An enhanced technique for strongly nonlinear oscillators with a harmonic restoring force

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Abstract

An enhanced analytical technique for nonlinear oscillators having a harmonic restoring force is proposed. The approach is passed on the change of the auxiliary operator by another suitable one leads to obtain a periodic solution. The fundamental idea of the new approach is based on obtaining an alternative equation free of the harmonic restoring forces. This method is a modification of the homotopy perturbation method. The approach allows not only an actual periodic solution, but also the frequency of the problem as a function of the amplitude of oscillation. Three nonlinear oscillators including restoring force, the simple pendulum motion, the cubic Duffing oscillator, the Sine-Gordon equation are offered to clarify the effectiveness and usefulness of the proposed technique. This approach allows an effective mathematical approach to noise and uncertain properties of nonlinear vibrations arising in physics and engineering.

Keywords: Homotopy Perturbation Method, Frequency Expansion, Periodic Solution, Pendulum Equation, Sine-Gordon Equation, Traveling Wave Solution.

1. INTRODUCTION

The motion of nanoparticles in the capillary fluid moves periodically with an extremely restoring force, and it plays an important role in enhancing mass, energy, and charge transfer in many nano/micro phenomena, from lithium batteries in micro/nanodevices, which are the footstone for nano-industriation. The restoring force of capillary vibration plays an important role in both nature and engineering, especially in nano/microdevices [1,2]; the vibration is a balance of the force produced by the capillary's geometric potential[3-5] and the gravity. The relationship between the frequency and the amplitude shows that an extremely restoring force is extremely helpful for mass and heat transfer through the nanofiber membrane[6-9], and it is especially important for nutrition and air transfer in a living body.

Nonlinear oscillations are a significant fact in mechanical structures, engineering problems, and physical science. All differential equations covering physical and engineering phenomena are nonlinear. The techniques of solutions of linear differential equations are relatively available and well determined. On the opposite, in the nonlinear differential equations, the methods of solutions are lowest available and therefore no exact solution and, overall, linear approximations are extremely used. A specific type of analytical solution specified nonlinear oscillator with a harmonic restoring force has a great quantity of importance, because, nearly of the phenomena that appear in mathematical physics and engineering scope can be described by it. Therefore, inspect strongly nonlinear oscillators with cubic and harmonic restoring force is becoming increasingly engaged in nonlinear sciences. Moreover, gain exact solutions for nonlinear oscillatory problems has more difficulties. It is very difficult to gain the solution of nonlinear problems and in general, it is often more complicated to get an analytic approximate solution than a numerical one for a given nonlinear problem. To overcome the shortcomings, many new analytical techniques have been successfully developed. He and Jin[2] supply a short review of analytical methods for the capillary oscillator in a nanoscale deformable tube. In their letter reviews some effective methods to solve analytically the frequency-amplitude relation of the capillary oscillator, including the variational principle, the variational iteration method, the homotopy perturbation method, He's frequency formulation and Taylor series method.

Yuste and Sánchez[10] used the so-called cubication approach, which consists of replacing the system of restoring force $f(x)$ by an equivalent cubic polynomial expression βx^3, where the value of β is determined by using a weighted mean-square method or by using the principle of harmonic balance [11,12]. Beléndez et al [13] used this idea and replaced the original second-order differential equation with the well-known Duffing equation. Uwe Starossek[14] studied the strongly nonlinear oscillator by assuming that the restoring force has a purely cubic function of the displacement variable. The investigation in nonlinear oscillators with cubic and harmonic restoring force solutions is becoming increasingly attractive in nonlinear sciences[15-18]. Moreover, obtaining exact solutions for nonlinear oscillatory problems has many difficulties. It is very problematic to solve nonlinear problems and overall, it is often more complicated to get an analytic approximation than a numerical one for an offered nonlinear problem. Only analytical approximate solutions are available, many new analytical methods have been successfully developed. There are some approximation techniques have been investigated. These include the Akbari-Ganji's [19], the cubication technique[13], the pseudo-spectral method[20], the frequency-amplitude formulation[21], the rational variational approach[22], and the closed-form numerical[23] methods and the iteration method[24-26]. Besides, the harmonic balance [27-29,16] has been used to derive periodic solutions to strongly nonlinear oscillatory problems. Traditional perturbation methods [30-37] are the most widely used analytical methods for solving nonlinear equations, which is the most flexible tool available for nonlinear analysis of science and engineering problems.

Here in this paper, the main goal is to obtain a periodic solution by an analytical method for the strongly nonlinear oscillation including a harmonic restoring force. We proposed a new technique to relax such a restoring force. This approach based on obtaining an alternative system free of the trigonometric functions. The outcome system is easier to handle by any analytical perturbation method. Here, we apply the enhanced homotopy perturbation method[38-40], which including the methodology of the expanded parameter[41,42]. The technology of two homotopy expanded parameters is used[34, 45] to construct the homotopy equation. One of these parameters used to expand the homotopy equation and the other used to expand the frequency-amplitude equation. To illustrate the effectiveness of the current method, three test examples are considered in this proposal.

