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Dynamics of a Leslie-Gower type predation model with a non-monotonic functional response †

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Summary

In the ecological literature, many models for the predator-prey interactions consider the monotonic functional responses to describe the action of the predators. However, there exist antipredator behaviors which are best represented by non-monotonic functions.

The mathematical results on the predator-prey models provide very useful information to understand the complex food webs; they also help to the insight of the mechanisms that govern the evolution of ecological systems.

The aim of this paper is to show, the dynamics of a modified Leslie-Gower model, assuming a rational non-monotonic functional response or Holling type IV. A principal target is to compare the obtained properties with other cases, in which different non-monotonic functional responses are incorporated.

The model is described by an autonomous bi-dimensional ordinary differential equation system (ODEs), assuming that the prey and predator growth functions are the logistic type.

The proposed model is not defined in $(0,0)$; considering a topological equivalent system, it is possible that to prove the origin is a non-hyperbolic saddle point.

We also have established, there are subsets of the parameter space in which: i) there exists a unique positive equilibrium point, ii) a heteroclinic curve exists. iii) two concentric limit cycles exist, the innermost unstable and the outermost stable.

Numerical simulations are given to endorse the analytical results and to exhibit the richness of the dynamics in the system.

KEYWORDS:
Predator-prey model, functional response, bifurcation, limit cycle, separatrix curve, stability

1 | INTRODUCTION

The dynamical relationship between the predators and their prey has been and it will follow to be one of the dominant themes in both Population Dynamics and Mathematical Ecology [1], due to its universal existence and importance in nature [2], constituting an important scope of study on Applied Mathematics. Many types of modelling have been formulated for this important interaction, from the seminal Lotka-Volterra model [3] in 1925, which obey a mass-action principle [2].

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In this work a deterministic continuous-time predator-prey model of Leslie-Gower type[4,5], is analyzed, which is described by nonlinear bidimensional ordinary differential equation systems (ODEs), considering two important aspects for specifying the interaction:

1. The functional response or predator consumption rate is a of Holling type IV, or non-monotonic.
2. The predators growth function is of logistic type.

The second aspect characterizes a *Leslie-Gower type predator-prey model*[6,9] also known as *logistic predator prey model*[1,3]. In this type of model, the conventional environmental carrying capacity OF predators Ky is proportional to prey abundance, i.e., $Ky = K(x) = nx$, a function of the available prey quantity[1,3], as in the *May-Holling-Tanner model*[7,8]. Moreover, the prey growth equation is preserved as the Volterra model[3].

Clearly, the model proposed by the English ecologist Patrick H. Leslie in 1948[4] does not fit to the Lotka-Volterra scheme[3]. It has been strongly criticized by presenting anomalies in their predictions; it vaticinates that even at very low prey density, when the consumption rate of an individual predator is essentially zero, predator population can increase, if predator prey ratio is very small[2,3]; nevertheless, these models are employed to describe some predator-prey interactions in some field studies[11].

It has been observed in the Nature that, in the case of critical scarcity, some predator species can switch over to other available food. This ability of the predators can be modeled by adding a positive constant c in the carrying capacity $K_y(x)$, being described now by $K(x) = nx + c$, a function of the available prey population[12]. This algebraic form permits to avoid the objections formulated to the original model proposed by Leslie, assuming the predator is *generalist*[13,14]. Then, it is said that the model is represented by a *Leslie-Gower scheme* or a *modified Leslie-Gower model*[6,12].

On the other hand, the *predator functional response* or *consumption function* refers to the change in the attacked prey density per unit of time per predator, when the prey density changes[15]; they are classified into several types, relying on the prey population size or the both population sizes. Crawford. S. Holling in 1969[16] described three types of saturated functions, based in experiments realized in the laboratory, and depending only in the prey population size (prey-dependent functional response). They are named as *Holling type, I , II or III*[3]; later, Robert J. Taylor[17] in 1984 described the Holling type IV or non-monotonic functional response[17].

