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} = 0.\end{aligned}$$

We conclude from (2.14) again that

$$\begin{aligned}(1-2\mu)\int_{\mathbb{R}^N} |\nabla u|^2 + (N+2)[1-(N+2)\mu]\int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \\ - \frac{N(N+2)}{4(N+1)}[1-(N+2)\mu]\int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} = 0,\end{aligned}$$

i.e.

$$G(u) - \mu \left[ 2\int_{\mathbb{R}^N} |\nabla u|^2 + (N+2)^2 \left( \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{\frac{4(N+1)}{N}} \right) \right] = 0.$$

Then combing $G(u) = 0$ with $\mu \neq 0$, we conclude that $\int_{\mathbb{R}^N} |\nabla u|^2 = 0$, which contradicts Lemma 2.6 (3). So it follows that $\mu = 0$ and then $u$ is a critical point of $I(u)$ constrained on $S_c$. □

The following Lemma is similar to that in [3], so we omit its proof.

**Lemma 2.9.** *Let $\{u_n\} \subset H$ be a bounded sequence of Schwartz Symmetric functions satisfying $u_n \rightharpoonup u$ in $H$, then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 + (N+2)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 \leq \liminf_{n\to+\infty} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 + (N+2)\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 \right).$$

**Proof of Theorem 1.2**

*Proof.* Let $\{u_n\} \subset M_c$ be a minimizing sequence of $m_c$, then by Lemma 2.6 (1), $\{u_n\}$ is uniformly bounded in $H$. To obtain a minimizer of $m_c$, let $\{v_n\}$ be the sequence of Schwartz Symmetric functions for $\{u_n\}$, then by the Pólya-Szegö inequality (see also Lemma 4.3 in [3]), we have

$$\begin{gathered} \int_{\mathbb{R}^N} |\nabla v_n|^2 \leq \int_{\mathbb{R}^N} |\nabla u_n|^2, \quad \int_{\mathbb{R}^N} |v_n|^2 = \int_{\mathbb{R}^N} |u_n|^2 = c^2, \\ \int_{\mathbb{R}^N} |v_n|^{\frac{4(N+1)}{N}} = \int_{\mathbb{R}^N} |u_n|^{\frac{4(N+1)}{N}}, \\ \int_{\mathbb{R}^N} |\nabla v_n|^2 + (N+2)\int_{\mathbb{R}^N} |v_n|^2|\nabla v_n|^2 \leq \int_{\mathbb{R}^N} |\nabla u_n|^2 + (N+2)\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2, \end{gathered} \tag{2.15}$$

hence the sequence $\{v_n\}$ is also uniformly bounded in $H$. Moreover, we have

$$G(v_n) \leq G(u_n) = 0. \tag{2.16}$$

Since $\{v_n\}$ is uniformly bounded, up to a subsequence, there exists $v \in H$ such that

$$\begin{cases} v_n \rightharpoonup v, & \text{in } H, \\ v_n \to v, & \text{in } L^p(\mathbb{R}^N), \qquad \forall\, p \in (2, 2^*). \end{cases} \tag{2.17}$$

In particular, since $\frac{4(N+1)}{N} < 2^*$, by (2.15) and Lemma 2.6 we see that

$$\int_{\mathbb{R}^N} |v|^{\frac{4(N+1)}{N}} = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^{\frac{4(N+1)}{N}} = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{\frac{4(N+1)}{N}} \geq C_0 > 0, \tag{2.18}$$

which implies that $v \neq 0$, where $C_0 > 0$ is a constant given in Lemma 2.6 (2). Set $\alpha := |v|_2$, then $\alpha \in (0, c]$. We conclude from Lemma 2.9 and (2.16)-(2.18) that

$$G(v) \leq \liminf_{n \to +\infty} G(v_n) \leq 0,$$

i.e. $v \in N_\alpha$ and $G(v) \leq 0$. So it follows from Corollary 2.2 that $\alpha \in (c_*, c]$. By Lemma 2.5, there exists a unique $t \in (0, 1]$ such that $v^t \in M_\alpha$. Then by Lemma 2.7 we have

