NEW DECAY RATES FOR CAUCHY PROBLEM OF TIMOSHENKO THERMOELASTIC SYSTEMS WITH PAST HISTORY: CATTANEO AND FOURIER LAW

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Abstract. In this paper, we investigate the decay properties of the thermoelastic Timoshenko system with past history in the whole space where the thermal effects are given by Cattaneo and Fourier laws. We obtain that both systems, Timoshenko-Fourier and Timoshenko-Cattaneo, have the same rate of decay $(1+t)^{-\frac{1}{4}}$ and the regularity-loss type property is not present in some cases. Moreover, new stability number χ is introduced, such new number controls the decay rate of the solution with respect to the regularity of the initial data. To prove our results, we use the energy method in Fourier space to build an appropriate Lyapunov functionals that give the desired results.

Keywords: Timoshenko system, Cattaneo law, Fourier Law, new stability number, history term, energy method, Lyapunov functional

1. Introduction

In this paper, we investigate the decay properties of the thermoelastic Timoshenko system with past history in the whole space where the thermal effects are given by Cattaneo and Fourier laws. We consider two systems, the first one is the coupling between Timoshenko beam with the heat conduction described by Cattaneo law with a history term, given by:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x - \psi\right)_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_2\psi_{tt} - b\psi_{xx} + m\int_0^\infty g\left(s\right)\psi_{xx}\left(t-s,x\right)ds - k\left(\varphi_x-\psi\right) + \delta\theta_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{1.1}$$

with the initial data

$$\left(\varphi,\varphi_t,\psi,\psi_t,\theta,q\right)\left(x,0\right) = \left(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,,q_0\right), \tag{1.2}$$

where $b,k,m,\delta,\beta,\rho_1,\rho_2,\rho_3$ and τ are positive constants, with φ, ψ, θ and q denoting the transversal displacement, the rotation angle of the beam, the temperature and the heat flow, respectively. The integral term represents a history term with kernel g satisfying the following hypotheses:

- (H_1) $g\left(.\right)$ is a non negative function.
- (H_2) There exist positive constants k_1 and k_2, such that, $-\,k_1\,g\left(s\right) \le g'\left(s\right) \le -k_2\,g\left(s\right).$
- (H_3) $a := b - mb_0 > 0,$ where $b_0 = \int_0^\infty g\left(s\right)ds.$

The second system of interest is the coupling between Timoshenko beam with the heat conduction described by Fourier law with a history term given by:

(1.3)
$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x - \psi\right)_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_2\psi_{tt} - b\psi_{xx} + m\int_0^\infty g\left(s\right)\psi_{xx}\left(t-s,x\right)ds - k\left(\varphi_x - \psi\right) + \delta\theta_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_3\theta_t - \dfrac{1}{\beta}\theta_{xx} + \delta\psi_{xt} = 0, & \text{in } (0,\infty)\times\mathbb{R}, \end{cases}$$

with the initial data

$$\left(\varphi,\varphi_t,\psi,\psi_t,\theta\right)(x,0) = \left(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0\right). \tag{1.4}$$

We will introduce the stability number given by

$$\chi = \left(\frac{b\rho_1}{k} - \rho_2\right).$$

The main purpose of this article is to investigate the relationship between damping terms, the stability numbers χ and their influence on the decay rate of solutions of systems (1.1)-(1.2) and (1.3)-(1.4). Using the stability number, we show a new estimates for the solution of the thermoelastic Timoshenko systems (1.1)-(1.3).

The main idea in our proof is to construct new functionals to capture the dissipation of all components in the solution of (1.1)-(1.3). Then, we build an appropriate Lyapunov functionals which gives the desired dissipation of all the components in the solution of (1.1)-(1.3). It is well known in the literature that the behavior of $\lambda_i(\xi)$ (see (3.4) and (3.39)) in the low frequencies determines the rate of decay of the solution, while its behavior for high frequencies gives the regularity restriction on the initial data see ([4], [5], [6], [7], [8], [9], [10]).

We need to mention here that the same systems have been considered recently by [1], the authors showed that the solution decays as follow:

- *If* $\chi_{0,\tau} = 0$ *(resp, if* $\chi_0 = 0$*)*, then
$$\left\|\partial_x^k U\left(t\right)\right\|_{L^2} \le C\left(1+t\right)^{-\frac{1}{8}-\frac{k}{4}}\left\|U_0\right\|_{L^1} + C\left(1+t\right)^{-\frac{l}{2}}\left\|\partial_x^{k+l}U_0\right\|_{L^2};\ t\ge 0.$$
- *If* $\chi_{0,\tau} \ne 0$ *(resp, if* $\chi_0 \ne 0$*)*
$$\left\|\partial_x^k U\left(t\right)\right\|_{L^2} \le C\left(1+t\right)^{-\frac{1}{8}-\frac{k}{4}}\left\|U_0\right\|_{L^1} + C\left(1+t\right)^{-\frac{l}{4}}\left\|\partial_x^{k+l}U_0\right\|_{L^2};\ t\ge 0.$$

Where $\chi_{0,\tau} = \left(\left(\tau - \dfrac{\rho_1}{k\rho_3}\right)\left(\rho_2 - \dfrac{b\rho_1}{k}\right) - \dfrac{\tau\rho_1\delta^2}{\rho_3 k}\right)$ and $\chi_0 = \left(\rho_2 - \dfrac{b\rho_1}{k}\right)$.

It's clear that our estimates in Theorem 4.1 and Theorem 4.2 improve the decay rates in [1], we need to mention here that in the case where $\chi = 0$ we don't have the regularity loss phenomena like in [1].

This paper is organized as follows. In Section 2 we state the problem. Section 3, is devoted for the construction of Lyapunov functionals using the energy method in the Fourier space. The last section is dedicated to the statements and the proof of our main results.

2. Statement of the problem

In this section and in order to establish the decay rates of the Timoshenko systems (1.1) and (1.3), we have to transform the original problems to a first-order systems. Here we need to define a new variables as in [1].

2.1. **The Cattaneo Model.** We consider Timoshenko system with history and Cattaneo law. Following the same change of variable as in [3]:

$$\eta(t,s,x) := \psi(t,x) - \psi(t-s,x) \quad (t,x) \in (0,\infty)\times\mathbb{R},\ s \geq 0. \tag{2.1}$$

System (1.1), can be rewritten as:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x-\psi)_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_2\psi_{tt} - a\psi_{xx} - m\int_0^\infty g(s)\,\eta_{xx}(s)ds - k(\varphi_x-\psi) + \delta\theta_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \eta_t + \eta_s - \psi_t = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ \eta(.,0,.) = 0, & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{2.2}$$

where a is a positive constant given by (H_3) and the operator $T\eta = -\eta_s$ is the usual operator defined in systems with history terms, see for instance ([2]) and references therein. Here, the last two equations of system (2.2) are obtained differentiating equation (2.1). We define also the initial data

$$\begin{aligned} &(\varphi,\varphi_t,\psi,\psi_t,\theta,q)(x,0) = (\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,q_0), \\ &\eta(0,s,.) = \psi(0,.) - \psi(-s,.). \end{aligned}$$

As in [1], we can rewrite system (2.2) as a first-order system, by considering the following change of variables:

$$u = \varphi_t,\ z = \psi_x,\ y = \psi_t,\ v = \varphi_x - \psi,$$

Then, (2.2) takes the form:

$$\begin{cases} v_t - u_x + y = 0, \\ \rho_1 u_t - k\,v_x = 0, \\ z_t - y_x = 0, \\ \rho_2\,y_t - a\,z_x - m\int_0^\infty g(s)\,\eta_{xx}(s)\,ds - k\,v + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta\,y_x = 0, \\ \tau q_t + \beta q + \theta_x = 0, \\ \eta_t + \eta_s - y = 0. \end{cases} \tag{2.3}$$

We define the solution of (2.3) by introducing the vector U given by:

$$U(x) = (v,u,z,y,\theta,q,\eta)^T.$$

The initial condition can be written as

$$U_0(x) = U(x,0) = (v_0,u_0,z_0,y_0,\theta_0,q_0,\eta_0)^T, \tag{2.4}$$

where

$$u_0 = \varphi_1,\ z_0 = \psi_{0,x},\ y_0 = \psi_1,\ v_0 = \varphi_{0,x} - \psi_0.$$

2.2. **The Fourier Model.** Here, we consider Timoshenko system (1.3) with history and the

Fourier law. Introducing η as in the Cattaneo model, we have the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x - \psi\right)_x = 0, & in\ (0,\infty)\times\mathbb{R}, \\ \rho_2\psi_{tt} - a\psi_{xx} - m\int_0^\infty g\left(s\right)\eta_{xx}(s)ds - k\left(\varphi_x - \psi\right) + \delta\theta_x = 0, & in\ (0,\infty)\times\mathbb{R}, \\ \rho_3\theta_t + \dfrac{1}{\beta}\theta_{xx} + \delta\psi_{xt} = 0, & in\ (0,\infty)\times\mathbb{R}, \\ \eta_t + \eta_s - \psi_t = 0, & in\ (0,\infty)\times\mathbb{R}, \\ \eta\left(.,0,.\right) = 0, & in\ (0,\infty)\times\mathbb{R}, \end{cases} \tag{2.5}$$

with initial data

$$\begin{array}{l} \left(\varphi,\varphi_t,\psi,\psi_t,\theta\right)\left(x,0\right) = \left(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0\right), \\ \eta\left(0,s,.\right) = \psi\left(0,.\right) - \psi\left(-s,.\right), \end{array}$$

As in the previous section, we can rewrite the system as a first-order system, by defining the following new variables:

$$u = \varphi_t,\ z = \psi_x,\ y = \psi_t,\ v = \varphi_x - \psi.$$

Then, (2.5) takes the form

$$\begin{cases} v_t - u_x + y = 0, \\ \rho_1 u_t - k\,v_x = 0, \\ z_t - y_x = 0, \\ \rho_2\,y_t - a\,z_x - m\int_0^\infty g\left(s\right)\eta_{xx}\left(s\right)ds - k\,v + \delta\theta_x = 0, \\ \rho_3\theta_t + \dfrac{1}{\beta}\theta_{xx} + \delta\,y_x = 0, \\ \eta_t + \eta_s - y = 0. \end{cases} \tag{2.6}$$

We define the solution of (2.6) by introducing the vector V given by:

$$V\left(x\right) = \left(v,u,z,y,\theta,\eta\right)^T.$$

The initial condition can be written as

$$V_0\left(x\right) = V\left(x,0\right) = \left(v_0,u_0,z_0,y_0,\theta_0,\eta_0\right)^T, \tag{2.7}$$

where

$$u_0 = \varphi_1,\ z_0 = \psi_{0,x},\ y_0 = \psi_1,\ v_0 = \varphi_{0,x} - \psi_0.$$

3. The energy method in the Fourier space

In this section, we establish some decay rates for the Fourier image of the solutions of Timoshenko-Cattaneo Law and Timoshenko-Fourier Law systems. For each model we use the energy method to build an appropriate Lyapunov functionals in the Fourier space. These estimates will play a crucial role in proving our results in Theorem 4.1 and Theorem 4.2.