In a great majority of predator-prey models considered in the ecological literature, the predator response to the prey density is assumed to be increasing monotonic[1,3]; this is an inherent assumption meaning that the more prey animals there are in the environment, the better off the predator[18,19]

However, there is evidence that indicates that this need not always the case. For instance, when there exists an antipredator behavior (APB), called *defence group formation*[18,19,20,30,24]. This term is used to describe the phenomenon whereby predators decrease, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their number are large enough[18,19,30,24]. In this case a Holling type IV consumption function or non-monotonic functional response is more adequate to describe that phenomenon[17].

Another manifestation of an APB in which a non-monotonic functional response can be used, is the *aggregation*, a social behavior of prey, in which they congregate on a finer scale relative to the predator; thus, the hunting of the predators is not spatially homogeneous[17], such as it happens with miles long schools of certain classes of fishes.

Here we will use a rational form for the non-monotonic functional response, which is described by the unimodal function $h(x) = \frac{qx}{x^3+a}$, similar to the function considered in[20,21,22,23,24], a particular case of the so-called Monod-Haldane function[25], which is similar to the Monod (i.e., the Michaelis-Menten) function for low concentrations but includes the inhibitory effect at high agglutinations[25].

The function considered in this work can be a rational generalized by the functional response described by

$h(x) = \frac{qx^m}{x^n+a}$, with n and $m \in N$, where $n > m \geq 1$

as in[28]. A question is if the exponents n and m have incidence in the dynamic of a predation model.

Nonetheless, other mathematical form has been formulated to describe a non-monotonic functional response as the following:

i) $h(x) = \frac{qx}{x^2+bx+a}$, with q, $a > 0$ and $b < 0$, Rothe and Schafer[31], Zhu et al.,[32,33].

ii) $h(x) = \frac{qx^2}{x^2-bx+a}$, with q, a and $b > 0$, Lamontagne et al.[27].

iii) $h(x) = qxe^{-bx}$, with q and $b > 0$[30].

iv) $h(x) = k(e^{-ax} - e^{-bx})$, with $b > a > 0$[26].

Then, considering that distinct mathematical forms proposed to represent the group defense formation[26,27,28,29,30,33], a comparison between the different formulated models is necessary. An interesting future work should be to verify the equivalence of the dynamical behaviors of the obtained systems; establishing the common and diverse properties of these models will permit the election of the simplest form to represent that ecological phenomenon[2].

Recently, another type of an antipredator behavior (APB) has received the attention of modelers which is called *prey herd behavior*. A prey species exhibit herd behavior when the individuals demonstrate collective social conduct to avoid predation, effecting an instinctive reaction equal to that realized by the majority of the other members[34].

This behavior can take various forms, including schools of fish staying close together and swimming in the same direction, flocks of migrating birds in formation, large herbivores populating the savannas gathering together in huge herds, and so on.

Generally, the strongest individuals are on the border, and the weakest are concentrated in the middle of the group. A group of predators might be more effective at taking down a herd of prey than a single animal.

This phenomenon has been modeled by the monotonic function $h(x) = \frac{q\sqrt{x}}{\sqrt{x}+a}$, a Holling type II functional response[3], but non-differentiable at $x = 0$. It is based on the so-called *square root functional response*, $h(x) = q\sqrt{x}$, which was proposed by the Russian biologist Georgii F. Gause in 1934[37]; this collective behavior can be also described by the generalized monotonic function $h(x) = \frac{qx^{\alpha}}{x^{\alpha}+a}$, with $0 < \alpha < 1$, named *Rosenzweig type II functional response* and also non-differentiable at $x = 0$[34].

Some articles have tried to homologate the way to model this phenomenon with the defense group formation[35]. We believe these phenomena could differentiate due to the distinct dynamics originated on each system when both kinds of functions are considered, i.e., the non-differentiable and the non-monotonic functional responses[28,34].

The existence and number of limit cycles are important topics to a better understanding of many real world oscillatory phenomena[24]. For predator-prey systems, the existence of limit cycles is related to the existence, stability, and bifurcation of a positive equilibrium point[24].

