FRACTIONAL SCHRÖDINGER-POISSON SYSTEM WITH LOW ORDER TERM

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Abstract. In this paper, we consider the following fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta)^s u + u + \lambda K(x)\phi u = a(x)|u|^{p-2}u + b(x)|u|^2 u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0,1)$, $\lambda > 0$, $2 < p < 4$, $K(x)$, $a(x)$ and $b(x)$ are nonnegative functions satisfying some suitable conditions. We establish the existence of nontrivial solutions by using a refinement constrained minimization methods combining with compactness-concentration arguments.

1. Introduction

In this paper, we consider the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + u + \lambda K(x)\phi u = a(x)|u|^{p-2}u + b(x)|u|^2 u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $2 < p < 4$, $s, t \in (0,1)$, $2s + 2t > 3$ and $K(x)$, $a(x)$, $b(x)$ are nonnegative functions. we assume that the function $a(x)$ and $b(x)$, $K(x)$ satisfy the following assumptions:

(H_1) $a(x)$ is a positive continuous function on $\mathbb{R}^3$, then we have

$$\lim_{x\to\infty} a(x) = a_\infty > 0, \quad a_{max} := \sup_{x\in\mathbb{R}^3} a(x) < \frac{a_\infty}{A(p)^{\frac{p-2}{2}}},$$

where

$$A(p) = \begin{cases} \left(\dfrac{4-p}{2}\right)^{\frac{1}{p-2}}, & \text{if } 2 < p \le 3, \\ \dfrac{1}{2}, & \text{if } 3 < p < 4; \end{cases}$$

(H_2) $K(x) \in L^\infty(\mathbb{R}^3)\backslash\{0\}$ is a non-negative function on $\mathbb{R}^3$ and

$$\lim_{x\to\infty} K(x) = K_\infty \ge 0;$$

(H_3) $b(x) \in L^\infty(\mathbb{R}^3)$ is a non-negative continuous function on $\mathbb{R}^3$ such that

$$\lim_{x\to\infty} b(x) = 0.$$

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When $s = t = 1$, problem (1.1) reduces to the following system

$$\begin{cases} -\Delta u + u + K(x)\phi u = a(x)|u|^{p-2}u + b(x)|u|^2 u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

and its more general form is written as

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $f(x,u) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Problem (1.3) can be regarded as a model to describe the interaction between charged particles and electrostatic field, which was proposed by Benci-Fortunato [5] in quantum mechanics. The nonlinearity $f(x,u)$ represents the particles interacting with each other, u and ϕ represent the wave functions associated with the particle and the electric potential. We refer the interesting readers to see [1] for more detailed information on mathematical and physical backgrounds.

In recent years, Schrödinger-Poisson systems with a local nonlinear term $f(x,u)$ have been studied in depth. For system (1.3), there have been extensive studies under different assumptions of V, K and f. For example, see [2, 3, 5, 7, 8, 9, 6, 14, 15, 19, 30, 28, 32, 31] and the references therein. In the case of the critical $f = |u|^{p-2}u + u^5$ with $4 < p < 6$ and the subcritical $3 < p < 6$, the existence of ground state solutions were proved in [3]. For the case $p \leq 2$ or $p \geq 6$, the reader may see [9] and for the case $2 < p < 6$, can see [2, 3, 7, 8, 19]. When V is non-radial, $K \equiv 1$ and $f = |u|^{p-2}u$, system (1.3) has a ground state solution in [3] and [30] for $4 < p < 6$ and $3 < p \leq 4$. In [26], when $V \equiv 1$ and $f = a(x)|u|^{p-2}u$ with $4 < p < 6$ and $\lambda \in \mathbb{R}\backslash\{0\}$, Varia proved the exist of ground state solutions in the case of $4 < p < 6$ if $\lambda > 0$ and $2 < p < 6$ if $\lambda < 0$.

On the other hand, for problem (1.1), we give its more general form by

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

In recent years, the system (1.4) are receving a great attention. For example, in [29], Zhang, Do ó and Squassina considered the existence of radial ground state solution to the fractional Schrödinger-poisson system with a general subcritical or critical nonlinearity as $V(x) = 0$,$K(x) = \lambda > 0$. In [23], when $f(x) = |u|^{p-1}u$, Teng established the existence of ground state solution to the nonlinear fractional Södinger-Poisson system (1.4) when $2 < p < 2_s^*$, and system (1.4) has a trivial solution when $1 < p \leq 2$, $K(x) = \lambda \geq \frac{1}{4}$ or $p = 2_s^* - 1$. In [22], Teng studied the existence of a nontrivial ground state solution through using the method of Pohozaev-Nehari manifold and the arguments of Brezis-Nirenberg, the monotonic trick and global compactness Lemma for $f(x,u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u$ with $q \in (1, 2_s^* - 1)$. For other related works, we refer the readers to see [13, 18, 20, 24, 25] and so on.

As far as we know, there are few results about the case of $2 < p \leq 4$ and $\lambda > 0$. In very recent, Sun and Wu, Feng [21] established the existence of ground state solutions and positive solutions to the non-autonomous fractional Schrödinger-poisson system (1.3) when $f(x,u) = a(x)|u|^{p-2}u$. Motivated by the above mentioned works, the purpose of this study is to prove the existence results of positive solutions for system (1.1) under $2 < p < 4$. Observed that the usual Nehari manifold is not ideal choices because the energy functional I constrained on its Nehari manifold is not bounded below when $2 < p < 4$. To kill this obstacle, based on recent study [21], through introducing a new set $N(c)$, which is sub-level set of the Nehari manifold

$$N(c) = \{u \in \mathcal{N} : I(u) < c\},$$

where N is the Nehari manifold and $c \in \mathbb{R}$, $N(c)$ is a subset of the Nehari manifold and it can be divided into two parts

$$N(c) = N^{(1)}(c) \cup N^{(2)}(c), \ N^{(1)}(c) = \{u \in N(c) : \|u\|_{H^s} < C_1\}$$

and

$$N^{(2)}(c) = \{u \in N(c) : \|u\|_{H^s} > C_2\}.$$

Moreover, local minimum of I on each set is a critical point of I in $H^s(\mathbb{R}^3)$. The advantage of this subset is that the functional I constrained on $N^{(1)}(c)$ is bounded below so that we can consider the corresponding minimum problem $\inf\limits_{u\in N^{(1)}(c)} I(u)$. Applying this approach, we can prove the existence of nontrivial solutions of system (1.1). Compare with the work of [21], we introduce a perturbation term $b(x)u^3$, this will make more careful analysis, except that our problem (1.1) is a class of nonlocal problem.

Now, we introduce some notations.
(a): S is the best constants for the embedding of $H^s(\mathbb{R}^3)$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3)$.
(b): $\overline{S}_t$ is the best constants for the embedding of $D^{t,2}(\mathbb{R}^3)$ in $L^{2_t^*}(\mathbb{R}^3)$.
(c): S_p is the best Sobolev constant for the embedding of $H^s(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$.

Remark 1.1. *For $2 < p < 4$, it is not difficult to show that*

$$A(p) < \frac{1}{\sqrt{e}} < 1 \quad \textit{and} \quad A(p)(\frac{2}{4-p})^{\frac{2}{p-2}} > 1.$$

Our main results are stated as follows.

Theorem 1.2. *Suppose that $2 < p < 4$, $K(x) = K_\infty > 0$ and $a(x) = a_\infty > 0$, $b(x) = b_\infty = 0$. Then for each $0 < \lambda < \Lambda$, system (1.1) has a positive solution $(w_\lambda, \phi^t_{w_\lambda})$, and it satisfies*

$$0 < \|w_\lambda\|_{H^s} < \Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{1}{p-2}}$$

and

$$\alpha_0^\infty < \alpha_\lambda^{\infty,-} := J_\lambda^\infty(w_\lambda) < \frac{A(p)(p-2)}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$

Theorem 1.3. *Suppose that $2 < p < 4$, $K_\infty > 0$ and conditions (H_1)-(H_3) hold. In addition, we assume that*
(H_4): $\int_{\mathbb{R}^3}[a(x) - a_\infty]w_\lambda^p dx \geq 0$ and $\int_{\mathbb{R}^3} K(x)\phi^t_{w_\lambda} w_\lambda^2 dx \leq \int_{\mathbb{R}^3} K_\infty \phi^t_{w_\lambda} w_\lambda^2 dx$ but the equality signs can not hold at the same time, where w_λ is the positive solution is given in Theorem 1.2.

Then for each $0 < \lambda < \Lambda$, system (1.1) has a nontrivial solution $(v_\lambda, \phi^t_{v_\lambda}) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ and it satisfies

$$0 < \|v_\lambda\|_{H^s} < \Big(\frac{2S_p^p}{a_{max}(4-p)}\Big)^{\frac{1}{p-2}}$$

and

$$\frac{p-2}{4p}C_0 \leq J_\lambda(v_\lambda) < \alpha_\lambda^{\infty,-}.$$

Theorem 1.4. *Suppose that $2 < p < 4$, $K_\infty \geq 0$ and conditions (H_1)-(H_3) hold. In addition, we assume that*
(H_5): $\int_{\mathbb{R}^3}[a(x)-a_\infty]w_0^p dx > 0$ where w_0 is the unique positive solution of equation

$$(-\Delta)^s u + u = a_\infty |u|^{p-2}u \quad in\ \mathbb{R}^3. \quad (E_0^\infty)$$

Then there exists $0 < \widehat{\Lambda} < \Lambda$ such that for each $0 < \lambda < \widehat{\Lambda}$, system (1.1) has a nontrivial solution $(u_0, \phi_{u_0}^t) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ and it satisfies

$$0 < \|u_0\|_{H^s} < \Big(\frac{2S_p^p}{a_{\max}(4-p)}\Big)^{\frac{1}{p-2}}$$

and

$$\frac{p-2}{4p}C_0 \leq J_\lambda(u_0) < \begin{cases} \alpha_\lambda^{\infty,-}, & if\ K_\infty > 0, \\ \alpha_0^\infty, & if\ K_\infty = 0. \end{cases}$$

The structure of this paper is organized as follows. In section 2, we introduce some technical Lemmas. In section 3, we prove Theorem1.2. Section 4 and Section 5 are devoted to proving Theorem1.3 and Theorem1.4.

2. Preliminaries

For the second equation of system (1.1), applying the Lax-Milgram theorem, for each $u \in H^s(\mathbb{R}^3)$, when $4s + 2t \geq 3$, there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that

$$\phi_u^t(x) = C_t \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|^{3-2t}}dy \quad \text{where} \quad C_t = \pi^{-\frac{3}{2}}2^{-2t}\frac{\Gamma(\frac{3-2t}{2})}{\Gamma(t)}, \qquad (2.1)$$

and then $(-\Delta)^t\phi = K(x)|u|^2$ in $\mathbb{R}^3$. Replacing it into the first equation of system (1.1), we get that

$$(-\Delta)^s u + u + \lambda K(x)\phi_u^t u = a(x)|u|^{p-2}u + b(x)|u|^2 u \quad \text{in } \mathbb{R}^3. \quad (E_\lambda)$$

Equation (E_λ) is variational, and its solutions are the critical points of the functional $J_\lambda(u)$ defined in $H^s(\mathbb{R}^3)$ as

$$J_\lambda(u) = \frac{1}{2}\|u\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{1}{4}\int_{\mathbb{R}^3} b(x)|u|^4 dx.$$

Obviously, $J_\lambda \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and

$$\begin{aligned}\langle J_\lambda'(u), \varphi\rangle = &\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi + u\varphi dx + \lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u\varphi dx \\ &- \int_{\mathbb{R}^3} a(x)|u|^{p-2}u\varphi dx - \int_{\mathbb{R}^3} b(x)|u|^2 u\varphi dx,\end{aligned}$$

for any $\varphi \in H^s(\mathbb{R}^3)$, where J_λ' denotes the Fréchet derivative of J_λ. Note that $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ is a solution of system (1.1) if and only if u is a critical point of J_λ and $\phi = \phi_u^t$.

Define the Nehari manifold for the functional J_λ as follows

$$M_\lambda := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : \langle J_\lambda'(u), u\rangle = 0\},$$

clearly, $u \in M_\lambda$ if and only if

$$\|u\|_{H^s}^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^p dx - \int_{\mathbb{R}^3} b(x)|u|^4 dx = 0.$$

Let $h_{\lambda,u} : t \to J_\lambda(tu)$ for $t > 0$, such map was introduced by Drábek-Pohozaev in [10], which is known as fibering map. For $u \in H^s(\mathbb{R}^3)$, we have

$$h_{\lambda,u}(t) = \frac{t^2}{2}\|u\|_{H^s}^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{t^p}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{t^4}{4}\int_{\mathbb{R}^3} b(x)|u|^4 dx,$$

$$h'_{\lambda,u}(t) = t\|u\|_{H^s}^2 + \lambda t^3 \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - t^{p-1}\int_{\mathbb{R}^3} a(x)|u|^p dx - t^3 \int_{\mathbb{R}^3} b(x)|u|^4 dx,$$

$$h''_{\lambda,u}(t) = \|u\|_{H^s}^2 + 3\lambda t^2 \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - (p-1)t^{p-2}\int_{\mathbb{R}^3} a(x)|u|^p dx - 3t^2 \int_{\mathbb{R}^3} b(x)|u|^4 dx.$$

Obviously, $h'_{\lambda,u}(t) = 0$ if and only if $tu \in M_\lambda$. Particularly, $h'_{\lambda,u}(1) = 0$ if and only if $u \in M_\lambda$. As usual, in order to find the local minimizer of J_λ, we split the set M_λ into three parts as follows

$$M_\lambda^+ = \{u \in M_\lambda : h''_{\lambda,u}(1) > 0\},$$
$$M_\lambda^0 = \{u \in M_\lambda : h''_{\lambda,u}(1) = 0\},$$
$$M_\lambda^- = \{u \in M_\lambda : h''_{\lambda,u}(1) < 0\}.$$

Lemma 2.1. *For $2 < p < 4$, $J_\lambda(u)$ is coercive and bounded below on M_λ^-. For any $u \in M_\lambda^-$, $J_\lambda(u) \geq \frac{p-2}{4p}C_0$.*

Proof. For all $u \in M_\lambda$, by the Sobolev inequality, we have that

$$\begin{aligned}\|u\|_{H^s}^2 \leq \|u\|_{H^s}^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx &= \int_{\mathbb{R}^3} a(x)|u|^p dx + \int_{\mathbb{R}^3} b(x)|u|^4 dx \\ &\leq a_{max} S_p^{-p}\|u\|_{H^s}^p + b_{max} S_4^{-4}\|u\|_{H^s}^4,\end{aligned}$$

thus

$$\begin{aligned}1 \leq a_{max} S_p^{-p}\|u\|_{H^s}^{p-2} + b_{max} S_4^{-4}\|u\|_{H^s}^2 &\leq \max\{a_{max}S_p^{-p}, b_{max}S_4^{-4}\}(\|u\|_{H^s}^{p-2} + \|u\|_{H^s}^2) \\ &\leq 2\max\{a_{max}S_p^{-p}, b_{max}S_4^{-4}\}\max\{\|u\|_{H^s}^{p-2}, \|u\|_{H^s}^2\},\end{aligned}$$

which implies that $\|u\|_{H^s}^2$ is bounded below, i.e.,

$$\|u\|_{H^s}^2 \geq C_0 > 0, \tag{2.2}$$

where C_0 is a positive constant only dependent of $a_{\max}$, $b_{\max}$, S_4 and S_p. Thus, for each $u \in M_\lambda^-$, by (2.2) and the definition of $h''_{\lambda,u}(1)$, we deduce that

$$J_\lambda(u) = \frac{1}{4}\|u\|_{H^s}^2 - \frac{4-p}{4p}\int_{\mathbb{R}^3} a(x)|u|^p dx \geq \frac{p-2}{4p}\|u\|_{H^s}^2 \geq \frac{p-2}{4p}C_0.$$

□

Lemma 2.2. *Assume that u_0 is a local minimizer for J_λ on M_λ and $u_0 \notin M_\lambda^0$, then $J'_\lambda(u_0) = 0$.*

Proof. Let

$$G(u) = \|u\|_{H^s}^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^p dx - \int_{\mathbb{R}^3} b(x)|u|^4 dx.$$

Using the Lagrange multiplier theorem, there exists $\mu \in \mathbb{R}$ such that

$$J'_\lambda(u) + \mu G'(u) = 0.$$

Next, we prove that $\mu = 0$. Otherwise, we have

$$\langle J'_\lambda(u), u\rangle + \mu\langle G'(u), u\rangle = 0,$$

that is

$$\begin{aligned}
&\|u\|^2_{H^s} + \lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^p dx - \int_{\mathbb{R}^3} b(x)|u|^4 dx \\
&+ \mu\Big[2\|u\|^2_{H^s} + 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - p\int_{\mathbb{R}^3} a(x)|u|^p dx - 4\int_{\mathbb{R}^3} b(x)|u|^4 dx\Big] = 0.
\end{aligned}$$

Since $u \in M_\lambda$, then $\langle J'_\lambda(u), u\rangle = 0$. So we deduce

$$2\|u\|^2_{H^s} + 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - p\int_{\mathbb{R}^3} a(x)|u|^p dx - 4\int_{\mathbb{R}^3} b(x)|u|^4 dx = 0,$$

that is

$$\|u\|^2_{H^s} + 3\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - (p-1)\int_{\mathbb{R}^3} a(x)|u|^p dx - 3\int_{\mathbb{R}^3} b(x)|u|^4 dx = 0,$$

which means that $u \in M^0_\lambda$, contradiction with $u \notin M^0_\lambda$. Thus $\mu = 0$ and then $J'_\lambda(u) = 0$.

□

Lemma 2.3. *For each $u \in H^s(\mathbb{R}^3)$, the following two inequalities are true.*
$(i)\phi_u^t \geq 0$;
$(ii)\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 \leq K^2_{max}\overline{S}_t^{-1} S^{-2}\|u\|^4_{H^s}$.

Proof. It follows from (2.1) that $\phi_u^t \geq 0$ holds. We will prove (ii) as follows.

By the definition of ϕ_u^t and Sobolev's inequality, we obtain

$$\begin{aligned}
\|\phi_u^t\|^2_{D^{t,2}} = \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx &\leq K_{max}\Big(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} dx\Big)^{\frac{1}{2_t^*}}\Big(\int_{\mathbb{R}^3} |u(x)|^{\frac{12}{3+2t}} dx\Big)^{\frac{3+2t}{6}} \\
&\leq K_{max}\overline{S}_t^{-\frac{1}{2}}\|\phi_u^t\|_{D^{t,2}}\Big(\int_{\mathbb{R}^3} |u(x)|^{\frac{12}{3+2t}} dx\Big)^{\frac{3+2t}{6}} \\
&\leq K_{max}\overline{S}_t^{-\frac{1}{2}} S^{-1}\|u\|^2_{H^s}\|\phi_u^t\|_{D^{t,2}},
\end{aligned}$$

which yields the following

$$\|\phi_u^t\|_{D^{t,2}} \leq K_{max}\overline{S}_t^{-\frac{1}{2}} S^{-1}\|u\|^2_{H^s}.$$

Thus, by Höder's inequality, we have that

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 \leq K_{max}\|\phi_u^t\|_{L^{2_t^*}}\|u\|^2_{L^{\frac{12}{3+2t}}} &\leq K_{max}\overline{S}_t^{-\frac{1}{2}} S^{-1}\|u\|^2_{H^s}\|\phi_u^t\|_{D^{t,2}} \\
&\leq K^2_{max}\overline{S}_t^{-1} S^{-2}\|u\|^4_{H^s}.
\end{aligned}$$

Consequently, the conclusion (ii) follows. □

For $2 < p < 4$ and any $u \in M_\lambda$ with $J_\lambda(u) < A(p)(\frac{p-2}{2p})(\frac{2S_p^p}{a_\infty(4-p)})^{\frac{2}{p-2}}$, we deduce that

$$\begin{aligned}
&A(p)(\frac{p-2}{2p})(\frac{2S_p^p}{a_\infty(4-p)})^{\frac{2}{p-2}} > J_\lambda(u) \\
=&\frac{1}{2}\|u\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{1}{4}\int_{\mathbb{R}^3} b(x)|u|^4 dx \\
=&\frac{p-2}{2p}\|u\|_{H^s}^2 - \frac{p-4}{4p}\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx + \frac{4-p}{4p}\int_{\mathbb{R}^3} b(x)|u|^4 dx \qquad (2.3)\\
\geq&\frac{p-2}{2p}\|u\|_{H^s}^2 - \frac{p-4}{4p}\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx \\
\geq&\frac{p-2}{2p}\|u\|_{H^s}^2 - \frac{4-p}{4p}\lambda K_{max}^2 \overline{S}_t^{-1} S^{-2}\|u\|_{H^s}^4.
\end{aligned}$$

Let

$$\Lambda = \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}}$$

and

$$\Lambda_0 = \Big[1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}}\Big](\frac{a_\infty}{S_p^p})^{\frac{2}{p-2}}\frac{\overline{S}_t S^2}{K_{max}^2}, \qquad (2.4)$$

where $K_{max} = \sup_{x\in\mathbb{R}^3} K(x)$.