3.1. **Cattaneo Model.** Taking Fourier transform in (2.3), we obtain the following ODE system:

$$
\begin{cases}
\widehat{v}_t - i\xi\,\widehat{u} + \widehat{y} = 0, \\
\rho_1 \widehat{u}_t - ik\xi\,\widehat{v} = 0, \\
\widehat{z}_t - i\xi\,\widehat{y} = 0, \\
\rho_2\,\widehat{y}_t - ia\xi\,\widehat{z} + m\xi^{\,2}\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds - k\,\widehat{v} + i\delta\xi\,\widehat{\theta} = 0, \\
\rho_3 \widehat{\theta}_t + i\xi\,\widehat{q} + i\delta\xi\,\widehat{y} = 0, \\
\tau \widehat{q}_t + \beta \widehat{q} + i\xi\,\widehat{\theta} = 0, \\
\widehat{\eta}_t + \widehat{\eta}_s - \widehat{y} = 0.
\end{cases}
\tag{3.1}
$$

The solution vector and initial data are given by $\widehat{U}\left(\xi,t\right) = \left(\widehat{v},\widehat{u},\widehat{z},\widehat{y},\widehat{\theta},\widehat{q},\widehat{\eta}\right)^T$ and $\widehat{U}\left(\xi,0\right) = \widehat{U}_0\left(\xi\right)$.

First, we define the corresponding energy as:

$$
\widehat{E}\left(\xi,t\right) = \rho_1 \left|\widehat{u}\right|^2 + \rho_2 \left|\widehat{y}\right|^2 + \rho_3 \left|\widehat{\theta}\right|^2 + k\left|\widehat{v}\right|^2 + a\left|\widehat{z}\right|^2 + \tau\left|\widehat{q}\right|^2 + m\xi^2 \int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2 ds.
\tag{3.2}
$$

The energy (3.2) satisfies the following estimate (Lemma 3.1, [1]):

$$
\frac{d}{dt}\widehat{E}\left(\xi,t\right) = -2\beta\left|\widehat{q}\right|^2 + m\xi^2\int_0^\infty g'\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2 ds,
$$

using (H_2), we have

$$
\frac{d}{dt}\widehat{E}\left(\xi,t\right) \leq -2\beta\left|\widehat{q}\right|^2 - k_1 m\xi^2\int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2 ds.
$$

Here, we give the pointwise estimates of the functional $\widehat{E}\left(\xi,t\right)$. This estimate will play the essential role in proving our main theorem. The result is stated below:

Proposition 3.1. *For any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following estimates*

$$
\widehat{E}\left(\xi,t\right) \leq \begin{cases} C\,e^{-c\lambda_1(\xi)\,t}\widehat{E}\left(\xi,0\right); & if\ \chi = 0, \\ C\,e^{-c\lambda_2(\xi)\,t}\widehat{E}\left(\xi,0\right); & if\ \chi \neq 0, \end{cases}
\tag{3.3}
$$

where

$$
\lambda_1\left(\xi\right) = \frac{\xi^2}{1+\xi^2}; \qquad \lambda_2\left(\xi\right) = \frac{\xi^2}{\left(1+\xi^2\right)^2}.
\tag{3.4}
$$

Here C and c are two positive constants.

We show that the decay rate of the solution will depend on the value of χ. The proof will be given through several lemmas.

First, we will use the following notation for the product of complex numbers:

$$
\begin{cases} \langle \widehat{z}_1,\,\widehat{z}_2\rangle = \widehat{z}_1\,\overline{\widehat{z}_2}, \\ \left|\widehat{z}_1\right|^2 = \widehat{z}_1\,\overline{\widehat{z}_1}. \end{cases}
$$

Lemma 3.2. *Let the functional*

$$\mathcal{F}_1(\xi,t) = \frac{2\tau\rho_3}{(1+\xi^2)} \operatorname{Re}\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle .$$

We have the following estimate:

$$\frac{\xi^2}{(1+\xi^2)}\rho_3 \left|\widehat{\theta}\right|^2 + \frac{d}{dt}\mathcal{F}_1(\xi,t) \le C(\varepsilon_1)\tau\left|\widehat{q}\right|^2 + \varepsilon_1 \frac{\xi^2}{(1+\xi^2)}\rho_2 \left|\widehat{y}\right|^2, \tag{3.5}$$

for any $\varepsilon_1 > 0$, C and $C(\varepsilon_1)$ are positive constants.

Proof. Multiplying $(3.1)_6$ by $i\xi\widehat{\theta}$ and using $(3.1)_5$, we get

$$\begin{aligned} 0 &= \tau\frac{d}{dt}\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle + \tau\left\langle i\xi\widehat{q}, \widehat{\theta}_t \right\rangle + \beta\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle + \xi^2\left|\widehat{\theta}\right|^2 \\ &= \tau\frac{d}{dt}\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle - \frac{\tau}{\rho_3}\xi^2\left|\widehat{q}\right|^2 - \frac{\tau\delta}{\rho_3}\xi^2\left\langle \widehat{q},\ \widehat{y}\right\rangle + \beta\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle + \xi^2\left|\widehat{\theta}\right|^2, \end{aligned}$$

then, we deduce

$$\rho_3\xi^2\left|\widehat{\theta}\right|^2 + \tau\rho_3\frac{d}{dt}\operatorname{Re}\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle = \tau\xi^2\left|\widehat{q}\right|^2 + \tau\delta\xi^2\operatorname{Re}\left\langle \widehat{q},\ \widehat{y}\right\rangle - \rho_3\beta\operatorname{Re}\left\langle \widehat{q}, i\xi\widehat{\theta} \right\rangle . \tag{3.6}$$

Multiplying (3.6) by $\frac{1}{(1+\xi^2)}$ and applying Young's inequality, to the terms on the right-hand side of (3.6), then (3.5) holds. This finishes the proof of Lemma 3.2. □

Lemma 3.3. *Let the functional*

$$\mathcal{F}_2(\xi,t) = -\frac{2\rho_2\xi^2}{b_0(1+\xi^2)}\operatorname{Re}\left\langle \int_0^\infty g(s)\,\widehat{\eta}(s)\,ds, \widehat{y} \right\rangle .$$

We have the following estimate, for any $\varepsilon_2 > 0$,

$$\begin{aligned} \frac{\xi^2}{(1+\xi^2)}\rho_2\left|\widehat{y}\right|^2 + \frac{d}{dt}\mathcal{F}_2(\xi,t) &\le C\frac{\xi^2}{(1+\xi^2)}\rho_3\left|\widehat{\theta}\right|^2 + \varepsilon_2\frac{\xi^2}{(1+\xi^2)}a\left|\widehat{z}\right|^2 \\ &\quad + C(\varepsilon_2)\,m\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2 ds + \varepsilon_2\frac{\xi^2}{(1+\xi^2)^2}k\left|\widehat{v}\right|^2, \end{aligned} \tag{3.7}$$

where C and $C(\varepsilon_2)$ are positive constants.

Proof. Multiplying $(3.1)_7$ by $g(s)\,\widehat{y}$ and using $(3.1)_4$, we get

$$\begin{aligned} 0 &= \frac{d}{dt}\left\langle \widehat{\eta}, g(s)\,\widehat{y}\right\rangle - \left\langle g(s)\,\widehat{\eta}, \widehat{y}_t\right\rangle + \left\langle g(s)\,\widehat{\eta}_s, \widehat{y}\right\rangle - g(s)\left|\widehat{y}\right|^2 \\ &= \left\langle g(s)\,\widehat{\eta}_s, \widehat{y}\right\rangle + \frac{d}{dt}\left(\left\langle \widehat{\eta}, g(s)\,\widehat{y}\right\rangle\right) - \left|\widehat{y}\right|^2 + \frac{a}{\rho_2}\left\langle i\xi g(s)\,\widehat{\eta},\ \widehat{z}\right\rangle \\ &\quad + \frac{m}{\rho_2}\xi^2\left\langle g(s)\,\widehat{\eta}, \int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right\rangle - \frac{k}{\rho_2}\left\langle g(s)\,\widehat{\eta},\ \widehat{v}\right\rangle + \frac{\delta}{\rho_2}\left\langle g(s)\,\widehat{\eta}, i\xi\,\widehat{\theta}\right\rangle, \end{aligned}$$

integrating with respect to s over $(0,\infty)$, using the fact that $b_0=\int_0^\infty g(s)\,ds$ and using integrating by parts, we get

$$
\begin{aligned}
&\rho_2\left|\widehat{y}\right|^2-\frac{\rho_2}{b_0}\frac{d}{dt}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{y}\right\rangle\\
=\;&-\frac{\rho_2}{b_0}\operatorname{Re}\left\langle\int_0^\infty g'(s)\,\widehat{\eta}(s)\,ds,\widehat{y}\right\rangle+\frac{a}{b_0}\operatorname{Re}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \widehat{z}\right\rangle+\frac{m}{b_0}\xi^{\,2}\left|\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right|^2\\
&-\frac{k}{b_0}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \widehat{v}\right\rangle+\frac{\delta}{b_0}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,i\xi\,\widehat{\theta}\right\rangle,
\end{aligned}
\tag{3.8}
$$

where the following inequalities have been used:

$$
\begin{aligned}
&\left|\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right|^2\le b_0\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds,\\
&\left|\int_0^\infty g'(s)\,\widehat{\eta}(s)\,ds\right|^2\le b_0\max\{k_1,k_2\}\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds.\\
&\frac{k\xi^2}{b_0\left(1+\xi^2\right)}\left|\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \widehat{v}\right\rangle\right|=\xi^2\left|\left\langle\frac{\sqrt{k}}{b_0}\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \frac{\sqrt{k}}{\left(1+\xi^2\right)}\widehat{v}\right\rangle\right|\\
&\qquad\qquad\le C(\varepsilon_2)\,m\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds+\varepsilon_2\frac{\xi^2}{\left(1+\xi^2\right)^2}k\left|\widehat{v}\right|^2.
\end{aligned}
$$

Now, multiplying (3.8) by $\dfrac{\xi^2}{\left(1+\xi^2\right)}$ and applying Young's inequality, to the terms on the right-hand side of (3.8), then (3.7) holds. This finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let the functional*

$$
\begin{aligned}
\mathcal{F}_3(\xi,t)\;=\;&-\frac{2\rho_2}{\left(1+\xi^2\right)}\operatorname{Re}\left\langle\widehat{y},i\xi\ \widehat{z}\right\rangle-\frac{2\rho_1}{\left(1+\xi^2\right)}\operatorname{Re}\left\langle\widehat{u},\widehat{z}\right\rangle-\frac{2\rho_1\rho_3}{\left(1+\xi^2\right)}\operatorname{Re}\left\langle\widehat{\theta},\widehat{u}\right\rangle\\
&+\frac{2\rho_2\rho_3}{k\left(1+\xi^2\right)}\operatorname{Re}\left\langle\widehat{\theta},i\xi\,\widehat{y}\right\rangle.
\end{aligned}
$$

We have the following estimate, for any $\varepsilon_3>0$,

$$
\begin{aligned}
\frac{\xi^2}{\left(1+\xi^2\right)}a\left|\widehat{z}\right|^2+\frac{d}{dt}\mathcal{F}_3(\xi,t)\;\le\;&C\rho_2\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2+C\rho_3\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{\theta}\right|^2\\
&+Cm\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds+C(\varepsilon_3)\,\tau\left|\widehat{q}\right|^2\\
&+\varepsilon_3\frac{\xi^2}{\left(1+\xi^2\right)^2}\rho_1\left|\widehat{u}\right|^2
\end{aligned}
\tag{3.9}
$$

where C and $C(\varepsilon_3)$ are positive constants.