The problem of determining conditions, which guarantee the uniqueness of a limit cycle or the global stability of the unique positive equilibrium in predator-prey systems, has been extensively studied over the last decades. This study starts with the work by Kuo-Shung Cheng in 1981[39], who was the first to prove the uniqueness of a limit cycle for a specific predator-prey model, using the symmetry of the prey isocline; he assumes the hyperbolic functional response, a Holling type II functional response[1].

To establish the quantity of limit cycles which can be born throughout the bifurcation of a center-type focus[42] is not an easy task; this question is related with the Hilbert 16th Problem[40,41], proposed by the German mathematician David Hilbert in 1900; it is referred to the maximum number and relative position of the limit cycles in a polynomial ODEs[40,41].

One of the main goals of this work is to describe the behaviour of the model, involving the description of the dynamical systems. An important issue is to establish the quantity of limit cycles that systems can exhibit, using the Lyapunov method to estimate that number[42,44]. The obtained results will be compared with those obtained in the analysis of similar models, such as the May-Holling-Tanner[8] and the Leslie-Gower model with a particular rational non-monotonic functional response[22,23,36],

This work is organized as follows: The model is presented in the next section 2; in Section 3 the main properties of the model are established; in Section 4 some simulations are shown, and in the last section we present a discussion of the obtained results, given the respective ecological interpretations.

2 | THE MODEL

The predator-prey model that will be analyzed is described by the autonomous bidimensional differential equations system of Kolmogorov type[15,38] given by

$$X_v(x,y) : \begin{cases} \frac{dx}{dt} = \left(r\left(1-\frac{x}{K}\right) - \frac{q\,y}{x^3+a}\right)x \\ \frac{dy}{dt} = s\left(1-\frac{y}{n\,x}\right)y \end{cases} \tag{1}$$

with $x(0) \geq 0$ and $y(0) \geq 0$, where $x = x(t)$ and $y = y(t)$ indicate the prey and predator population sizes respectively for $t \geq 0$, measured as density or biomass; all the parameters are positives, i.e., $v = (r, q, a, s, K, n) \in \mathbb{R}_+^6$, having the following biological meanings:

- r and s represent the intrinsic growth rate of the prey and the predators, respectively,
- K indicates the prey environmental carrying capacity,

- q is the maximal per capita consumption rate,
- $\sqrt[3]{\frac{a}{2}}$ is the amount of prey for which the predation effect is maximun, and
- n represents a measure of the quality the prey as food for the predators.

System (2.1) is defined in the first quadrant, except for $x = 0$, i.e., in the set:

$$\Psi = \left\{(x, y) \in \mathbb{R}^2 / x > 0,\ y \geq 0\right\} = \mathbb{R}^+ \times \mathbb{R}_0^+$$

The equilibrium points of system (2.1) or singularities of vector field $X_\upsilon(x, y)$ are: $(K, 0)$ and $\left(x_e, y_e\right)$ satisfying the equations of the isoclines $y = nx$ and $y = \frac{r}{q}\left(1 - \frac{x}{k}\right)\left(x^3 + a\right)$.

We note that:

i) System (2.1) is not defined for $x = 0$, but the point $(0, 0)$ has a strong influence in the behaviour of system as it will see in this work.

ii) The point $\left(x_e, y_e\right)$ lies in the interior of the first quadrant, if and only if, $x_e < K$.

iii) The point $\left(x_e, y_e\right)$ can lie in the fourth quadrant, if and only if, $x_e > K$; then the unique equilibrium is the point $(K, 0)$.

To simplify the calculations and to make an adequate description of behavior of system (2.1), following the methodology used in[6,7,8,28], which involved a change of variable and a time rescaling[42] given by the following function:

$$\Upsilon : \bar{\Psi} \times \mathbb{R} \to \Psi \times \mathbb{R}$$

such us,

$$\Upsilon(u, v, \tau) = \left(ku, knv, \frac{u\left(u^3 + \frac{a}{K^3}\right)\tau}{r}\right) = (x, y, t)$$

with,

$$\bar{\Psi} = \left\{(u, v) \in \mathbb{R}^2 /\ u \geq 0,\ v \geq 0\right\} = \mathbb{R}_0^+ \times \mathbb{R}_0^+.$$

Clearly, $\det D\Upsilon(u, v, \tau) = \frac{1}{Kr}\left(nK^3u^4 + anu\right) > 0$.