Moreover, consider the following quadratic equation

$$\frac{1}{4}\Big(1 - A(p)\Big(\frac{a_{max}}{a_\infty}\Big)^{\frac{2}{p-2}}\Big)\Big(\frac{a_\infty(4-p)}{pS_p^p}\Big)^{\frac{2}{p-2}} x^2 - x + A(p)\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} = 0, \quad (2.5)$$

where $x = \|u\|_{H^s}^2$. It is not difficult to get its solutions,

$$x_0 = \frac{2\Big(1 + \sqrt{1 - A(p)\Big(1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}}\Big)(\frac{2}{p})^{\frac{2}{p-2}}}\Big)}{(1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}})(\frac{2}{p})^{\frac{2}{p-2}}}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}, \qquad (2.6)$$

$$x_1 = \frac{2\Big(1 - \sqrt{1 - A(p)\Big(1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}}\Big)(\frac{2}{p})^{\frac{2}{p-2}}}\Big)}{(1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}})(\frac{2}{p})^{\frac{2}{p-2}}}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}. \qquad (2.7)$$

We infer that

$$\begin{aligned}
x_0 >&2\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}, \\
\Big(\frac{2S_p^p}{a_{max}(4-p)}\Big)^{\frac{2}{p-2}} > x_1 >&A(p)\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.
\end{aligned} \qquad (2.8)$$

In fact, we denote $G = (1 - A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}})(\frac{2}{p})^{\frac{2}{p-2}}$. By (H_1), we can easily see that $0 < G < 1$. By Remark1.1, we have

$$1 - A(p)G > (1 - A(p)G)^2.$$

By (H_1), we have $A(p) < \Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{2}{p-2}}$. Directly calculations, we have

$$1 - \sqrt{1 - A(p)G} < \frac{G}{2}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{2}{p-2}},$$

thus, we have

$$\begin{aligned}x_1 &= \frac{2\Big(1-\sqrt{1-A(p)G}\Big)}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}\\ &< \frac{G}{2}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{2}{p-2}}\frac{2}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} = \Big(\frac{2S_p^p}{a_{max}(4-p)}\Big)^{\frac{2}{p-2}}.\end{aligned}$$

On the other hand, by calculations, we have

$$2(1-\sqrt{1-A(p)G}) > A(p)G,$$

and then

$$\begin{aligned}x_1 &= \frac{2\Big(1-\sqrt{1-A(p)G}\Big)}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}\\ &> \frac{A(p)G}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} = A(p)\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.\end{aligned}$$

By using the fact of $1-A(p)G > (1-A(p)G)^2$ and $0<G<1$, we have

$$\begin{aligned}x_0 &= \frac{2\Big(1+\sqrt{1-A(p)G}\Big)}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} > \frac{2\Big(1+1-A(p)G\Big)}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}\\ &= \frac{2\Big(2-A(p)G\Big)}{G}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} \geq 2\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}},\end{aligned}$$

which leads to $x_0 > 2\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}$. Thus, (2.8) is proved.

For $0<\lambda<\frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, according to (2.3) and (2.6), (2.7), (2.8), then there exist two positive numbers $\widehat{D}_1$ and $\widehat{D}_2$ satisfying

$$\sqrt{A(p)}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{1}{p-2}} < \widehat{D}_1 < \Big(\frac{2S_p^p}{a_{max}(4-p)}\Big)^{\frac{1}{p-2}} < \sqrt{2}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{1}{p-2}} < \widehat{D}_2,$$

such that

$$\|u\|_{H^s} < \widehat{D}_1 \quad \text{or} \quad \|u\|_{H^s} > \widehat{D}_2.$$

Clearly, we can get that $\widehat{D}_1 \to \infty$ as $p \to 4^-$.
Therefore, we define

$$\widetilde{M_\lambda} = \Big\{u\in M_\lambda : J_\lambda(u) < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}\Big\} = M_\lambda^{(1)}\cup M_\lambda^{(2)}, \quad (2.9)$$

where

$$M_\lambda^{(1)} := \Big\{u\in\widetilde{M_\lambda} : \|u\|_{H^s} < \widehat{D}_1\Big\}, \quad M_\lambda^{(2)} := \Big\{u\in\widetilde{M_\lambda} : \|u\|_{H^s} > \widehat{D}_2\Big\}.$$

Furthermore, it is easy to see that

$$\|u\|_{H^s} < \widehat{D}_1 < \Big(\frac{2S_p^p}{a_{max}(4-p)}\Big)^{\frac{1}{p-2}}, \quad \text{for any } u\in M_\lambda^{(1)}, \quad (2.10)$$

$$\|u\|_{H^s} > \widehat{D}_2 > \sqrt{2}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{1}{p-2}}, \quad \text{for any } u\in M_\lambda^{(2)}. \quad (2.11)$$

Lemma 2.4. *For $0 < \lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, we have $M_\lambda^{(1)} \subset M_\lambda^-$, $M_\lambda^{(2)} \subset M_\lambda^+$ are C^1 sub-manifolds. Furthermore, each local minimizer of the functional J_λ in the sub-manifolds $M_\lambda^{(1)}$ and $M_\lambda^{(2)}$ is a critical point of J_λ in $H^s(\mathbb{R}^3)$.*

Proof. For $u \in M_\lambda^{(1)}$, by (2.10) and Sobolev's inequality, we have that

$$h''_{\lambda,u}(1) = -2\|u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx \le -2\|u\|_{H^s}^2 + (4-p)S_p^{-p}a_{max}\|u\|_{H^s}^p < 0,$$

which means that $M_\lambda^{(1)} \subset M_\lambda^-$.

For $u \in M_\lambda^{(2)}$, by (2.11), we deduce that

$$\begin{aligned}\frac{1}{4}\|u\|_{H^s}^2 - \frac{(4-p)}{4p}\int_{\mathbb{R}^3} a(x)|u|^p dx = J_\lambda(u) < A(p)\frac{p-2}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}\\ < \frac{p-2}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} < \frac{p-2}{4p}\|u\|_{H^s}^2,\end{aligned}$$

which implies

$$2\|u\|_{H^s}^2 < (4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx, \quad \text{for any } u \in M_\lambda^{(2)}. \tag{2.12}$$

From (2.12), it follows that

$$h''_{\lambda,u}(1) = -2\|u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx > 0.$$

Thus, $M_\lambda^{(2)} \subset M_\lambda^+$. According to Lemma 2.2 , we know that each local minimizer of the functional J_λ in the sub-manifolds $M_\lambda^{(1)}$ and $M_\lambda^{(2)}$ is a critical point of J_λ in $H^s(\mathbb{R}^3)$. The proof is completed. □

For $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, we define

$$T_a(u) = \Big(\frac{\|u\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|u|^p dx}\Big)^{\frac{1}{p-2}}.$$

Lemma 2.5. *For each $\lambda > 0$ and $u \in H^s(\mathbb{R}^3) \setminus \{0\}$ satisfying*

$$(H_6): \quad \int_{\mathbb{R}^3} a(x)|u|^p dx > \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_{max}^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}}\|u\|_{H^s}^p$$

and

$$\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx > \int_{\mathbb{R}^3} b(x)|u|^4 dx,$$

there exists a constant $\widehat{t}_\lambda^{(0)} > (\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)$ such that

$$\inf_{t\ge 0} J_\lambda(tu) = \inf_{(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)<t<\widehat{t}_\lambda^{(0)}} J_\lambda(tu) < 0. \tag{2.13}$$

Proof. For any $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, and $t > 0$, one has

$$\begin{aligned}J_\lambda(tu) =& \frac{t^2}{2}\|u\|_{H^s}^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{t^p}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{t^4}{4}\int_{\mathbb{R}^3} b(x)|u|^4 dx\\ =& t^4\Big[g(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} b(x)|u|^4 dx\Big] = h_{\lambda,u}(t),\end{aligned}$$

where
$$g(t)=\frac{t^{-2}}{2}\|u\|^2_{H^s}-\frac{t^{p-4}}{p}\int_{\mathbb{R}^3}a(x)|u|^pdx.$$
It is obvious that $J_\lambda(tu)=0$ if and only if
$$g(t)+\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^tu^2dx-\frac{1}{4}\int_{\mathbb{R}^3}b(x)|u|^4dx=0.$$
A direct calculation shows that
$$g(\widehat{t_a})=0,\quad \lim_{t\to0^+}g(t)=\infty \text{ and } \lim_{t\to\infty}g(t)=0,$$
where $\widehat{t_a}=(\frac{p}{2})^{\frac{1}{p-2}}T_a(u)$. Hence,
$$g'(t)=-t^{-3}\|u\|^2_{H^s}-\frac{p-4}{p}t^{p-5}\int_{\mathbb{R}^3}a(x)|u|^pdx=t^{-3}\Big[-\|u\|^2_{H^s}+\frac{4-p}{p}t^{p-2}\int_{\mathbb{R}^3}a(x)|u|^pdx\Big],$$
which implies that $g(t)$ is decreasing when $0<t<(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)$ and is increasing when $t>(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)$. Thus, we have that
$$\inf_{t>0}g(t)=g\Big[(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)\Big]=-\frac{p-2}{2(4-p)}\Big(\frac{p\|u\|^2_{H^s}}{(4-p)\int_{\mathbb{R}^3}a(x)|u|^pdx}\Big)^{-\frac{2}{p-2}}\|u\|^2_{H^s}.$$
By Lemma 2.3 and Sobolev's inequality, for each $u\in H^s(\mathbb{R}^3)\setminus\{0\}$ satisfying
$$\int_{\mathbb{R}^3}a(x)|u|^pdx>\frac{p}{4-p}\Big(\frac{2\lambda(4-p)K^2_{max}}{(p-2)\overline{S}_tS^2}\Big)^{\frac{p-2}{2}}\|u\|^p_{H^s},$$
we obtain
$$\begin{aligned}\inf_{t>0}g(t)=&-\frac{p-2}{2(4-p)}\Big(\frac{p\|u\|^2_{H^s}}{(4-p)\int_{\mathbb{R}^3}a(x)|u|^pdx}\Big)^{-\frac{2}{p-2}}\|u\|^2_{H^s}\\ <&-\lambda K^2_{max}\overline{S}_t^{-1}S^{-2}\|u\|^4_{H^s}\le-\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^tu^2dx\\ <&-\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^tu^2dx+\frac{1}{4}\int_{\mathbb{R}^3}b(x)|u|^4dx.\end{aligned}$$
From the hypothesis, there exist $\widehat{t}_\lambda^{(0)}$ and $\widehat{t}_\lambda^{(1)}$ satisfying
$$0<\widehat{t}_\lambda^{(1)}<(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)<\widehat{t}_\lambda^{(0)},\tag{2.14}$$
such that
$$g(\widehat{t}_\lambda^{(j)})+\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^tu^2dx-\frac{1}{4}\int_{\mathbb{R}^3}b(x)|u|^4dx=0 \text{ for } j=0,1,$$
that is,
$$J_\lambda(\widehat{t}_\lambda^{(j)}u)=0 \text{ for } j=0,1.$$
Therefore, for each $\lambda>0$ and $u\in H^s(\mathbb{R}^3)\setminus\{0\}$ satisfying
$$\int_{\mathbb{R}^3}a(x)|u|^pdx>\frac{p}{4-p}\Big(\frac{2\lambda(4-p)K^2_{max}}{(p-2)\overline{S}_tS^2}\Big)^{\frac{p-2}{2}}\|u\|^p_{H_s},$$

we have

$$\begin{aligned}&J_\lambda\Big[(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)u\Big]\\=&\Big[(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)\Big]^4\Big[g\Big((\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)\Big)+\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^t u^2dx-\frac{1}{4}\int_{\mathbb{R}^3}b(x)|u|^4dx\Big]<0.\end{aligned}$$

Thus

$$\inf_{t\geq 0}J_\lambda(tu)<0.$$

Note that

$$h'_{\lambda,u}(t)=4t^3\Big[g(t)+\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_u^t u^2dx-\frac{1}{4}\int_{\mathbb{R}^3}b(x)|u|^4dx\Big]+t^4g'(t),$$

which means that

$$h'_{\lambda,u}(t)<0 \text{ for all } t\in\Big(\widehat{t}_\lambda^{(1)},(\frac{p}{4-p})^{\frac{1}{p-2}}T_a(u)\Big],$$

and

$$h'_{\lambda,u}(t)(\widehat{t}_\lambda^{(0)})>0.$$

Thus, (2.13) is proved. □

Lemma 2.6. *For $\lambda>0$ and $u\in H^s(\mathbb{R}^3)\backslash\{0\}$ satisfying (H_6), then the following two statements are true.*
(i) If $\lambda\int_{\mathbb{R}^3}K(x)\phi_u^t u^2dx>\int_{\mathbb{R}^3}b(x)|u|^4dx$, then there exist two constants t_λ^+ and t_λ^- which satisfy

$$T_a(u)<t_\lambda^-<\sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}}<\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}}T_a(u)<t_\lambda^+,$$

such that

$$t_\lambda^\pm u\in M_\lambda^\pm,\quad J_\lambda(t_\lambda^-u)=\sup_{0\leq t\leq t_\lambda^+}J_\lambda(tu)$$

and

$$J_\lambda(t_\lambda^+u)=\inf_{t\geq t_\lambda^-}J_\lambda(tu)=\inf_{t\geq 0}J_\lambda(tu)<0.$$

(ii) If $\lambda\int_{\mathbb{R}^3}K(x)\phi_u^t u^2dx\leq\int_{\mathbb{R}^3}b(x)|u|^4dx$, then there exist a constant t_λ^0 which satisfies

$$0<t_\lambda^0<T_a(u),$$

such that

$$t_\lambda^0u\in M_\lambda^-,\quad J_\lambda(t_\lambda^0u)=\sup_{0\leq t\leq T_a(u)}J_\lambda(tu).$$

Proof. Define

$$f(t)=t^{-2}\|u\|_{H^s}^2-t^{p-4}\int_{\mathbb{R}^3}a(x)|u|^pdx \text{ for } t>0.$$

It is easy to see that $tu\in M_\lambda$ if and only if

$$f(t)+\lambda\int_{\mathbb{R}^3}K(x)\phi_u^t u^2dx-\int_{\mathbb{R}^3}b(x)|u|^4dx=0.$$

Directly computation, we have that

$$\begin{aligned} f(T_a(u)) &= \Big(\frac{\|u\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|u|^p dx}\Big)^{\frac{-2}{p-2}} \|u\|_{H^s}^2 - \Big(\frac{\|u\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|u|^p dx}\Big)^{\frac{p-4}{p-2}} \int_{\mathbb{R}^3} a(x)|u|^p dx \\ &= \frac{(\|u\|_{H^s}^2)^{\frac{p-4}{p-2}}}{(\int_{\mathbb{R}^3} a(x)|u|^p dx)^{\frac{-2}{p-2}}} - \frac{(\|u\|_{H^s}^2)^{\frac{p-4}{p-2}}}{(\int_{\mathbb{R}^3} a(x)|u|^p dx)^{\frac{-2}{p-2}}} \\ &= 0 \end{aligned}$$

and

$$\lim_{t\to 0^+} f(t) = \infty, \quad \lim_{t\to\infty} f(t) = 0.$$

From $2 < p < 4$ and

$$f'(t) = t^{-3}\Big(- 2\|u\|_{H^s}^2 + (4-p)t^{p-2} \int_{\mathbb{R}^3} a(x)|u|^p dx\Big),$$

we know that $f(t)$ is decreasing if $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(u)$ and is increasing if $t > (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(u)$. Thus

$$\inf_{t>0} f(t) = f\Big[(\frac{2}{4-p})^{\frac{1}{p-2}} T_a(u)\Big]. \tag{2.15}$$

For each $\lambda > 0$ and $u \in H^s(\mathbb{R}^3) \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^3} a(x)|u|^p dx > \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_{max}^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}} \|u\|_{H_s}^p.$$

By Lemma 2.3, Sobolev's inequality, and $(\frac{p}{2})^{\frac{2}{p-2}} > 1$, we have

$$\begin{aligned} f\Big((\frac{2}{4-p})^{\frac{1}{p-2}} T_a(u)\Big) &= -(\frac{p-2}{4-p})\Big(\frac{2\|u\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx}\Big)^{-\frac{2}{p-2}} \|u\|_{H^s}^2 \\ &< -2\Big(\frac{2}{p}\Big)^{-\frac{2}{p-2}} \lambda K_{max}^2 \overline{S}_t^{-1} S^{-2} \|u\|_{H^s}^4 \\ &< -\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx < -\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx + \int_{\mathbb{R}^3} b(x)|u|^4 dx. \end{aligned}$$

And, for $2 < p < 4$, by Remark 1.1 we have

$$T_a(u) < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(u) < \Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(u), \tag{2.16}$$

and it is not difficult to show that

$$\frac{(\frac{2}{4-p})A(p)^{\frac{p-2}{2}} - 1}{A(p)(\frac{2}{4-p})^{\frac{2}{p-2}}} > \frac{p-2}{2(4-p)}\Big(\frac{4-p}{p}\Big)^{\frac{2}{p-2}}. \tag{2.17}$$

By using (2.15)-(2.17), we deduce that

$$\begin{aligned} f\Big(\sqrt{A(p)}(\frac{2}{4-p})^{\frac{1}{p-2}} T_a(u)\Big) &= -\frac{\Big[(\frac{2}{4-p})A(p)^{\frac{p-2}{2}} - 1\Big]}{A(p)(\frac{2}{4-p})^{\frac{2}{p-2}}}\Big(\frac{\|u\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|u|^p dx}\Big)^{-\frac{2}{p-2}} \|u\|_{H^s}^2 \\ &< -\lambda K_{max}^2 \overline{S}_t^{-1} S^{-2} \|u\|_{H^s}^4 \le -\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx \\ &< -\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx + \int_{\mathbb{R}^3} b(x)|u|^4 dx. \end{aligned}$$

Now, we need to distinguish two cases.

(i) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx > \int_{\mathbb{R}^3} b(x)|u|^4 dx$ holds, there exist two constants t_λ^+ and $t_\lambda^- > 0$ which satisfy

$$T_a(u) < t_\lambda^- < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(u) < \Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(u) < t_\lambda^+, \tag{2.18}$$

such that

$$f(t_\lambda^\pm) + \lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \int_{\mathbb{R}^3} b(x)|u|^4 dx = 0,$$

which implies that $t_\lambda^\pm u \in M_\lambda$. Moreover, we have

$$h''_{\lambda, t_\lambda^- u}(1) = -2\|t_\lambda^- u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|t_\lambda^- u|^p dx = (t_\lambda^-)^5 f'(t_\lambda^-) < 0$$

and

$$h''_{\lambda, t_\lambda^+ u}(1) = -2\|t_\lambda^+ u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|t_\lambda^+ u|^p = (t_\lambda^+)^5 f'(t_\lambda^+) > 0.$$

These yield that $t_\lambda^\pm u \in M_\lambda^\pm$. It is not difficult to verify that $h'_{\lambda,u}(t) > 0$ if $t \in (0, t_\lambda^-)\bigcup(t_\lambda^+, \infty)$ and $h'_{\lambda,u}(t) < 0$ if $t \in (t_\lambda^-, t_\lambda^+)$. Hence, we obtain

$$J_\lambda(t_\lambda^- u) = \sup_{0 \le t \le t_\lambda^+} J_\lambda(tu), \quad J_\lambda(t_\lambda^+ u) = \inf_{t \ge t_\lambda^-} J_\lambda(tu)$$

and $J_\lambda(t_\lambda^+ u) < J_\lambda(t_\lambda^- u)$. By Lemma 2.5, we get

$$J_\lambda(t_\lambda^+ u) = \inf_{t \ge 0} J_\lambda(tu) < 0.$$

(ii) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx \le \int_{\mathbb{R}^3} b(x)|u|^4 dx$ holds, similar to the proof of *(i)*, there exists a constant t_λ^0 which satisfies

$$0 < t_\lambda^0 < T_a(u) \tag{2.19}$$

such that

$$f(t_\lambda^0) + \lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \int_{\mathbb{R}^3} b(x)|u|^4 dx = 0,$$

which implies that $t_\lambda^0 u \in M_\lambda$. Moreover,

$$h''_{\lambda, t_\lambda^0 u}(1) = -2\|t_\lambda^0 u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|t_\lambda^0 u|^p dx = (t_\lambda^0)^5 f'(t_\lambda^-) < 0.$$

This means that $t_\lambda^0 u \in M_\lambda^-$ and $h'_{\lambda,u}(t) > 0$ if $t \in (0, t_\lambda^0)$, $h'_{\lambda,u}(t) < 0$ if $t \in (t_\lambda^0, +\infty)$. Hence,

$$J_\lambda(t_\lambda^0 u) = \sup_{0 \le t \le T_a(u)} J_\lambda(tu).$$

This proof is completed. $\square$

3. Proof of Theorem 1.2

In this section, suppose that $K(x) \equiv K_\infty > 0$ and $a(x) \equiv a_\infty > 0$, $b(x) \equiv b_\infty = 0$.

Now, we consider the following limit problem associated to problem (E_λ):

$$(-\Delta)^s u + u + \lambda K_\infty \phi_u^t u = a_\infty |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \quad (E_\lambda^\infty)$$

The energy functional $J_\lambda^\infty : H^s(\mathbb{R}^3) \to \mathbb{R}$ corresponding to (E_λ^∞) is defined by

$$J_\lambda^\infty(u) = \frac{1}{2}\|u\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_u^t u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |u|^p dx,$$

clearly, $J_\lambda^\infty \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions of (E_λ^∞).