$$\begin{aligned} m_\alpha \leq I(v^t) = I(v^t) - \frac{1}{N+2} G(v^t) &= \frac{N}{2(N+2)} t^2 \int_{\mathbb{R}^N} |\nabla v|^2 \\ &\leq \frac{N}{2(N+2)} \liminf_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \\ &\leq \frac{N}{2(N+2)} \liminf_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \\ &= \liminf_{n \to +\infty} \left( I(u_n) - \frac{1}{N+2} G(u_n) \right) \\ &= m_c \leq m_\alpha, \end{aligned}$$

where the equality holds only for $\alpha = c$ and $t = 1$. So $\alpha = c$ and $I(v) = m_c$. Therefore we obtain a minimizer $v \in M_c$ of $m_c$. By Lemma 2.8, we see that $v$ is a critical point of $I(u)$ constrained on $S_c$. That is to say, there exists $\lambda_c \in \mathbb{R}$ such that $I'(v) - \lambda_c v = 0$. Hence by $G(v) = 0$, we have

$$\begin{aligned} \lambda_c c^2 &= \int_{\mathbb{R}^N} |\nabla v|^2 + 4 \int_{\mathbb{R}^N} |v|^2 |\nabla v|^2 - \int_{\mathbb{R}^N} |v|^{\frac{4(N+1)}{N}} \\ &= \frac{N^2 - 2N - 4}{N(N+2)} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{4}{N} \int_{\mathbb{R}^N} |v|^2 |\nabla v|^2 < 0, \end{aligned}$$

i.e. $\lambda_c < 0$. So $(v, \lambda_c) \in S_c \times \mathbb{R}_-$ is a couple of solution to the problem (1.7). The theorem is proved. □

**Remark 2.10.** *When $N \geq 4$, similarly to the proof of Lemma 2.5, we see that $m_c$ is also well defined. However, it seems impossible to show the strict monotonicity of $m_c$ by the scaling arguments as Lemma 2.7. Indeed, for any $c > c_*$ and any $u \in M_c$, set $u_\theta(x) := \theta^\alpha u(\theta^\beta x)$, $\forall\, \theta > 1$, where $\alpha, \beta \in \mathbb{R}$ are to be undetermined so that $u_\theta \in N_{\theta c}$. Then it should require that*

$$\begin{cases} 2\alpha - N\beta = 2, \\ 2\alpha + (2 - N)\beta \leq 0, \\ 4\alpha + (2 - N)\beta \leq 0, \\ \frac{4(N+1)}{N}\alpha - N\beta \geq 0. \end{cases}$$

*Hence we conclude that*
$$-\frac{N}{N+2}\leq\alpha\leq\frac{2-N}{2}$$
*and then we obtain a necessary condition:* $-N^2+2N+4\geq 0$, *which is impossible since* $N\geq 4$.

# 3 Proof of Proposition 1.4

**Proof of Proposition 1.4**

*Proof.* For any $c>c_*$, let $(u_c,\lambda_c)\in M_c\times\mathbb{R}_-$ be the solution of (1.7) obtained in Theorem 1.2, then $G(u_c)=0$,
$$m_c=I(u_c)=\frac{N}{2(N+2)}\int_{\mathbb{R}^N}|\nabla u_c|^2 \tag{3.1}$$
and
$$\lambda_c c^2=\frac{N-2}{N+2}\int_{\mathbb{R}^N}|\nabla u_c|^2-\frac{1}{N+1}\int_{\mathbb{R}^N}|u_c|^{\frac{4(N+1)}{N}}. \tag{3.2}$$
We complete the proof in three steps.

(1) We claim that the function $c\mapsto m_c$ is continuous on $(c_*,+\infty)$.

To prove that the function $c\mapsto m_c$ is continuous at $c\in(c_*,+\infty)$, by Lemma 2.7 it is enough to show that $\limsup\limits_{c_n\to c^-} m_{c_n}\leq m_c$ for any sequence $c_n\to c^-$.