2. THE PROPOSAL METHOD

We aim to apply the enhanced approach to obtain a periodic solution of the simple pendulum equation that has a restoring force. Thus, we consider the following equation:

$$\ddot{\theta} + a\theta = b\sin\theta; \quad \theta(0) = A, \dot{\theta}(0) = 0, \tag{1}$$

where a and b are real parameters. Also, this equation is used to describe the capillary oscillator and a detailed derivation was given in Jin et al.[1] Several approximate solutions of (1) have been derived by using different techniques[2]. Here, we deal with the approximate periodic solutions to (1) that have been not derived before. Our approach doesn't depend on expanded the sine-function, but to derive an alternative form of it, free of the harmonic function.

By expanding the sine-function reveals that the parameter a is not the full natural frequency. At this end, a suitable primary periodic solution cannot be found. Because of the operator ($L \equiv D^2 + a$), which is the highest order derivative and is assumed to be easily invertible, does not the actual auxiliary operator. For convenience, the indeed linear natural frequency must be formulated as

$$\omega_0^2 = a - b. \tag{2}$$

To perform the periodic solution of equation (1), we need to remove the wrong auxiliary operator L and replacing it by the correct auxiliary one. Without resorting to the expanded technique for the harmonic force, equation (1) can be re-arranged in another form. To illustrate the basic concept of the present proposal, let's begin with the integration of equation (1) by applying the operator L^{-1} on both sides, we get

$$\theta(t) = \theta(0) + \frac{b}{D^2 + a}\sin\theta. \tag{3}$$

To obtain an alternative form of equation (1), we re-build it so that the actual natural frequency ω_0^2 is working to get an analytical periodic solution. To illustrate this suggestion, the following process is offered:

Differentiating equation (3) twice to the variable t, we find

$$D^2\theta(t) = \frac{b}{D^2 + a}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \tag{4}$$

By the bits of help of the original equation (1) and its first-order derivative, one can remove the harmonic functions $\sin\theta$ and $\cos\theta$ from equation (4). At this end, the pendulum equation (1) is converted to the form

$$D^2\theta(t) = \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]. \tag{5}$$

This formulation is free of the restoring forces, the parameter b is disappearing through the process. In formulating the modified equation, one can reset the role of the parameter, b, through the including of the natural frequency ω_0^2. At this end, the alternative form of the pendulum equation (1) is presented having a primary periodic solution when $a > b$. To analyze, such highly nonlinear equation, the perturbation technique is urgent. By applying the homotopy perturbation method [45], the homotopy equation can be constructed in the form

$$D^2\theta + \omega_0^2\theta = \rho\left\{\omega_0^2\theta + \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]\right\}; \quad \rho \in [0,1] \tag{6}$$

The additional frequency ω is introduced through the frequency extension technology [41] as follows:

$$\omega^2 = \omega_0^2 + \rho\omega_1 + \rho^2\omega_2 + \dots\,, \tag{7}$$

where the additional frequency ω is unknown to be determined later. Employing the expansion (7) into the homotopy equation (6), the result is

$$\left(D^2 + \omega^2\right)\theta = \rho\left\{\omega_1\theta + \left(\omega^2 - \rho\omega_1 + \dots\right)\theta + \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]\right\}. \tag{8}$$

Consider the solution $\theta(t)$ has been expanded in the form

$$\theta(t) = \theta_0(t) + \rho\theta_1(t) + \rho^2\theta_2(t) + \dots \tag{9}$$

Substituting (9) into the above homotopy equation, equating the identical powers of ρ on both sides, we obtain the first-two unknowns $\theta_0(t)$ and $\theta_1(t)$ in the form

$$\theta_0(t) = A\cos\omega t, \tag{10}$$

$$\left(D^2+\omega^2\right)\theta_1 = \left(\omega^2+\omega_1\right)\theta_0 + \frac{1}{D^2+a}\left[\left(\frac{\dddot{\theta}_0}{\dot{\theta}_0}+a\right)\ddot{\theta}_0 - \dot{\theta}_0^2\left(\ddot{\theta}_0 + a\theta_0\right)\right]. \tag{11}$$

Insert (10) into the first-order equation (11) becomes

$$\left(D^2+\omega^2\right)\theta_1 = \left(\omega_1 - \frac{1}{4}A^2\omega^2\right)A\cos\omega t - \frac{1}{4}A^3\omega^2\frac{\left(\omega^2-a\right)}{a-9\omega^2}\cos 3\omega t. \tag{12}$$