Proof. Multiplying $(3.1)_4$ by $i\xi\widehat{z}$ and using $(3.1)_3$, we get

$$
\begin{aligned}
\xi^2a\left|\widehat{z}\right|^2-\rho_2\frac{d}{dt}\left\langle\widehat{y},i\xi\ \widehat{z}\right\rangle\;=\;&\xi^2\rho_2\left|\widehat{y}\right|^2+m\xi^{\,2}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{z}\right\rangle\\
&+\delta\xi^{\,2}\left\langle\widehat{\theta},\widehat{z}\right\rangle-k\left\langle\widehat{v},i\xi\widehat{z}\right\rangle.
\end{aligned}
$$

Using $(3.1)_2$ and $(3.1)_3$, we have

$$-k\left\langle \widehat{v}, i\xi\widehat{z}\right\rangle = \rho_1\left\langle \widehat{u}_t, \widehat{z}\right\rangle = \rho_1\frac{d}{dt}\left\langle \widehat{u}, \widehat{z}\right\rangle + \rho_1\left\langle i\xi\widehat{u}, \widehat{y}\right\rangle, \tag{3.10}$$

then, (3.10) becomes

$$\begin{aligned} & \xi^2 a\left|\widehat{z}\right|^2 - \rho_2\frac{d}{dt}\left\langle \widehat{y}, i\xi\ \widehat{z}\right\rangle - \rho_1\frac{d}{dt}\left(\left\langle \widehat{u}, \widehat{z}\right\rangle\right) \\ = {} & \xi^2\rho_2\left|\widehat{y}\right|^2 + m\xi^{\,2}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds, \widehat{z}\right\rangle \\ & +\delta\xi^{\,2}\left\langle \widehat{\theta}, \widehat{z}\right\rangle + \rho_1\left\langle i\xi\widehat{u}, \widehat{y}\right\rangle. \end{aligned} \tag{3.11}$$

Multiplying $(3.1)_5$ by $\widehat{u}$ and using $(3.1)_2$, we get

$$\delta\left\langle \widehat{y}, i\xi\widehat{u}\right\rangle = \rho_3\frac{d}{dt}\left\langle \widehat{\theta}, \widehat{u}\right\rangle + \left\langle i\xi\widehat{q}, \widehat{u}\right\rangle + \frac{\rho_3}{\rho_1}\left\langle i\xi\widehat{\theta}, \widehat{v}\right\rangle, \tag{3.12}$$

by using $(3.1)_4$ in (3.12), we obtain

$$\begin{aligned} \delta\left\langle \widehat{y}, i\xi\widehat{u}\right\rangle = {} & \rho_3\frac{d}{dt}\left\langle \widehat{\theta}, \widehat{u}\right\rangle + \left\langle i\xi\widehat{q}, \widehat{u}\right\rangle + \frac{\rho_2\rho_3}{k\rho_1}\left\langle i\xi\widehat{\theta}, \widehat{y}_t\right\rangle - \frac{a\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, \ \widehat{z}\right\rangle \\ & -\frac{m\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right\rangle + \frac{\delta\rho_3}{k\rho_1}\xi^2\left|\widehat{\theta}\right|^2 \\ = {} & \rho_3\frac{d}{dt}\left\langle \widehat{\theta}, \widehat{u}\right\rangle + \left\langle i\xi\widehat{q}, \widehat{u}\right\rangle - \frac{a\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, \ \widehat{z}\right\rangle \\ & -\frac{m\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right\rangle + \frac{\delta\rho_3}{k\rho_1}\xi^2\left|\widehat{\theta}\right|^2 \\ & -\frac{\rho_2\rho_3}{k\rho_1}\frac{d}{dt}\left\langle \widehat{\theta}, i\xi\,\widehat{y}\right\rangle + \frac{\rho_2}{k\rho_1}\left\langle \rho_3\widehat{\theta}_t, i\xi\,\widehat{y}\right\rangle, \end{aligned}$$

using $(3.1)_5$, we deduce

$$\begin{aligned} \delta\left\langle \widehat{y}, i\xi\widehat{u}\right\rangle = {} & \rho_3\frac{d}{dt}\left\langle \widehat{\theta}, \widehat{u}\right\rangle + \left\langle i\xi\widehat{q}, \widehat{u}\right\rangle - \frac{a\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, \ \widehat{z}\right\rangle \\ & -\frac{m\rho_3}{k\rho_1}\xi^{\,2}\left\langle \widehat{\theta}, i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right\rangle + \frac{\delta\rho_3}{k\rho_1}\xi^2\left|\widehat{\theta}\right|^2 \\ & -\frac{\rho_2\rho_3}{k\rho_1}\frac{d}{dt}\left\langle \widehat{\theta}, i\xi\,\widehat{y}\right\rangle - \frac{\rho_2}{k\rho_1}\xi^2\left\langle \widehat{q}, \widehat{y}\right\rangle - \frac{\delta\rho_2}{k\rho_1}\xi^2\left|\widehat{y}\right|^2, \end{aligned} \tag{3.13}$$

by using (3.13) in (3.11), we obtain

$$
\begin{aligned}
&\xi^2 a\left|\widehat{z}\right|^2-\rho_2\frac{d}{dt}\left(\operatorname{Re}\left\langle \widehat{y},i\xi\ \widehat{z}\right\rangle\right)-\rho_1\frac{d}{dt}\left(\operatorname{Re}\left\langle \widehat{u},\widehat{z}\right\rangle\right)\\
&-\rho_1\rho_3\frac{d}{dt}\operatorname{Re}\left\langle \widehat{\theta},\widehat{u}\right\rangle+\frac{\rho_2\rho_3}{k}\frac{d}{dt}\left(\operatorname{Re}\left\langle \widehat{\theta},i\xi\ \widehat{y}\right\rangle\right)\\
=\ &\left(1-\frac{\delta}{k}\right)\rho_2\xi^2\left|\widehat{y}\right|^2+\frac{\delta\rho_3}{k}\xi^2\left|\widehat{\theta}\right|^2+m\xi^{\,2}\operatorname{Re}\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{z}\right\rangle\\
&+\left(\delta-\frac{a\rho_3}{k}\right)\xi^{\,2}\operatorname{Re}\left\langle \widehat{\theta},\widehat{z}\right\rangle-\rho_1\operatorname{Re}\left\langle \widehat{q},i\xi\widehat{u}\right\rangle\\
&-\frac{m\rho_3}{k}\xi^{\,2}\operatorname{Re}\left\langle \widehat{\theta},i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds\right\rangle-\frac{\rho_2}{k}\xi^2\operatorname{Re}\left\langle \widehat{q},\widehat{y}\right\rangle.
\end{aligned}\tag{3.14}
$$

Multiplying (3.14) by $\frac{1}{\left(1+\xi^2\right)}$, and applying Young's inequality, to the terms on the right-hand side of (3.14) and the fact that

$$
\frac{1}{\left(1+\xi^2\right)}\left|\left\langle \widehat{q},i\xi\widehat{u}\right\rangle\right|=\left|\left\langle \widehat{q},i\frac{\xi}{\left(1+\xi^2\right)}\widehat{u}\right\rangle\right|\preceq C\left(\varepsilon_3\right)\tau\left|\widehat{q}\right|^2+\varepsilon_3\frac{\xi^2}{\left(1+\xi^2\right)^2}\rho_1\left|\widehat{u}\right|^2;
$$

then (3.9) holds. This finishes the proof of Lemma 3.4. □

Lemma 3.5. *Let the functionals*

$$
\begin{aligned}
\mathcal{G}\left(\xi,t\right)\ &=-2\rho_2\operatorname{Re}\left\langle \widehat{y},\widehat{v}\right\rangle+2\tau\delta\operatorname{Re}\left\langle \widehat{q},\widehat{v}\right\rangle-\frac{2a\rho_1}{k}\operatorname{Re}\left\langle \widehat{u},\widehat{z}\right\rangle\\
&-\frac{2m\rho_1}{k}\operatorname{Re}\left\langle i\xi\textstyle\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle,\\
\mathcal{H}\left(\xi,t\right)&=2\rho_2\operatorname{Re}\left\langle \widehat{y},\widehat{v}\right\rangle.
\end{aligned}
$$

- *If $\chi=0$: Let the functional $\mathcal{F}_4\left(\xi,t\right)=\frac{\xi^2}{\left(1+\xi^2\right)}\mathcal{G}\left(\xi,t\right)$, then we have the following estimate*

$$
\begin{aligned}
\frac{\xi^2}{\left(1+\xi^2\right)}k\left|\widehat{v}\right|^2+\frac{d}{dt}\mathcal{F}_4\left(\xi,t\right)\ &\leq\ C\left(\varepsilon_4\right)m\xi^2\int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2ds\\
&+C\left(\varepsilon_4\right)\tau\left|\widehat{q}\right|^2+C\frac{\xi^2\rho_2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2+\varepsilon_4\frac{\xi^2\rho_1}{\left(1+\xi^2\right)}\left|\widehat{u}\right|^2.
\end{aligned}\tag{3.15}
$$

- *If $\chi\neq0$: Let the functional $\mathcal{F}_4\left(\xi,t\right)=\frac{\xi^2}{\left(1+\xi^2\right)^2}\mathcal{H}\left(\xi,t\right)$, then we have the following estimate*

$$
\begin{aligned}
&\frac{\xi^2}{\left(1+\xi^2\right)^2}k\left|\widehat{v}\right|^2+\frac{d}{dt}\mathcal{F}_4\left(\xi,t\right)\ \leq C\left(\varepsilon_4\right)m\xi^2\int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2ds+C\frac{\xi^2\rho_3}{\left(1+\xi^2\right)}\left|\widehat{\theta}\right|^2\\
&+C\left(\varepsilon_4\right)\frac{\xi^2\rho_2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2+C\frac{a\,\xi^2}{\left(1+\xi^2\right)}\left|\widehat{z}\right|^2+\varepsilon_4\frac{\xi^2\rho_1}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2,
\end{aligned}\tag{3.16}
$$

for any $\varepsilon_4>0$, C and $C\left(\varepsilon_4\right)$ are positive constants.

Proof. Using $(3.1)_6$ and$(3.1)_1$, we get

$$\begin{aligned}
\delta\left\langle i\xi\widehat{\theta},\widehat{v}\right\rangle &= -\tau\delta\left\langle \widehat{q}_t,\widehat{v}\right\rangle-\beta\delta\left\langle \widehat{q},\widehat{v}\right\rangle\\
&= -\tau\delta\frac{d}{dt}\left\langle \widehat{q},\widehat{v}\right\rangle+\tau\delta\left\langle \widehat{q},\widehat{v}_t\right\rangle-\beta\delta\left\langle \widehat{q},\widehat{v}\right\rangle\\
&= -\tau\delta\frac{d}{dt}\left\langle \widehat{q},\widehat{v}\right\rangle-\tau\delta\left\langle i\xi\widehat{q},\,\widehat{u}\right\rangle-\tau\delta\left\langle \widehat{q},\widehat{y}\right\rangle-\beta\delta\left\langle \widehat{q},\widehat{v}\right\rangle,
\end{aligned}$$

then, we have

$$\delta\left\langle i\xi\widehat{\theta},\widehat{v}\right\rangle=-\tau\delta\frac{d}{dt}\left\langle \widehat{q},\widehat{v}\right\rangle-\tau\delta\left\langle i\xi\widehat{q},\,\widehat{u}\right\rangle-\tau\delta\left\langle \widehat{q},\widehat{y}\right\rangle-\beta\delta\left\langle \widehat{q},\widehat{v}\right\rangle. \tag{3.17}$$