Then Υ is a diffeomorphism preserving the orientation of time[42]. The vector field $X_\upsilon(x, y)$ in the new system of coordinates is topologically equivalent to the vector field $Y_\eta(u, v) = \Upsilon \circ X_\upsilon(x, y)$[43]; it takes the form $Y_\eta(u, v) = P(u, v)\frac{\partial}{\partial u} + Q(u, v)\frac{\partial}{\partial v}$ [43]; the associated differential equations is given by a sixth order polynomial system:

$$Y_\eta(u, v) : \begin{cases} \frac{du}{d\tau} = \left((1 - u)\left(u^3 + A\right) - Qv\right)\ u^2 \\ \frac{dv}{d\tau} = B(u - v)\left(u^3 + A\right)v \end{cases} \tag{2}$$

with $A = \frac{a}{K^3}, Q = \frac{qn}{r}, B = \frac{s}{r}$; the system (2) is defined in

$$\bar{\Psi} = \left\{(u, v) \in \mathbb{R}^2 / u \geq 0,\ v \geq 0\right\}.$$

The equilibrium point of system (2) or singularities of vector field $Y_\eta(u, v)$ are: $(0, 0), (1, 0)$ and $\left(u_e, v_e\right)$, which is determined by the intersection of isoclines:

$$v = u \text{ and } v = \frac{1}{Q}(1 - u)\left(u^3 + A\right).$$

Then, the abscissa u of this point at $\bar{\Psi}$, is solution of the fourth degree equation:

$$P(u) = u^4 - u^3 + (A + Q)u - A = 0. \tag{3}$$

According to the Descartes'rule of sign, the polynomial $P(u)$ may have one or three different real positive roots, or two different being one of them with multiplicity two, since the sign of the coefficient $(A + Q)$ is always positive

Let $u_e = H$ be, the real positive root that always exists for equation (2.3) and (H, H) the equilibrium point that always exist at $\bar{\Psi}$ for system (2.2).

Dividing the polynomial $P(u)$ by $(u - H)$ is obtained the polynomial

$$P_1(u) = u^3 - (1 - H)u^2 - H(1 - H)u + A + Q - H^2(1 - H)$$

being a factor of $P(u)$; the rest of the division is

$$R(H) = H^4 - H^3 + (A+Q)H - A.$$

If $R(H) = 0$; then

$$Q = \frac{1}{H}(1-H)\left(A+H^3\right). \tag{4}$$

Replacing Q in $P_1(u)$, we have that

$$P_1(u) = u^3 - (1-H)u^2 - H(1-H)u + \frac{A}{H}. \tag{5}$$

Assuming $H < 1$, then $H(1-H) > 0$ and so $P_1(u)$ has two change of sign; therefore, equation (5) would have up two different real positive roots, and as consequences, equation (3) would have up three different real positive roots as is shown in the following picture.

HERE figure 1

As

$P_1(-u) = -u^3 - (1-H)u^2 + H(1-H)u + \frac{A}{H}$

then equation (5) has a unique negative real root, and equation (3) has one negative real root.

Let $u = -L$, with $L > 0$, such root.

Dividing $P_1(-u)$ by $u + L$, i.e., $\frac{u^3-(1-H)u^2-H(1-H)u+\frac{A}{H}}{u+L}$.

Thus, it is obtained,

$$P_2(u) = u^2 - (1-H+L)u + \left(L^2 + (1-H)L - H(1-H)\right). \tag{6}$$

and the rest is

$R(H) = L^3 + (1-H)L^2 - H(1-H)L - \frac{A}{H} = 0.$

Let $\Delta = (1-H+L)^2 - 4\left(L^2 + (1-H)L - H(1-H)\right)$.

Lemma 1. A. Supposing $a_0 = L^2 + (1-H)L - H(1-H) > 0$.