Define

$$M_\lambda^\infty := \{u \in H^s(\mathbb{R}^3) \setminus \{0\} : \langle (J_\lambda^\infty)'(u), u\rangle = 0\},$$

where $(J_\lambda^\infty)'$ is the Fréchet derivative of J_λ^∞. Then, $u \in M_\lambda^\infty$ if and only if

$$\|u\|_{H^s}^2 + \lambda \int_{\mathbb{R}^3} K_\infty \phi_u^t u^2 dx - \int_{\mathbb{R}^3} a_\infty |u|^p dx = 0.$$

In particularly, when $\lambda = 0$, equation E_λ^∞ reduces to following fractional Schrödinger equation

$$(-\Delta)^s u + u = a_\infty |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \quad (E_0^\infty)$$

Let w_0 be the unique positive solution of E_0^∞ (see [12]), we know that

$$w_0(0) = \max_{x\in\mathbb{R}^3} w_0(x),$$

$$\|w_0\|_{H^s}^2 = \int_{\mathbb{R}^3} a_\infty |w_0|^p dx = \Big(\frac{S_p^p}{a_\infty}\Big)^{\frac{2}{p-2}} \tag{3.1}$$

and

$$\alpha_0^\infty := \inf_{u\in M_0^\infty} J_0^\infty(w_0) = \frac{p-2}{2p}\Big(\frac{S_p^p}{a_\infty}\Big)^{\frac{2}{p-2}},$$

where J_0^∞ is the energy functional of equation (E_0^∞) in $H^s(\mathbb{R}^3)$ as follows

$$J_0^\infty(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |u|^p dx, \tag{3.2}$$

and

$$M_0^\infty = \{u \in H^s(\mathbb{R}^3)\backslash\{0\} | \langle (J_0^\infty)'(u), u\rangle = 0\}.$$

Since $K(x) = K_\infty$ and $a(x) = a_\infty$, we have

$$\Lambda_0 = [1 - A(p)](\frac{a_\infty}{S_p^p})^{\frac{2}{p-2}} \frac{\overline{S}_t S^2}{K_\infty^2}.$$

For $0 < \lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, by (3.1), w_0 is the unique positive solution of equation (E_0^∞), we have

$$\int_{\mathbb{R}^3} a_\infty |w_0|^p dx = a_\infty S_p^{-p}\|w_0\|_{H^s}^p > \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}}\|w_0\|_{H^s}^p.$$

By (H_3), similar to the proof of conclusion (i) of Lemma 2.6, there exist two constants $t_\lambda^{\infty,+}$ and $t_\lambda^{\infty,-}$ satisfying

$$1 < t_\lambda^{\infty,-} < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} < t_\lambda^{\infty,+},$$

such that $t_\lambda^{\infty,\pm} w_0 \in M_\lambda^{\infty,\pm}$. Moreover, we have that

$$
\begin{aligned}
J_\lambda^\infty(t_\lambda^{\infty,-} w_0) &= \sup_{0\le t\le t_\lambda^{\infty,+}} J_\lambda^\infty(tw_0),\\
J_\lambda^\infty(t_\lambda^{\infty,+} w_0) &= \inf_{t\ge t_\lambda^{\infty,-}} J_\lambda^\infty(tw_0) = \inf_{t\ge 0} J_\lambda^\infty(tw_0) < 0.
\end{aligned}
$$

Lemma 3.1. *$J_\lambda^\infty(u)$ is coercive and bounded below on $M_\lambda^{\infty,-}$. Furthermore, for all $u \in M_\lambda^{\infty,-}$, there holds $J_\lambda^\infty(u) \ge \frac{p-2}{4p}(\frac{S_p^p}{a_{max}})^{\frac{2}{p-2}}$.*

Proof. Let $u \in M_\lambda^\infty$, by Sobolev's inequality, we have that

$$
\|u\|_{H^s}^2 \le \|u\|_{H^s}^2 + \lambda \int_{\mathbb{R}^3} K_\infty \phi_u^t u^2 dx = \int_{\mathbb{R}^3} a_\infty |u|^p dx \le S_p^{-p} a_{max} \|u\|_{H^s}^p,
$$

which leads to

$$
\int_{\mathbb{R}^3} a_\infty |u|^p dx \ge \|u\|_{H^s}^2 \ge \Big(\frac{S_p^p}{a_{max}}\Big)^{\frac{2}{p-2}}. \tag{3.3}
$$

For $u \in M_\lambda^{\infty,-}$, there holds

$$
\begin{aligned}
(h_{\lambda,u}^\infty)''(1) =& \|u\|_{H^s}^2 + 3\lambda \int_{\mathbb{R}^3} K_\infty \phi_u^t u^2 dx - (p-1)\int_{\mathbb{R}^3} a_\infty |u|^p dx\\
=& -2\|u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a_\infty |u|^p dx < 0,
\end{aligned}
$$

thus, we have

$$
J_\lambda^\infty(u) = \frac{1}{4}\|u\|_{H^s}^2 - \frac{4-p}{4p}\int_{\mathbb{R}^3} a_\infty |u|^p dx \ge \frac{p-2}{4p}\|u\|_{H^s}^2 \ge \frac{p-2}{4p}\Big(\frac{S_p^p}{a_{max}}\Big)^{\frac{2}{p-2}}. \tag{3.4}
$$

This completes the proof. □

By a simple computation, we deduce that

$$
\begin{aligned}
J_\lambda^\infty(t_\lambda^{\infty,-} w_0) =& \frac{1}{2}\|t_\lambda^{\infty,-} w_0\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi^t_{t_\lambda^{\infty,-} w_0} (t_\lambda^{\infty,-} w_0)^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |t_\lambda^{\infty,-} w_0|^p dx\\
=& \frac{1}{4}(t_\lambda^{\infty,-})^2 \|w_0\|_{H^s}^2 - \frac{4-p}{4p}(t_\lambda^{\infty,-})^p \int_{\mathbb{R}^3} a_\infty |w_0|^p dx\\
=& \frac{1}{4}(t_\lambda^{\infty,-})^2 \|w_0\|_{H^s}^2 - \frac{4-p}{4p}(t_\lambda^{\infty,-})^p \|w_0\|_{H^s}^2 = \frac{(t_\lambda^{\infty,-})^2}{4}\Big[1 - \frac{4-p}{4p}(t_\lambda^{\infty,-})^{p-2}\Big]\|w_0\|_{H^s}^2\\
<& A(p)\Big(\frac{2}{4-p}\Big)^{\frac{2}{p-2}} \frac{p-2}{2p}\|w_0\|_{H^s}^2 = A(p)\frac{p-2}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}},
\end{aligned} \tag{3.5}
$$

and

$$
\|t_\lambda^{\infty,-} w_0\|_{H^s} < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}}\Big(\frac{S_p^p}{a_\infty}\Big)^{\frac{1}{p-2}} < \widehat{D}_1.
$$

Thus, $t_\lambda^{\infty,-} w_0 \in M_\lambda^{\infty,(1)}$, which implies that $M_\lambda^{\infty,(1)}$ is nonempty.

Similar to Lemma 2.4, we know that $M_\lambda^{\infty,(1)} \subset M_\lambda^{\infty,-}$, $M_\lambda^{\infty,(2)} \subset M_\lambda^{\infty,+}$. On the other hand, $M_\lambda^{\infty,(1)}$ or $M_\lambda^{\infty,(2)}$ is a sublevel set, and thus the infimums of

$J_\lambda^\infty(u)$ constrained on these two sets are equal. Hence, we can define the following minimum problem

$$\alpha_\lambda^{\infty,-} = \inf_{u\in M_\lambda^{\infty,(1)}} J_\lambda^\infty(u) = \inf_{u\in M_\lambda^{\infty,-}} J_\lambda^\infty(u),$$
$$\alpha_\lambda^{\infty,+} = \inf_{u\in M_\lambda^{\infty,(2)}} J_\lambda^\infty(u) = \inf_{u\in M_\lambda^{\infty,+}} J_\lambda^\infty(u).$$

By Lemma 3.1, (3.5) and Lemma 6.1 in Appendix A, we have that

$$\frac{p-2}{4p}\Big(\frac{S_p^p}{a_{max}}\Big)^{\frac{2}{p-2}} \le \alpha_\lambda^{\infty,-} < A(p)\frac{p-2}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}} \tag{3.6}$$

and

$$\alpha_\lambda^{\infty,+} = -\infty. \tag{3.7}$$

Proof of Theorem 1.2.

Let $\{u_n\} \in M_\lambda^{\infty,(1)}$ satisfy

$$J_\lambda^\infty(u_n) = \alpha_\lambda^{\infty,-} + o(1) \quad \text{and} \quad (J_\lambda^\infty)'(u_n) = o(1) \text{ in } H^{-s}(\mathbb{R}^3). \tag{3.8}$$

By virtue of Lemma 6.2 in Appendix B, we get

$$0 < \lambda < \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}},$$

compactness holds for the sequence $\{u_n\}$. Then for each $\theta > 0$ there exist a positive constant $R = R(\theta)$ and a sequence $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{[B(z_n;R)]^c} (|(-\Delta)^{\frac{s}{2}} u_n(x)|^2 + u_n^2(x))dx < \theta \text{ uniformly for } n \ge 1. \tag{3.9}$$

Let

$$v_n := u_n(\cdot + z_n) \in H^s(\mathbb{R}^3),$$

then $\{v_n\} \subset M_\lambda^{\infty,(1)}$, and

$$\phi_{v_n}^t = \phi_{u_n(\cdot+z_n)}^t \quad \text{and} \quad J_\lambda^\infty(v_n) = \alpha_\lambda^{\infty,-} + o(1).$$

By (3.9), for each $\theta > 0$, there exists $R = R(\theta) > 0$ such that

$$\int_{[B(0;R)]^c} (|(-\Delta)^{\frac{s}{2}} v_n(x)|^2 + v_n^2(x))dx < \theta \text{ uniformly for } n \ge 1. \tag{3.10}$$

Since $\{v_n\}$ is bounded in $H^s(\mathbb{R}^3)$, up to a subsequence, we can assume that there exists $w_\lambda \in H^s(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup w_\lambda \text{ weakly in } H^s(\mathbb{R}^3), \tag{3.11}$$

$$v_n \to w_\lambda \text{ strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 \le r < 2_s^*, \tag{3.12}$$

$$v_n \to w_\lambda \; a.e. \text{ in } \mathbb{R}^3.$$

By (3.10)-(3.12) and Fatou's Lemma, for any $\theta > 0$ and sufficiently large $n(\geq 1)$, there exists a constant $R > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |v_n - w_\lambda|^p dx \\
\leq &\int_{[B(0;R)]} |v_n - w_\lambda|^p dx + \int_{[B(0;R)]^c} |v_n - w_\lambda|^p dx \\
\leq &\theta + \Big(\int_{[B(0;R)]^c} v_n^2 dx \Big)^{\frac{\tau p}{2}} \Big(\int_{[B(0;R)]^c} v_n^{2_s^*} dx \Big)^{\frac{(1-\tau)p}{2_s^*}} + \Big(\int_{[B(0;R)]^c} w_\lambda^2 dx \Big)^{\frac{\tau p}{2}} \Big(\int_{[B(0;R)]^c} w_\lambda^{2_s^*} dx \Big)^{\frac{(1-\tau)p}{2_s^*}} \\
\leq &\theta + 2\widetilde{C}\theta^{\frac{\tau p}{2}},
\end{aligned}$$

which means that for every $r \in (2, 2_s^*)$, there holds

$$v_n \to w_\lambda \text{ strongly in } L^r(\mathbb{R}^3). \tag{3.13}$$

Since $\phi : L^{\frac{12}{3+2t}} \to D^{t,2}(\mathbb{R}^3)$ is a continuous function, we get that

$$\phi_{v_n}^t \to \phi_{w_\lambda}^t \text{ in } D^{t,2}(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} \phi_{v_n}^t v_n^2 dx \to \int_{\mathbb{R}^3} \phi_{w_\lambda}^t w_\lambda^2 dx. \tag{3.14}$$

Since $v_n \in M_\lambda^{\infty,(1)}$, by (3.3) and (3.13), we obtain

$$\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx \geq \Big(\frac{S_p^p}{a_{max}} \Big)^{\frac{2}{p-2}} > 0.$$

This implies that $w_\lambda \neq 0$ and

$$\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx \geq \|w_\lambda\|_{H^s}^2 > 0. \tag{3.15}$$

Next, we prove that

$$v_n \to w_\lambda \text{ strongly in } H^s(\mathbb{R}^3),$$

suppose by the contrary that

$$\|w_\lambda\|_{H^s} < \liminf_{n\to\infty} \|v_n\|_{H^s}. \tag{3.16}$$

Similar to the argument of Lemma 2.6, we have

$$J_\lambda^\infty(tw_\lambda) = (h_{\lambda,w_\lambda}^\infty)(t) = \frac{t^2}{2}\|w_\lambda\|_{H^s}^2 + \frac{\lambda t^4}{4} \int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx - \frac{t^p}{p} \int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx,$$

and

$$(h_{\lambda,w_\lambda}^\infty)'(t) = t^3 \Big(d^\infty(t) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx \Big) \text{ for } t > 0,$$

where $d^\infty(t) = t^{-2}\|w_\lambda\|_{H^s}^2 - t^{p-4} \int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx$. Clearly, $d^\infty(T_{a_\infty}(w_\lambda)) = 0$. By (3.15), we have $T_{a_\infty}(w_\lambda) = (\frac{\|w_\lambda\|_{H^s}^2}{\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx})^{\frac{1}{p-2}} < 1$. Then, by (3.13)-(3.16)

$$(h_{\lambda,w_\lambda}^\infty)'(1) < 0,$$

$$(h_{\lambda,w_\lambda}^\infty)'(T_{a_\infty}(w_\lambda)) = (T_{a_\infty}(w_\lambda))^3 \lambda \int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx > 0.$$

Hence, there exists $T_{a_\infty}(w_\lambda) < t_\lambda^- < 1$ such that

$$t_\lambda^- w_\lambda \in M_\lambda^\infty \quad \text{and} \quad (h_{\lambda,w_\lambda}^\infty)'(t_\lambda^-) = 0. \tag{3.17}$$

Since $v_n \in M_\lambda^{\infty,(1)}$, by (3.13), (3.14) and (3.16), we have

$$(h^\infty_{\lambda,w_\lambda})''(1) < 0.$$

This implies that $t_\lambda^- w_\lambda \in M_\lambda^{\infty,-}$.

By (3.13), (3.14) and (3.16), we know that $(h^\infty_{\lambda,v_n})'(t_\lambda^-) > 0$ for sufficiently large n. Owing to $v_n \in M_\lambda^{\infty,(1)}$, we get

$$(h^\infty_{\lambda,v_n})'(1) = 0. \tag{3.18}$$

Similar to the proof of Lemma 2.6, we have

$$(h^\infty_{\lambda,v_n})'(t) = t^3\Big(f^\infty(t) + \lambda\int_{\mathbb{R}^3} K_\infty \phi^t_{v_n} v_n^2 dx\Big) \text{ for } t > 0,$$

where

$$f^\infty(t) = t^{-2}\|v_n\|^2_{H^s} - t^{p-4}\int_{\mathbb{R}^3} a_\infty |v_n|^p dx.$$

$$\begin{cases} f^\infty(t) \text{is decreasing}, & 0 < t < (\dfrac{2}{4-p})^{\frac{1}{p-2}}(\dfrac{\|v_n\|^2_{H^s}}{\int_{\mathbb{R}^3} a_\infty |v_n|^p dx})^{\frac{1}{p-2}}, \\ f^\infty(t) \text{is increasing}, & t > (\dfrac{2}{4-p})^{\frac{1}{p-2}}(\dfrac{\|v_n\|^2_{H^s}}{\int_{\mathbb{R}^3} a_\infty |v_n|^p dx})^{\frac{1}{p-2}}, \end{cases}$$

by using (2.16) and (3.18), we have $(\frac{2}{4-p})^{\frac{1}{p-2}}(\frac{\|v_n\|^2_{H^s}}{\int_{\mathbb{R}^3} a_\infty |v_n|^p dx})^{\frac{1}{p-2}} > 1$. This means that $(h^\infty_{\lambda,v_n})'(t) > 0$ when $0 < t < 1$, which implies that (h^∞_{λ,v_n}) is increasing on $(t_\lambda^-, 1)$ for sufficiently large n. Therefore, $(h^\infty_{\lambda,v_n})(t_\lambda^-) < (h^\infty_{\lambda,v_n})(1)$ for sufficiently large n. That is

$$J_\lambda^\infty(t_\lambda^- v_n) < J_\lambda^\infty(v_n) \text{ for sufficiently large } n.$$

By (3.13)-(3.16), we deduce that

$$J_\lambda^\infty(t_\lambda^- w_\lambda) < \liminf_{n\to\infty} J_\lambda^\infty(t_\lambda^- v_n) \le \liminf_{n\to\infty} J_\lambda^\infty(v_n) = \alpha_\lambda^{\infty,-},$$

we get a contradiction. Thus we have $v_n \to w_\lambda$ strongly in $H^s(\mathbb{R}^3)$ and

$$J_\lambda^\infty(v_n) \to J_\lambda^\infty(w_\lambda) = \alpha_\lambda^{\infty,-} \text{ as } n \to \infty.$$

Furthermore, we obtain that

$$\Lambda = \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}} < \frac{p-2}{2(4-p)}\Big(\frac{4-p}{p}\Big)^{\frac{2}{p-2}}\Lambda_0. \tag{3.19}$$

Therefore, w_λ is a minimizer for J_λ^∞ on $M_\lambda^{\infty,-}$ for each $0 < \lambda < \Lambda$. By (3.5), we deduce that

$$J_\lambda^\infty(w_\lambda) = \alpha_\lambda^{\infty,-} \le J_\lambda^\infty(t_\lambda^{\infty,-} w_0) < A(p)\frac{p-2}{2p}\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}},$$

which implies that $w_\lambda \in M_\lambda^{\infty,(1)}$. Therefore, by Lemma 2.4, we see that w_λ is a nontrivial solution of problem (E_λ^∞). By standard argument as [23], it follows that $w_\lambda(x) > 0$ in $\mathbb{R}^3$. This indicates that $(w_\lambda, \phi^t_{w_\lambda})$ is a positive solution of system (1.1). In addition, since

$$(4-p)\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx < 2\|w_\lambda\|^2_{H^s} \text{ and } t_{a_\infty}(w_\lambda) w_\lambda \in M_0^\infty,$$

where

$$(\frac{4-p}{2})^{\frac{1}{p-2}} < t_{a_\infty}(w_\lambda) := \Big(\frac{\|w_\lambda\|^2_{H^s}}{\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx}\Big)^{\frac{1}{p-2}} < 1, \tag{3.20}$$

similar argument as Lemma 2.6, there holds

$$J^\infty_\lambda(w_\lambda) = \sup_{0\le t\le t^+_\lambda} J^\infty_\lambda(tw_\lambda), \tag{3.21}$$

where $t^+_\lambda > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda) > 1$ by (3.20). Thus, we obtain

$$\alpha^{\infty,-}_\lambda = J^\infty_\lambda(w_\lambda) > J^\infty_\lambda(t_{a_\infty}(w_\lambda) w_\lambda) \ge \alpha^\infty_0 + \frac{\lambda\Big[t_{a_\infty}(w_\lambda)\Big]^4}{4} \int_{\mathbb{R}^3} K_\infty \phi^t_{w_\lambda} w^2_\lambda dx > \alpha^\infty_0.$$

Thus, the proof is completed.

4. Proof of Theorem 1.3

Proposition 4.1. *Let $\{u_n\}$ be a bounded $(PS)_\beta -$ sequence in $H^s(\mathbb{R}^3)$ for J_λ. There exist a subsequence $\{u_n\}$, a number $m \in \mathbb{N}$, a sequences $\{y^i_n\}^\infty_{n=1}$ in $\mathbb{R}^3$, a function $v_0 \in H^s(\mathbb{R}^3)$, and $0 \ne w_i \in H^s(\mathbb{R}^3)$ when $1 \le i \le m$ such that*
(i) $|y^i_n| \to \infty$ and $|y^i_n - y^j_n| \to \infty$ as $n \to \infty$, $1 \le i \ne j \le m$;
(ii) $(-\Delta)^s v_0 + v_0 + \lambda K(x)\phi^t_{v_0} v_0 = a(x)|v_0|^{p-2}v_0 + b(x)|v_0|^2 v_0$;
(iii) $(-\Delta)^s w^i + w^i + \lambda K_\infty \phi^t_{w^i} w^i = a_\infty |w^i|^{p-2} w^i$;
(iv) $u_n = v_0 + \sum_{i=1}^m w^i(\cdot + y^i_n) + o(1)$ strongly in $H^s(\mathbb{R}^3)$;
(v)$J_\lambda(u_n) = J_\lambda(v_0) + \sum_{i=1}^m J^\infty_\lambda(w^i) + o(1)$.

Proof. (1) Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, up to a subsequence, there exists a $v_0 \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup v_0$ in $H^s(\mathbb{R}^3)$. Next, we will prove $J'_\lambda(v_0) = 0$. It is suffice to prove $\langle J'_\lambda(u_n), \varphi\rangle \to \langle J'_\lambda(v_0), \varphi\rangle$ for all $\varphi \in H^s(\mathbb{R}^3)$. Indeed,

$$\begin{aligned}&\langle J'_\lambda(u_n), \varphi\rangle \\ &= \langle u_n, \varphi\rangle_{H^s} + \int_{\mathbb{R}^3} K(x)\phi^t_{u_n} u_n \varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} a(x)|u_n|^{p-2} u_n\varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} b(x)|u_n|^2 u_n \varphi \,\mathrm{d}x,\end{aligned}$$

$$\langle J'_\lambda(v_0), \varphi\rangle = \langle v_0, \varphi\rangle_{H^s} + \int_{\mathbb{R}^3} K(x)\phi^t_{v_0} v_0 \varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} a(x)|v_0|^{p-2} v_0\varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} b(x)|v_0|^2 v_0 \varphi \,\mathrm{d}x.$$

By the $u_n \rightharpoonup v_0$ in $H^s(\mathbb{R}^3)$, we can conclude that $\langle u_n, \varphi\rangle_{H^s} \to \langle v_0, \varphi\rangle_{H^s}$. Since $|u_n|^{p-2}u_n$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ and combining with $u_n \to u$ almost everywhere in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} a(x)|u_n|^{p-2} u_n \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^3} a(x)|v_0|^{p-2} v_0 \varphi \,\mathrm{d}x, \quad \forall \varphi \in L^p(\mathbb{R}^3).$$

Similar to the above argument, we also have

$$\int_{\mathbb{R}^3} b(x)|u_n|^2 u_n \varphi dx \to \int_{\mathbb{R}^3} b(x)|v_0|^2 v_0 \varphi dx, \qquad \forall \varphi \in L^p(\mathbb{R}^3).$$

From the fact that $u_n \rightharpoonup v_0$ in $H^s(\mathbb{R}^3)$, we can deduce that

$$\int_{\mathbb{R}^3} K(x)\phi^t_{u_n} u_n \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^3} K(x)\phi^t_{v_0} v_0 \varphi \,\mathrm{d}x, \quad \forall \varphi \in H^s(\mathbb{R}^3).$$

Hence, $J'_\lambda(v_0) = 0$.