Since $c_n\to c^-$, for $n$ large enough, $\frac{c_n}{c}u_c\in N_{c_n}$ and by Lemma 2.5 there exists a sequence $\{t_n\}\subset\mathbb{R}_+$ such that $\frac{c_n}{c}u_c^{t_n}\in M_{c_n}$, moreover, $\lim\limits_{c_n\to c^-}t_n=1$. So
$$m_{c_n}\leq I(\frac{c_n}{c}u_c^{t_n})\to I(u_c)=m_c,$$
which implies the conclusion.

(2) $\begin{cases} m_c\to+\infty, \\ \int_{\mathbb{R}^N}|\nabla u_c|^2\to+\infty, & \text{as } c\to(c_*)^+. \\ \lambda_c\to-\infty, \end{cases}$

We conclude from Lemma 2.6 (3) and (3.1) that $\int_{\mathbb{R}^N}|\nabla u_c|^2\geq\frac{2(N+2)}{N}C_1>0$, where $C_1>0$ is a positive constant. Hence by (2.2) and (2.7) we have
$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla u_c|^2 &\leq \frac{N(N+2)}{4(N+1)}\left[1-\left(\frac{c_*}{c}\right)^{\frac{4}{N}}\right]\int_{\mathbb{R}^N}|u_c|^{\frac{4(N+1)}{N}}\\ &\leq \frac{N(N+2)}{4(N+1)}C\left[1-\left(\frac{c_*}{c}\right)^{\frac{4}{N}}\right]c^{\frac{-N^2+2N+4}{N}}\left(\int_{\mathbb{R}^N}|\nabla u_c|^2\right)^{\frac{N+2}{2}},\end{aligned}$$
which implies that $\int_{\mathbb{R}^N}|\nabla u_c|^2\to+\infty$ and $\frac{\int_{\mathbb{R}^N}|u_c|^{\frac{4(N+1)}{N}}}{\int_{\mathbb{R}^N}|\nabla u_c|^2}\to+\infty$ as $c\to(c_*)^+$. So the results follow from (3.1) and (3.2).

$$(3)\quad \begin{cases} m_c \to 0, \\ \int_{\mathbb{R}^N} |\nabla u_c|^2 \to 0, & \text{as } c \to +\infty. \\ \lambda_c \to 0, \end{cases}$$

Let $c_0 > c_*$ be fixed and $u_{c_0} \in M_{c_0}$ be the minimizer of $m_{c_0}$. For any $c > c_0$, then $w(x) := u_{c_0}((\frac{c_0}{c})^{\frac{2}{N}} x) \in N_c$, hence by Lemma 2.5 there exists

$$t = \left( \frac{\frac{1}{N+2} \int_{\mathbb{R}^N} |\nabla w|^2}{\frac{N}{4(N+1)} \int_{\mathbb{R}^N} |w|^{\frac{4(N+1)}{N}} - \int_{\mathbb{R}^N} |w|^2 |\nabla w|^2} \right)^{\frac{1}{N}}$$

such that $w^t \in M_c$. So

$$\begin{aligned} m_c \le I(w^t) &= \frac{N \int_{\mathbb{R}^N} |\nabla w|^2}{2(N+2)} \left( \frac{\frac{1}{N+2} \int_{\mathbb{R}^N} |\nabla w|^2}{\frac{N}{4(N+1)} \int_{\mathbb{R}^N} |w|^{\frac{4(N+1)}{N}} - \int_{\mathbb{R}^N} |w|^2 |\nabla w|^2} \right)^{\frac{2}{N}} \\ &= m_{c_0} \left( \frac{\frac{1}{N+2} (\frac{c}{c_0})^{N-2} \int_{\mathbb{R}^N} |\nabla u_{c_0}|^2}{(\frac{c}{c_0})^{\frac{4}{N}} \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u_{c_0}|^{\frac{4(N+1)}{N}} - \int_{\mathbb{R}^N} |u_{c_0}|^2 |\nabla u_{c_0}|^2} \right)^{\frac{2}{N}} \\ &\longrightarrow 0 \end{aligned}$$

as $c \to +\infty$, which implies that $\lim_{c \to +\infty} m_c = 0$ and by (3.1) we have $\int_{\mathbb{R}^N} |u_c|^2 \to 0$ as $c \to +\infty$. Then by (3.2) and (2.7) we see that $\lim_{c \to +\infty} \lambda_c = 0$. $\square$