Multiplying $(3.1)_7$ by $i\xi\widehat{u}\,g\left(s\right)$, then we get

$$0=\left\langle g\left(s\right)\widehat{\eta}_s,i\xi\widehat{u}\right\rangle+\frac{d}{dt}\left\langle g\left(s\right)\widehat{\eta},i\xi\widehat{u}\right\rangle-\left\langle g\left(s\right)\widehat{\eta},i\xi\widehat{u}_t\right\rangle-\left\langle g\left(s\right)\widehat{y},i\xi\widehat{u}\right\rangle,$$

integrating with respect to s over $(0,\infty)$ and using integrating by parts, we obtain

$$\begin{aligned}
0 &= \left\langle i\xi\int_0^\infty g'\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle-\frac{d}{dt}\left(i\xi\left\langle\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle\right)\\
&\quad+\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}_t\right\rangle-\left(\int_0^\infty g\left(s\right)ds\right)\left\langle \widehat{y},i\xi\widehat{u}\right\rangle,
\end{aligned}$$

using $(3.1)_2$ and (H_3), we have

$$\begin{aligned}
0 &= \left\langle i\xi\int_0^\infty g'\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle-\frac{d}{dt}\left(\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle\right)\\
&\quad+\frac{k}{\rho_1}\xi^2\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\ \widehat{v}\right\rangle-b_0\left\langle \widehat{y},i\xi\widehat{u}\right\rangle,
\end{aligned}$$

then, we deduce

$$\begin{aligned}
\frac{k}{\rho_1}\xi^2\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\ \widehat{v}\right\rangle &= -\left\langle i\xi\int_0^\infty g'\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle\\
&\quad+\frac{d}{dt}\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle+b_0\left\langle \widehat{y},i\xi\widehat{u}\right\rangle.
\end{aligned} \tag{3.18}$$

Multiplying $(3.1)_4$ by $\widehat{v}$, and using $(3.1)_1$, then we obtain

$$\begin{aligned}
k\left|\widehat{v}\right|^2 &= \rho_2\left(\left\langle \widehat{y},\widehat{v}\right\rangle\right)+\rho_2\left\langle i\xi\widehat{y},\widehat{u}\right\rangle+\rho_2\left|\widehat{y}\right|^2-a\left\langle i\xi\ \widehat{z},\widehat{v}\right\rangle\\
&\quad+m\xi^2\left\langle\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{v}\right\rangle+\delta\left\langle i\xi\widehat{\theta},\widehat{v}\right\rangle,
\end{aligned} \tag{3.19}$$

using (3.10), (3.17) and (3.18), we get

$$\begin{aligned}
&k\left|\widehat{v}\right|^2-\rho_2\frac{d}{dt}\operatorname{Re}\left\langle \widehat{y},\widehat{v}\right\rangle+\tau\delta\frac{d}{dt}\left(\operatorname{Re}\left\langle \widehat{q},\widehat{v}\right\rangle\right)\\
&-\frac{a\rho_1}{k}\frac{d}{dt}\operatorname{Re}\left\langle \widehat{u},\widehat{z}\right\rangle-\frac{m\rho_1}{k}\frac{d}{dt}\operatorname{Re}\left\langle i\xi\int_0^\infty g\left(s\right)\widehat{\eta}\left(s\right)ds,\widehat{u}\right\rangle=
\end{aligned}$$

$$
\rho_2 \left|\widehat{y}\right|^2 - \left(\frac{b\rho_1}{k} - \rho_2\right) \operatorname{Re}\left\langle i\xi \widehat{y}, \widehat{u}\right\rangle - \frac{m\rho_1}{k} \operatorname{Re}\left\langle i\xi \int_0^\infty g'(s)\, \widehat{\eta}(s)\, ds, \widehat{u} \right\rangle
$$

$$
-\tau\delta \operatorname{Re}\left\langle i\xi \widehat{q},\, \widehat{u}\right\rangle - \tau\delta \operatorname{Re}\left\langle \widehat{q}, \widehat{y}\right\rangle - \beta\delta \operatorname{Re}\left\langle \widehat{q}, \widehat{v}\right\rangle . \tag{3.20}
$$

Multiplying (3.20) by $\dfrac{\xi^2}{\left(1+\xi^2\right)}$ $if\ \chi = 0$, and (3.19) by $\dfrac{\xi^2}{\left(1+\xi^2\right)^2}$ $if\ \chi \neq 0$ and applying Young's inequality, to the terms on the right-hand side of (3.20) and (3.19), then (3.15) and (3.16) holds. This finishes the proof of our Lemma 3.5. $\square$

Lemma 3.6. *Let the functional*

$$
\begin{cases}
\mathcal{G}(\xi, t) = -\rho_1 \operatorname{Re}\left\langle \widehat{v}\,, i\xi \widehat{u}\,\right\rangle - \dfrac{\rho_1\rho_3}{\delta} \operatorname{Re}\left\langle \widehat{\theta}, \widehat{u}\right\rangle + \dfrac{\rho_2\rho_3}{k\delta} \operatorname{Re}\left\langle \widehat{\theta}, i\xi\, \widehat{y}\right\rangle , \\
\mathcal{H}(\xi, t) = -2\rho_1 \operatorname{Re}\left\langle \widehat{v}\,, i\xi \widehat{u}\,\right\rangle .
\end{cases}
$$

- *If $\chi = 0$: Let the functional $\mathcal{F}_5(\xi, t) = \dfrac{1}{\left(1+\xi^2\right)} \mathcal{G}(\xi, t)$, then we have the following estimate*

$$
\begin{aligned}
\rho_1 \frac{\xi^2}{\left(1+\xi^2\right)} \left|\widehat{u}\right|^2 + \frac{d}{dt}\mathcal{F}_5(\xi, t) \ \leq\ & C\frac{\xi^2}{\left(1+\xi^2\right)} k\left|\widehat{v}\right|^2 + C\frac{\xi^2}{\left(1+\xi^2\right)} \rho_2 \left|\widehat{y}\right|^2 \\
& + C\frac{\xi^2}{\left(1+\xi^2\right)} \rho_3 \left|\widehat{\theta}\right|^2 + C\frac{\xi^2}{\left(1+\xi^2\right)} a\left|\widehat{z}\right|^2 \\
& + C\tau \left|\widehat{q}\right|^2 + Cm\xi^2 \int_0^\infty g(s) \left|\widehat{\eta}(s)\right|^2 ds.
\end{aligned} \tag{3.21}
$$

- *If $\chi \neq 0$: Let the functional $\mathcal{F}_5(\xi, t) = \dfrac{1}{\left(1+\xi^2\right)^2} \mathcal{H}(\xi, t)$, then we have the following estimate*

$$
\rho_1 \frac{\xi^2}{\left(1+\xi^2\right)^2} \left|\widehat{u}\right|^2 + \frac{d}{dt}\mathcal{F}_5(\xi, t) \leq C\frac{\xi^2}{\left(1+\xi^2\right)^2} k\left|\widehat{v}\right|^2 + C\frac{\xi^2}{\left(1+\xi^2\right)} \rho_2 \left|\widehat{y}\right|^2 . \tag{3.22}
$$

Here C is positive constant.

Proof. Multiplying $(3.1)_1$ by $i\xi\widehat{u}$ and using $(3.1)_2$, we get

$$
\rho_1 \xi^2 \left|\widehat{u}\right|^2 - \rho_1 \frac{d}{dt} \left\langle \widehat{v}\,, i\xi \widehat{u}\,\right\rangle = \rho_1 \xi^2 \left|\widehat{v}\right|^2 + \rho_1 \left\langle \widehat{y}, i\xi \widehat{u}\,\right\rangle \tag{3.23}
$$

using (3.13), we obtain

$$
\begin{aligned}
& \rho_1 \xi^2 \left|\widehat{u}\right|^2 - \rho_1 \frac{d}{dt} \left\langle \widehat{v}\,, i\xi \widehat{u}\,\right\rangle \\
& - \frac{\rho_1\rho_3}{\delta} \frac{d}{dt} \left(\left\langle \widehat{\theta}, \widehat{u}\right\rangle \right) + \frac{\rho_2\rho_3}{k\delta} \frac{d}{dt} \left(\left\langle \widehat{\theta}, i\xi\, \widehat{y}\right\rangle \right) \\
= \ & \rho_1 \xi^2 \left|\widehat{v}\right|^2 - \frac{\rho_2}{k} \xi^2 \left|\widehat{y}\right|^2 - \delta\rho_1 \left\langle \widehat{q}, i\xi \widehat{\theta}\right\rangle - \frac{a\rho_3}{k\delta} \xi^2 \left\langle \widehat{\theta},\ \widehat{z}\right\rangle \\
& - \frac{m\rho_3}{k\delta} \xi^2 \left\langle \widehat{\theta}, i\xi \int_0^\infty g(s)\, \widehat{\eta}(s)\, ds \right\rangle + \frac{\rho_3}{k} \xi^2 \left|\widehat{\theta}\right|^2 - \frac{\rho_2}{k\delta} \xi^2 \left\langle \widehat{q}, \widehat{y}\right\rangle .
\end{aligned} \tag{3.24}
$$

Multiplying (3.24) by $\dfrac{1}{(1+\xi^2)}$ $if\ \chi=0$ and (3.23) by $\dfrac{1}{(1+\xi^2)^2}$ $if\ \chi\neq 0$ and applying Young's inequality, to the terms on the right-hand side of (3.24) and (3.23), then (3.21) and (3.22) holds. This finishes the proof of Lemma 3.6. □

Proof of Proposition 3.1. In this step, we build the Lyapunov functional $\mathcal{L}(\xi,t)$. In order to construct this functional, we need to take into account two main things. First, this functional should satisfy the estimate (3.27) and second, it should verify another estimate of the form

$$c_1\,\widehat{E}(\xi,t)\le \mathcal{L}(\xi,t)\le c_2\,\widehat{E}(\xi,t),$$

where c_1 and c_2 are two positive constants.

Now, we introduce the Lyapunov functional $\mathcal{L}(\xi,t)$

$$\mathcal{L}(\xi,t)=N\,\hat{E}(\xi,t)+N_1\mathcal{F}_1(\xi,t)+N_2\,\mathcal{F}_2(\xi,t)+N_3\,\mathcal{F}_3(\xi,t)+N_4\,\mathcal{F}_4(\xi,t)+\mathcal{F}_5(\xi,t)\,, \tag{3.25}$$

where N, N_i for $i=1..4$, are positive constants that will be fixed later.

- $If\ \ (\chi=0):$

  Taking the derivative of $\mathcal{L}(\xi,t)$ with respect to t and making use of (3.5), (3.7), (3.9), (3.15) and (3.21), we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{L}(\xi,t)+(N_1-N_2C-N_3C-C)\frac{\xi^2}{(1+\xi^2)}\rho_3\left|\widehat{\theta}\right|^2\\
&+(N_2-N_1\varepsilon_1-N_3C-N_4C-C)\frac{\xi^2}{(1+\xi^2)}\rho_2\left|\widehat{y}\right|^2\\
&+(N_3-N_2\varepsilon_2-C)\frac{\xi^2}{(1+\xi^2)}a\left|\widehat{z}\right|^2\\
&+(N_4-N_2\varepsilon_2-C)\frac{\xi^2}{(1+\xi^2)}k\left|\widehat{v}\right|^2+(1-N_4\varepsilon_4-N_3\varepsilon_3)\frac{\xi^2}{(1+\xi^2)}\rho_1\left|\widehat{u}\right|^2\\
\le\ &-\left(\frac{2\beta}{\tau}N-(N_1C(\varepsilon_1)+N_3C(\varepsilon_3)+N_4C(\varepsilon_4)+C)\right)\tau\left|\widehat{q}\right|^2\\
&-(Nk_1-(N_2C(\varepsilon_2)+N_3C+N_4C(\varepsilon_4)+C))\,m\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds.
\end{aligned}\tag{3.26}$$

  Here we have used

$$\varepsilon_3\frac{\xi^2}{(1+\xi^2)^{\mathbf{2}}}\rho_1\left|\widehat{u}\right|^2\le\varepsilon_3\frac{\xi^2}{(1+\xi^2)}\rho_1\left|\widehat{u}\right|^2,$$

  and

$$\varepsilon_2\frac{\xi^2}{(1+\xi^2)^{\mathbf{2}}}k\left|\widehat{v}\right|^2\le\varepsilon_2\frac{\xi^2}{(1+\xi^2)}k\left|\widehat{v}\right|^2.$$