(2) We will prove that $(J_\lambda^\infty)'(w_n^1) = o(1)$ and $J_\lambda(u_n) = J_\lambda(v_0) + J_\lambda^\infty(w_n^1) + o(1)$, where $w_n^1 = u_n - v_0$. By the Brezis-Lieb Lemma, we deduce that

$$\begin{aligned} &\|w_n^1\|_{H^s}^2 = \|u_n\|_{H^s}^2 - \|v_0\|_{H^s}^2 + o(1), \\ &\int_{\mathbb{R}^3} a(x)|w_n^1|^p \,\mathrm{d}x = \int_{\mathbb{R}^3} a(x)|u_n|^p \,\mathrm{d}x - \int_{\mathbb{R}^3} a(x)|v_0|^p \,\mathrm{d}x + o(1), \\ &\int_{\mathbb{R}^3} b(x)|w_n^1|^4 \,\mathrm{d}x = \int_{\mathbb{R}^3} b(x)|u_n|^4 \,\mathrm{d}x - \int_{\mathbb{R}^3} b(x)|v_0|^4 \,\mathrm{d}x + o(1). \end{aligned} \tag{4.1}$$

From (H_1), $w_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^3)$ and (4.1), we conclude that

$$\int_{\mathbb{R}^3} a_\infty |w_n^1|^p \,\mathrm{d}x = \int_{\mathbb{R}^3} a(x)|u_n|^p \,\mathrm{d}x - \int_{\mathbb{R}^3} a(x)|v_0|^p \,\mathrm{d}x + o(1). \tag{4.2}$$

By (H_3), $\forall \varepsilon > 0$, there exist $R(\varepsilon) > 0$ such that

$$\int_{|x| \geq R(\varepsilon)} b(x)|w_n^1|^4 dx \leq \varepsilon,$$

and from the fact that $w_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^3)$, we have that

$$\lim_{n\to\infty} \int_{|x| \leq R(\varepsilon)} b(x)|w_n^1|^4 dx = 0.$$

Thus,

$$\int_{\mathbb{R}^3} b(x)|u_n|^4 dx - \int_{\mathbb{R}^3} b(x)|v_0|^4 dx = o(1). \tag{4.3}$$

By Lemma 2.4 in [22] $(2s + 2t > 3)$, we obtain

$$\int_{\mathbb{R}^3} K_\infty \phi^t_{w_n^1} (w_n^1)^2 dx = \int_{\mathbb{R}^3} K(x)\phi^t_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} K(x)\phi^t_{v_0} v_0^2 dx = o(1). \tag{4.4}$$

Combining (4.2)-(4.4), we get that

$$J_\lambda(u_n) = J_\lambda(v_0) + J_\lambda^\infty(w_n^1) + o(1). \tag{4.5}$$

By Lemma 8.1 in [27], we have that

$$\Big| \int_{\mathbb{R}^3} a(x)(|u_n|^{p-2}u_n - |v_0|^{p-2}v_0 - |w_n^1|^{p-2}w_n^1)\varphi dx \Big| = o(1)\|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}^3).$$

From the condition (H_1), we have

$$\Big| \int_{\mathbb{R}^3} (a(x) - a_\infty)|w_n^1|^{p-2} w_n^1 \varphi dx \Big| = o(1)\|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}^3).$$

Thus,

$$\Big| \int_{\mathbb{R}^3} \big[a(x)(|u_n|^{p-2}u_n - |v_0|^{p-2}v_0) - a_\infty |w_n^1|^{p-2} w_n^1\big]\varphi dx \Big| = o(1)\|\varphi\|_{H^s}, \forall \varphi \in H^s(\mathbb{R}^3). \tag{4.6}$$

Similarly, we have

$$\int_{\mathbb{R}^3} \big[K(x)(\phi^t_{u_n} u_n - \phi^t_{v_0} v_0) - K_\infty \phi^t_{w_n^1} w_n^1\big]\varphi dx = o(1)\|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}^3). \tag{4.7}$$

By (H_3), and $\forall \varphi \in H^s(\mathbb{R}^3)$ we have that

$$\int_{\mathbb{R}^3} b(x)|w_n^1|^2 w_n^1 \varphi dx = o(1)\|\varphi\|_{H^s},$$

and

$$\Big|\int_{\mathbb{R}^3} b(x)[|w_n^1|^2 w_n^1 - |u_n|^2 u_n - |v_0|^2 v_0]\varphi dx\Big| = o(1)\|\varphi\|_{H^s}.$$

Thus,

$$\Big|\int_{\mathbb{R}^3} b(x)[|u_n|^2 u_n - |v_0|^2 v_0]\varphi dx\Big| = o(1)\|\varphi\|_{H^s}. \tag{4.8}$$

Combining (4.6)-(4.8), we have

$$\big|\langle J_\lambda'(u_n) - J_\lambda'(v_0), \varphi\rangle - \langle (J_\lambda^\infty)'(w_n^1), \varphi\rangle\big| = o(1)\|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}^3).$$

Therefore,

$$(J_\lambda^\infty)'(w_n^1) = o(1). \tag{4.9}$$

Next, we will consider the following two cases.
Case 1.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |w_n^1|^2 dx = 0.$$

By the vanishing Lemma, we have that

$$w_n^1 \to 0 \text{ in } L^t(\mathbb{R}^3), \quad \forall t \in (2, 2_s^*). \tag{4.10}$$

Combining (4.5) and (4.10), there holds

$$J_\lambda(u_n) - J_\lambda(v_0) = \frac{1}{2}\|w_n^1\|_{H^s}^2 + \int_{\mathbb{R}^3} K_\infty \phi_{w_n^1}^t (w_n^1)^2 - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |w_n^1|^p dx + o(1)$$

and

$$\int_{\mathbb{R}^3} K_\infty \phi_{w_n^1}^t (w_n^1)^2 dx \le \widehat{C}\|w_n^1\|_{\frac{12}{3+2s}}^4.$$

Thus,

$$J_\lambda(u_n) - J_\lambda(v_0) = \frac{1}{2}\|w_n^1\|_{H^s}^2 + o(1).$$

By $(J_\lambda^\infty)'(w_n^1) = 0$, we have

$$\|w_n^1\|_{H^s}^2 + \int_{\mathbb{R}^3} K_\infty \phi_{w_n^1}^t (w_n^1)^2 dx - \int_{\mathbb{R}^3} a_\infty |w_n^1|^p dx = 0$$

and by (4.10), we have

$$\int_{\mathbb{R}^3} K_\infty \phi_{w_n^1}^t (w_n^1)^2 dx \to 0, \quad \int_{\mathbb{R}^3} a_\infty |w_n^1|^p dx \to 0,$$

which yields $\|w_n^1\|_{H^s}^2 = o(1)$. Thus, $J_\lambda(u_n) = J_\lambda(v_0)$.
Case 2. There is $\gamma_1 > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^1|^2 \, dx \ge \gamma_1 > 0.$$

In this case, there exists $y_n^1 \in \mathbb{R}^3$ with $|y_n^1| \to \infty$ such that $\int_{B_1(y_n^1)} |w_n^1|^2 \, dx \ge \frac{\gamma_1}{2} > 0$. Up to a subsequence, we assume that $w_n^1(. + y_n^1) \rightharpoonup w^1 \ne 0$ weakly in $H^s(\mathbb{R}^3)$. Thus,

$$\begin{aligned} &J_\lambda(u_n) - J_\lambda(v_0) = J_\lambda^\infty(w_n^1(. + y_n^1)) + o(1), \\ &(J_\lambda^\infty)'(w_n^1(. + y_n^1)) = o(1). \end{aligned} \tag{4.11}$$

Therefore $(J_\lambda^\infty)'(w^1) = 0$. Let $w_n^2 = w_n^1(. + y_n^1) - w^1$, then we have that $\|w_n^2\|_{H^s}^2 = \|w_n^1\|_{H^s}^2 - \|w^1\|_{H^s}^2 + o(1)$. Combining the first equality of (4.1), we have

$$\|w_n^2\|_{H^s}^2 = \|u_n\|_{H^s}^2 - \|v_0\|_{H^s}^2 - \|w^1\|_{H^s}^2 + o(1). \tag{4.12}$$

Similar argument as (4.5) and (4.9), we deduce that

$$\begin{aligned} &J_\lambda(u_n) - J_\lambda(v_0) - J_\lambda^\infty(w^1) + o(1) = J_\lambda^\infty(w_n^2), \\ &(J_\lambda^\infty)'(w_n^2) = o(1). \end{aligned} \tag{4.13}$$

Next, similarly argue as above, we note that either

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |w_n^2|^2 dx = 0, \tag{4.14}$$

or there is $\gamma_2 > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |w_n^2|^2 dx \geq \gamma_2 > 0. \tag{4.15}$$

If (4.14) holds, it is similar to the case 1, we can show that $J_\lambda(u_n) = J_\lambda(v_0) + J_\lambda^\infty(w^1)$.

If (4.15) holds, it is similar to the case 2. There is $y_n^2 \in \mathbb{R}^3$ with $|y_n^2| \to \infty$ satisfying $\int_{B_1(y_n^2)} |w_n^2|^2 \, dx \geq \frac{\gamma_2}{2} > 0$. Up to a subsequence, we can assume that $w_n^2(. + y_n^2) \rightharpoonup w^2 \neq 0$ weakly in $H^s(\mathbb{R}^3)$, $(J_\lambda^\infty)'(w^2) = 0$ and

$$\begin{aligned} &J_\lambda(u_n) - J_\lambda(v_0) - J_\lambda^\infty(w^1) - J_\lambda^\infty(w^2) + o(1) = J_\lambda^\infty(w_n^3), \\ &(J_\lambda^\infty)'(w_n^3) = o(1), \\ &\|w_n^3\|_{H^s}^2 = \|u_n\|_{H^s}^2 - \|v_0\|_{H^s}^2 - \|w^1\|_{H^s}^2 - \|w^2\|_{H^s}^2 + o(1), \end{aligned}$$

where $w_n^3 = w_n^2(. + y_n^2) - w^2$. Continuing this process, we have $w_n^i \in H^s(\mathbb{R}^3)$, $y_n^i \in \mathbb{R}^3$ with $|y_n^i| \to \infty$ satisfying $w_n^i(. + y_n^i) \rightharpoonup w^i \neq 0$ weakly in $H^s(\mathbb{R}^3)$ and

$$(J_\lambda^\infty)'(w^i) = 0, \tag{4.16}$$

where $w_n^{j+1} = w_n^j(. + y_n^j) - w^j$, $j \in \mathbb{N}$.

(3) From the above argument, we have that

$$\begin{aligned} &J_\lambda(u_n) - J_\lambda(v_0) - \sum_{i=1}^{j} J_\lambda^\infty(w^i) + o(1) = J_\lambda^\infty(w_n^{j+1}), \\ &(J_\lambda^\infty)'(w_n^{j+1}) = o(1), \\ &\|w_n^{j+1}\|_{H^s}^2 = \|u_n\|_{H^s}^2 - \|v_0\|_{H^s}^2 - \sum_{i=1}^{j} \|w^i\|_{H^s}^2 + o(1). \end{aligned} \tag{4.17}$$

Combining the fact that $\langle (J_\lambda^\infty)'(w^i), w^i \rangle = 0$ and Sobolev embedding theorem, we can find $\kappa > 0$ independent of i such that

$$\|w^i\|^2 \geq \kappa > 0.$$

From (4.17), it is obvious that $w_n^{j+1} \to 0$ at some $j = m$. Hence, we conclude that

$$J_\lambda(u_n) = J_\lambda(v_0) + \sum_{i=1}^{m} J_\lambda^\infty(w^i) + o(1). \tag{4.18}$$

□

Corollary 4.2. *If $\{u_n\} \subset M_\lambda^{(1)}$ is a $(PS)_\beta$ − sequence in $H^s(\mathbb{R}^3)$ for J_λ and $0 < \beta < \alpha_\lambda^{\infty,-}$. Then there exist a subsequence $\{u_n\}$ and a nonzero u_0 in $H^s(\mathbb{R}^3)$ such that $u_n \to u_0$ strongly in $H^s(\mathbb{R}^3)$ and $J_\lambda(u_0) = \beta$. Furthermore, (u_0, ϕ_{u_0}) is a nonzero solution of equation (E_λ).*

By Theorem 1.2, we see that problem (E_λ^∞) admits a positive solution $w_\lambda \in M_\lambda^{\infty,-}$, and

$$J_\lambda^\infty(w_\lambda) = \alpha_\lambda^{\infty,-} \quad , \quad \frac{4-p}{2}\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx < \|w_\lambda\|_{H^s}^2.$$

Define $T_a(w_\lambda)$ as follows

$$\Big(\frac{(4-p)a_\infty}{2a_{max}}\Big)^{\frac{1}{p-2}} < T_a(w_\lambda) := \Big(\frac{\|w_\lambda\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|w_\lambda|^p dx}\Big)^{\frac{1}{p-2}}.$$

Lemma 4.3. *For $0<\lambda<\Lambda$, then there exists $t_\lambda^\infty > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda) > 1$ such that*

$$J_\lambda^\infty(w_\lambda) = \sup_{0\le t\le t_\lambda^\infty} J_\lambda^\infty(tw_\lambda) = \alpha_\lambda^{\infty,-}, \tag{4.19}$$

where $t_{a_\infty}(w_\lambda)$ is given in (3.20).

Proof. Since

$$(h_{\lambda,u}^\infty)'(t) = t^3\Big[b_\lambda^\infty(t) + \lambda\int_{\mathbb{R}^3} K_\infty \phi_u^t u^2 dx\Big],$$

where

$$b_\lambda^\infty(t) = t^{-2}\|w_\lambda\|_{H^s}^2 - t^{p-4}\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx \text{ for } t>0. \tag{4.20}$$

Observe that

$$b_\lambda^\infty(1) + \lambda\int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx = 0, \tag{4.21}$$

for all $0<\lambda<\Lambda$, it is easy to show that

$$b_\lambda^\infty(t_{a_\infty}(w_\lambda)) = 0, \quad \lim_{t\to 0^+} b_\lambda^\infty(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} b_\lambda^\infty(t) = 0.$$

Hence,

$$\begin{aligned}(b_\lambda^\infty)'(t) =& -2t^{-3}\|w_\lambda\|_{H^s}^2 + (4-p)t^{p-5}\int_{\mathbb{R}^3} a_\infty|w_\lambda|^p dx \\ =& t^{-3}\Big(-2\|w_\lambda\|_{H^s}^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p dx\Big).\end{aligned}$$

A straightforward calculation gives that

$$\begin{cases} b_\lambda^\infty(t) \text{ is decreasing}, & 0<t<(\dfrac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda), \\ b_\lambda^\infty(t) \text{ is increasing}, & t>(\dfrac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda). \end{cases}$$

Thus, we get

$$\inf_{t>0} b_\lambda^\infty(t) = b_\lambda^\infty\Big((\frac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda)\Big). \tag{4.22}$$

In view of (3.20), we know that

$$\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda) > 1. \tag{4.23}$$

Hence, by (4.21)-(4.23), we obtain

$$\inf_{t>0} b_\lambda^\infty(t) < b_\lambda^\infty(1) = -\lambda\int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx \tag{4.24}$$

which implies that there exists $t_\lambda^\infty > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda) > 1$ such that

$$b_\lambda^\infty(t_\lambda^\infty) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{w_\lambda}^t w_\lambda^2 dx = 0.$$

By a similar argument as the proof of Lemma 2.6, we get (4.19). $\square$

Lemma 4.4. *Assume that* $0 < \lambda < \Lambda$ *and* (H_1)*-*(H_3), (H_4) *hold, the following two statements are true:*
(1) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_{w_\lambda}^t w_\lambda^2 dx > \int_{\mathbb{R}^3} b(x)|w_\lambda|^4 dx$, *then there exist two constants* $t_\lambda^{(1)}$ *and* $t_\lambda^{(2)}$ *satisfying*

$$T_a(w_\lambda) < t_\lambda^{(1)} < (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(w_\lambda) < t_\lambda^{(2)},$$

such that

$$t_\lambda^{(i)} w_\lambda \in M_\lambda^{(i)} \ (i = 1, 2),$$

$$J_\lambda(t_\lambda^{(1)} w_\lambda) = \sup_{0 \le t \le t_\lambda^{(2)}} J_\lambda(t w_\lambda) < \alpha_\lambda^{\infty,-} < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}$$

and

$$J_\lambda(t_\lambda^{(2)} w_\lambda) = \inf_{t \ge t_\lambda^{(1)}} J_\lambda(t w_\lambda).$$

(2) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_{w_\lambda}^t w_\lambda^2 \mathrm{d}x \le \int_{\mathbb{R}^3} b(x)|w_\lambda|^4 \mathrm{d}x$, *then there exist a constant* $t_\lambda^{(3)}$ *satisfying*

$$0 < t_\lambda^{(3)} < T_a(w_\lambda),$$

such that

$$t_\lambda^{(3)} w_\lambda \in M_\lambda^{(1)},$$

$$J_\lambda(t_\lambda^{(3)} w_\lambda) = \sup_{0 \le t \le T_a(w_\lambda)} J_\lambda(t w_\lambda) < \alpha_\lambda^{\infty,-}.$$

Proof. Let

$$b_\lambda(t) = t^{-2}\|w_\lambda\|_{H^s}^2 - t^{p-4} \int_{\mathbb{R}^3} a(x)|w_\lambda|^p \mathrm{d}x \text{ for } t > 0.$$

Clearly, $t w_\lambda \in M_\lambda$ if and only if

$$b_\lambda(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{w_\lambda}^t w_\lambda^2 \mathrm{d}x - \int_{\mathbb{R}^3} b(x) w_\lambda^4 \mathrm{d}x = 0. \tag{4.25}$$

It is not difficult to verify that

$$b_\lambda(T_a(w_\lambda)) = 0, \quad \lim_{t \to 0^+} b_\lambda(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} b_\lambda(t) = 0,$$

$b_\lambda(t)$ is decreasing when $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(w_\lambda)$ and is increasing when $t > (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(w_\lambda)$. From (H_4), we get

$$T_a(w_\lambda) \le T_{a_\infty}(w_\lambda) < 1 \text{ and } b_\lambda(t) \le b_\lambda^\infty(t),$$

where b_λ^∞ is given in (4.20). By condition (H_4) and (4.24), we have

$$\begin{aligned}
\inf_{t>0} b_\lambda(t) =& b_\lambda\Big(\big(\frac{2}{4-p}\big)^{\frac{1}{p-2}}T_a(w_\lambda)\Big)\\
=&-\frac{p-2}{4-p}\Big(\frac{4-p}{2}\Big)^{\frac{2}{p-2}}\|w_\lambda\|_{H^s}^2\Big(\frac{\|w_\lambda\|_{H^s}^2}{\int_{\mathbb{R}^3}a(x)|w_\lambda|^p\mathrm{d}x}\Big)^{-\frac{2}{p-2}}\\
\le&-\frac{p-2}{4-p}\Big(\frac{4-p}{2}\Big)^{\frac{2}{p-2}}\|w_\lambda\|_{H^s}^2\Big(\frac{\|w_\lambda\|_{H^s}^2}{\int_{\mathbb{R}^3}a_\infty|w_\lambda|^p dx}\Big)^{-\frac{2}{p-2}}\\
=&\inf_{t>0} b_\lambda^\infty(t) < -\lambda\int_{\mathbb{R}^3}K_\infty\phi_{w_\lambda}^t w_\lambda^2 dx\\
\le&-\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2 dx\\
<&-\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2 dx+\int_{\mathbb{R}^3}b(x)|w_\lambda|^4 dx.
\end{aligned}$$

Similar to the argument of Lemma 2.6, we need to distinguish two cases.
Case 1:

$$\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2 dx > \int_{\mathbb{R}^3}b(x)|w_\lambda|^4 dx,$$

then there are two constants $t_\lambda^{(1)}$ and $t_\lambda^{(2)}$ satisfying $T_a(w_\lambda) < t_\lambda^{(1)} < (\frac{2}{4-p})^{\frac{1}{p-2}}T_a(w_\lambda) < t_\lambda^{(2)}$ such that

$$b_\lambda(t_\lambda^{(i)})+\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2 dx-\int_{\mathbb{R}^3}b(x)|w_\lambda|^4 dx=0 \text{ for } i=1,2$$

which implies that $t_\lambda^{(i)}w_\lambda \in M_\lambda (i=1,2)$. Taking the derivative of $h'_{\lambda,t_\lambda^{(i)}w_\lambda}(t)$, we have

$$\begin{aligned}
h''_{\lambda,t_\lambda^{(1)}w_\lambda}(1) =& -2\|t_\lambda^{(1)}w_\lambda\|_{H^s}^2+(4-p)\int_{\mathbb{R}^3}a(x)|t_\lambda^{(1)}w_\lambda|^p dx\\
=&(t_\lambda^{(1)})^5(b_\lambda)'(t_\lambda^{(1)})<0,
\end{aligned}$$

and

$$\begin{aligned}
h''_{\lambda,t_\lambda^{(2)}w_\lambda}(1) =& -2\|t_\lambda^{(1)}w_\lambda\|_{H^s}^2+(4-p)\int_{\mathbb{R}^3}a(x)|t_\lambda^{(2)}w_\lambda|^p dx\\
=&(t_\lambda^{(2)})^5(b_\lambda)'(t_\lambda^{(2)})>0,
\end{aligned}$$

which implies that $t_\lambda^{(1)}w_\lambda \in M_\lambda^-$, $t_\lambda^{(2)}w_\lambda \in M_\lambda^+$ and

$$t_\lambda^{(1)} < (\frac{2}{4-p})^{\frac{1}{p-2}}T_a(w_\lambda) \le (\frac{2}{4-p})^{\frac{1}{p-2}}t_{a_\infty}(w_\lambda) < \min\{(\frac{2}{4-p})^{\frac{1}{p-2}}, t_\lambda^\infty\},$$

where t_λ^∞ is given in Lemma 4.3. By (3.6), Lemma 4.3 and (H_4), for every $0<\lambda<\Lambda$, we have

$$\begin{aligned}
J_\lambda\Big(t_\lambda^{(1)}w_\lambda\Big) =&\frac{(t_\lambda^{(1)})^2}{2}\|w_\lambda\|_{H^s}^2+\frac{\lambda(t_\lambda^{(1)})^4}{4}\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx\\
&-\frac{(t_\lambda^{(1)})^p}{p}\int_{\mathbb{R}^3}a(x)|w_\lambda|^pdx-\frac{(t_\lambda^{(1)})^4}{4}\int_{\mathbb{R}^3}b(x)|w_\lambda|^4dx\\
=&J_\lambda^\infty(t_\lambda^{(1)}w_\lambda)+\frac{\lambda(t_\lambda^{(1)})^4}{4}\Big(\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx-\int_{\mathbb{R}^3}K_\infty\phi_{w_\lambda}^t w_\lambda^2dx\Big)\\
&-\frac{(t_\lambda^{(1)})^p}{p}\int_{\mathbb{R}^3}(a(x)-a_\infty)|w_\lambda|^pdx-\frac{(t_\lambda^{(1)})^4}{4}\int_{\mathbb{R}^3}b(x)|w_\lambda|^4dx\\
<&J_\lambda^\infty(t_\lambda^{(1)}w_\lambda)+\frac{\lambda(t_\lambda^{(1)})^4}{4}(\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx-\int_{\mathbb{R}^3}K_\infty\phi_{w_\lambda}^t w_\lambda^2dx)\\
&-\frac{(t_\lambda^{(1)})^p}{p}\int_{\mathbb{R}^3}(a(x)-a_\infty)|w_\lambda|^pdx\\
\le&\sup_{0\le t\le t_\lambda^\infty}J_\lambda^\infty(tw_\lambda)+\frac{\lambda(t_\lambda^{(1)})^4}{4}(\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx-\int_{\mathbb{R}^3}K_\infty\phi_{w_\lambda}^t w_\lambda^2dx)\\
&-\frac{(t_\lambda^{(1)})^p}{p}\int_{\mathbb{R}^3}(a(x)-a_\infty)|w_\lambda|^pdx\\
<&\alpha_\lambda^{\infty,-}<A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.
\end{aligned}\tag{4.26}$$

Therefore, $t_\lambda^{(1)}w_\lambda\in M_\lambda^{(1)}$ and $J_\lambda(t_\lambda^{(1)}w_\lambda)<\alpha_\lambda^{\infty,-}$ and

$$(h_{\lambda,w_\lambda})'(t)=t^3\Big(b_\lambda(t)+\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx-\int_{\mathbb{R}^3}b(x)|w_\lambda|^4dx\Big).$$

By computation, we get

$$\begin{aligned}
&(h_{\lambda,w_\lambda})'(t)>0 \text{ for all } t\in(0,t_\lambda^{(1)})\bigcup(t_\lambda^{(2)},\infty),\\
&(h_{\lambda,w_\lambda})'(t)<0 \text{ for all } t\in(t_\lambda^{(1)},t_\lambda^{(2)}).
\end{aligned}\tag{4.27}$$

Therefore, we deduce that

$$J_\lambda(t_\lambda^{(1)}w_\lambda)=\sup_{0\le t\le t_\lambda^{(2)}}J_\lambda(tw_\lambda) \text{ and } J_\lambda(t_\lambda^{(2)}w_\lambda)=\inf_{t\ge t_\lambda^{(1)}}J_\lambda(tw_\lambda).$$

By (4.26) and (4.27), we have $J_\lambda(t_\lambda^{(2)}w_\lambda)\le J_\lambda(t_\lambda^{(1)}w_\lambda)<\alpha_\lambda^{\infty,-}$, $t_\lambda^{(2)}w_\lambda\in M_\lambda^{(2)}$, which yields the conclusion.