# 4 Proof of Theorem 1.5

Recall from the well-known Gagliardo-Nirenberg inequality (1.8)-(1.10) in section 1 that for any $u \in S_c$, we have

$$\int_{\mathbb{R}^N} |u|^{\frac{2N+4}{N}} \le \frac{N+2}{N} \left( \frac{c}{c^*} \right)^{\frac{4}{N}} \int_{\mathbb{R}^N} |\nabla u|^2, \qquad \forall\, u \in H^1(\mathbb{R}^N), \tag{4.1}$$

with equality only for $u = Q_{\frac{2N+4}{N}}$, where $c^* := |Q_{\frac{2N+4}{N}}|_2$. Moreover, we conclude from (1.9) and the associated Pohozaev identity that

$$\int_{\mathbb{R}^N} |\nabla Q_{\frac{2N+4}{N}}|^2 = \int_{\mathbb{R}^N} |Q_{\frac{2N+4}{N}}|^2 = \frac{N}{N+2} \int_{\mathbb{R}^N} |Q_{\frac{2N+4}{N}}|^{\frac{2N+4}{N}}. \tag{4.2}$$

**Proof of Theorem 1.5**

*Proof.* (1) For any $c > 0$ and $u \in S_c$, set $u^t(x) := t^{\frac{N}{2}} u(tx)$, $t > 0$. Then $u^t \in S_c$ and $I_{\frac{2N+4}{N}}(u^t) \to 0$ as $t \to 0^+$, then $i_{\frac{2N+4}{N},c} \le 0$.

If $0 < c \le c^*$, then by (4.1) we have $\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \ge \frac{N}{2N+4} \int_{\mathbb{R}^N} |u|^{\frac{2N+4}{N}}$ and then

$$I_{\frac{2N+4}{N}}(u) \ge \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 > 0,$$

which implies that $i_{\frac{2N+4}{N},c} \geq 0$. So $i_{\frac{2N+4}{N},c} = 0$ for each $0 < c \leq c^*$.

If $c > c^*$, set $Q_c^t(x) := \frac{ct^{\frac{N}{2}}}{c^*} Q_{\frac{2N+4}{N}}(tx)$, $\forall t > 0$, then $Q_c^t \in S_c$ and by (1.10) and (4.2) we see that

$$I(Q_c^t) = \left(\frac{c}{c^*}\right)^4 \int_{\mathbb{R}^N} |Q_{\frac{2N+4}{N}}|^2 |\nabla Q_{\frac{2N+4}{N}}|^2 t^{N+2} - \frac{c^2}{2}\left[\left(\frac{c}{c^*}\right)^{\frac{4}{N}} - 1\right] t^2 := f(t)$$

Hence

$$i_{\frac{2N+4}{N},c} \leq \inf_{t>0} I(Q_c^t) = -\frac{N}{2(N+2)} \left[\frac{(c^*)^4}{(N+2)\int_{\mathbb{R}^N} |Q_{\frac{2N+4}{N}}|^2 |\nabla Q_{\frac{2N+4}{N}}|^2}\right]^{\frac{2}{N}} c^{2-\frac{4}{N}} \left[\left(\frac{c}{c^*}\right)^{\frac{4}{N}} - 1\right]^{1+\frac{2}{N}} < 0.$$

(2) For any $0 < c \leq c^*$ and any $u \in S_c$, by (1) we see that $I_{\frac{2N+4}{N}}(u) > 0$. So there exists no minimizer for $i_{\frac{2N+4}{N},c}$.