  Now, we fix the constants in (3.26) as follows

$$\begin{aligned}
&\varepsilon_4=\frac{1}{4N_4},\ \varepsilon_3=\frac{1}{4N_3},\ \varepsilon_2=\frac{1}{2N_2},\ \ N_4=1+C,\ N_3=1+C\ ,\\
&\varepsilon_1=\frac{1}{2N_1},\ \ N_2=1+N_3C+N_4C+C\ \text{and}\ N_1=\frac{1}{2}\ +N_2C+N_3C+C\ \ .
\end{aligned}$$

Finally, we choose N large enough such that

$$N > \max\left(\frac{\tau}{2\beta}\left(N_1C\left(\varepsilon_1\right) + N_3C\left(\varepsilon_3\right) + N_4C\left(\varepsilon_4\right) + C\right), \frac{1}{k_1}\left(N_2C\left(\varepsilon_2\right) + N_3C + N_4C\left(\varepsilon_4\right) + C\right)\right).$$

With these choices, (3.26) takes the following form

$$\frac{\partial}{\partial t}\mathcal{L}\left(\xi, t\right) + c_0\mathcal{F}\left(\xi, t\right) \leq 0, \tag{3.27}$$

where c_0 is a positive constant, and

$$\begin{aligned} \mathcal{F}\left(\xi, t\right) &= \frac{\xi^2}{\left(1+\xi^2\right)}k\left|\widehat{v}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\rho_1\left|\widehat{u}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\rho_2\left|\widehat{y}\right|^2 \\ &\quad + \frac{\xi^2}{\left(1+\xi^2\right)}a\left|\widehat{z}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\rho_3\left|\widehat{\theta}\right|^2 + m\xi^2\int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2 ds + \tau\left|\widehat{q}\right|^2. \end{aligned} \tag{3.28}$$

Since N is large enough and by using (3.25) then there exist two positive constants c_1 and c_2

$$c_1\hat{E}\left(\xi, t\right) \leq \mathcal{L}\left(\xi, t\right) \leq c_2\hat{E}\left(\xi, t\right). \tag{3.29}$$

From (3.28), we deduce that

$$\mathcal{F}\left(\xi, t\right) \geq \lambda\left(\xi\right)\ \hat{E}\left(\xi, t\right), \tag{3.30}$$

where $\lambda\left(\xi\right) = \dfrac{\xi^2}{\left(1+\xi^2\right)}$. Consequently, from(3.27), (3.29) and (3.30), we can find C and c such that

$$\widehat{E}\left(\xi, t\right) \leq C\,\hat{E}\left(\xi, 0\right)e^{-c\lambda(\xi)\,t}.$$

- $If\ \ (\chi \neq 0):$ Taking the derivative of $\mathcal{L}\left(\xi, t\right)$ with respect to t and making use of (3.5), (3.7), (3.9), (3.16) and (3.22), we find

$$\begin{aligned} &\frac{\partial}{\partial t}\mathcal{L}\left(\xi, t\right) + \left(N_1 - N_2C - N_3C - N_4C\right)\frac{\xi^2}{\left(1+\xi^2\right)}\rho_3\left|\widehat{\theta}\right|^2 \\ &+ \left(N_2 - N_1\varepsilon_1 - N_3C - N_4C\left(\varepsilon_4\right) - C\right)\frac{\xi^2}{\left(1+\xi^2\right)}\rho_2\left|\widehat{y}\right|^2 \\ &+ \left(N_3 - N_2\varepsilon_2 - N_4C\right)\frac{\xi^2}{\left(1+\xi^2\right)}a\left|\widehat{z}\right|^2 \\ &+ \left(N_4 - N_2\varepsilon_2 - C\right)\frac{\xi^2}{\left(1+\xi^2\right)^2}k\left|\widehat{v}\right|^2 + \left(1 - N_4\varepsilon_4 - N_3\varepsilon_3\right)\frac{\xi^2}{\left(1+\xi^2\right)^2}\rho_1\left|\widehat{u}\right|^2 \\ \leq\ & -\left(\frac{2\beta}{\tau}N - \left(N_1C\left(\varepsilon_1\right) + N_3C\left(\varepsilon_3\right)\right)\right)\tau\left|\widehat{q}\right|^2 \\ &- \left(Nk_1 - \left(N_3C + N_2C\left(\varepsilon_2\right) + N_4C\left(\varepsilon_4\right)\right)\right)m\xi^2\int_0^\infty g\left(s\right)\left|\widehat{\eta}\left(s\right)\right|^2 ds. \end{aligned} \tag{3.31}$$

Now, we fix the constants in (3.31) as follows

$$\varepsilon_3 = \frac{1}{4N_3},\ \varepsilon_4 = \frac{1}{4N_4}\ ,\ \varepsilon_2 = \frac{1}{2N_2},\ N_4 = 1 + C,$$
$$N_3 = 1 + N_4 C\ ,\ \varepsilon_1 = \frac{1}{2N_1},\ N_2 = 1 + N_3 C + N_4 C(\varepsilon_4) + C \text{ and } N_1 = \frac{1}{2} + N_2 C + N_3 C + N_4 C.$$

Finally, we choose N large enough such that

$$N > \max\left(\frac{\tau}{2\beta}\left(N_1 C(\varepsilon_1) + N_3 C(\varepsilon_3)\right), \frac{1}{k_1}\left(N_3 C + N_2 C(\varepsilon_2) + N_4 C(\varepsilon_4)\right)\right).$$

With these choices, (3.31) takes the form

$$\frac{\partial}{\partial t}\mathcal{L}(\xi,t) + c_0 \mathcal{F}(\xi,t) \le 0, \tag{3.32}$$

where c_0 is a positive constant, and

$$\begin{aligned} \mathcal{F}(\xi,t) \;=\; & \frac{\xi^2}{(1+\xi^2)^2} k\,|\widehat{v}|^2 + \frac{\xi^2}{(1+\xi^2)^2}\rho_1\,|\widehat{u}|^2 + \frac{\xi^2}{(1+\xi^2)}\rho_2\,|\widehat{y}|^2 \\ & + \frac{\xi^2}{(1+\xi^2)} a\,|\widehat{z}|^2 + \frac{\xi^2}{(1+\xi^2)}\rho_3\left|\widehat{\theta}\right|^2 + m\xi^2\int_0^\infty g(s)\,|\widehat{\eta}(s)|^2\,ds + \tau\,|\widehat{q}|^2. \end{aligned} \tag{3.33}$$

Since N is large enough and using (3.25) then there exist two positive constants c_1 and c_2

$$c_1 \hat{E}(\xi,t) \le \mathcal{L}(\xi,t) \le c_2 \hat{E}(\xi,t). \tag{3.34}$$

From (3.33), we deduce that

$$\mathcal{F}(\xi,t) \ge \lambda(\xi)\ \hat{E}(\xi,t), \tag{3.35}$$

where $\lambda(\xi) = \dfrac{\xi^2}{(1+\xi^2)^2}$. Consequently, from(3.32), (3.34) and (3.35), we can find C and c such that

$$\widehat{E}(\xi,t) \le C\,\hat{E}(\xi,0)\,e^{-c\,\lambda(\xi)\,t}$$

This finishes the proof of the Proposition.

□

3.2. **Fourier Model.** Taking Fourier transform in (2.6), we obtain the following ODE system:

$$\begin{cases} \widehat{v}_t - i\xi\,\widehat{u} + \widehat{y} = 0, \\ \rho_1 \widehat{u}_t - ik\xi\ \widehat{v} = 0, \\ \widehat{z}_t - i\xi\,\widehat{y} = 0, \\ \rho_2\,\widehat{y}_t - ia\xi\ \widehat{z} + m\xi^{\,2}\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds - k\,\widehat{v} + i\delta\xi\,\widehat{\theta} = 0, \\ \rho_3 \widehat{\theta}_t + \dfrac{1}{\beta}\xi^{\,2}\,\widehat{\theta} + i\delta\xi\ \widehat{y} = 0, \\ \widehat{\eta}_t + \widehat{\eta}_s - \widehat{y} = 0. \end{cases} \tag{3.36}$$

The solution vector and initial data are given by $\widehat{V}(\xi,t) = \left(\widehat{v},\widehat{u},\widehat{z},\widehat{y},\widehat{\theta},\widehat{\eta}\right)^T$ and $\widehat{V}(\xi,0) = \widehat{V}_0(\xi)$.

The energy functional corresponding to the above system is defined as:

$$\widehat{\mathcal{E}}(\xi,t)=\rho_1|\widehat{u}|^2+\rho_2|\widehat{y}|^2+\rho_3\left|\widehat{\theta}\right|^2+k|\widehat{v}|^2+a|\widehat{z}|^2+m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2\,ds. \tag{3.37}$$

The energy (3.37) satisfies the following estimate (see Lemma 3.7, [1]):

$$\frac{d}{dt}\widehat{\mathcal{E}}(\xi,t)=-\frac{2}{\beta}\xi^2\left|\widehat{\theta}\right|^2+m\xi^2\int_0^\infty g'(s)|\widehat{\eta}(s)|^2\,ds,$$

using (H_2), we have

$$\frac{d}{dt}\widehat{\mathcal{E}}(\xi,t)\le-\frac{2}{\beta}\xi^2\left|\widehat{\theta}\right|^2-k_1m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2\,ds.$$

The pointwise estimates of the functional $\widehat{\mathcal{E}}(\xi,t)$ are given in the proposition below. This estimate will play an important role in proving our main result.

Proposition 3.7. *For any $t\ge 0$ and $\xi\in\mathbb{R}$, we have the following estimates*

$$\widehat{\mathcal{E}}(\xi,t)\le\begin{cases}C\,e^{-c\lambda_1(\xi)\,t}\,\widehat{\mathcal{E}}(\xi,0); & if\ \chi=0\\ C\,e^{-c\lambda_2(\xi)\,t}\,\widehat{\mathcal{E}}(\xi,0); & if\ \chi\neq 0\end{cases} \tag{3.38}$$

where

$$\lambda_1(\xi)=\frac{\xi^2}{1+\xi^2};\qquad \lambda_2(\xi)=\frac{\xi^2}{\left(1+\xi^2\right)^2}, \tag{3.39}$$

and C and c are two positive constants.

As in the previous subsection, we show that the decay rate of the solution will depend on the value of χ. The proof will be given through several lemmas.

Lemma 3.8. *Let the functional*

$$\mathcal{K}_0(\xi,t)=-\frac{2\rho_2\xi^2}{b_0\left(1+\xi^2\right)}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{y}\right\rangle.$$

we have the following estimate: for any $\varepsilon_0>0$,

$$\begin{aligned}\frac{\xi^2}{\left(1+\xi^2\right)}\rho_2|\widehat{y}|^2+\frac{d}{dt}\mathcal{K}_0(\xi,t)\ &\le\ C\xi^2\rho_3\left|\widehat{\theta}\right|^2+\varepsilon_0\frac{\xi^2}{\left(1+\xi^2\right)}a|\widehat{z}|^2\\ &\quad+C(\varepsilon_0)\,m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2\,ds+\varepsilon_0\frac{\xi^2}{\left(1+\xi^2\right)^2}k|\widehat{v}|^2,\end{aligned} \tag{3.40}$$

where C and $C(\varepsilon_0)$ are positive constants.