Case 2:

$$\lambda\int_{\mathbb{R}^3}K(x)\phi_{w_\lambda}^t w_\lambda^2dx\le\int_{\mathbb{R}^3}b(x)|w_\lambda|^4dx.$$

Similar to the discussion of Case 1, there exists a constant $t_\lambda^{(3)}$ satisfying

$$0<t_\lambda^{(3)}<T_a(w_\lambda)$$

such that
$$b_\lambda(t_\lambda^{(3)}) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{w_\lambda}^t w_\lambda^2 dx - \int_{\mathbb{R}^3} b(x)|w_\lambda|^4 dx = 0$$
which implies that $t_\lambda^{(3)} u \in M_\lambda$.

Taking the second order derivatives of $h_{\lambda, t_\lambda^{(3)} w_\lambda}(t)$, we have
$$\begin{aligned} h''_{\lambda, t_\lambda^{(3)} u}(1) =& -2\|t_\lambda^{(3)} w_\lambda\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|t_\lambda^{(3)} w_\lambda|^p dx \\ =& (t_\lambda^{(3)})^5 b'_\lambda(t_\lambda^{(3)}) < 0, \end{aligned}$$
thus, $t_\lambda^{(3)} w_\lambda \in M_\lambda^-$. Moreover,
$$t_\lambda^{(3)} < T_a(w_\lambda) < (\frac{2}{4-p})^{\frac{1}{p-2}} T_a(w_\lambda) \le (\frac{2}{4-p})^{\frac{1}{p-2}} t_{a_\infty}(w_\lambda) < \min\{(\frac{2}{4-p})^{\frac{1}{p-2}}, t_\lambda^\infty\}.$$
Similarly, we have
$$J_\lambda\Big(t_\lambda^{(3)} w_\lambda\Big) < \alpha_\lambda^{\infty,-} < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$
Therefore, $t_\lambda^{(3)} w_\lambda \in M_\lambda^{(1)}$ and $J_\lambda(t_\lambda^{(3)} w_\lambda) < \alpha_\lambda^{\infty,-}$. Since
$$(h_{\lambda,w_\lambda})'(t) = t^3\Big(b_\lambda(t) + \lambda\int_{\mathbb{R}^3} K(x)\phi_{w_\lambda}^t w_\lambda^2 dx - \int_{\mathbb{R}^3} b(x)|w_\lambda|^4 dx\Big).$$
By computation, we know that $h'_{\lambda,w_\lambda}(t) > 0$ holds if $t \in (0, t_\lambda^{(3)})$ and $h'_{\lambda,w_\lambda}(t) < 0$ holds if $t \in (t_\lambda^{(3)}, +\infty)$. This implies that
$$J_\lambda(t_\lambda^{(3)} w_\lambda) = \sup_{0 \le t \le T_a(w_\lambda)} J_\lambda(t w_\lambda).$$
□

Lemma 4.5. *For $2 < p < 4$ and $0 < \lambda < \Lambda$, for each $u \in M_\lambda^{(1)}$, there exist $\sigma > 0$ and a differentiable function:*
$$t^* : B(0;\sigma) \subset H^s(\mathbb{R}^3) \to \mathbb{R}^+$$
such that
$$t^*(0) = 1 \text{ and } t^*(v)(u-v) \in M_\lambda^{(1)}$$
for all $v \in B(0;\sigma)$, and there holds

$\langle (t^*)'(0), \varphi\rangle$
$$= \frac{2\int_{\mathbb{R}^3}((-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi + u\varphi) + 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u\varphi dx - p\int_{\mathbb{R}^3} a(x)|u|^{p-2}u\varphi dx - 4\int_{\mathbb{R}^3} b(x)|u|^2 u\varphi dx}{-2\|u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx}$$
for $\varphi \in H^s(\mathbb{R}^3)$.

Proof. For any $u \in M_\lambda^{(1)}$, we define the function $F_u : \mathbb{R} \times H^s(\mathbb{R}^3) \to \mathbb{R}$ by
$$\begin{aligned} F_u(t,v) =& \langle (J_\lambda)'(t(u-v)), t(u-v)\rangle \\ =& t^2\|u-v\|_{H^s}^2 + \lambda t^4 \int_{\mathbb{R}^3} K(x)\phi_{u-v}^t (u-v)^2 dx \\ & - t^p\int_{\mathbb{R}^3} a(x)|u-v|^p dx - t^4\int_{\mathbb{R}^3} b(x)|u-v|^4 dx. \end{aligned}$$

Clearly, $F_u(1,0) = \langle (J_\lambda)'(u), u\rangle = 0$, and

$$\begin{aligned}\frac{d}{dt}F_u(1,0) =& 2\|u\|_{H^s}^2 + 4\lambda \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - p\int_{\mathbb{R}^3} a(x)|u|^p dx - 4\int_{\mathbb{R}^3} b(x)|u|^4 dx\\ =& -2\|u\|_{H^s}^2 - (p-4)\int_{\mathbb{R}^3} a(x)|u|^p dx < 0.\end{aligned}$$

By applying the implicit function theorem, there exist $\sigma > 0$ and a differentiable function $t^* : B(0;\sigma) \subset H^s(\mathbb{R}^3) \to \mathbb{R}$ such that $t^*(0) = 1$,

$$\begin{aligned}&\langle (t^*)'(0), \varphi\rangle\\ &= \frac{2\int_{\mathbb{R}^3}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi + u\varphi)dx + 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u^t u\varphi dx - p\int_{\mathbb{R}^3} a(x)|u|^{p-2}u\varphi dx - 4\int_{\mathbb{R}^3} b(x)|u|^2 u\varphi dx}{-2\|u\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|u|^p dx},\end{aligned}$$

and

$$F_u(t^*(v), v) = 0 \text{ for all } v \in B(0;\sigma),$$

that is,

$$\langle (J_\lambda)'(t^*(v)(u-v)), t^*(v)(u-v)\rangle = 0 \text{ for all } v \in B(0;\sigma).$$

From the continuity of the map t^*, if σ is sufficiently small, we have

$$h''_{\lambda,t^*(v)(u-v)}(1) = -2\|t^*(v)(u-v)\|_{H^s}^2 - (p-4)\int_{\mathbb{R}^3} a(x)|t^*(v)(u-v)|^p dx < 0,$$

and

$$J_\lambda(t^*(v)(u-v)) < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$

Hence, $t^*(v)(u-v) \in M_\lambda^{(1)}$ for all $v \in B(0;\sigma)$. □

Similar reason as we define $\alpha_\lambda^{\infty,-}$ and $\alpha_\lambda^{\infty,+}$ in Section 3, we can define the following minimum problem

$$\alpha_\lambda^- = \inf_{u\in M_\lambda^{(1)}} J_\lambda(u) = \inf_{u\in M_\lambda^-} J_\lambda(u),$$
$$\alpha_\lambda^+ = \inf_{u\in M_\lambda^{(2)}} J_\lambda(u) = \inf_{u\in M_\lambda^+} J_\lambda(u).$$

Proposition 4.6. *For $0 < \lambda < \Lambda$, there exists $\{u_n\} \subset M_\lambda^{(1)}$ such that*

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1) \text{ and } (J_\lambda)'(u_n) = o(1). \tag{4.28}$$

Proof. First, we will show that $M_\lambda^{(1)}$ is a complete metric space. It is obvious that $M_\lambda^{(1)}$ is a metric space. Then take any sequence $u_n \in M_\lambda^{(1)}$ is a cauchy sequence. We have $d(u_n, u_m) \to 0$, that is $\|u_n - u_m\|_{H^s} \to 0$. Then there exists $u \in H^s(\mathbb{R}^3)$, such that $u_n \to u$ in $H^s(\mathbb{R}^3)$. It is easy to show that

$$\|u\|_{H^s} \le \liminf_{n\to\infty} \|u_n\|_{H^s} \le \widehat{D}_1, \quad J_\lambda(u) < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$

So, we have $u \in M_\lambda^{(1)}$. Hence, $M_\lambda^{(1)}$ is a complete metric space. Then, by Lemma2.1, we know $J_\lambda(u)$ is bounded below on $M_\lambda^{(1)}$. Using the Ekeland variational principle [11], there exists a minimizing sequence $\{u_n\} \subset M_\lambda^{(1)}$ such that

$$J_\lambda(u_n) < \alpha_\lambda^- + \frac{1}{n},$$

and

$$J_\lambda(u_n) \leq J_\lambda(w) + \frac{1}{n}\|w-u_n\|_{H^s} \text{ for all } w \in M_\lambda^{(1)}. \tag{4.29}$$

By applying Lemma 4.5 with $u = u_n$, there exists a function $t_n^* : B(0;\epsilon_n) \to \mathbb{R}$ for some ϵ_n such that $t_n^*(w)(u_n - w) \in M_\lambda^{(1)}$. Let $0 < \delta < \epsilon_n$ and $u \in H^s(\mathbb{R}^3)$ with $u \not\equiv 0$. Set

$$w_\delta = \frac{\delta u}{\|u\|_{H^s}} \text{ and } z_\delta = t_n^*(w_\delta)(u_n - w_\delta).$$

Clearly, $z_\delta \in M_\lambda^{(1)}$, and by (4.29), we have

$$J_\lambda(z_\delta) - J_\lambda(u_n) \geq -\frac{1}{n}\|z_\delta - u_n\|_{H^s}.$$

Using the mean value theorem, we have

$$\langle (J_\lambda)'(u_n), z_\delta - u_n\rangle + o(\|z_\delta - u_n\|_{H^s}) \geq -\frac{1}{n}\|z_\delta - u_n\|_{H^s},$$

and

$$\begin{aligned}&\langle (J_\lambda)'(u_n), -w_\delta\rangle + (t_n^*(w_\delta)-1)\langle (J_\lambda)'(u_n), u_n - w_\delta\rangle\\ &\geq -\frac{1}{n}\|z_\delta - u_n\|_{H^s} + o(\|z_\delta - u_n\|_{H^s}).\end{aligned} \tag{4.30}$$

Observe that $t_n^*(w_\delta)(u_n - w_\delta) \in M_\lambda^{(1)}$. By (4.30), we deduce that

$$\begin{aligned}&-\delta\Big\langle (J_\lambda)'(u_n), \frac{u}{\|u\|_{H^s}}\Big\rangle + \frac{t_n^*(w_\delta)-1}{t_n^*(w_\delta)}\langle (J_\lambda)'(z_\delta), t_n^*(w_\delta)(u_n - w_\delta)\rangle\\ &+(t_n^*(w_\delta)-1)\langle (J_\lambda)'(u_n) - (J_\lambda)'(z_\delta), u_n - w_\delta\rangle\\ &\geq \frac{1}{n}\|z_\delta - u_n\|_{H^s} + o(\|z_\delta - u_n\|_{H^s}),\end{aligned}$$

that is,

$$\begin{aligned}\Big\langle (J_\lambda)'(u_n), \frac{u}{\|u\|_{H^s}}\Big\rangle \leq& \frac{t_n^*(w_\delta)-1}{\delta}\langle (J_\lambda)'(u_n) - (J_\lambda)'(z_\delta), u_n - w_\delta\rangle\\ &+\frac{\|z_\delta - u_n\|_{H^s}}{\delta_n} + \frac{o(\|z_\delta - u_n\|_{H^s})}{\delta}.\end{aligned} \tag{4.31}$$

There exists a constant $C > 0$ independent of δ such that

$$\begin{aligned}\|z_\delta - u_n\|_{H^s} &= \|t_n^*(w_\delta)(u_n - w_\delta) - u_n\|_{H^s}\\ &= \|(t_n^*(w_\delta)-1)(u_n - w_\delta) - w_\delta\|_{H^s}\\ &\leq \|w_\delta\|_{H^s} + |t_n^*(w_\delta)-1|\|u_n - w_\delta\|_{H^s} \leq \delta + C|t_n^*(w_\delta)-1|,\end{aligned}$$

and

$$\lim_{\delta\to 0}\frac{|t_n^*(w_\delta)-1|}{\delta} = \lim_{\delta\to 0}|\frac{t_n^*(w_\delta)-t_n^*(0)}{w_\delta}| = (t_n^*)'(0) \leq \|(t_n^*)'(0)\| \leq C.$$

Owing to $\lim\limits_{\delta\to 0}\|z_\delta - u_n\|_{H^s} = 0$, and letting $\delta \to 0$ in (4.31), we have that

$$\Big\langle (J_\lambda)'(u_n), \frac{u}{\|u\|_{H^s}}\Big\rangle \leq \frac{C}{n},$$

thus, (4.28) holds true. $\square$

Proof of Theorem 1.3.
By Proposition 4.6, there exists $\{u_n\} \subset M_\lambda^{(1)}$ such that

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1) \text{ and } (J_\lambda)'(u_n) = o(1).$$

By Corollary 4.2, Lemma 4.3, and Lemma 4.4, let $v_\lambda \in M_\lambda^-$ be a nontrivial solution of equation (E_λ) such that $J_\lambda(v_\lambda) = \alpha_\lambda^-$. Hence, J_λ has a minimizer v_λ on M_λ^-. By $\alpha_\lambda^- < \alpha_\lambda^{\infty,-} < A(p)(\frac{p-2}{2p})(\frac{2S_p^p}{a_\infty(4-p)})^{\frac{2}{p-2}}$, we obtain $v_\lambda \in M_\lambda^{(1)}$. By Lemma 2.2, problem (E_λ) has a nontrivial solution v_λ. Therefore, system (1.1) has a nontrivial solution $(v_\lambda, \phi_{v_\lambda}^t)$.

5. PROOF OF THE THEOREM 1.4

Recall that $w_0(x)$ is a unique positive solution of equation (E_0^∞) (up to translation) such that $J_0^\infty = \alpha_0^\infty = \frac{p-2}{2p}(\frac{S_p^p}{a_\infty})^{\frac{2}{p-2}}$ and $w_0(0) = \max_{x\in\mathbb{R}^3} w_0(x)$.

Define $T_a(w_0)$ as follows

$$\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{1}{p-2}} < T_a(w_0) := \Big(\frac{\|w_0\|_{H^s}^2}{\int_{\mathbb{R}^3} a(x)|w_0|^p dx}\Big)^{\frac{1}{p-2}} < 1, \tag{5.1}$$

where $\Big(\frac{\|w_0\|_{H^s}^2}{\int_{\mathbb{R}^3} a_\infty|w_0|^p dx}\Big)^{\frac{1}{p-2}} = 1$.

From (H_5) and $0 < \lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, we have

$$\int_{\mathbb{R}^3} a(x)|w_0|^p > \int_{\mathbb{R}^3} a_\infty|w_0|^p dx = a_\infty S_p^{-p}\|w_0\|_{H^s}^p > \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}}\|w_0\|_{H^s}^p.$$

Similar argument as Lemma 2.6, we can prove the following Lemma holds true.

Lemma 5.1. *If conditions (H_1)-(H_3) and (H_5) hold. Then there exists a positive number $\widehat{\Lambda} = \min\{\overline{\Lambda}, \widetilde{\Lambda}\} < \Lambda$, such that for every $0 < \lambda < \widehat{\Lambda}$, the following two statements are true.*
(1) If $\lambda\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx > \int_{\mathbb{R}^3} b(x)|w_0|^4 dx$, there exist two constants $\widetilde{t}_\lambda^+$ and $\widetilde{t}_\lambda^-$ satisfying

$$T_a(w_0) < \widetilde{t}_\lambda^- < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(w_0) < \Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(w_0) < \widetilde{t}_\lambda^+$$

such that $\widetilde{t}_\lambda^- w_0 \in M_\lambda^{(1)}$, $\widetilde{t}_\lambda^+ w_0 \in M_\lambda^{(2)}$, and

$$J_\lambda(\widetilde{t}_\lambda^- w_0) = \sup_{0\le t\le \widetilde{t}_\lambda^+} J_\lambda(tw_0) < \begin{cases} \alpha_\lambda^{\infty,-}, & \text{if } K_\infty > 0,\\ \alpha_0^\infty, & \text{if } K_\infty = 0,\end{cases}$$

and

$$J_\lambda(\widetilde{t}_\lambda^+ w_0) = \inf_{t\ge \widetilde{t}_\lambda^-} J_\lambda(tw_0) < 0.$$

(2) If $\lambda\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx \le \int_{\mathbb{R}^3} b(x)|w_0|^4 dx$, there exist a constant $\widetilde{t}_\lambda^0$ satisfying

$$0 < \widetilde{t}_\lambda^0 < T_a(w_0)$$

such that $\widetilde{t}_\lambda^0 w_0 \in M_\lambda^{(1)}$ and

$$J_\lambda(\widetilde{t}_\lambda^0 w_0) = \sup_{0\le t\le T_a(w_0)} J_\lambda(tw_0) < \begin{cases} \alpha_\lambda^{\infty,-}, & \text{if } K_\infty > 0,\\ \alpha_0^\infty, & \text{if } K_\infty = 0.\end{cases}$$

Proof. By (3.19), we have $\Lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$. For each $0 < \lambda < \Lambda$, similar to the proof of Lemma 2.6:
(1) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx > \int_{\mathbb{R}^3} b(x)|w_0|^4 dx$, there exist two constants $\widetilde{t}_\lambda^+$ and $\widetilde{t}_\lambda^-$ satisfying

$$T_a(w_0) < \widetilde{t}_\lambda^- < \sqrt{A(p)}\Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(w_0) < \Big(\frac{2}{4-p}\Big)^{\frac{1}{p-2}} T_a(w_0) < \widetilde{t}_\lambda^+,$$

such that $\widetilde{t}_\lambda^\pm w_0 \in M_\lambda^\pm$,

$$J_\lambda(\widetilde{t}_\lambda^- w_0) = \sup_{0 \le t \le \widetilde{t}_\lambda^+} J_\lambda(tw_0),$$

and

$$J_\lambda(\widetilde{t}_\lambda^+ w_0) = \inf_{t \ge \widetilde{t}_\lambda^-} J_\lambda(tw_0) = \inf_{t \ge 0} J_\lambda(tw_0) < 0.$$

(2) If $\lambda \int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx \le \int_{\mathbb{R}^3} b(x)|w_0|^4 dx$, there exists $\widetilde{t}_\lambda^0$ satisfying

$$0 < \widetilde{t}_\lambda^0 < T_a(w_0),$$

such that $\widetilde{t}_\lambda^0 w_0 \in M_\lambda^-$ and

$$J_\lambda(\widetilde{t}_\lambda^0 w_0) = \sup_{0 \le t \le T_a(w_0)} J_\lambda(tw_0).$$

From (H_5) and (5.1), it follows that

$$\begin{aligned}
J_\lambda(\widetilde{t}_\lambda^- w_0) = &\frac{(\widetilde{t}_\lambda^-)^2}{2}\|w_0\|_{H^s}^2 + \frac{\lambda(\widetilde{t}_\lambda^-)^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx \\
&- \frac{(\widetilde{t}_\lambda^-)^p}{p}\int_{\mathbb{R}^3} a(x)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^-)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\
=&\frac{(\widetilde{t}_\lambda^-)^2}{2}\|w_0\|_{H^s}^2 + \frac{\lambda(\widetilde{t}_\lambda^-)^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx - \frac{(\widetilde{t}_\lambda^-)^p}{p}\int_{\mathbb{R}^3} a(x)|w_0|^p dx \\
&- \frac{(\widetilde{t}_\lambda^-)^p}{p}\int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^-)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\
<&\alpha_0^\infty + \frac{\lambda}{4}A(p)^2\Big(\frac{2}{4-p}\Big)^{\frac{4}{p-2}} K_{max}^2 \overline{S}_t^{-1} S^{-2}\|w_0\|_{H^s}^4 \\
&- \frac{1}{p}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{p}{p-2}}\int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^-)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\
<&\alpha_0^\infty + \frac{\lambda}{4}A(p)^2\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{4}{p-2}} K_{max}^2 \overline{S}_t^{-1} S^{-2} \\
&- \frac{1}{p}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{p}{p-2}}\int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx.
\end{aligned}$$

This, together with $\alpha_0^\infty < \alpha_\lambda^{\infty,-}$(see Theorem 1.2), we obtain that there exists a positive number $\overline{\Lambda} \le \Lambda$ such that for every $\lambda < \overline{\Lambda}$, we have

$$J_\lambda(\widetilde{t}_\lambda^- w_0) < \begin{cases} \alpha_\lambda^{\infty,-}, & \text{if } K_\infty > 0, \\ \alpha_0^\infty, & \text{if } K_\infty = 0. \end{cases}$$

In fact, if $K_\infty > 0$, then $K_{max} > 0$, there exists a positive number $\overline{\Lambda} \le \Lambda$ such that for every $\lambda < \overline{\Lambda}$, we have

$$\begin{aligned} J_\lambda(\widetilde{t}_\lambda^- w_0) <& \alpha_0^\infty + \frac{\lambda}{4} A(p)^2 \Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{4}{p-2}} K_{max}^2 \overline{S}_t^{-1} S^{-2} \|w_0\|_{H^s}^4 \\ & - \frac{1}{p}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{p}{p-2}} \int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx < \alpha_\lambda^{\infty,-}. \end{aligned}$$

If $K_\infty = 0$, then we need to distinguish two cases.
(1): when $K_{max} = 0$, we have $J_\lambda(\widetilde{t}_\lambda^- w_0) < \alpha_0^\infty$.
(2): when $K_{max} > 0$, for every $\lambda < \overline{\Lambda}$, we have

$$\begin{aligned} J_\lambda(\widetilde{t}_\lambda^- w_0) <& \alpha_0^\infty + \frac{\lambda}{4} A(p)^2 \Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{4}{p-2}} K_{max}^2 \overline{S}_t^{-1} S^{-2} \|w_0\|_{H^s}^4 \\ & - \frac{1}{p}\Big(\frac{a_\infty}{a_{max}}\Big)^{\frac{p}{p-2}} \int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx \\ & < \alpha_\lambda^{\infty,-} = \alpha_0^\infty. \end{aligned}$$

It implies that

$$J_\lambda(\widetilde{t}_\lambda^- w_0) < \max\{\alpha_\lambda^{\infty,-}, \alpha_0^\infty\} < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$

We know $\widetilde{t}_\lambda^- w_0 \in M_\lambda^{(1)}$. Since $J_\lambda(\widetilde{t}_\lambda^+ w_0) < 0$, we obtain $\widetilde{t}_\lambda^+ w_0 \in M_\lambda^{(2)}$.