Thus, $a_0 > 0$, if and only if, $L > L_1$, being

$L_1 = \frac{1}{2}\left(-(1-H) + \sqrt{(3H+1)(1-H)}\right) > 0.$

For the equation (6) it has

A1) two real positive roots, if and only if,

$\Delta = -3L^2 - 2(1-H)L + (3H+1)(1-H) > 0,$

which are given by

$u_1 = \frac{1}{2}\left((1-H+L) - \sqrt{\Delta}\right)$ and $u_2 = \frac{1}{2}\left((1-H+L) + \sqrt{\Delta}\right)$.

Moreover, $\Delta > 0$, if and only if, $L < L_3$, o $L_4 < L$, being

$L_3 = \frac{1}{3}\left(-(1-H) - 2\sqrt{-2H^2+H+1}\right)$ and $L_4 = \frac{1}{3}\left(-(1-H) + 2\sqrt{-2H^2+H+1}\right)$

A2) a unique positive real root, if and only if, $\Delta = 0$, which is $u_* = \frac{1}{2}(1-H+L)$.

A3) has not positive real roots, if and only if, $\Delta < 0$.

B. Supposing $a_0 = 0$, there exists a unique positive real root given by $u = (1-H+L)$. Moreover, $u = 0$ is a root of multiplicity two.

C. Supposing $a_0 < 0$, there exists a unique positive real root given by $u_2 = \frac{1}{2}\left((1-H+L) + \sqrt{\Delta}\right)$; moreover, $u_1 < 0$.

Proof. It is immediate □

For determine the nature of the hyperbolic equilibrium points, the Jacobian matrix is required, being:

$$DY_\eta(u,v) = \begin{pmatrix} DY_\eta(u,v)_{11} & -Qu^2 \\ Bv\left(4u^3 - 3vu^2 + A\right) & B(u-2v)\left(u^3+A\right) \end{pmatrix},$$

with $DY_\eta(u,v)_{11} = 2u\left((1-u)\left(u^3+A\right) - Qv\right) + u^2\left(-4u^3 + 3u^2 - A\right)$.

Remark 1. In the following, we consider the case when there exists a unique positive equilibrium point (H, H). The dynamics of the system considering two or three positive equilibrium point will be analyzed in a future paper. However, some simulations will be shown in order to make evident the dynamical richness of the system. (2)

3 | MAIN RESULTS

For system (2) or vector field $Y_\eta(u,v)$, it has the following results:

Lemma 2. The set $\overline{\Gamma} = \left\{(u,v) \in \mathbb{R}^2 / 0 \le u \le 1, v \ge 0\right\}$ is a region positively invariant .

Proof. As system (2) is of Kolmogorov type[15,38], the coordinates axis are invariant sets[42].
Let $u = 1$ be; we have that $\frac{du}{d\tau} = -Qv < 0$, and any let be the sign of $\frac{dv}{d\tau}$ the trajectories enter to the region $\bar{\Gamma}$. □

We note that in the system (1) the set
$\Gamma = \left\{(x,y) \in \mathbb{R}^2 / 0 < x \le K, y \ge 0\right\}$
is a positively invariant region.

Lemma 3. The solutions are bounded.

Proof. Using Poincaré compactification[45]
Let be $X = \frac{u}{v}$ and $Y = \frac{1}{v}$, then,

$$\frac{dX}{d\tau} = \frac{1}{v^2}\left(v\frac{du}{d\tau} - u\frac{dv}{d\tau}\right), \quad \frac{dY}{d\tau} = -\frac{1}{v^2}\frac{dv}{d\tau};$$

then, the system takes the form:

$$\widehat{Y}_\eta : \begin{cases} \dfrac{dX}{d\tau} = -\frac{1}{Y^5}\left(-X^5Y + X^6 - ABXY^4 - AX^2Y^4 - BX^4Y + BX^5Y + QX^2Y^3 + AX^3Y^3 + ABX^2Y^4\right) \\ \dfrac{dY}{d\tau} = -B(X-1)\frac{AY^2+X^2}{Y^2} \end{cases}.$$