Proof. Multiplying $(3.36)_6$ by $g(s)\,\widehat{y}$ and using $(3.1)_4$, we get

$$\begin{aligned}0&=\frac{d}{dt}\langle\widehat{\eta},g(s)\,\widehat{y}\rangle-\langle g(s)\,\widehat{\eta},\widehat{y}_t\rangle+\langle g(s)\,\widehat{\eta}_s,\widehat{y}\rangle-g(s)|\widehat{y}|^2\\ &=\langle g(s)\,\widehat{\eta}_s,\widehat{y}\rangle+\frac{d}{dt}\left(\langle\widehat{\eta},g(s)\,\widehat{y}\rangle\right)-|\widehat{y}|^2+\frac{a}{\rho_2}\langle i\xi g(s)\,\widehat{\eta},\ \widehat{z}\rangle\\ &\quad+\frac{m}{\rho_2}\xi^2\left\langle g(s)\,\widehat{\eta},\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right\rangle-\frac{k}{\rho_2}\langle g(s)\,\widehat{\eta},\ \widehat{v}\rangle+\frac{\delta}{\rho_2}\left\langle g(s)\,\widehat{\eta},i\xi\,\widehat{\theta}\right\rangle,\end{aligned}$$

integrating with respect to s over $(0,\infty)$ and integrating by parts the first term on the right-hand side of the above equality, we get

$$\begin{aligned}
&\rho_2\left|\widehat{y}\right|^2-\frac{\rho_2}{b_0}\frac{d}{dt}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{y}\right\rangle\\
=\;&-\frac{\rho_2}{b_0}\operatorname{Re}\left\langle\int_0^\infty g'(s)\,\widehat{\eta}(s)\,ds,\widehat{y}\right\rangle+\frac{a}{b_0}\operatorname{Re}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \widehat{z}\right\rangle\\
&+\frac{m}{b_0}\xi^{\,2}\left|\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right|^2-\frac{k}{b_0}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\ \widehat{v}\right\rangle\\
&+\frac{\delta}{b_0}\operatorname{Re}\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,i\xi\,\widehat{\theta}\right\rangle,
\end{aligned}\tag{3.41}$$

where the following inequalities have been used:

$$\begin{aligned}
&\left|\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds\right|^2\le b_0\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds,\\
&\left|\int_0^\infty g'(s)\,\widehat{\eta}(s)\,ds\right|^2\le b_0\max\{k_1,k_2\}\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds.
\end{aligned}$$

Now, multiplying (3.41) by $\dfrac{\xi^2}{(1+\xi^2)}$, using the fact that

$$\frac{k}{b_0}\xi^2\left|\left\langle\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\frac{1}{(1+\xi^2)}\,\widehat{v}\right\rangle\right|\preceq C(\varepsilon_0)\,m\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds+\varepsilon_0\frac{\xi^2}{(1+\xi^2)^2}k\left|\widehat{v}\right|^2$$

and applying Young's inequality, to the terms on the right-hand side of (3.41), then (3.40) holds. This finishes the proof of Lemma 3.8. $\square$

Lemma 3.9. *Let the functional*

$$\mathcal{G}(\xi,t)=-2\rho_2\operatorname{Re}\langle\widehat{y},i\xi\ \widehat{z}\rangle-2\rho_1\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle-\frac{2\rho_1\rho_3}{\delta}\operatorname{Re}\left\langle\widehat{u},\widehat{\theta}\right\rangle-\frac{2\rho_2\rho_3}{\delta}\operatorname{Re}\left\langle\ \widehat{y},i\xi\widehat{\theta}\right\rangle.$$

Let the functional $\mathcal{K}_1(\xi,t)=\dfrac{1}{(1+\xi^2)}\mathcal{G}(\xi,t)$, *then we have the following estimate*

$$\begin{aligned}
\frac{\xi^2}{(1+\xi^2)}a\left|\widehat{z}\right|^2+\frac{d}{dt}\mathcal{K}_1(\xi,t)\;\le\;&C(\varepsilon_5)\,\rho_3\xi^{\,2}\left|\widehat{\theta}\right|^2+C\frac{\rho_2\xi^2}{(1+\xi^2)}\left|\widehat{y}\right|^2\\
&+Cm\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2ds+\varepsilon_5\frac{\xi^2\rho_1}{(1+\xi^2)^2}\left|\widehat{u}\right|^2.
\end{aligned}\tag{3.42}$$

for any $\varepsilon_5>0$, C *and* $C(\varepsilon_5)$ *are positive constants.*

Proof. Multiplying $(3.36)_2$ by $\widehat{\theta}$ and Using $(3.36)_5$, we get

$$\begin{aligned}
\langle i\xi\widehat{u},\widehat{y}\rangle&=\frac{\rho_3}{\delta}\frac{d}{dt}\left\langle\widehat{u},\widehat{\theta}\right\rangle-\frac{\rho_3}{\delta}\left\langle\widehat{u}_t,\widehat{\theta}\right\rangle+\frac{1}{\beta\delta}\xi^{\,2}\left\langle\widehat{u},\ \widehat{\theta}\right\rangle\\
&=\frac{\rho_3}{\delta}\frac{d}{dt}\left\langle\widehat{u},\widehat{\theta}\right\rangle+\frac{1}{\beta\delta}\xi^{\,2}\left\langle\widehat{u},\ \widehat{\theta}\right\rangle+\frac{\rho_3}{\rho_1\delta}\left\langle k\ \widehat{v},i\xi\widehat{\theta}\right\rangle,
\end{aligned}$$

using $(3.36)_4$, we obtain

$$\begin{aligned}\langle i\xi\widehat{u},\widehat{y}\rangle &= \frac{\rho_3}{\delta}\frac{d}{dt}\left\langle \widehat{u},\widehat{\theta}\right\rangle + \frac{1}{\beta\delta}\xi^{\,2}\left\langle \widehat{u},\,\widehat{\theta}\right\rangle + \frac{\rho_3}{\rho_1}\xi^{\,2}\left|\widehat{\theta}\right|^2 - \frac{a\rho_3}{\rho_1\delta}\xi^{\,2}\left\langle \widehat{z},\widehat{\theta}\right\rangle \\ &\quad + \frac{m\rho_3}{\rho_1\delta}\xi^{\,2}\left\langle \int_0^\infty g(s)\,\widehat{\eta}(s)\,ds, i\xi\widehat{\theta}\right\rangle + \frac{\rho_2\rho_3}{\rho_1\delta}\frac{d}{dt}\left\langle\ \widehat{y}, i\xi\widehat{\theta}\right\rangle + \frac{\rho_2}{\rho_1\delta}\left\langle\ i\xi\widehat{y},\rho_3\widehat{\theta}_t\right\rangle,\end{aligned}$$

and finally by using $(3.36)_5$, we deduce

$$\begin{aligned}\langle i\xi\widehat{u},\widehat{y}\rangle &= \frac{\rho_3}{\delta}\frac{d}{dt}\left\langle \widehat{u},\widehat{\theta}\right\rangle + \frac{\rho_2\rho_3}{\rho_1\delta}\frac{d}{dt}\left\langle\ \widehat{y}, i\xi\widehat{\theta}\right\rangle + \frac{1}{\beta\delta}\xi^{\,2}\left\langle \widehat{u},\,\widehat{\theta}\right\rangle \\ &\quad + \frac{\rho_3}{\rho_1}\xi^{\,2}\left|\widehat{\theta}\right|^2 - \frac{a\rho_3}{\rho_1\delta}\xi^{\,2}\left\langle \widehat{z},\widehat{\theta}\right\rangle + \frac{m\rho_3}{\rho_1\delta}\xi^{\,2}\left\langle \int_0^\infty g(s)\,\widehat{\eta}(s)\,ds, i\xi\widehat{\theta}\right\rangle \\ &\quad + \frac{\rho_2}{\beta\rho_1\delta}\xi^{\,2}\left\langle \widehat{y}, i\xi\,\widehat{\theta}\right\rangle - \frac{\rho_2}{\rho_1}\xi^{\,2}\left|\,\widehat{y}\right|^2. \qquad (3.43)\end{aligned}$$

Using (3.11) and (3.43), we obtain

$$\begin{aligned}& a\xi^2\left|\widehat{z}\right|^2 - \rho_2\frac{d}{dt}\operatorname{Re}\langle\widehat{y}, i\xi\ \widehat{z}\rangle - \rho_1\frac{d}{dt}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle \\ &-\frac{\rho_1\rho_3}{\delta}\frac{d}{dt}\operatorname{Re}\left\langle \widehat{u},\widehat{\theta}\right\rangle - \frac{\rho_2\rho_3}{\delta}\frac{d}{dt}\operatorname{Re}\left\langle\ \widehat{y}, i\xi\widehat{\theta}\right\rangle \\ =\ & \rho_3\xi^{\,2}\left|\widehat{\theta}\right|^2 + \left(\delta - \frac{a\rho_3}{\delta}\right)\xi^{\,2}\operatorname{Re}\left\langle \widehat{z},\widehat{\theta}\right\rangle \\ &+\frac{\rho_1}{\beta\delta}\xi^{\,2}\operatorname{Re}\left\langle \widehat{u},\,\widehat{\theta}\right\rangle + \frac{\rho_2}{\beta\delta}\xi^{\,2}\operatorname{Re}\left\langle \widehat{y}, i\xi\,\widehat{\theta}\right\rangle + m\xi^{\,2}\operatorname{Re}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{z}\right\rangle \\ &+\frac{m\rho_3}{\delta}\xi^{\,2}\operatorname{Re}\left\langle \int_0^\infty g(s)\,\widehat{\eta}(s)\,ds, i\xi\widehat{\theta}\right\rangle. \qquad (3.44)\end{aligned}$$

Multiplying (3.44) by $\dfrac{1}{(1+\xi^2)}$, using the fact that

$$\frac{\rho_1}{\beta\delta}\frac{\xi^{\,2}}{(1+\xi^2)}\left|\left\langle \widehat{u},\,\widehat{\theta}\right\rangle\right| = \xi^{\,2}\left|\left\langle \frac{\sqrt{\rho_1}}{(1+\xi^2)}\widehat{u},\,\frac{\sqrt{\rho_1}}{\beta\delta}\,\widehat{\theta}\right\rangle\right| \preceq C(\varepsilon_5)\,\rho_3\xi^{\,2}\left|\widehat{\theta}\right|^2 + \varepsilon_5\frac{\xi^2}{(1+\xi^2)^2}\rho_1\left|\widehat{u}\right|^2$$

and applying Young's inequality, to the terms on the right-hand side of (3.44), then (3.42) holds. This finishes the proof of Lemma 3.9. □

Lemma 3.10. *Let the functionals*

$$\begin{cases}\mathcal{G}(\xi,t) = -2\rho_2\operatorname{Re}\langle\widehat{y},\widehat{v}\rangle - \dfrac{2a\rho_1}{k}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle - \dfrac{2m\rho_1}{k}\operatorname{Re}\left\langle i\xi\int_0^\infty g(s)\,\widehat{\eta}(s)\,ds,\widehat{u}\,\right\rangle, \\ \mathcal{H}(\xi,t) = 2\rho_2\operatorname{Re}\langle\widehat{y},\widehat{v}\rangle.\end{cases}$$

- *If* $\chi = 0$: *Let the functional* $\mathcal{K}_2(\xi,t) = \dfrac{\xi^2}{(1+\xi^2)}\mathcal{G}(\xi,t)$, *then we have the following estimate*

$$\begin{aligned}\frac{\xi^2}{(1+\xi^2)}k\left|\widehat{v}\right|^2 + \frac{d}{dt}\mathcal{K}_2(\xi,t) &\le C(\varepsilon_6)\,m\xi^2\int_0^\infty g(s)\left|\widehat{\eta}(s)\right|^2 ds \\ &\quad + C\rho_3\xi^{\,2}\left|\widehat{\theta}\right|^2 + C\frac{\rho_2\xi^2}{(1+\xi^2)}\left|\widehat{y}\right|^2 + \varepsilon_6\frac{\xi^2\rho_1}{(1+\xi^2)}\left|\widehat{u}\right|^2. \qquad (3.45)\end{aligned}$$