Similarly, we have

$$\begin{aligned} J_\lambda(\widetilde{t}_\lambda^0 w_0) =& \frac{(\widetilde{t}_\lambda^0)^2}{2}\|w_0\|_{H^s}^2 + \frac{\lambda(\widetilde{t}_\lambda^0)^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx \\ & - \frac{(\widetilde{t}_\lambda^0)^p}{p}\int_{\mathbb{R}^3} a(x)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^0)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\ =& \frac{(\widetilde{t}_\lambda^0)^2}{2}\|w_0\|_{H^s}^2 + \frac{\lambda(\widetilde{t}_\lambda^0)^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_0}^t w_0^2 dx - \frac{(\widetilde{t}_\lambda^0)^p}{p}\int_{\mathbb{R}^3} a_\infty|w_0|^p dx \\ & - \frac{(\widetilde{t}_\lambda^0)^p}{p}\int_{\mathbb{R}^3} (a(x)-a_\infty)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^0)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\ <& \alpha_0^\infty + \frac{\lambda}{4}K_{max}^2 \overline{S}_t^{-1} S^{-2}\|w_0\|_{H^s}^4 - \frac{(\widetilde{t}_\lambda^0)^p}{p}\int_{\mathbb{R}^3}(a(x)-a_\infty)|w_0|^p dx - \frac{(\widetilde{t}_\lambda^0)^4}{4}\int_{\mathbb{R}^3} b(x)|w_0|^4 dx \\ <& \alpha_0^\infty + \frac{\lambda}{4}K_{max}^2 \overline{S}_t^{-1} S^{-2}\|w_0\|_{H^s}^4 - \frac{\vartheta^p}{p}\int_{\mathbb{R}^3}(a(x)-a_\infty)|w_0|^p dx, \end{aligned}$$

where $\vartheta > 0$ sufficiently small. Using $\alpha_0^\infty < \alpha_\lambda^{\infty,-}$(see Theorem 1.2), Similar argument as above, there exists a positive number $\widetilde{\Lambda} \le \Lambda$ such that for every $\lambda < \widetilde{\Lambda}$, we have

$$J_\lambda(\widetilde{t}_\lambda^0 w_0) < \begin{cases} \alpha_\lambda^{\infty,-}, & \text{if } K_\infty > 0, \\ \alpha_0^\infty, & \text{if } K_\infty = 0, \end{cases}$$

that is

$$J_\lambda(\widetilde{t}_\lambda^0 w_0) < \max\{\alpha_\lambda^{\infty,-}, \alpha_0^\infty\} < A(p)(\frac{p-2}{2p})\Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{2}{p-2}}.$$

We see $\widetilde{t}_\lambda^0 w_0 \in M_\lambda^{(1)}$. □

Proof of Theorem 1.4.
Let $\{u_n\} \subset M_\lambda^{(1)}$ be a sequence satisfying

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1) \text{ and } (J_\lambda)'(u_n) = o(1).$$

In terms of Lemma 2.1 and Lemma 5.1, we obtain

$$\frac{p-2}{4p} C_0 \leq \alpha_\lambda^- < \begin{cases} \alpha_\lambda^{\infty,-}, & \text{if } K_\infty > 0, \\ \alpha_0^\infty, & \text{if } K_\infty = 0. \end{cases}$$

Because $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, we assume that there exists $u_0 \in H^s(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } H^s(\mathbb{R}^3), \tag{5.2}$$

$$u_n \to u_0 \text{ strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 < r < 2_s^*, \tag{5.3}$$

$$u_n \to u_0 \ a.e. \text{ in } \mathbb{R}^3. \tag{5.4}$$

We claim that $u_0 \not\equiv 0$. Otherwise, $u_0 \equiv 0$. Since $\{u_n\} \subset M_\lambda^{(1)}$ and $\alpha_\lambda^- > 0$, we have that

$$0 < \alpha_\lambda^- < J_\lambda(u_n) = \frac{1}{4}\|u_n\|_{H^s}^2 - \frac{4-p}{4p}\int_{\mathbb{R}^3} a(x)|u_n|^p dx \leq \frac{1}{4}\|u_n\|_{H^s}^2.$$

which yields that $\|u_n\|_{H^s} > \nu > 0$ for some constant ν and for all n. By concentration compactness principle [16], there are positive constants R, θ and a sequence $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(0;R)} |u_n(x+z_n)|^p dx \geq \theta \text{ for sufficiently large } n, \tag{5.5}$$

which implies that sequence $\{z_n\}$ is unbounded in $\mathbb{R}^3$. By contradiction, suppose that $z_n \to z_0$ for some $z_0 \in \mathbb{R}^3$. Using (5.3), (5.5) we have

$$\int_{B(z_0;R)} |u_0|^p dx \geq \theta,$$

which contradicts with $u_0 \equiv 0$.

Set

$$\widetilde{u}_n(x) = u_n(x+z_n),$$

it is not difficult to prove that

$$\begin{cases} J_0^\infty(\widetilde{u}_n) \to \alpha_\lambda^- \text{ and } (J_0^\infty)'(\widetilde{u}_n) = o(1) \text{ in } H^{-s}(\mathbb{R}^3), \text{ if } K_\infty = 0, \\ J_\lambda^\infty(\widetilde{u}_n) \to \alpha_\lambda^- \text{ and } (J_\lambda^\infty)'(\widetilde{u}_n) = o(1) \text{ in } H^{-s}(\mathbb{R}^3), \text{ if } K_\infty > 0. \end{cases} \tag{5.6}$$

In fact, up to a subsequence, we may assume that $\lim\limits_{n\to\infty} |x+z_n| = +\infty$. Hence, it is easy to check that

$$\int_{\mathbb{R}^3} a(x)|u_n|^p dx = \int_{\mathbb{R}^3} a(x+z_n)|\widetilde{u}_n|^p dx \to \int_{\mathbb{R}^3} a_\infty |\widetilde{u}_n|^p dx,$$

$$\int_{\mathbb{R}^3} b(x)|u_n|^4 dx = \int_{\mathbb{R}^3} b(x+z_n)|\widetilde{u}_n|^4 dx \to \int_{\mathbb{R}^3} b_\infty |\widetilde{u}_n|^4 dx = 0,$$

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3} K(x+z_n)\phi_{\widetilde{u}_n}^t \widetilde{u}_n^2 dx \to \int_{\mathbb{R}^3} K_\infty \phi_{\widetilde{u}_n}^t \widetilde{u}_n^2 dx.$$

Thus, when $K_\infty > 0$, $J_\lambda^\infty(\widetilde{u}_n) \to \alpha_\lambda^-$, when $K_\infty = 0$, $J_0^\infty(\widetilde{u}_n) \to \alpha_\lambda^-$.

In virtue of $\{u_n\}$ bounded in $H^s(\mathbb{R}^3)$, we may assume that there exists $\widetilde{u}_0 \in H^s(\mathbb{R}^3)$ such that

$$\widetilde{u}_n \rightharpoonup \widetilde{u}_0 \text{ weakly in } H^s(\mathbb{R}^3).$$

Similar argument as above can lead to the followings

$$\begin{cases} (J_0^\infty)'(\widetilde{u}_0) = 0 \ \text{ if } K_\infty = 0, \\ (J_\lambda^\infty)'(\widetilde{u}_0) = 0 \ \text{ if, } K_\infty > 0. \end{cases}$$

But, using Theorem 1.2, we see that $\widetilde{u}_0 \not\equiv 0$ in $\mathbb{R}^3$ and

$$\begin{cases} \widetilde{u}_0 \in M_0^\infty \text{ and } J_0^\infty(\widetilde{u}_0) \leq \alpha_\lambda^-, \text{ if } K_\infty = 0, \\ \widetilde{u}_0 \in M_\lambda^\infty \text{ and } J_\lambda^\infty(\widetilde{u}_0) \leq \alpha_\lambda^-, \text{ if } K_\infty > 0. \end{cases}$$

If $K_\infty = 0$, it is easily seen that $\alpha_0^\infty \leq J_0^\infty(\widetilde{u}_0) \leq \alpha_\lambda^-$, we achieve a contradiction with $\alpha_0^\infty < \alpha_\lambda^-$.

If $K_\infty > 0$, similarly argument as the proof of Theorem 1.2, it is not difficult to obtain that $\widetilde{u}_0 \in M_\lambda^{\infty,-}$, which indicates that $J_\lambda^\infty(\widetilde{u}_0) \geq \alpha_\lambda^{\infty,-}$, and so $\alpha_\lambda^{\infty,-} \leq J_\lambda^\infty(\widetilde{u}_0) \leq \alpha_\lambda^-$, this is impossible.

Hence, $u_0 \not\equiv 0$ and $(J_\lambda)'(u_0) = 0$, this means that equation (E_λ) has a nontrivial solution u_0.

Due to $\{u_n\} \in M_\lambda^{(1)}$ and

$$\|u_n\|_{H^s} < \widehat{D}_1 < \Big[\frac{2S_p^p}{a_{max}(4-p)}\Big]^{\frac{1}{p-2}} \text{ for all } n = 1, 2...,$$

using Fatou′s lemma, we can easily get that

$$\|u_0\|_{H^s} \leq \liminf_{n\to\infty} \|u_n\|_{H^s} \leq \widehat{D}_1 < \Big[\frac{2S_p^p}{a_{max}(4-p)}\Big]^{\frac{1}{p-2}}.$$

By Sobolev's inequality, we have that

$$\begin{aligned} (h_{\lambda,u_0})''(1) =& -2\|u_0\|_{H^s}^2 + (4-p)\int_{\mathbb{R}^3} a(x)|u_0|^p dx \\ \leq& -2\|u_0\|_{H^s}^2 + (4-p)a_{max}S_p^{-p}\|u_0\|_{H^s}^p \\ <& 0. \end{aligned}$$

Thus, $u_0 \in M_\lambda^-$ and then

$$J_\lambda(u_0) \geq \alpha_\lambda^-. \tag{5.7}$$

Now we are in a position to prove $u_n \to u_0$ strongly in $H^s(\mathbb{R}^3)$. By contradiction, suppose that there exists $c_0 > 0$ such that $\|u_n - u_0\|_{H^s} > c_0$. Let $v_n = u_n - u_0$. By (5.2)-(5.4), up to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup 0 \text{ weakly in } H^s(\mathbb{R}^3), \\ &v_n \to 0 \text{ strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 < r < 2_s^*, \\ &v_n \to 0 \ a.e. \text{ in } \mathbb{R}^3. \end{aligned}$$

In terms of conditions (H_1)-(H_3) and (5.6), we have

$$\|v_n\|_{H^s}^2 + \lambda\int_{\mathbb{R}^3} K_\infty \phi_{v_n}^t v_n^2 dx = \int_{\mathbb{R}^3} a_\infty |v_n|^p dx + o(1),$$

it means that

$$J_\lambda^\infty(v_n) \geq \begin{cases} \alpha_\lambda^{\infty,-}, \text{ if } K_\infty > 0, \\ \alpha_0^\infty, \text{ if } K_\infty = 0. \end{cases} \tag{5.8}$$

By (5.7), (5.8) and Brezis-Lieb Lemma in [4], we have

$$\begin{aligned} J_\lambda(u_n) =& \frac{1}{2}\|u_n\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^t u_n^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u_n|^p dx - \frac{1}{4}\int_{\mathbb{R}^3} b(x)|u_n|^4 dx \\ =& \frac{1}{2}\|u_0\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_0}^t u_0^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u_0|^p dx - \frac{1}{4}\int_{\mathbb{R}^3} b(x)|u_0|^4 dx \\ &+ \frac{1}{2}\|v_n\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_{v_n}^t v_n^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |v_n|^p dx + o(1) \\ =& J_\lambda(u_0) + J_\lambda^\infty(v_n) + o(1) \geq \begin{cases} \alpha_\lambda^- + \alpha_0^\infty + o(1), & \text{if } K_\infty = 0, \\ \alpha_\lambda^- + \alpha_\lambda^{\infty,-} + o(1), & \text{if } K_\infty > 0, \end{cases} \end{aligned}$$

which implies that

$$\lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^- \geq \begin{cases} \alpha_\lambda^- + \alpha_0^\infty + o(1) \text{ if } K_\infty = 0, \\ \alpha_\lambda^- + \alpha_\lambda^{\infty,-} + o(1) \text{ if } K_\infty > 0. \end{cases}$$

Thus, we get a contradiction. So, $u_n \to u_0$ strongly in $H^s(\mathbb{R}^3)$ and $J_\lambda(u_0) = \alpha_\lambda^-$. For $2 < p < 4$, we have

$$\alpha_\lambda^- < \max\{\alpha_\lambda^{\infty,-}, \alpha_0^\infty\} < A(p)(\frac{p-2}{2p})(\frac{2S_p^p}{a_\infty(4-p)})^{\frac{2}{p-2}}.$$

That is $u_0 \in M_\lambda^{(1)}$. Similarly, we obtain that $u_0 \in M_\lambda^-$ and $J_\lambda(u_0) = \alpha_\lambda^-$. By Lemma 2.2, we see that equation (E_λ) has a nontrivial solution u_0. Hence, system (1.1) has a nontrivial solution $(u_0, \phi_{u_0}^t)$.

6. Appendix

6.1. **Appendix A.** As we know that the following fractional Schrödinger equation

$$(-\Delta)^s u + u = a_\infty |u|^{p-2}u \qquad (E_0^\infty)$$

has a unique positive solution w_0 with $w_0(0) = \max_{x\in\mathbb{R}^3} w_0(x)$.

If $0 < \lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, $K(x) = K_\infty$ and $a(x) = a_\infty$, $b(x) = b_\infty = 0$, we have

$$T_a(w_0) = T_{a_\infty}(w_0) = \Big(\frac{\|w_0\|_{H^s}^2}{\int_{\mathbb{R}^3} a_\infty |w_0|^p dx}\Big)^{\frac{1}{p-2}} = 1,$$

and

$$\int_{\mathbb{R}^3} a_\infty |w_0|^p dx = a_\infty S_p^{-p}\|w_0\|_{H^s}^p > \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}}\|w_0\|_{H^s}^p. \tag{6.1}$$

For $2 < p < 4$, using Lemmas 2.5 and 2.6, there exists a constant $t_\lambda^+ > 0$ satisfying

$$(\frac{p}{4-p})^{\frac{1}{p-2}} < t_\lambda^+ < \hat{t}_\lambda^{(0)},$$

such that

$$J_\lambda^\infty(t_\lambda^+ w_0) = \inf_{(\frac{p}{4-p})^{\frac{1}{p-2}} < t < \hat{t}_\lambda^{(0)}} J_\lambda^\infty(tw_0) = \inf_{t\geq 0} J_\lambda^\infty(tw_0) < 0,$$

where $\widehat{t}_\lambda^{(0)}$ is given in Lemma 2.5. For $R>1$, we define a function $\psi_R \in C^1(\mathbb{R}^3,[0,1])$ as

$$\psi_R(x)=\begin{cases}1 & |x|<\frac{R}{2},\\ 0 & |x|>R,\end{cases}$$

and $|\nabla\psi_R|\le 1$ in $\mathbb{R}^3$. Let $u_R(x)=w_0(x)\psi_R(x)$. So, we have

$$\int_{\mathbb{R}^3}|u_R|^p dx \to \int_{\mathbb{R}^3}|w_0|^p dx \ \text{ as } \ R\to\infty, \tag{6.2}$$

$$\int_{\mathbb{R}^3}|u_R|^4 dx \to \int_{\mathbb{R}^3}|w_0|^4 dx \ \text{ as } \ R\to\infty,$$

$$\|u_R\|_{H^s} \to \|w_0\|_{H^s} \ \text{ as } \ R\to\infty, \tag{6.3}$$

$$\int_{\mathbb{R}^3}K_\infty\phi_{u_R}^t u_R^2 dx \to \int_{\mathbb{R}^3}K_\infty\phi_{w_0}^t w_0^2 dx \ \text{ as } \ R\to\infty. \tag{6.4}$$

Since $J_\lambda^\infty\in C^1(H^s(\mathbb{R}^3),\mathbb{R}^3)$, and using (6.1)-(6.4) there exists an $R_0>0$ such that

$$\int_{\mathbb{R}^3}a_\infty|u_{R_0}|^p dx > \frac{2}{p}a_\infty S_p^{-p}\|u_{R_0}\|_{H^s}^p, \tag{6.5}$$

and

$$J_\lambda^\infty(t_\lambda^+ u_{R_0})<0.$$

Let

$$u_{R_0,N}^{(i)}(x)=w_0(x+iN^3e)\psi_{R_0}(x+iN^3e)$$

for $e\in\mathbb{S}^2$ and $i=1,2,...,N$,where $N^3>2R_0$. In terms of condition (H_1), we get

$$\|u_{R_0,N}^{(1)}\|_{H^s}^2=\|u_{R_0}\|_{H^s}^2 \ \text{ for all } N. \tag{6.6}$$

In fact,

$$\begin{aligned}
\|u_{R_0,N}^{(1)}\|_{H^s}^2 &= C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_{R_0,N}^{(1)}(x)-u_{R_0,N}^{(1)}(y)|^2}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}|u_{R_0,N}^{(1)}|^2dx\\
&= C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|w_0(x+N^3e)\psi_{R_0}(x+N^3e)-w_0(y+N^3e)\psi_{R_0}(y+N^3e)|^2}{|x-y|^{3+2s}}dxdy\\
&\quad+\int_{\mathbb{R}^3}|w_0(x+N^3e)\psi_{R_0}(x+N^3e)|^2dx\\
&= C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|w_0(x)\psi_{R_0}(x)-w_0(y)\psi_{R_0}(y)|^2}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}|w_0(x)\psi_{R_0}(x)|^2dx\\
&= C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_{R_0}(x)-u_{R_0}(y)|^2}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}|u_{R_0}(x)|^2dx\\
&= \|u_{R_0}\|_{H^s}^2.
\end{aligned}$$

$$\int_{\mathbb{R}^3}a(x)|u_{R_0,N}^{(1)}|^p dx \to \int_{\mathbb{R}^3}a_\infty|u_{R_0}|^p dx \ \ as\ N\to\infty.$$

$$\begin{aligned}
\int_{\mathbb{R}^3}K_\infty\phi^t_{u_{R_0,N}^{(1)}}[u_{R_0,N}^{(1)}]^2dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u_{R_0,N}^{(1)}(x)]^2[u_{R_0,N}^{(1)}(y)]^2}{|x-y|^{3-2t}}dxdy\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x-N^3e)K(y-N^3e)[u_{R_0}(x)]^2[u_{R_0}(y)]^2}{|x-y|^{3-2t}}dxdy\\
&\to K_\infty^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_t[u_{R_0}(x)]^2[u_{R_0}(y)]^2}{|x-y|^{3-2t}}dxdy \text{ as } N\to\infty.
\end{aligned}$$