Avoiding the secular terms, we get

$$\omega_1 = \frac{1}{4}A^2\omega^2. \tag{13}$$

Insert (13) into the expansion (7), letting $\rho \to 1$, we obtain

$$\omega^2 = \omega_0^2\left(1-\frac{1}{4}A^2\right)^{-1}. \tag{14}$$

The first-order approximate solution of the pendulum equation (1) is found in the form

$$\theta(t) = A\cos\omega t - \frac{1}{32}A^3\frac{\left(\omega^2-a\right)}{\left(9\omega^2-a\right)}\left(\cos 3\omega t - \cos\omega t\right). \tag{15}$$

It is seen from (14) that the periodic solution is available when the following conditions have been satisfied:

$$a > b \quad \text{and} \quad A^2 < 4. \tag{16}$$

3. CUBIC NONLINEAR OSCILLATION HAVING THE HARMONIC RESTORING FORCE

A highly nonlinear oscillator with a cubic and harmonic restoring force is derived in the form

$$\ddot{\theta} + a\theta + Q\theta^3 = b\sin\theta; \qquad \theta(0) = A, \dot{\theta}(0) = 0, \tag{17}$$

Mathematically, this equation is considered as a modification of the equation (1), which characterized by included the Duffing parameter. Therefore, we follow the same procedure as the previous item. So, the alternative form of the equation (17) is found in the form

$$D^2\theta(t) = \frac{1}{D^2+a}\left[\frac{\dddot{\theta}}{\dot{\theta}}\ddot{\theta} + a\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta}+a\theta+Q\theta^3\right) - 6Q\theta\dot{\theta}^2\right]. \tag{18}$$

The modified homotopy equation includes the frequency expansion (7) with the two small parameters $\rho \in [0,1]$ and $\varepsilon \in [0,1]$, is coming in the form

$$\left(D^2+\omega^2\right)\theta(t) = \rho\left\{\left(\omega_1+\omega^2\right)\theta + \frac{1}{D^2+a}\left[\frac{\dddot{\theta}}{\dot{\theta}}\ddot{\theta} + a\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta}+a\theta\right) - \varepsilon Q\dot{\theta}^2\theta\left(\theta^2+6\right)\right]\right\}. \tag{19}$$

On using the expansion (9), the zero-order solution (10) is available and the first-order equation has the following configuration:

$$\left(D^2+\omega^2\right)\theta_1(t)=\left[\omega_1-\frac{1}{4}A^2\omega^2-\varepsilon QA^2\omega^2\frac{\left(12+A^2\right)}{8\left(a-\omega^2\right)}\right]A\cos\omega t+\frac{A^3\omega^2}{4\left(9\omega^2-a\right)}\left(\omega^2-a-\frac{1}{4}\varepsilon Q\left(24+A^2\right)\right)\cos 3\omega t$$
$$+\frac{\varepsilon A^5\omega^2 Q}{16\left(a-25\omega^2\right)}\cos 5\omega t. \tag{20}$$

The requiring condition for the uniform solution is

$$\omega_1(\varepsilon)=\frac{1}{4}A^2\omega^2-\varepsilon QA^2\omega^2\frac{\left(12+A^2\right)}{8\left(\omega^2-a\right)}. \tag{21}$$

The final first-order approximate solution for equation (17), where $\rho\to 1$ and $\varepsilon\to 1$, is performed as

$$\theta(t)=A\cos\omega t-\frac{A^3}{32\left(9\omega^2-a\right)}\left(\omega^2-a-\frac{1}{4}QA^2\left(24+A^2\right)\right)\left(\cos 3\omega t-\cos\omega t\right)+\frac{A^5 Q}{384\left(25\omega^2-a\right)}\left(\cos 5\omega t-\cos\omega t\right). \tag{22}$$

The frequency-amplitude equation can be derived by insert (21) into the expansion (7) yields

$$\omega_0^2+\left(\frac{1}{4}A^2-1\right)\omega^2-\varepsilon QA^4\omega^2\frac{\left(12+A^2\right)}{8\left(\omega^2-a\right)}=0. \tag{23}$$

This is a complicated frequency-amplitude equation. The absence of the Duffing coefficient Q yields the same frequency formula (14). The perturbation technique is very suitable for analysis of the above frequency-amplitude equation[46]. To derive an approximate solution of the above equation, we use the following expansion:

$$\omega^2(\varepsilon)=\varpi_0^2+\varepsilon\varpi_1+\varepsilon^2\varpi_2+\ldots \tag{24}$$

Substituting this expansion into the equation (23), equating the identical powers of ε on both sides we get

$$\varpi_0^2=\omega_0^2\left(1-\frac{1}{4}A^2\right)^{-1}, \tag{25}$$

$$\varpi_1=-\frac{\varpi_0^2 QA^2}{2\left(4-A^2\right)}\frac{\left(12+A^2\right)}{\left(\varpi_0^2-a\right)}. \tag{26}$$