- *If* $\chi \neq 0$: *Let the functional* $\mathcal{K}_2(\xi,t) = \dfrac{\xi^2}{(1+\xi^2)^2}\mathcal{H}(\xi,t)$, *then we have the following estimate*

$$\frac{\xi^2}{(1+\xi^2)^2}k|\widehat{v}|^2 + \frac{d}{dt}\mathcal{K}_2(\xi,t) \leq C(\varepsilon_6)m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2 ds + C(\varepsilon_6)\frac{\xi^2\rho_2}{(1+\xi^2)}|\widehat{y}|^2 + C\xi^2\rho_3\left|\widehat{\theta}\right|^2 + \varepsilon_6\frac{\xi^2\rho_1}{(1+\xi^2)^2}|\widehat{u}|^2 + C\frac{a\,\xi^2}{(1+\xi^2)}|\widehat{z}|^2, \tag{3.46}$$

for any $\varepsilon_6 > 0$, C *and* $C(\varepsilon_6)$ *are positive constants.*

Proof. Multiplying $(3.36)_4$ by $\widehat{v}$, and using $(3.36)_1$, then we get

$$k|\widehat{v}|^2 = \rho_2\frac{d}{dt}\langle\widehat{y},\widehat{v}\rangle + \rho_2\langle i\xi\widehat{y},\widehat{u}\rangle + \rho_2|\widehat{y}|^2 - a\langle i\xi\,\widehat{z},\widehat{v}\rangle + m\xi^2\left\langle\int_0^\infty g(s)\widehat{\eta}(s)\,ds,\widehat{v}\right\rangle + \delta\left\langle i\xi\widehat{\theta},\widehat{v}\right\rangle. \tag{3.47}$$

Using the same steps as in (3.18), we have

$$\frac{k}{\rho_1}\xi^2\left\langle i\xi\int_0^\infty g(s)\widehat{\eta}(s)\,ds,\ \widehat{v}\right\rangle = -\left\langle i\xi\int_0^\infty g'(s)\widehat{\eta}(s)\,ds,\widehat{u}\right\rangle + b_0\langle\widehat{y},i\xi\widehat{u}\rangle + \frac{d}{dt}\left\langle i\xi\int_0^\infty g(s)\widehat{\eta}(s)\,ds,\widehat{u}\right\rangle. \tag{3.48}$$

Using the same steps as in (3.10), we have

$$-k\langle\widehat{v},i\xi\widehat{z}\rangle = \rho_1\frac{d}{dt}\langle\widehat{u},\widehat{z}\rangle + \rho_1\langle i\xi\widehat{u},\widehat{y}\rangle. \tag{3.49}$$

Using (3.47), (3.48) and (3.49), we get

$$\begin{aligned} &k|\widehat{v}|^2 - \rho_2\frac{d}{dt}(\mathrm{Re}\langle\widehat{y},\widehat{v}\rangle) - \frac{a\rho_1}{k}\frac{d}{dt}\mathrm{Re}\langle\widehat{u},\widehat{z}\rangle - \frac{m\rho_1}{k}\frac{d}{dt}\mathrm{Re}\left\langle i\xi\int_0^\infty g(s)\widehat{\eta}(s)\,ds,\widehat{u}\right\rangle \\ &= \rho_2|\widehat{y}|^2 - \left(\frac{b\rho_1}{k}-\rho_2\right)\mathrm{Re}\langle i\xi\widehat{y},\widehat{u}\rangle + \delta\,\mathrm{Re}\left\langle i\xi\widehat{\theta},\widehat{v}\right\rangle - \frac{m\rho_1}{k}\mathrm{Re}\left\langle i\xi\int_0^\infty g'(s)\widehat{\eta}(s)\,ds,\widehat{u}\right\rangle. \end{aligned} \tag{3.50}$$

Multiplying (3.50) by $\dfrac{\xi^2}{(1+\xi^2)}$ *if* $\chi = 0$ and (3.47) by $\dfrac{\xi^2}{(1+\xi^2)^2}$ *if* $\chi \neq 0$ and applying Young's inequality, to the terms on the right-hand side of (3.50) and (3.47), then (3.45) and (3.46) holds. This finishes the proof of Lemma 3.10. □

Lemma 3.11. *Let the functional*

$$\begin{cases} \mathcal{G}(\xi,t) = -2\rho_1\langle\widehat{v},i\xi\widehat{u}\rangle - \dfrac{2\rho_1\rho_3}{\delta}\left\langle\widehat{u},\widehat{\theta}\right\rangle - \dfrac{2\rho_2\rho_3}{\delta}\left\langle\ \widehat{y},i\xi\widehat{\theta}\right\rangle, \\ \mathcal{H}(\xi,t) = -2\rho_1\langle\widehat{v},i\xi\widehat{u}\rangle. \end{cases}$$

- *If* $\chi = 0$: *Let the functional* $\mathcal{K}_3(\xi,t) = \dfrac{1}{(1+\xi^2)}\mathcal{G}(\xi,t)$, *then we have the following estimate*

$$\rho_1 \frac{\xi^2}{(1+\xi^2)}|\widehat{u}|^2 + \frac{d}{dt}\mathcal{K}_3(\xi,t) \le C\frac{\xi^2}{(1+\xi^2)}k|\widehat{v}|^2 + C\frac{\xi^2}{(1+\xi^2)}\rho_2|\widehat{y}|^2 + C\xi^2\rho_3\left|\widehat{\theta}\right|^2 + C\frac{\xi^2}{(1+\xi^2)}a|\widehat{z}|^2 + Cm\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2 ds. \tag{3.51}$$

- *If* $\chi \neq 0$: *Let the functional* $\mathcal{K}_3(\xi,t) = \dfrac{1}{(1+\xi^2)^2}\mathcal{H}(\xi,t)$, *then we have the following estimate*

$$\rho_1\frac{\xi^2}{(1+\xi^2)^2}|\widehat{u}|^2 + \frac{d}{dt}\mathcal{K}_3(\xi,t) \le C\frac{\xi^2}{(1+\xi^2)^2}k|\widehat{v}|^2 + C\frac{\xi^2}{(1+\xi^2)}\rho_2|\widehat{y}|^2. \tag{3.52}$$

Here C *is positive constant.*

Proof. Multiplying $(3.36)_1$ by $i\xi\widehat{u}$ and using $(3.36)_2$, we get

$$\rho_1\xi^2|\widehat{u}|^2 - \rho_1\frac{d}{dt}\langle \widehat{v}, i\xi\widehat{u}\rangle = \rho_1\xi^2|\widehat{v}|^2 + \rho_1\langle \widehat{y}, i\xi\widehat{u}\rangle, \tag{3.53}$$

using (3.43), we obtain

$$\begin{aligned} &\rho_1\xi^2|\widehat{u}|^2 - \rho_1\frac{d}{dt}\langle \widehat{v}, i\xi\widehat{u}\rangle - \frac{\rho_1\rho_3}{\delta}\frac{d}{dt}\left(\left\langle \widehat{u}, \widehat{\theta}\right\rangle\right) - \frac{\rho_2\rho_3}{\delta}\frac{d}{dt}\left(\left\langle \widehat{y}, i\xi\widehat{\theta}\right\rangle\right) \\ = \;& \rho_1\xi^2|\widehat{v}|^2 - \rho_2\xi^2|\widehat{y}|^2 + \rho_3\xi^2\left|\widehat{\theta}\right|^2 + \frac{\rho_1}{\beta\delta}\xi^2\left\langle \widehat{u}, \widehat{\theta}\right\rangle \\ &-\frac{a\rho_3}{\delta}\xi^2\left\langle \widehat{z}, \widehat{\theta}\right\rangle + \frac{m\rho_3}{\delta}\xi^2\left\langle \int_0^\infty g(s)\widehat{\eta}(s)\,ds, i\xi\widehat{\theta}\right\rangle. \end{aligned} \tag{3.54}$$

Multiplying (3.54) by $\dfrac{1}{(1+\xi^2)}$ *if* $\chi = 0$ and (3.53) by $\dfrac{1}{(1+\xi^2)^2}$ *if* $\chi \neq 0$ and applying Young's inequality, to the terms on the right-hand side of (3.54) and (3.53), then (3.51) and (3.52) holds. This finishes the proof of Lemma 3.11. □

Proof of Proposition 3.7. Here, we introduce the Lyapunov functional $\mathcal{L}(\xi,t)$ as follows

$$\mathcal{L}(\xi,t) = N\,\widehat{\mathcal{E}}(\xi,t) + N_0\,\mathcal{K}_0(\xi,t) + N_5\,\mathcal{K}_1(\xi,t) + N_6\,\mathcal{K}_2(\xi,t) + \mathcal{K}_3(\xi,t). \tag{3.55}$$

Here N, N_0, N_5 and N_6 are positive constants that will be fixed later.

- *if* $(\chi = 0)$: Taking the derivative of $\mathcal{L}(\xi, t)$ with respect to t and making use of (3.40), (3.42), (3.45) and (3.51), we find

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{L}(\xi,t) + (N_0 - N_5C - N_6C - C)\frac{\xi^2}{(1+\xi^2)}\rho_2 |\widehat{y}|^2 \\
&+ (N_5 - N_0\varepsilon_0 - C)\frac{\xi^2}{(1+\xi^2)}a|\widehat{z}|^2 + (N_6 - N_0\varepsilon_0 - C)\frac{\xi^2}{(1+\xi^2)}k|\widehat{v}|^2 \\
&+ (1 - N_5\varepsilon_5 - N_6\varepsilon_6)\rho_1\frac{\xi^2}{(1+\xi^2)}|\widehat{u}|^2 \\
\leq\; & -\left(\frac{2}{\beta\rho_3}N - (N_0C + N_5C(\varepsilon_5) + N_6C + C)\right)\rho_3\xi^2\left|\widehat{\theta}\right|^2 \\
&- (Nk_1 - (N_0C(\varepsilon_0) + N_5C + N_6C(\varepsilon_6) + C))\, m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2 ds.
\end{aligned}
\tag{3.56}
$$

Here we have used

$$
\varepsilon_5\frac{\xi^2}{(1+\xi^2)^{\mathbf{2}}}\rho_1|\widehat{u}|^2 \leq \varepsilon_5\frac{\xi^2}{(1+\xi^2)}\rho_1|\widehat{u}|^2,
$$

and

$$
\varepsilon_0\frac{\xi^2}{(1+\xi^2)^{\mathbf{2}}}k|\widehat{v}|^2 \leq \varepsilon_0\frac{\xi^2}{(1+\xi^2)}k|\widehat{v}|^2.
$$

Now, we fix the constants in (3.56) as follows

$$
\begin{aligned}
&\varepsilon_5 = \frac{1}{4N_5},\ \varepsilon_6 = \frac{1}{4N_6},\ \varepsilon_0 = \frac{1}{2N_0},\ N_6 = 1 + C, \\
&N_5 = 1 + C \text{ and } N_0 = \frac{1}{2} + N_5C + N_6C + C.
\end{aligned}
$$

Finally, we choose N large enough such that

$$
N > \max\left(\frac{\beta\rho_3}{2}(N_0C + N_5C(\varepsilon_5) + N_6C + C), \frac{1}{k_1}(N_0C(\varepsilon_0) + N_5C + N_6C(\varepsilon_6) + C)\right).
$$