If $a(x) \geq a_\infty$, $K(x) \leq K_\infty$ and $b(x) \geq b_\infty = 0$, then there exists an N_0 with $N_0^3 > 2R_0$ such that for every $N > N_0$, we have

$$\int_{\mathbb{R}^3} a(x)|u^{(i)}_{R_0,N}|^p dx \geq \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx > \frac{2a_\infty}{pS_p^p}\|u_{R_0}\|^p_{H^s} = \frac{2a_\infty}{pS_p^p}\|u^{(i)}_{R_0,N}\|^p_{H^s},$$

and

$$\inf_{t\geq 0} J_\lambda(tu^{(i)}_{R_0,N}) \leq J_\lambda(t^+_\lambda u^{(i)}_{R_0,N}) \leq J^\infty_\lambda(t^+_\lambda u_{R_0}),$$

for $e \in \mathbb{S}^2$ and $i = 1, 2, ..., N$.
Let

$$w_{R_0,N}(x) = \sum_{i=1}^{N} u^{(i)}_{R_0,N}.$$

When $N^3 \geq N_0^3 > 2R_0$, by (6.6), we have

$$\begin{aligned}
\|w_{R_0,N}\|^2_{H^s} &= \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_{R_0,N}|^2 + (w_{R_0,N})^2 dx \\
&= C_s \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|w_{R_0,N}(x) - w_{R_0,N}(y)|^2}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} |w_{R_0,N}|^2 dx \\
&= C_s \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\sum_{i=1}^N u^{(i)}_{R_0,N}(x) - \sum_{i=1}^N u^{(i)}_{R_0,N}(y)|^2}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} |\sum_{i=1}^N u^{(i)}_{R_0,N}|^2 dx,
\end{aligned}$$

where

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\sum_{i=1}^N u^{(i)}_{R_0,N}(x) - \sum_{i=1}^N u^{(i)}_{R_0,N}(y)|^2}{|x-y|^{3+2s}} dxdy \\
&= \sum_{i=1}^N \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u^{(i)}_{R_0,N}(x) - u^{(i)}_{R_0,N}(y)|^2}{|x-y|^{3+2s}} dxdy \\
&+ 2\sum_{i\neq j}^N \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u^{(i)}_{R_0,N}(x) - u^{(i)}_{R_0,N}(y))(u^{(j)}_{R_0,N}(x) - u^{(j)}_{R_0,N}(y))}{|x-y|^{3+2s}} dxdy \\
&= N \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_{R_0}(x) - u_{R_0}(y)|^2}{|x-y|^{3+2s}} dxdy \\
&+ \sum_{i\neq j}^N \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2}{|x-y|^{3+2s}} \Big[u_{R_0}(x+iN^3e)u_{R_0}(x+jN^3e) - u_{R_0}(x+iN^3e)u_{R_0}(y+jN^3e) \\
&- u_{R_0}(x+jN^3e)u_{R_0}(y+iN^3e) + u_{R_0}(y+iN^3e)u_{R_0}(y+jN^3e)\Big] dxdy.
\end{aligned}$$

Let

$$\begin{aligned}
T = &\sum_{i\neq j}^N \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2}{|x-y|^{3+2s}} \Big[u_{R_0}(x+iN^3e)u_{R_0}(x+jN^3e) - u_{R_0}(x+iN^3e)u_{R_0}(y+jN^3e) \\
&- u_{R_0}(x+jN^3e)u_{R_0}(y+iN^3e) + u_{R_0}(y+iN^3e)u_{R_0}(y+jN^3e)\Big] dxdy.
\end{aligned}$$

By $|\psi_{R_0}| \leq 1$, we have

$$\begin{aligned}
&\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{R_0}(x+iN^3e)u_{R_0}(y+jN^3e)}{|x-y|^{3+2s}}dxdy\\
&=\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{R_0}(x)u_{R_0}(y)}{|x-y+(j-i)N^3e|^{3+2s}}dxdy\\
&\leq\sum_{i\neq j}^{N}\frac{1}{|N^3-2R_0|^{3+2s}}(\int_{\mathbb{R}^3}w_{R_0}(x)dx)^2\\
&\to 0 \ \text{ as } \ N\to\infty.
\end{aligned}$$

Similarly,

$$\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{R_0}(x+jN^3e)u_{R_0}(y+iN^3e)}{|x-y|^{3+2s}}dxdy\to 0 \ \text{ as } \ N\to\infty.$$

By Hölder inequality, we have

$$\begin{aligned}
&\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{R_0}(x+iN^3e)u_{R_0}(x+jN^3e)+u_{R_0}(y+iN^3e)u_{R_0}(y+jN^3e)}{|x-y|^{3+2s}}dxdy\\
&\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{R_0}(x)u_{R_0}(x+(j-i)N^3e)+u_{R_0}(y)u_{R_0}(y+(i-j)N^3e)}{|x-y+(j-i)N^3e|^{3+2s}}dxdy\\
&\leq\sum_{i\neq j}^{N}\frac{2}{|N^3-2R_0|^{3+2s}}\int_{\mathbb{R}^3}w_{R_0}(x)w_{R_0}(x+(j-i)N^3e)dx\\
&\leq\sum_{i\neq j}^{N}\frac{2}{|N^3-2R_0|^{3+2s}}(\int_{\mathbb{R}^3}|w_{R_0}(x)|^2dx)^{\frac{1}{2}}(\int_{\mathbb{R}^3}|w_{R_0}(x+(j-i)N^3e)|^2dx)^{\frac{1}{2}}\\
&\leq\sum_{i\neq j}^{N}\frac{2}{|N^3-2R_0|^{3+2s}}\int_{\mathbb{R}^3}|w_{R_0}(x)|^2dx \quad\to 0 \ \text{ as } \ N\to\infty,
\end{aligned}$$

where

$$|x-iN^3e-(y-jN^3e)|^{3+2s}\geq\Big||i-j|N^3e-|x|-|y|\Big|^{3+2s}\geq|N^3-2R_0|^{3+2s}.$$

Hence, we have

$$\begin{aligned}
&\|w_{R_0,N}\|_{H^s}^2\\
&=NC_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_{R_0}(x)-u_{R_0}(y)|^2}{|x-y|^{3+2s}}dxdy+N\int_{\mathbb{R}^3}|u_{R_0}|^2dx+T\\
&=N\|u_{R_0}\|_{H^s}^2+T.
\end{aligned}$$

So, we obtain

$$\|w_{R_0,N}\|_{H^s}^2=N\|u_{R_0}\|_{H^s}^2+T, \tag{6.7}$$

$$\int_{\mathbb{R}^3}a(x)|w_{R_0,N}|^pdx=\sum_{i=1}^{N}\int_{\mathbb{R}^3}a(x)|u_{R_0,N}^{(i)}|^pdx, \tag{6.8}$$

$$\int_{\mathbb{R}^3} b(x)|w_{R_0,N}|^4dx = \sum_{i=1}^{N}\int_{\mathbb{R}^3} b(x)|u^{(i)}_{R_0,N}|^4dx, \tag{6.9}$$

$$\begin{aligned}\int_{\mathbb{R}^3} K_\infty\phi^t_{w_{R_0,N}}[w_{R_0,N}]^2dx =& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[w_{R_0,N}(x)]^2[w_{R_0,N}(y)]^2}{|x-y|^{3-2t}}dxdy\\ =&\sum_{i=1}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u^{(i)}_{R_0,N}(x)]^2[u^{(i)}_{R_0,N}(y)]^2}{|x-y|^{3-2t}}dxdy\\ &+\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u^{(i)}_{R_0,N}(x)]^2[u^{(i)}_{R_0,N}(y)]^2}{|x-y|^{3-2t}}dxdy.\end{aligned} \tag{6.10}$$

By simple calculation, we obtain

$$\begin{aligned}&\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u^{(i)}_{R_0,N}(x)]^2[u^{(i)}_{R_0,N}(y)]^2}{|x-y|^{3-2t}}dxdy\\ \leq&\frac{C_tK^2_{max}(N^2-N)}{(N^3-2R_0)^{3-2t}}\Big(\int_{\mathbb{R}^3} w_0^2(x)dx\Big)^2,\end{aligned}$$

which indicates that

$$\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u^{(i)}_{R_0,N}(x)]^2[u^{(i)}_{R_0,N}(y)]^2}{|x-y|^{3-2t}}dxdy \to 0 \quad \text{as} \quad N\to\infty. \tag{6.11}$$

Lemma 6.1. *For $2<p<4$ and $0<\lambda<\frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$. Let $K(x)\leq K_\infty$ and $a(x)\geq a_\infty$, $b(x)=b_\infty=0$. Then we obtain*

$$\alpha^+_\lambda = \inf_{u\in M^{(2)}_\lambda} J_\lambda(u) = \inf_{u\in M^+_\lambda} J_\lambda(u) = -\infty. \tag{6.12}$$

Proof. For $N\in\mathbb{N}$, and let

$$f_N(t) = t^{-2}\|w_{R_0,N}\|^2_{H^s} - t^{p-4}\int_{\mathbb{R}^3} a(x)|w_{R_0,N}|^p dx \text{ for } t>0,$$

and

$$f^\infty(t) = t^{-2}\|u_{R_0}\|^2_{H^s} - t^{p-4}\int_{\mathbb{R}^3} a_\infty|u_{R_0}|^p dx \text{ for } t>0.$$

According to (6.7) and (6.8), it is easy to get

$$\begin{aligned}f_N(t) =& t^{-2}(N\|u_{R_0}\|^2_{H^s}+T) - t^{p-4}\sum_{i=1}^{N}\int_{\mathbb{R}^3} a(x)|u^{(i)}_{R_0,N}|^p dx\\ \leq& t^{-2}N\|u_{R_0}\|^2_{H^s} - t^{p-4}N\int_{\mathbb{R}^3} a_\infty|u_{R_0}|^p dx + t^{-2}T\\ =& Nf^\infty(t)+t^{-2}T.\end{aligned} \tag{6.13}$$

So we observe that $tw_{R_0,N}\in M_\lambda$ if and only if

$$f_N(t)+\lambda\int_{\mathbb{R}^3} K(x)\phi^t_{w_{R_0,N}}w^2_{R_0,N}dx - \int_{\mathbb{R}^3} b(x)|w_{R_0,N}|^4dx = 0.$$

By a direct computation, we can deduce that

$$f^\infty(T_{a_\infty}(u_{R_0}))=0,\ \lim_{t\to 0^+} f^\infty(t)=\infty \text{ and } \lim_{t\to\infty} f^\infty(t)=0,$$

where

$$T_{a_\infty}(u_{R_0}) = \Big(\frac{\|u_{R_0}\|_{H^s}^2}{\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx}\Big)^{\frac{1}{p-2}}.$$

For $2 < p < 4$ and

$$(f^\infty)'(t) = -2t^{-3}\|u_{R_0}\|_{H^s}^2 - (p-4)t^{p-5}\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx,$$

we can easily show that $f^\infty(t)$ is decreasing when $0 < t < (\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{1}{p-2}}$ and is increasing when $t > (\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{1}{p-2}}$. In view of (6.5) we can deduce that

$$\begin{aligned}
\inf_{t>0} f^\infty(t) =& f^\infty\Big((\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{1}{p-2}}\Big)\\
=& -\frac{p-2}{4-p}\Big(\frac{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx}{2\|u_{R_0}\|_{H^s}^2}\Big)^{\frac{2}{p-2}}\|u_{R_0}\|_{H^s}^2\\
<& -\frac{p-2}{4-p}\Big(\frac{(4-p)a_\infty\|u_{R_0}\|_{H^s}^p}{pS_p^p\|u_{R_0}\|_{H^s}^2}\Big)^{\frac{2}{p-2}}\|u_{R_0}\|_{H^s}^2\\
=& -\frac{p-2}{4-p}\Big(\frac{(4-p)a_\infty}{pS_p^p}\Big)^{\frac{2}{p-2}}\|u_{R_0}\|_{H^s}^4.
\end{aligned}$$

For $0 < \lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, by (2.4) we have

$$\begin{aligned}
&\lambda < \frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Big[1-A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}}\Big](\frac{a_\infty}{S_p^p})^{\frac{2}{p-2}}\frac{\overline{S}_t S^2}{K_{max}^2}\\
&\frac{p-2}{4-p}(\frac{(4-p)a_\infty}{pS_p^p})^{\frac{2}{p-2}} > 2\lambda\frac{K_{max}^2}{\overline{S}_t S^2}\frac{1}{\Big[1-A(p)(\frac{a_{max}}{a_\infty})^{\frac{2}{p-2}}\Big]}\\
&\frac{p-2}{4-p}(\frac{(4-p)a_\infty}{pS_p^p})^{\frac{2}{p-2}} > 2\lambda\frac{K_{max}^2}{\overline{S}_t S^2} > \lambda K_{max}^2\overline{S}_t^{-1}S^{-2},
\end{aligned}$$

hence, in terms of Lemma 2.3 and (6.13),

$$\begin{aligned}
\inf_{t>0} f_N(t) \le& f_N\Big((\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{1}{p-2}}\Big)\\
\le& -\frac{N(p-2)}{4-p}\Big(\frac{(4-p)a_\infty}{pS_p^p}\Big)^{\frac{2}{p-2}}\|u_{R_0}\|_{H^s}^4 + (\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{-2}{p-2}}T\\
<& -\lambda N K_{max}^2\overline{S}_t^{-1}S^{-2}\|u_{R_0}\|_{H_s}^4 + (\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{-2}{p-2}}T\\
<& -\lambda N\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_t K(x)K(y)[u_{R_0}(x)]^2[u_{R_0}(y)]^2}{|x-y|^{3-2t}}dxdy + (\frac{2\|u_{R_0}\|_{H^s}^2}{(4-p)\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p dx})^{\frac{-2}{p-2}}T.
\end{aligned}$$

By (6.11), we can deduce that

$$\begin{aligned}\inf_{t>0} f_N(t) <& -\lambda N\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u_{R_0}(x)]^2[u_{R_0}(y)]^2}{|x-y|^{3-2t}}dxdy\\ &-\lambda\sum_{i\neq j}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{C_tK(x)K(y)[u^{(i)}_{R_0}(x)]^2[u^{(i)}_{R_0}(y)]^2}{|x-y|^{3-2t}}dxdy+\Big(\frac{2\|u_{R_0}\|^2_{H^s}}{(4-p)\int_{\mathbb{R}^3}a_\infty|u_{R_0}|^pdx}\Big)^{\frac{-2}{p-2}}T\\ =& -\lambda\int_{\mathbb{R}^3}K(x)\phi^t_{w_{R_0,N}}w^2_{R_0,N}dx+\Big(\frac{2\|u_{R_0}\|^2_{H^s}}{(4-p)\int_{\mathbb{R}^3}a_\infty|u_{R_0}|^pdx}\Big)^{\frac{-2}{p-2}}T.\end{aligned}$$

From the above proof, we know that $T\to 0$ as $N\to\infty$, hence, we get that

$$\inf_{t>0} f_N(t) < -\lambda\int_{\mathbb{R}^3}K(x)\phi^t_{w_{R_0,N}}w^2_{R_0,N}dx+\int_{\mathbb{R}^3}b(x)w^4_{R_0,N}dx \text{ for sufficiently large } N.$$

Hence, when $0<\lambda<\frac{p-2}{2(4-p)}(\frac{4-p}{p})^{\frac{2}{p-2}}\Lambda_0$, by Lemma 2.6, we need to distinguish two cases.
(1) If $\lambda\int_{\mathbb{R}^3}K(x)\phi^t_{w_{R_0,N}}w^2_{R_0,N}dx>\int_{\mathbb{R}^3}b(x)w^4_{R_0,N}dx$ there exist two constants $t^{(1)}_{\lambda,N}$ and $t^{(2)}_{\lambda,N}$ satisfying

$$1<t^{(1)}_{\lambda,N}<\Big(\frac{2\|u_{R_0}\|^2_{H^s}}{(4-p)\int_{\mathbb{R}^3}a_\infty|u_{R_0}|^pdx}\Big)^{\frac{1}{p-2}}<t^{(2)}_{\lambda,N},$$

such that

$$f_N(t^{(i)}_{\lambda,N})+\lambda\int_{\mathbb{R}^3}K(x)\phi^t_{w_{R_0,N}}w^2_{R_0,N}dx-\int_{\mathbb{R}^3}b(x)w^4_{R_0,N}dx=0,$$

for $i=1,2$ and for all $N\in\mathbb{N}$. So, $t^{(i)}_{\lambda,N}w_{R_0,N}\in M_\lambda$ for $i=1,2$ and for all $N\in\mathbb{N}$. Taking the derivative of $h'_{\lambda,t^{(i)}_{\lambda,N}w_{R_0,N}}(t)$, we have

$$\begin{aligned}h''_{\lambda,t^{(1)}_{\lambda,N}w_{R_0,N}}(1)=&-2\|t^{(1)}_{\lambda,N}w_{R_0,N}\|^2_{H^s}+(4-p)\int_{\mathbb{R}^3}a(x)|t^{(1)}_{\lambda,N}w_{R_0,N}|^pdx=(t^{(1)}_{\lambda,N})^5f'_N(t^{(1)}_{\lambda,N})\\ &<0,\end{aligned}$$

and

$$\begin{aligned}h''_{\lambda,t^{(2)}_{\lambda,N}w_{R_0,N}}(1)=&-2\|t^{(2)}_{\lambda,N}w_{R_0,N}\|^2_{H^s}+(4-p)\int_{\mathbb{R}^3}a(x)|t^{(2)}_{\lambda,N}w_{R_0,N}|^pdx=(t^{(2)}_{\lambda,N})^5f'_N(t^{(2)}_{\lambda,N})\\ &>0.\end{aligned}$$

Hence, we can easily deduce that

$$t^{(1)}_{\lambda,N}w_{R_0,N}\in M^-_\lambda \text{ and } t^{(2)}_{\lambda,N}w_{R_0,N}\in M^+_\lambda.$$

In terms of (6.7)-(6.11), we get

$$\begin{aligned}J_\lambda(t^{(2)}_{\lambda,N}w_{R_0,N})&=\inf_{t>0}J_\lambda(tw_{R_0,N})\le J_\lambda(t^+_\lambda w_{R_0,N})\\ &\le NJ^\infty_\lambda(t^+_\lambda u_{R_0})+(t^+_\lambda u_{R_0})^{-2}T+C_0 \text{ for some } C_0>0,\end{aligned}$$

and

$$J_\lambda(t^{(2)}_{\lambda,N}w_{R_0,N})\to-\infty \text{ as } N\to\infty.$$

Hence, (6.12) is proved.

(2) If $\lambda\int_{\mathbb{R}^3} K(x)\phi^t_{w_{R_0,N}} w^2_{R_0,N}dx \le \int_{\mathbb{R}^3} b(x)w^4_{R_0,N}dx$ there exist a constant $t^{(0)}_{\lambda,N}$ satisfying

$$0 < t^{(0)}_{\lambda,N} < T_{a_\infty}(w_0) = 1,$$

such that

$$f_N(t^{(0)}_{\lambda,N}) + \lambda\int_{\mathbb{R}^3} K(x)\phi^t_{w_{R_0,N}} w^2_{R_0,N}dx - \int_{\mathbb{R}^3} b(x)w^4_{R_0,N}dx = 0.$$

Similar to the argument above we have $t^{(0)}_{\lambda,N} w_{R_0,N} \in M^-_\lambda$. □

6.2. **Appendix B.** In order to prove (3.9), we will apply the concentration-compactness lemma due to [16, 17], to get the compactness.

For $2 < p < 4$, let $\{u_n\} \subset M^{\infty,(1)}_\lambda$ be a sequence as follows

$$\lim_{n\to\infty} J^\infty_\lambda(u_n) = \alpha^{\infty,-}_\lambda > 0. \tag{6.14}$$

Define the functional $\Phi^\infty_\lambda : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Phi^\infty_\lambda(u) = \frac{p-2}{2p}\|u\|^2_{H^s} - \frac{\lambda(4-p)}{4p}\int_{\mathbb{R}^3} K_\infty \phi^t_u u^2 dx. \tag{6.15}$$

Using Lemma 2.1, for any $u \in M^{\infty,(1)}_\lambda \subset M^{\infty,-}_\lambda$ we obtain

$$J^\infty_\lambda(u) = \Phi^\infty_\lambda(u) > 0.$$

In view of $\{u_n\} \subset M^{\infty,(1)}_\lambda$, we have that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, it indicates that there exist a subsequence $\{u_n\}$ and $u_\infty \in H^s(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u_\infty \text{ weakly in } H^s(\mathbb{R}^3), \\ u_n \to u_\infty \text{ strongly in } L^r_{loc} \text{ for } 2 \le r < 2^*_s, \\ u_n \to u_\infty \ a.e. \text{ in } \mathbb{R}^3. \end{cases} \tag{6.16}$$

We need that sequence $\{u_n\}$ has compactness. Finally, using a concentration-compactness argument on the positive measures which are defined as follows. For every $u_n \in M^{\infty,(1)}_\lambda$, we define the measure $v_n(\Omega)$ by

$$v_n(\Omega) = \frac{p-2}{2p}\int_\Omega \Big(\frac{C_s}{2}\int_{\mathbb{R}^3} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dx + u_n^2\Big)dx - \frac{\lambda(4-p)}{4p}\int_\Omega K_\infty \phi^t_{u_n} u_n^2 dx, \tag{6.17}$$

where C_s is a normalized constant. By $u_n \in M_\lambda^{\infty,(1)}$, we obtain $\|u_n\|_{H^s} < \widehat{D}_1$. By Lemma 2.3 and $K(x) = K_\infty$, we have

$$\begin{aligned}
v_n(\Omega) &= \frac{p-2}{2p}\int_\Omega \Big(\frac{C_s}{2}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dx + u_n^2\Big)dx - \frac{\lambda(4-p)}{4p}\int_\Omega K_\infty \phi_{u_n}^t u_n^2 dx \\
&\geq \frac{p-2}{2p}\int_\Omega \Big(\frac{C_s}{2}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dx + u_n^2\Big)dx - \frac{\lambda(4-p)}{4p}K_{max}^2 \overline{S}_t^{-1} S^{-2}\|u_n\|_{H^s(\Omega)}^2 \|u_n\|_{H^s(\mathbb{R}^3)}^2 \\
&\geq \frac{p-2}{2p}\int_\Omega \Big(\frac{C_s}{2}\int_\Omega\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dx + u_n^2\Big)dx \\
&- \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}}\frac{4-p}{4p}K_{max}^2\overline{S}_t^{-1}S^{-2}\|u_n\|_{H^s(\Omega)}^2 \|u_n\|_{H^s(\mathbb{R}^3)}^2 \\
&= \frac{p-2}{2p}\|u_n\|_{H^s(\Omega)}^2 - \frac{(p-2)}{8p}\frac{K_{max}^2}{K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}}\|u_n\|_{H^s(\Omega)}^2 \|u_n\|_{H^s(\mathbb{R}^3)}^2 \\
&\geq \frac{p-2}{2p}\|u_n\|_{H^s(\Omega)}^2\Big[1 - \frac{1}{2}\frac{K_{max}^2}{K_\infty^2}\Big(\frac{4-p}{p}\Big)^{\frac{2}{p-2}}\Big] \\
&= \frac{p-2}{2p}\|u_n\|_{H^s(\Omega)}^2\Big[1 - \frac{1}{2}\Big(\frac{4-p}{p}\Big)^{\frac{2}{p-2}}\Big] \\
&\geq 0.
\end{aligned}$$

On the other hand, it is not difficult to check that $v_n(\Omega)$ possesses the subadditivity and monotonicity, Thus, $v_n(\Omega)$ is a positive measure on $\mathbb{R}^3$. In terms of (6.14), we have

$$v_n(\mathbb{R}^3) = \Phi_\lambda^\infty(u_n) = \alpha_\lambda^{\infty,-} + o(1),$$

and we have three possibilities as follows:
(a) Vanishing: for all $r > 0$,

$$\lim_{n\to\infty}\sup_{\xi\in\mathbb{R}^3}\int_{B_r(\xi)} dv_n = 0, \tag{6.18}$$

where $B_r(\xi) = \{x \in \mathbb{R}^3 : |x-\xi| < r\}$.
(b) Dichotomy: there exist a constant $\alpha \in (0, \alpha_\lambda^{\infty,-})$, two sequences $\{\xi_n\}$ and $\{r_n\}$, with $r_n \to \infty$ and two nonnegative measures v_n^1 and v_n^2 such that

$$v_n - (v_n^1 + v_n^2) \to 0, \quad v_n^1(\mathbb{R}^3) \to \alpha, \quad v_n^2(\mathbb{R}^3) \to \alpha_\lambda^{\infty,-} - \alpha, \tag{6.19}$$

and

$$supp(v_n^1) \subset B_{r_n}(\xi_n), \quad supp(v_n^2) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n). \tag{6.20}$$

(c) Compactness: there exists a sequence $\{\xi_n\} \in \mathbb{R}^3$ with the following property: for any $\delta > 0$, there exists an $r = r(\delta) > 0$ such that

$$\int_{B_r(\xi_n)} dv_n \geq \alpha_\lambda^{\infty,-} - \delta, \text{ for large } n. \tag{6.21}$$

Lemma 6.2. *For* $0 < \lambda < \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\big(\frac{a_\infty(4-p)^2}{2pS_p^p}\big)^{\frac{2}{p-2}}$, $2 < p < 4$, *compactness holds for the sequence of measures* $\{u_n\}$ *defined by* (6.17).