In one iteration operation, we insert (25) and (26) into (24) and setting $\varepsilon\to 1$, yields

$$\omega^2=\omega_0^2\left(1-\frac{1}{4}A^2\right)^{-1}\left[1-\frac{QA^2\left(A^2+12\right)}{2\left(aA^2-4b\right)}\right]. \tag{27}$$

The periodic solution is available whence the following condition is presented, with $a>b$:

$$\left(1-\frac{1}{4}A^2\right)\left(1-\frac{QA^2\left(A^2+12\right)}{2\left(aA^2-4b\right)}\right)>0. \tag{28}$$

4. DERIVING A PERIODIC SOLUTION OF THE SINE-GORDON EQUATION

The sine-Gordon equation is a nonlinear partial differential equation, including the d'Alembert operator and the sine-function of the unknown variable. The equation, as well as several solution techniques, was known in the two-century ago in the course of the study of various problems of differential geometry. The Sine-Gordon equation appears in several physical applications[47-49]. The Sine-Gordon equation has attracted wide interest over the years in the depiction of classical and quantum mechanical phenomena[50]. In the current section, we **c**onsider the well known Sine-Gordon equation that has the form

$$y_{tt} - Py_{xx} = \omega_0^2 \sin y, \tag{29}$$

where $y = y(x,t)$, with the initial conditions $y(x,0) = A(x),\ y_t(x,0) = 0.$ where $x -$ coordinate, $t -$ time, $y -$ the unknown function. The aim is to seek a modified equation free of the harmonic restoring force. To achieve this goal, we first remember that

$$\partial_{tt} \sin y = y_{tt} \cos y - y_t^2 \sin y. \tag{30}$$

Using the fact

$$\partial_x \sin y = y_x \cos y. \tag{31}$$

Thus, one can rewrite (30) in the form

$$\partial_{tt} \sin y = \left(\frac{y_{tt}}{y_x} \partial_x - y_t^2 \right) \sin y. \tag{32}$$

By the bits of help of the original equation (29), one can remove the function $\sin y$ from the formula (32), the result is

$$D_t^4 y = PD_t^2 y_{xx} + \left(\frac{y_{tt}}{y_x} \partial_x - y_t^2 \right) (y_{tt} - Py_{xx}). \tag{33}$$

Since the frequency ω_0^2 has disappeared through the replacing process, we restore it, by the addition method, as the auxiliary parameter. The corresponding homotopy equation is formulated as

$$\left(D_t^4 - \omega_0^4\right) y = \rho \left[-\omega_0^4 y + PD_t^2 y_{xx} + \frac{y_{tt}}{y_x} (y_{ttx} - Py_{xxx}) - y_t^2 (y_{tt} - Py_{xx}) \right];\ \rho \in [0,1] \tag{34}$$

To derive the periodic solution the parameter expansion technology is utilized so that

$$\Omega^2 = \omega_0^2 + \rho \Omega_1 + \rho \Omega_2 + \ldots \tag{35}$$

Employing the expansion (35) into the homotopy equation (34) becomes

$$\left(D_t^4 - \Omega^4\right) y = \rho \left[-\Omega^2 \left(\Omega^2 + 2\Omega_1\right) y + PD_t^2 y_{xx} + \frac{y_{tt}}{y_x} (y_{ttx} - Py_{xxx}) - y_t^2 (y_{tt} - Py_{xx}) \right];\ \rho \in [0,1] \tag{36}$$

Consider, as usual, the solution is given by

$$y(x,t) = y_0(x,t) + \rho y_1(x,t) + \rho^2 y_2(x,t) + \ldots \tag{37}$$

It is noted that the zero-order solution, when $\rho \to 0$, is satisfied with

$$y_0 = A(x) \cos \Omega t, \tag{38}$$

The first-order ρ is found to be

$$\left(D_t^4 - \Omega^4\right) y_1 = \left\{ -2\Omega_1 A + P\left[\frac{AA_{xxx}}{A_x} - \left(1 - \frac{1}{4} A^2\right) A_{xx} \right] + \frac{1}{4} A^3 \Omega^2 \right\} \Omega^2 \cos \Omega t - \frac{1}{4} A^2 \Omega^2 \left(\Omega^2 A + PA_{xx}\right) \cos 3\Omega t. \tag{39}$$

Its solution, without secular terms, has performed in the following configuration:

$$y_1(x,t) = -\frac{A^2\left(\Omega^2 A + PA_{xx}\right)}{320\Omega^2}\left(\cos 3\Omega t - \cos\Omega t\right). \tag{40}$$