With these choices, (3.56) takes the form

$$
\frac{\partial}{\partial t}\mathcal{L}(\xi,t) + c_0\mathcal{K}(\xi,t) \leq 0,
\tag{3.57}
$$

where c_0 is a positive constant, and

$$
\begin{aligned}
\mathcal{K}(\xi,t) \;=\; & \frac{\xi^2}{(1+\xi^2)}k|\widehat{v}|^2 + \frac{\xi^2}{(1+\xi^2)}\rho_1|\widehat{u}|^2 + \frac{\xi^2}{(1+\xi^2)}\rho_2|\widehat{y}|^2 \\
& + \frac{\xi^2}{(1+\xi^2)}a|\widehat{z}|^2 + \xi^2\rho_3\left|\widehat{\theta}\right|^2 + m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2 ds.
\end{aligned}
\tag{3.58}
$$

Since N is large enough and using (3.55) then there exist two positive constants c_1 and c_2

$$
c_1\widehat{\mathcal{E}}(\xi,t) \leq \mathcal{L}(\xi,t) \leq c_2\widehat{\mathcal{E}}(\xi,t).
\tag{3.59}
$$

From (3.58), we deduce that

$$
\mathcal{K}(\xi,t) \geq \lambda(\xi)\ \widehat{\mathcal{E}}(\xi,t),
\tag{3.60}
$$

where $\lambda(\xi) = \dfrac{\xi^2}{(1+\xi^2)}$. Consequently, from (3.57), (3.59) and (3.60), we can find C and c such that

$$\widehat{\mathcal{E}}(\xi,t) \leq C\,\widehat{\mathcal{E}}(\xi,0)\,e^{-c\,\lambda(\xi)\,t}.$$

- *If* $\chi \neq 0$: Taking the derivative of $\mathcal{L}(\xi,t)$ with respect to t and making use of (3.40), (3.42), (3.46) and (3.52), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{L}(\xi,t) + (N_0 - N_5C - N_6C(\varepsilon_6) - C)\frac{\xi^2}{(1+\xi^2)}\rho_2|\widehat{y}|^2 \\
&+ (N_5 - N_0\varepsilon_0 - N_6C)\frac{\xi^2}{(1+\xi^2)}a|\widehat{z}|^2 + (N_6 - N_0\varepsilon_0 - C)\frac{\xi^2}{(1+\xi^2)^2}k|\widehat{v}|^2 \\
&+ (1 - N_5\varepsilon_5 - N_6\varepsilon_6)\rho_1\frac{\xi^2}{(1+\xi^2)^2}|\widehat{u}|^2 \\
\leq\; & -\left(\frac{2}{\beta\rho_3}N - (N_0C + N_5C(\varepsilon_5) + N_6C)\right)\rho_3\xi^2\left|\widehat{\theta}\right|^2 \\
&- (Nk_1 - (N_0C(\varepsilon_0) + N_5C + N_6C(\varepsilon_6)))\,m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2\,ds.
\end{aligned} \tag{3.61}$$

Now, we fix the constants in (3.61) as follows

$$\begin{cases} \varepsilon_5 = \dfrac{1}{4N_5},\ \varepsilon_6 = \dfrac{1}{4N_6},\ \varepsilon_0 = \dfrac{1}{2N_0},\ N_6 = 1 + C, \\ N_5 = 1 + C,\ N_0 = \dfrac{1}{2} + N_5C + N_6(\varepsilon_6) + C. \end{cases}$$

Finally, we take N large enough such that

$$N > \max\left(\frac{\beta\rho_3}{2}(N_0C + N_5C(\varepsilon_5) + N_6C),\ \frac{1}{k_1}(N_0C(\varepsilon_0) + N_5C + N_6C(\varepsilon_6))\right).$$

With these choices, (3.61) takes the form

$$\frac{\partial}{\partial t}\mathcal{L}(\xi,t) + c_0\mathcal{K}(\xi,t) \leq 0, \tag{3.62}$$

where c_0 is a positive constant, and

$$\begin{aligned}
\mathcal{K}(\xi,t) = & \frac{\xi^2}{(1+\xi^2)^2}k|\widehat{v}|^2 + \frac{\xi^2}{(1+\xi^2)^2}\rho_1|\widehat{u}|^2 + \frac{\xi^2}{(1+\xi^2)}\rho_2|\widehat{y}|^2 \\
&+ \frac{\xi^2}{(1+\xi^2)}a|\widehat{z}|^2 + \xi^2\rho_3\left|\widehat{\theta}\right|^2 + m\xi^2\int_0^\infty g(s)|\widehat{\eta}(s)|^2\,ds.
\end{aligned} \tag{3.63}$$

Since N is large enough and by using (3.55) then there exist two positive constants c_1 and c_2

$$c_1\widehat{\mathcal{E}}(\xi,t) \leq \mathcal{L}(\xi,t) \leq c_2\widehat{\mathcal{E}}(\xi,t). \tag{3.64}$$

From (3.63), we deduce that

$$\mathcal{K}(\xi,t) \geq \lambda(\xi)\,\widehat{\mathcal{E}}(\xi,t), \tag{3.65}$$

where $\lambda(\xi) = \dfrac{\xi^2}{\left(1+\xi^2\right)^2}$ .Consequently, from(3.62), (3.64) and (3.65), we can find C, c such that

$$\widehat{\mathcal{E}}(\xi,t) \leq C\,\widehat{\mathcal{E}}(\xi,0)\, e^{-c\,\lambda(\xi)\,t}.$$

This finishes the proof of the Proposition

□

4. The decay estimates

In this section, using the previous estimates, we establish new decay rates of the solution $U(x,t)$ and $V(x,t)$ of the systems $(2.3)-(2.4)$ and $(2.6)-(2.7)$, respectively. We need to mention here that in the case of $\chi = 0$ we don't have the regularity loss phenomena.

Our first main result is stated as follow:

Theorem 4.1. *Let s be a nonnegative integer and*

$$\chi = \frac{b\rho_1}{k} - \rho_2.$$

Suppose that $U_0 \in H^s \cap L^1(\mathbb{R})$. Then, the solution U of the system (2.3), satisfies the following decay estimates:

- *If $(\chi = 0)$, then*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2};\ t \geq 0.$$

- *If $(\chi \neq 0)$, then*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_{L^1} + C(1+t)^{-\dfrac{l}{2}}\left\|\partial_x^{k+l} U_0\right\|_{L^2};\ t \geq 0,$$

where $k + l \leq s$, C and c are two positive constants.

Proof. Using the Fourier transform, the proof of Theorem 4.1 is reduced to the analysis of the behavior of the spectral parameter in low- frequency and in the high-frequency regions. The proof is based on the pointwise estimates in Proposition 3.1. Applying Plancherel theorem and making use of the inequality in (3.3) and as $c_1\left|\widehat{U}(\xi,t)\right|^2 \leq \widehat{E}(\xi,t) \leq c_2\left|\widehat{U}(\xi,t)\right|^2$, we obtain

$$\begin{aligned}
(4.1)\quad \left\|\partial_x^n U(t)\right\|_2^2 &= \int_{\mathbb{R}} \xi^{2n}\left|U(\xi,t)\right|^2 d\xi \\
&\leq C\int_{\mathbb{R}} \xi^{2n} e^{-c\,\lambda(\xi)\,t}\left|\widehat{U}(\xi,0)\right|^2 d\xi \\
&\leq C\int_{|\xi|\leq 1} \xi^{2n} e^{-c\,\lambda(\xi)\,t}\left|\widehat{U}(\xi,0)\right|^2 d\xi + C\int_{|\xi|\geq 1} \xi^{2n} e^{-c\,\lambda(\xi)\,t}\left|\widehat{U}(\xi,0)\right|^2 d\xi \\
&= I_1 + I_2.
\end{aligned}$$

The integral is spitted into two parts: the low-frequency part $(|\xi| \leq 1)$ and the high-frequency part $(|\xi| \geq 1)$.

- Estimation of I_1 :

– $if\ \ (\chi = 0)$ or $if\ \ (\chi \neq 0)$, with $\lambda(\xi) = \dfrac{\xi^2}{1+\xi^2}$ or $\lambda(\xi) = \dfrac{\xi^2}{(1+\xi^2)^2}$. Here, we see that $\lambda(\xi) \geq \dfrac{\xi^2}{2}$ or $\lambda(\xi) \geq \dfrac{\xi^2}{4}$, so that we have we infer that

$$I_1 \leq C \left\| \widehat{U}_0 \right\|_{L^\infty}^2 \int\limits_{|\xi|\leq 1} |\xi|^{2n}\, e^{-c\xi^2 t} d\xi \leq C\,(1+t)^{-\frac{1}{2}(1+2n)} \left\| U_0 \right\|_{L^1}^2,$$

where, we have used the following inequality

$$\int\limits_0^1 |\xi|^{\sigma}\, e^{-c\xi^2 t} d\xi \leq C\,(1+t)^{-\frac{(1+\sigma)}{2}}.$$

- Approximation of I_2 :
  – $If\ \ (\chi = 0)$: In high frequency region, we see that $\lambda(\xi) \geq C$ for $|\xi| \geq 1$. Therefore I_2 is estimated as follow:

$$\begin{aligned} I_2 &\leq Ce^{-c\,t} \int\limits_{|\xi|\geq 1} \xi^{2n} \left| \widehat{U}(\xi, 0) \right|^2 d\xi \\ &\leq Ce^{-c\,t} \left\| \partial_x^n U_0 \right\|_{L^2}^2. \end{aligned}$$

  – $If\ \ (\chi \neq 0)$: We have $\lambda(\xi) \geq \xi^{-2}$ for $|\xi| \geq 1$. Therefore we can estimate I_2 as

$$\begin{aligned} I_2 &\leq C \sup_{|\xi|\geq 1} \left\{ |\xi|^{-2\delta}\, e^{-c\,|\xi|^{-2}\,t} \right\} \int\limits_{|\xi|\geq 1} \xi^{2(n+\delta)} \left| \widehat{U}(\xi, 0) \right|^2 d\xi \\ &\leq C\,(1+t)^{-\delta} \left\| \partial_x^{n+\delta} U_0 \right\|_{L^2}^2. \end{aligned}$$

Substituting these estimates into (4.1) gives the desired estimate in Theorem.

□

Using a similar method of proof like in the previous theorem, we establish the decay estimates of the solution $V(x,t)$ solution of (2.6) – (2.7). The result is stated as follow:

Theorem 4.2. *Let s be a nonnegative integer and*

$$\chi = \frac{b\rho_1}{k} - \rho_2.$$

Suppose that $V_0 \in H^s \cap L^1(\mathbb{R})$. Then, the solution V of the system (2.6), satisfies the following decay estimates:

- *$If\ \ (\chi = 0)$, then*

$$\left\| \partial_x^k V(t) \right\|_{L^2} \leq C\,(1+t)^{-\frac{1}{4}-\frac{k}{2}} \left\| V_0 \right\|_{L^1} + Ce^{-ct} \left\| \partial_x^k V_0 \right\|_{L^2};\ t \geq 0.$$

- *$If\ \ (\chi \neq 0)$, then*

$$\left\| \partial_x^k V(t) \right\|_{L^2} \leq C\,(1+t)^{-\frac{1}{4}-\frac{k}{2}} \left\| V_0 \right\|_{L^1} + C\,(1+t)^{-\frac{l}{2}} \left\| \partial_x^{k+l} V_0 \right\|_{L^2};\ t \geq 0,$$

where $k+l \leq s$, C and c are two positive constants.

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Compliance with Ethical Standards

The authors declares no potential conflict of interest.

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