Proof. (I) Vanishing does not occur. Otherwise, for all $r > 0$, (6.18) holds. We can show that there exists an $\overline{r} > 0$ such that

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}^3} \int_{B_{\overline{r}}(\xi)} u_n^2 dx = 0.$$

It indicates that $u_n \to 0$ strongly in $L^s(\mathbb{R}^3)$ for $2 < s < 2_s^*$. Hence, for $u_n \in M_\lambda^{\infty,(1)}$, by Lemma 2.1, we have

$$\begin{aligned} 0 \le J_\lambda^\infty(u_n) =& \frac{1}{2}\|u_n\|_{H^s}^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_{u_n}^t u_n^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a_\infty |u_n|^p dx \\ =& -\frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_{u_n}^t u_n^2 dx + \frac{p-2}{2p}\int_{\mathbb{R}^3} a_\infty |u_n|^p dx \to 0. \end{aligned}$$

Hence, we get a contradiction.
(II) Dichotomy does not occur. Otherwise, there exist a constant $\alpha \in (0, \alpha_\lambda^{\infty,-})$, two sequences $\{\xi_n\}$ and $\{r_n\}$, with $r_n \to \infty$ and two nonnegative measures v_n^1 and v_n^2 such that (6.19) and (6.20) hold. Let $\rho_n \in C^1(\mathbb{R}^3)$ satisfy that

$$\begin{cases} \rho_n = 0 & \text{in} \quad \mathbb{R}^3 \backslash B_{2r_n}(\xi_n), \\ 0 < \rho_n < 1 & \text{in} \quad B_{2r_n}(\xi_n) \backslash B_{r_n}(\xi_n), \\ \rho_n = 1 & \text{in} \quad B_{r_n}(\xi_n), \end{cases}$$

and $|\nabla \rho_n| \le \frac{2}{r_n}$. Let

$$h_n := \rho_n u_n, \quad w_n := (1-\rho_n) u_n.$$

Similar argument in [23], we have

$$\liminf_{n\to\infty} \Phi_\lambda^\infty(h_n) \ge \alpha \text{ and } \liminf_{n\to\infty} \Phi_\lambda^\infty(w_n) \ge \alpha_\lambda^{\infty,-} - \alpha. \tag{6.22}$$

Therefore, let $\Omega_n := B_{2r_n}(\xi_n) \setminus B_{r_n}(\xi_n)$, then we have

$$v_n(\Omega_n) \to 0 \text{ as } n \to \infty,$$

that is

$$\int_{\Omega_n}\int_{\mathbb{R}^3} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}} dxdy + \int_{\Omega_n} u_n^2 dx \to 0 \text{ as } n \to \infty,$$

$$\int_{\Omega_n} K_\infty \phi_{u_n}^t u_n^2 dx \to 0 \text{ as } n \to \infty. \tag{6.23}$$

Similar arguments as that in [23], we have that

$$\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} h_n|^2 dx + \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} w_n|^2 dx + o_n(1), \tag{6.24}$$

$$\int_{\mathbb{R}^3} u_n^2 dx = \int_{\mathbb{R}^3} h_n^2 dx + \int_{\mathbb{R}^3} w_n^2 dx, \tag{6.25}$$

$$\int_{\mathbb{R}^3} |u_n|^p dx = \int_{\mathbb{R}^3} |h_n|^p dx + \int_{\mathbb{R}^3} |w_n|^p dx + o_n(1), \tag{6.26}$$

and

$$\int_{\mathbb{R}^3} K_\infty \phi_{u_n}^t u_n^2 dx \ge \int_{\mathbb{R}^3} K_\infty \phi_{h_n}^t h_n^2 dx - \int_{\mathbb{R}^3} K_\infty \phi_{w_n}^t w_n^2 dx + o_n(1). \tag{6.27}$$

Thus, by (6.24)-(6.27), we deduce that

$$\Phi_\lambda^\infty(u_n) \ge \Phi_\lambda^\infty(h_n) + \Phi_\lambda^\infty(w_n) + o_n(1).$$

Hence, we have

$$\alpha_\lambda^{\infty,-} \geq \lim_{n\to\infty} \Phi_\lambda^\infty(u_n) \geq \lim_{n\to\infty} \Phi_\lambda^\infty(h_n) + \lim_{n\to\infty} \Phi_\lambda^\infty(w_n) \geq \alpha + (\alpha_\lambda^{\infty,-} - \alpha) = \alpha_\lambda^{\infty,-}.$$

It indicates that $\lim_{n\to\infty} \Phi_\lambda^\infty(h_n) + \lim_{n\to\infty} \Phi_\lambda^\infty(w_n) = \alpha_\lambda^{\infty,-}$. By (6.22), we get

$$\lim_{n\to\infty} \Phi_\lambda^\infty(h_n) = \alpha > 0 \quad \text{and} \quad \lim_{n\to\infty} \Phi_\lambda^\infty(w_n) = \alpha_\lambda^{\infty,-} - \alpha. \tag{6.28}$$

By (6.24), (6.25) and (6.26), (6.27) we have

$$0 = \langle (J_\lambda^\infty)'(u_n), u_n\rangle \geq \langle (J_\lambda^\infty)'(h_n), h_n\rangle + \langle (J_\lambda^\infty)'(w_n), w_n\rangle + o_n(1). \tag{6.29}$$

So for all $n \geq 1$ and $2 < p < 4$, there holds

$$\max\{\|h_n\|_{H^s}, \|w_n\|_{H^s}\} < \widehat{D}_1 < \Big(\frac{2S_p^p}{a_\infty(4-p)}\Big)^{\frac{1}{p-2}} \tag{6.30}$$

and

$$\lambda < \frac{(p-2)\overline{S}_t S^2}{2(4-p)K_\infty^2}\Big(\frac{a_\infty(4-p)^2}{2pS_p^p}\Big)^{\frac{2}{p-2}}.$$

Furthermore, we get

$$\begin{aligned}
\frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}} \|h_n\|_{H^s}^p &= \frac{p}{4-p}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}} \|h_n\|_{H^s}^{p-2}\|h_n\|_{H^s}^2 \\
&\leq \frac{2pS_p^p}{a_\infty(4-p)^2}\Big(\frac{2\lambda(4-p)K_\infty^2}{(p-2)\overline{S}_t S^2}\Big)^{\frac{p-2}{2}} \|h_n\|_{H^s}^2 \\
&< \|h_n\|_{H^s}^2 \leq \|h_n\|_{H^s}^2 + \lambda\int_{\mathbb{R}^3} K_\infty \phi_{h_n}^t h_n^2 dx \\
&\leq \int_{\mathbb{R}^3} a_\infty |h_n|^p dx.
\end{aligned}$$

By Lemma 2.6, for any $n \geq 1$ there exists

$$T_{a_\infty}(h_n) < t_{\lambda,n}^- < \sqrt{A(p)}(\frac{2}{4-p})^{\frac{1}{p-2}} T_{a_\infty}(h_n),$$

such that $t_{\lambda,n}^- h_n \in M_\lambda^{\infty,-}$, where

$$T_{a_\infty}(h_n) = \Big(\frac{\|h_n\|_{H^s}^2}{\int_{\mathbb{R}^3} a_\infty |h_n|^p dx}\Big)^{\frac{1}{p-2}} > 0.$$

Next, we discuss the following three cases:

Case (i). Up to a subsequence, $\langle (J_\lambda^\infty)'(h_n), h_n\rangle \leq 0$. We claim that $t_{\lambda,n}^- \leq 1$. Since $t_{\lambda,n}^- h_n \in M_\lambda^{\infty,-}$, we have

$$\int_{\mathbb{R}^3} \lambda K_\infty \phi_{h_n}^t h_n^2 dx = -(t_{\lambda,n}^-)^{-2}\|h_n\|_{H^s}^2 + (t_{\lambda,n}^-)^{p-4}\int_{\mathbb{R}^3} a_\infty |h_n|^p dx. \tag{6.31}$$

By (6.31), we obtain

$$\begin{aligned}
0 \geq \langle (J_\lambda^\infty)'(h_n), h_n\rangle &= \|h_n\|_{H^s}^2 + \lambda\int_{\mathbb{R}^3} K_\infty \phi_{h_n}^t h_n^2 dx - \int_{\mathbb{R}^3} a_\infty |h_n|^p dx \\
=& \|h_n\|_{H^s}^2 - (t_{\lambda,n}^-)^{-2}\|h_n\|_{H^s}^2 + (t_{\lambda,n}^-)^{p-4}\int_{\mathbb{R}^3} a_\infty |h_n|^p dx - \int_{\mathbb{R}^3} a_\infty |h_n|^p dx \qquad (6.32) \\
=& [1-(t_{\lambda,n}^-)^{-2}]\|h_n\|_{H^s}^2 + [(t_{\lambda,n}^-)^{p-4} - 1]\int_{\mathbb{R}^3} a_\infty |h_n|^p dx.
\end{aligned}$$

Suppose by contradiction that $t^-_{\lambda,n} > 1$. By (6.31), we have that

$$\int_{\mathbb{R}^3} a_\infty |h_n|^p dx < \frac{2}{4-p}(t^-_{\lambda,n})^{2-p}\|h_n\|^2_{H^s}. \tag{6.33}$$

By virtue of (6.32) and (6.33), we have

$$\begin{aligned} 0 \geq &[1-(t^-_{\lambda,n})^{-2}]\|h_n\|^2_{H^s} + \frac{2}{4-p}[(t^-_{\lambda,n})^{-2} - (t^-_{\lambda,n})^{2-p}]\|h_n\|^2_{H^s} \\ =&[1+\frac{p-2}{4-p}(t^-_{\lambda,n})^{-2} - \frac{2}{4-p}(t^-_{\lambda,n})^{2-p}]\|h_n\|^2_{H^s} \\ =&(t^-_{\lambda,n})^{-2}[(t^-_{\lambda,n})^2 + \frac{p-2}{4-p} - \frac{2}{4-p}(t^-_{\lambda,n})^{4-p}]\|h_n\|^2_{H^s}, \end{aligned} \tag{6.34}$$

that is,

$$(t^-_{\lambda,n})^2 + \frac{p-2}{4-p} - \frac{2}{4-p}(t^-_{\lambda,n})^{4-p} \leq 0.$$

Nevertheless, for $2 < p < 4$, it is not difficult to get that

$$t^2 + \frac{p-2}{4-p} - \frac{2}{4-p}t^{4-p} > 0 \quad \text{for} \quad t > 1. \tag{6.35}$$

In fact, let $g(t) = t^2 + \frac{p-2}{4-p} - \frac{2}{4-p}t^{4-p}$, we have $g'(t) = 2t - 2t^{3-p}$. When $g'(t) = 0$, we obtain $t = 0$ or $t = 1$. Hence,

$$\begin{aligned} &f(t) > 0, \ \text{ if } \ t > 1, \\ &f(t) < 0, \ \text{ if } \ t < 1. \end{aligned}$$

The above facts gives a contradiction. Thus, $t^-_{\lambda,n} \leq 1$.

Now, we define function $\Phi^\infty_\lambda(th_n)$ by

$$\Phi^\infty_\lambda(th_n) = \frac{(p-2)t^2}{2p}\|h_n\|^2_{H^s} - \frac{(4-p)t^4}{4p}\lambda\int_{\mathbb{R}^3} K_\infty \phi^t_{h_n} h_n^2 dx \text{ for } t > 0.$$

Since $2 < p < 4$, it is not difficult to prove that there exists a constant

$$t^\infty_\lambda(h_n) = \Big[\frac{(p-2)\|h_n\|^2_{H^s}}{(4-p)\lambda\int_{\mathbb{R}^3} K_\infty \phi^t_{h_n} h_n^2 dx}\Big]^{\frac{1}{2}} > 0,$$

such that $\Phi^\infty_\lambda(th_n)$ is increasing when $t \in (0, t^\infty_\lambda(h_n))$ and is decreasing when $t \in (t^\infty_\lambda(h_n), \infty)$. By using Lemma 2.3 and (6.30), we deduce that

$$\begin{aligned} t^\infty_\lambda(h_n) =&\Big[\frac{(p-2)\|h_n\|^2_{H^s}}{(4-p)\lambda\int_{\mathbb{R}^3} K_\infty \phi^t_{h_n} h_n^2 dx}\Big]^{\frac{1}{2}} \geq \Big[\frac{p-2}{(4-p)\lambda\overline{S}_t^{-1}S^{-2}K^2_\infty\|h_n\|^2_{H^s}}\Big]^{\frac{1}{2}} \\ \geq&\Big[\frac{(p-2)\lambda\overline{S}_t S^2}{\lambda(4-p)K^2_\infty}\Big]^{\frac{1}{2}}(\frac{a_\infty(4-p)^2}{2S^p_p})^{\frac{1}{p-2}} \geq 1, \end{aligned}$$

where

$$\lambda < \frac{(p-2)\overline{S}_t S^2}{2(4-p)K^2_\infty}(\frac{a_\infty(4-p)^2}{2pS^p_p})^{\frac{2}{p-2}}.$$

Therefore, owing to $t^-_{\lambda,n} \leq 1$, we have that

$$\alpha^{\infty,-}_\lambda \leq J^\infty_\lambda(t^-_{\lambda,n}h_n) = \Phi^\infty_\lambda(t^-_{\lambda,n}h_n) \leq \Phi^\infty_\lambda(h_n) \to \alpha < \alpha^{\infty,-}_\lambda,$$

this is impossible.

Case(ii). $\langle (J_\lambda^\infty)'(w_n), w_n\rangle \le 0$. Similar argument as the proof of Case (i), we also get a contradiction.

Case(iii). Up to a subsequence, $\langle (J_\lambda^\infty)'(h_n), h_n\rangle > 0$, $\langle (J_\lambda^\infty)'(w_n), w_n\rangle > 0$.

By (6.29), we obtain that $\langle (J_\lambda^\infty)'(h_n), h_n\rangle = o_n(1)$, and $\langle (J_\lambda^\infty)'(w_n), w_n\rangle = o_n(1)$. Repeating the arguments of Case (i), suppose by contradiction that

$$\lim_{n\to\infty} t^-_{\lambda,n} = t^-_{\lambda,\infty} > 1. \tag{6.36}$$

By (6.32), we have that

$$o_n(1) = \langle (J_\lambda^\infty)'(h_n), h_n\rangle = [1-(t^-_{\lambda,n})^{-2}]\|h_n\|^2_{H^s} + [(t^-_{\lambda,n})^{p-4}-1]\int_{\mathbb{R}^3} a_\infty |h_n|^p dx.$$

From (6.34), we deduce that

$$o_n(1) \ge (t^-_{\lambda,n})^{-2}[(t^-_{\lambda,n})^2 + \frac{p-2}{4-p} - \frac{2}{4-p}(t^-_{\lambda,n})^{4-p}]\|h_n\|^2_{H^s},$$

which implies that

$$\|h_n\|^2_{H^s} \to 0 \text{ as } n\to\infty,$$

and then

$$\int_{\mathbb{R}^3} K_\infty \phi^t_{h_n} h_n^2 dx \to 0 \text{ as } n\to\infty.$$

Thus, it is easy to infer that $\Phi_\lambda^\infty(h_n)\to 0$ as $n\to\infty$, which contradicts with (6.28). Therefore, dichotomy does not occur. □

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References

[1] A. Ambrosetti, On Schrödinger-Poisson systems, *Milan J. Math.* 76 (2008) 257–274.

[2] A. Azzollini, P.d́Avenia, A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term,*Ann. Inst. H. Poincar Anal. Non Linéaire* 27 (2010) 779–791.

[3] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.* 345 (2008) 90–108.

[4] H. Bŕezis and E. H. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer Math. Soc.*, 88 (1983) 486–490.

[5] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Top. Methods. Nonlinear Anal.* 11 (1998) 283–293.

[6] G. Cerami and G. Vaira, Positive solutions for some non autonomous Schrödinger-Poisson Systems, *J. Differential Equations.* 248 (2010) 521–543.

[7] G.M. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, *Commun. Appl. Anal.* 7 (2003) 417–423.

[8] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrodinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A* 134 (2004) 893–906.

[9] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, *Adv. Nonlinear Stud.* 4 (2004) 307–322.

[10] P. Drábek and S. I. Pohozaev, Positive solutions for the p-Laplacian: Application of the fibering method, *Proc. Roy. Soc. Edinburgh Sect. A*, 127 (1997) 703–726.

[11] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, 47 (1974) 324–353.

[12] R. L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, Commu. Pure Appl. Math. LXIX (2016) 1671–1726.

[13] M. M. Fall, F. Mahmoudi and E. Valdinoci, Ground states and concentration phenomena for the fractional Schrödinger equation, *Nonlinearity* 28 (2015) 1937-1961.

[14] Y. Jiang, H. Zhou, Schrödinger-Poisson system with steep potential well, *J. Differential Equations* 251 (2011) 582–608.
[15] H. Kikuchi, On the existence of a solution for elliptic system related to the Maxwell-Schrödinger equations, *Nonlinear Anal.* 67 (2007) 1445–1456.
[16] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case I, *Ann. Inst. H. Poincar e Anal. Non Lińeaire*, 1 (1984) 109–145.
[17] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case II, *Ann. Inst. H. Poincar e Anal. Non Lińeaire*, 1 (1984) 223–283.
[18] Z. S. Liu and J. J. Zhang, Multiplicity and concentration of solutions for the following critical fractional Schrödinger-Poisson systems with critical growth, *ESAIM: Control, Optim. Calc. Var.* 23 (2017) 1515-1542.
[19] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* 237 (2006) 655–674.
[20] C. D. Ronaldo and A. S. S. Marco, Fractional Schrödinger-Poisson equations with general nonlinearity, *Elec. J. Differ. Equa.* Vol. 2016 (2016), 1–19.
[21] J. Sun, T. Wu and Z. Feng, Non-autonomous Schrödinger-Poisson System in $\mathbb{R}^3$, *Discrete and Continuous Dynamical Systems*, 38 (2018) 1889–1933.
[22] K. M. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent, *J. Differential Equations,* 261 (2016) 3061–3106.
[23] K. M. Teng, Ground state solutions for the nonlinear fractional Schrödinger-Poisson system, *Applicable Analysis,* (2018) 1–43.
[24] K. M. Teng, Multiple solutions for a class of fractional Schrödinger equation in $\mathbb{R}^n$, *Nonlinear Anal. Real World Appl.* 21 (2015) 76-86.
[25] K. M. Teng and X. M. He, Ground state solutions for fractional Schrödinger equations with critical Sobolev exponent, *Commu. Pure Appl. Anal.* 15 (2016) 991-1008.
[26] G. Vaira, Ground states for Schrödinger-Poisson type systems, *Ric. Mat*, 60 (2011) 263–297.
[27] M. Willem, Minimax Theorems, *Birkhäuser, Boston*, 1996.
[28] Z. Wang, H. Zhou, Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$, *Discrete Contin. Dyn. Syst.* 18 (2007) 809-816.
[29] J. Zhang, J. M. DO Ó and M. Squassina, Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity, *Adv. Nonlinear Stud.* 16 (2016) 15–30.
[30] L. Zhao, F. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, *J. Math. Anal. Appl.* 346 (2008) 155–169.
[31] L. Zhao, H. Liu, F. Zhao, Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential, *J. Differential Equations* 255 (2013) 1-23.
[32] J. Zhang, On the Schrödinger-Poisson equations with a general nonlinearity in the critical growth, *Nonlinear Anal.* 75 (2012) 6391-6401.

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