For any $c > c^*$, let $\{u_n\} \subset S_c$ be a minimizing sequence for $i_{\frac{2N+4}{N},c} < 0$. Let $u_n^\theta(x) = u_n(\theta^{-\frac{2}{N}}x)$ with $\forall\, \theta > 1$, then $u_n^\theta \in S_{\theta c}$ and $I_{\frac{2N+4}{N}}(u_n^\theta) \leq \theta^2 I_{\frac{2N+4}{N}}(u_n)$. Letting $n \to +\infty$, then

$$i_{\frac{2N+4}{N},\theta c} \leq \theta^2 i_{\frac{2N+4}{N},c} < i_{\frac{2N+4}{N},c},$$

which implies that $i_{\frac{2N+4}{N},c}$ is strictly decreasing on $(c^*, +\infty)$.

Since $I_{\frac{2N+4}{N}}(u_n) \to i_{\frac{2N+4}{N}} < 0$ as $n \to +\infty$, we conclude that for $n$ large enough $I_{\frac{2N+4}{N}}(u_n) \leq 1$. By the Hölder and Sobolev inequalities (see also (4.5) in [3]), there exists a positive constant $C$ depending only on $N$ such that

$$\int_{\mathbb{R}^N} |u_n|^{\frac{2N+4}{N}} \leq C c^{\frac{2N^2+8}{N^2+2N}} \left(\int_{\mathbb{R}^N} (1 + |u_n|^2)|\nabla u_n|^2\right)^{\frac{2}{N+2}}. \tag{4.3}$$

Then

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^N} (1 + u_n|^2)|\nabla u_n|^2 &\leq \frac{N}{2N+4}\int_{\mathbb{R}^N} |u_n|^{\frac{2N+4}{N}} + 1 \\ &\leq \frac{N}{2N+4} C c^{\frac{2N^2+8}{N^2+2N}} \left(\int_{\mathbb{R}^N} (1 + |u_n|^2)|\nabla u_n|^2\right)^{\frac{2}{N+2}} + 1, \end{aligned}$$

which implies that $\{u_n\}$ is uniformly bounded in $H$.

Similarly to the proof of Theorem 1.2, let $\{v_n\} \subset S_c$ be the sequence of Schwartz symmetric functions for $\{u_n\}$, then $\{v_n\}$ is a uniformly bounded minimizing sequence for $i_{\frac{2N+4}{N},c}$. Hence there exists $v \in H$ such that $v_n \rightharpoonup v$ in $H$ and

$$I(v) \leq \lim_{n\to+\infty} I(v_n) = i_{\frac{2N+4}{N},c} < 0,$$

which implies that $c^* < \alpha := |v|_2 < c$. Set $w(x) = v((\frac{\alpha}{c})^{\frac{2}{N}}x)$, then $w \in S_c$ and

$$i_{\frac{2N+4}{N},c} \leq I(w) \leq (\frac{c}{\alpha})^2 I(v) < i_{\frac{2N+4}{N},c},$$

which is a contradiction. So $v \in S_c$ and $I(v) = i_{\frac{2N+4}{N},c}$.

(3) By contradiction, if there exists some $c \in (0, c^*]$ and some $u_c \in S_c$ such that $(I|_{S_c})'(u_c) = 0$, then similarly to the proof of Lemma 2.4, we see that $u$ satisfies

$$\int_{\mathbb{R}^N} |\nabla u_c|^2 + (N+2)\int_{\mathbb{R}^N} |u_c|^2 |\nabla u_c|^2 = \frac{N}{N+2}\int_{\mathbb{R}^N} |u_c|^{\frac{2N+4}{N}},$$

hence $I_{\frac{2N+4}{N}}(u_c) = -\frac{N}{2}\int_{\mathbb{R}^N} |u_c|^2|\nabla u_c|^2 < 0$, which is a contradiction with (1). Then the theorem is proved. □

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