The first-order approximate solution has been getting in the form

$$y(x,t) = \lim_{\rho\to 1}\left(y_0 + \rho y_1\right) = A(x)\cos\Omega t - \frac{A^2\left(\Omega^2 A + PA_{xx}\right)}{320\Omega^2}\left(\cos 3\Omega t - \cos\Omega t\right). \tag{41}$$

This solution has been derived under the following condition:

$$\Omega_1 = \frac{1}{8}A^2\Omega^2 + \frac{1}{2}P\left[\frac{A_{xxx}}{A_x} - \left(1 - \frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]. \tag{42}$$

To formulate the frequency-amplitude equation, we insert (42) into the expansion (35) and letting $\rho \to 1$, yields

$$\Omega^2(x) = \left(1 - \frac{1}{8}A^2\right)^{-1}\left\{\omega_0^2 + \frac{1}{2}P\left[\frac{A_{xxx}}{A_x} - \left(1 - \frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]\right\}. \tag{43}$$

The periodic solution is available when the following condition is satisfied:

$$\left(1 - \frac{1}{8}A^2\right)\left\{\omega_0^2 + \frac{1}{2}P\left[\frac{A_{xxx}}{A_x} - \left(1 - \frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]\right\} > 0. \tag{44}$$

4.1 Traveling wave solution for the Sine-Gordon equation

In this section, we try to find the traveling wave solution to the above equation (29) assuming that the initial conditions have sought as $y(x,0) = A_0\cos kx$ and $y_t(x,0) = -A_0\omega_0\sin kx$, where the parameter k refers to the wave-number of the traveling wave and A_0 denote to a constant amplitude. Follow the above procedure, equation (29) can perform as

$$y(x,t) = \frac{\omega_0^2}{\partial_{tt} - P\partial_{xx}}\sin y. \tag{45}$$

As seen, it is a complicated nonlinear equation so we proceed as explained before, the harmonic function $\sin y$ can be relaxed so that the homotopy equation has the following configuration:

$$D_t^2 y + \omega_0^2 y = \rho\left\{\omega_0^2 y + \frac{1}{\partial_{tt} - P\partial_{xx}}\left(\frac{y_{tt}}{y_x}\partial_x - y_t^2\right)\left(y_{tt} - Py_{xx}\right)\right\};\ \rho \in [0,1] \tag{46}$$

Utilizing the approach of the parameter expansion as given by

$$\sigma = \omega_0 + \rho\sigma_1 + \rho^2\sigma_2 + \ldots \tag{47}$$

Employing (47) into the above equation yields:

$$\left(D_t^2 + \sigma^2\right)y = \rho\left\{\left(\sigma^2 + 2\sigma\sigma_1\right)y + \frac{1}{\partial_{tt} - P\partial_{xx}}\left(\frac{y_{tt}}{y_x}\partial_x - y_t^2\right)\left(y_{tt} - Py_{xx}\right)\right\}. \tag{48}$$

In using the expanded solution as given by (37), we have the primary solution as $\rho \to 0$, is satisfied with

$$y_0(x,t) = A_0\cos(\sigma t + kx). \tag{49}$$

By the help of the above zero-order solution, yields the equation that covered the first-order in has arranged in the form

$$\left(D_t^2+\sigma^2\right)y_1(x,t)=2\sigma\sigma_1 A_0\cos(\sigma t+kx)-\frac{1}{4}A_0^3\sigma^2\left[\cos(\sigma t+kx)-\frac{1}{9}\cos 3(\sigma t+kx)\right]; \qquad (50)$$
$$y_1(x,0)=0,\ y_{1t}(x,0)=A_0\sigma_1\sin kx$$

This equation has a bounded solution, because of the initial conditions, in the form

$$y_1=\frac{1}{288}A_0^3\left[2\cos(\sigma t+3kx)-\cos(\sigma t-3kx)-18\cos(\sigma t+kx)+18\cos(\sigma t-kx)-\cos 3(\sigma t+kx)\right]. \qquad (51)$$

The above solution is performed under the condition

$$\sigma_1=\frac{1}{8}A_0^2\sigma. \qquad (52)$$

The final first-order solution can be derived by insert (51) and (53) in the expansion (37), letting $\rho\to 1$, yields

$$\begin{aligned}y(x,t)&=A_0\cos(\sigma t+kx)+\frac{1}{288}A_0^3\cos 3(\sigma t+kx)\\&+\frac{1}{288}A_0^3\left[\cos(\sigma t-3kx)-2\cos(\sigma t+3kx)-18\cos(\sigma t+kx)+18\cos(\sigma t-kx))\right].\end{aligned} \qquad (53)$$

Also, the frequency-amplitude equation can be performed as

$$\sigma=\omega_0\left(1-\frac{1}{8}A_0^2\right)^{-1}. \qquad (55)$$

4.2 A periodic solution for a generalized sine-Gordon equation

In the present subsection, a generalized sine-Gordon equation is considered in the form

$$u_{tt}-\left(P+\omega_0^2P_0\cos u\right)u_{xx}=\omega_0^2\left(1-P_0u_x^2\right)\sin u, \qquad (56)$$

where the initial conditions are $u(x,0)=A_0\cos kx,\ u_t(x,0)=-A_0\omega_0\sin kx$. This equation can be rewritten in the form

$$u_{tt}-Pu_{xx}=\omega_0^2\left(1+P_0\partial_{xx}\right)\sin u, \qquad (57)$$

where the following formula is used:

$$\partial_{xx}\sin u=u_{xx}\cos u-u_x^2\sin u. \qquad (58)$$

Applying the same procedure as in the previous subsection (4.1) to replace the function $\sin u$ by its equivalent linear instruction in (57). Then equation (57) should be transformed into the following form:

$$\partial_{tt}u=\frac{\left(1+P_0\partial_{xx}\right)}{\left(\partial_{tt}-P\partial_{xx}\right)}\left(\frac{u_{tt}}{u_x}\partial_x-u_t^2\right)\left(1+P_0\partial_{xx}\right)^{-1}\left(u_{tt}-Pu_{xx}\right). \qquad (59)$$

Construct the corresponding homotopy equation with including the parameter ω_0^2 in it as an auxiliary linear part, yields

$$\left(\partial_{tt}+\omega_0^2\right)u=\rho\left[\omega_0^2u+\frac{\left(1+P_0\partial_{xx}\right)}{\left(\partial_{tt}-P\partial_{xx}\right)}\left(\frac{u_{tt}}{u_x}\partial_x-u_t^2\right)\left(1+P_0\partial_{xx}\right)^{-1}\left(u_{tt}-Pu_{xx}\right)\right];\ \rho\in[0,1] \tag{60}$$

Applying the homotopy perturbation technique to the above equation, yields

$$u_0(x,t)=A_0\cos(\tilde{\sigma}t+kx), \tag{61}$$

where $\tilde{\sigma}$ is unknown wave-frequency of the traveling wave solution and will determined later. This frequency is given similar to the expansion (47) in which each σ is replaced by $\tilde{\sigma}$. The equation that covers the first-order perturbation in ρ, is given by

$$\left(\partial_{tt}+\tilde{\sigma}^2\right)u_1=\left(\tilde{\sigma}^2+2\tilde{\sigma}\tilde{\sigma}_1\right)u_0+\frac{\left(1+P_0\partial_{xx}\right)}{\left(\partial_{tt}-P\partial_{xx}\right)}\left(\frac{u_{0tt}}{u_{0x}}\partial_x-u_{0t}^2\right)\left(1+P_0\partial_{xx}\right)^{-1}\left(u_{0tt}-Pu_{0xx}\right)$$
$$;\ u_1(x,0)=0,\ u_{1t}(x,0)=A_0\tilde{\sigma}_1\sin kx. \tag{62}$$

Employing (61) in (62), after simplification, we obtain its solution has the form

$$\begin{aligned}u_1(x,t)=&A_0^3\frac{\left(1-9P_0k^2\right)}{288\left(1-P_0k^2\right)}\left[2\cos(\tilde{\sigma}t+3kx)-\cos(\tilde{\sigma}t-3kx)-\cos 3(\tilde{\sigma}t+kx)\right]\\&+\frac{1}{16}A_0^3\left[\cos(\sin\tilde{\sigma}t-kx)-\cos(\sin\tilde{\sigma}t+kx)\right],\end{aligned} \tag{63}$$

where the frequency-amplitude formula (55) is still working. The final first-order approximate solution is found in the form

$$\begin{aligned}u(x,t)=&A_0^3\frac{\left(1-9P_0k^2\right)}{288\left(1-P_0k^2\right)}\left[2\cos(\tilde{\sigma}t+3kx)-\cos(\tilde{\sigma}t-3kx)-\cos 3(\tilde{\sigma}t+kx)\right]\\&+\frac{1}{16}A_0^3\left[\cos(\sin\tilde{\sigma}t-kx)-\cos(\sin\tilde{\sigma}t+kx)\right]+A_0\cos(\tilde{\sigma}t+kx).\end{aligned} \tag{64}$$

5. DISCUSSION AND CONCLUSIONS

The purpose of the article is to employ the modified HPM to find an analytical approximate periodic solution of a nonlinear oscillator with a harmonic restoring force. The approach developed here does not consist of the expanded of the harmonic restoring force, nor used the cubication approach, but to introduce an alternative form free of this force. The alternative equation is solvable by any perturbation method. In this proposal, we present some examples to illustrate the applicability and to establish the approximate analytical periodic solutions. Also, the traveling wave solution for the Sine-Gordon equation has been established. The frequency-amplitude equation has been performed in each case. Conditions for the validation of a periodic solution are performed. The method adopted here is a well-established procedure for determining analytical approximations to the periodic solutions of the nonlinear oscillators having a restoring force. The current work suggests an effective modification of the well-known homotopy perturbation method for solving differential equations having a restoring force, and some new findings were obtained. It can be concluded that this article gives an absolute new avenue of research in various fields such as mathematics, vibration theory, and engineering. This paper will open up a flood of opportunities for further research.

CONFLICT OF INTEREST

The author declares that there are no competing interests regarding the publication of this paper.

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References

1. Jin X, Liu MN, Pan F, Li Y, Fan J. Low frequency of a deforming capillary vibration, part 1: mathematical model. *J Low Freq Noise V A Control*. 2019;38(3-4):1676-1680.
2. He JH and Jin X., A short review on analytical methods for the capillary oscillator in a nanoscale deformable tube, *Math Meth Appl Sci.* 2020;1–8, DOI: 10.1002/mma.6321
3. Liu P, He JH. Geometrical potential: an explanation on of nanofibers wettability. *Therm Sci*. 2018;22:33-38.
4. Zhou CJ, Tian D, He JH. What factors affect lotus effect? *Therm Sci*. 2018;22:1737-1743.
5. Li XX, He JH. Nanoscale adhesion and attachment oscillation under the geometric potential, part 1: the formation mechanism of nanofiber membrane in the electrospinning. *Results Phys*. 2019;12:1405-1410.
6. He JH, Variational principle and periodic solution of the Kundu–Mukherjee–Naskar equation, *Results in Physics*, 2020. https://doi.org/10.1016/j.rinp.2020.103031
7. Fan J, Zhang Y, Liu Y, et al. Explanation of the cell orientation in a nanofiber membrane by the geometric potential theory. *Results in Physics*. 2019;15:102537.
8. Yang ZP, Dou F, Yu T, et al. On the cross-section of shaped fibers in the dry spinning process: physical explanation by the geometric potential theory. *Results in Physics*. 2019;14:102347.
9. Tian D, Li XX, He JH. Geometrical potential and nanofiber membrane's highly selective adsorption property. *Adsorpt Sci Technol*. 2019; 37(5-6):367-388.
10. Yuste SB and Sánchez ÁM , "A weighted mean-square method of "Cubication" for nonlinear oscillators", Journal of Sound and Vibration, 1989, **134**(3), 423–433
11. Yuste SB, "Cubication of nonlinear oscillators using the principle of harmonic balance," International Journal of Non-Linear Mechanics, 2002, **27**(3), 347–356
12. Sinha SC and Srinivasan P., "A weighted mean square method of linearization in non-linear oscillations," Journal of Sound and Vibration, 1971, **16**, 139–148
13. Beléndez A, Hernández A, Beléndez T., et al, Solutions for conservative nonlinear oscillators using an approximate method based on Chebyshev series expansion of the restoring force. Act Phy Pol A, 2016, 2016, **130**(3), 667-678
14. Starossek U., Exact analytical solutions for forced cubic restoring force oscillator, Nonlinear Dyn 2016,**83**:2349–2359
15. V Marinca, N Herisanu, An optimal iteration method for strongly nonlinear oscillators. J Appl Math, 2012,**11,** 906341.
16. Hosen MA, Chowdhury MSH., A new reliable analytical solution for strongly nonlinear oscillator with cubic and harmonic restoring force. Res Phy, 2015,**5,** 111-114
17. Junfeng L., Li M, The VIM-Pade technique for strongly nonlinear oscillators with cubic and harmonic restoring force. J Low Freq Noise Vib Active Control, DOI: 2018,10.1177/1461348418813612
18. Hosen MD, Ismail G, Yildirim A and Kamal MAS, A Modified Energy Balance Method for Oscillators with Cubic and Harmonic Restoring Force, Journal of Applied and Computational Mechanics, 2020, **6** (2) 320-331
19. Akbari MR, Nimafar M, Ganji DD, Chalmiani HK, Investigation on non-linear vibration in arched beam for bridges construction via AGM method. Appl. Math Comput, 2017,**298** ,95-110 .
20. Nhat LA, Using differentiation matrices for pseudo spectral method solve Duffing Oscillator. J Non Sci Appl, 2018, **11,** 1331-1336
21. Wang Q, Shi X, Li Z , A short remark on Ren-Hu's modification of He's frequency-amplitude formulation and the temperature oscillation in a polar bear hair. J Low Freq Noise Vib Active Control, DOI: 2019, 10.1177/1461348419831478
22. Yazdi MK, Tehrani PH, Rational variational approaches to strong nonlinear oscillations. Int. J Appl Comp Math,2017, **3** (2), 757-771
23. Shui X, Wang S, Closed-form numerical formulae for solutions of strongly nonlinear oscillators. Int. J Non Mech, 2018, **103,** 12-22
24. Hoang T , Duhamel D, Foret G., et al, Frequency dependent iteration method for forced nonlinear oscillators. Appl. Math Mod, 2017,**42,** 441-448
25. Javidi M, Iterative methods to nonlinear equations. Appl. Math Comput, **193,** 360-365, (2007) .
26. Razzak MA, A simple new iterative method for solving strongly nonlinear oscillator systems having a rational

and an irrational force. Alex Engg J, 2018, **57,** 1099-1107
27. Mickens RE, A generalization of the method of harmonic balance. J Sound Vib, **111,**115-518, (1986) .
28. Chowdhury MSH, Hosen MA, Ali MY, Ismail FA, An analytical technique to obtain higher-order approximate periods for nonlinear oscillator. IIUM Engg J, 2018, **19**(2), 182-191
29. Akbarzade M, Farshidianfar A. , Nonlinear transversely vibrating beams by the improved energy balance method and the global residue harmonic balance method. Appl Math Mod,2017, **45,** 393-404
30. El-Naggar AM, Ismail GM, Analytical solution of strongly nonlinear Duffing Oscillators. Alex Engg J, 2016,**55,** 1581-1585
31. Yao S, Cheng Z, The homotopy perturbation method for a nonlinear oscillator with a damping. J Low Freq Noise Vib Active Control, 2019, DOI: https://doi.org/10.1177/1461348419836344
32. El-Dib YO, Periodic solution of the cubic nonlinear Klein–Gordon equation and the stability criteria via the He-multiple-scales method, *Pramana – J. Phys.* 2019*,* **92**:7
33. Alam Ms, Yeasmin IA, Ahamed MS, Generalization of the modified Lindstedt-Poincare method for solving some strong nonlinear oscillators. Ain Shams Engg J, 2019, **10,** 195-201
34. Suleman M, Wu Q, Comparative solution of nonlinear quantic-cubic oscillator using modified homotopy perturbation method. Ad Math. Phy, 2015, **5,** 932905
35. Razzak MA, Alam MZ, Sharif MN, Modified multiple time scale method for solving strongly nonlinear damped forced vibration systems. Res Phy, 2018, **8,** 231-238
36. El-Dib YO and Moatimid GM, Stability Configuration of a Rocking Rigid Rod over a Circular Surface Using the Homotopy Perturbation Method and Laplace Transform, Arabian Journal for Science and Engineering, 2019,**44**:6581–6591
37. El-Dib YO, “Multiple scales homotopy perturbation method for nonlinear oscillators”, Nonlinear Sci Lett A, 2017, **8**: 352–364
38. Filobello-Nino U, Vazquez-Leal H, Jimenez-Fernandez VM, et al. , “Enhanced classical perturbation method”, Nonlinear Sci Lett A, 2018, **9**: 172–185
39. Li XX and He CH., “Homotopy perturbation method coupled with the enhanced perturbation method”, J Low Freq Noise VA, DOI: 20,19,10.1177/1461348418800554
40. Ren ZF, Yao SW, and He JH, “He’s multiple scales method for nonlinear vibrations”, Low Freq Noise VA., DOI: 2019,10.1177/1461348419861450
41. He JH, Homotopy Perturbation Method with an Auxiliary Term. Abstr. Appl. Anal **2012**; 857612
42. El-Dib YO, The stability conditions of the cubic damping Van Der Pol-Duffing oscillator using the HPM with the frequency-expansion technology, Journal of Applied Mathematics and Computational Mechanics, 2018,**17**(3), 31-44
43. El-Dib YO, Multi-homotopy perturbation technique for solving nonlinear partial differential equation with Laplace transforms, Nonlinear Sci. Lett. A, 2018, **9**(4),349-359
44. Yu DN, He JH and, Garcıa AG, Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators, J. of Low-Frequency Noise, Vibration and Active Control, DOI:2018,10.1177/1461348418811028,
45. He JH, “ Homotopy perturbation technique”, Comput Meth Appl, Mech Eng 1999, **178**: 257–262
46. Nayfeh AH, Perturbation methods. New York: Wiley, 1973.
47. Tabor T., The Sine-Gordon Equation An Introduction, Wiley, New York, NY, USA, 1989.
48. Gibbon JD, James IN, and Moroz IM, “The sine-Gordon equation as a model for a rapidly rotating baroclinic fluid,” Physica Scripta, 1979, **20** (3-4), 402–408
49. Sun Y, New Travelling Wave Solutions for Sine-Gordon Equation, Journal of Applied Mathematics, Article 2014, ID 841416, 4 pages
50. Zarmi Y, Nonlinear quantum-mechanical system associated with Sine-Gordon equation in (1+2) dimensions, Journal of Mathematical Physics 2014,**55** (10), 103510