To simplify the calculus, we make a time rescaling given by $T = \frac{1}{Y^5}\tau$ then,

$$\widetilde{Y}_\eta : \begin{cases} \dfrac{dX}{d\tau} = -\left(-X^5Y + X^6 - ABXY^4 - AX^2Y^4 - BX^4Y + BX^5Y + QX^2Y^3 + AX^3Y^3 + ABX^2Y^4\right) \\ \dfrac{dY}{d\tau} = -Y^3B(X-1)\left(AY^2 + X^2\right) \end{cases}$$

then,

$$D\widetilde{Y}_\eta(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For desingularizing the origen, we consider the blowing-up directional method[43], making $X = r$ and $Y = r^2 s$, then, we have:

$$V_\eta : \begin{cases} \dfrac{dr}{dT} = \dfrac{dr}{dT} \\ \dfrac{ds}{dT} = \frac{1}{r^2}\left(\dfrac{dY}{dT} - 2rs\dfrac{dr}{dT}\right) \end{cases}$$

so,

$$V_\eta : \begin{cases} \dfrac{dr}{dT} = r^6\left(Bs + rs - Ar^3s^3 + Ar^4s^4 - Qr^2s^3 - Brs + ABr^3s^4 - ABr^4s^4 - 1\right) \\ \dfrac{ds}{dT} = r^5 s\left(-2Bs - 2rs + Brs^2 + 2Ar^3s^3 - Br^2s^2 - 2Ar^4s^4 + 2Qr^2s^3 + 2Brs - ABr^3s^4 + ABr^4s^4 + 2\right) \end{cases}$$

Once again, making a time rescaling given by $\lambda = r^5 T$, the following new rescaled vector field is obtained:

$$\bar{V}_\eta : \begin{cases} \dfrac{dr}{d\lambda} = r\left(Bs + rs - Ar^3s^3 + Ar^4s^4 - Qr^2s^3 - Brs + ABr^3s^4 - ABr^4s^4 - 1\right) \\ \dfrac{ds}{d\lambda} = s\left(-2Bs - 2rs + Brs^2 + 2Ar^3s^3 - Br^2s^2 - 2Ar^4s^4 + 2Qr^2s^3 + 2Brs - ABr^3s^4 + ABr^4s^4 + 2\right) \end{cases}$$

so, evaluating the Jacobian matrix of $\bar{V}_\eta$ in $(0,0)$, we obtain:

$$D\bar{V}_\eta(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus, $(0,0)$ is a hyperbolic saddle point of vector field $\bar{V}_\eta$ since $\det D\bar{V}_\eta(0,0) = -2$; so, $(0,0)$ is a non-hyperbolic saddle point of vector field $\widehat{Y}_\eta$ and $\widetilde{Y}_\eta$, which is repelling over the positive s-axis; hence, $(0,\infty)$ is a nonhyperbolic saddle point of vector field Y_η, repelling negatively over the v - axis, Therefore, the solutions of the system (2) are bounded. □

Lemma 4. The singularity $(1,0)$ is a hyperbolic saddle point, for all parameter values.

Proof. Evaluating the Jacobian matrix at equilibrium point $(1,0)$,

$$DY_\eta(1,0) = \begin{pmatrix} -(A+1) & -Q \\ 0 & B(A+1) \end{pmatrix}.$$

Clearly, $\det DY_\eta(1,0) = -B(A+1)^2 < 0$, thus the point $(1,0)$ is a hyperbolic saddle point. □

Lemma 5. The point $(0,0)$ is a non-hyperbolic saddle point.

Proof. Evaluating the Jacobian matrix at the point $(0,0)$, we have that

$$DY_\eta(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the origin is a non-hyperbolic singularity. To desingularize the origin, we consider the vertical blowing-up method, that is, we consider the function given by $\Theta(p,q) = (p,pq) = (u,v)$.

We have that $\frac{dp}{d\tau} = \frac{du}{d\tau}$ and $\frac{dq}{d\tau} = \frac{1}{p}\left(\frac{dv}{d\tau} - q\frac{dp}{d\tau}\right)$, rescaling the time by $T = p\tau$, it becomes,

$$\bar{Z}_\eta : \begin{cases} \frac{dp}{d\tau} = pA - Ap^2 + p^4 - p^5 - Qp^2q \\ \frac{dq}{d\tau} = p^4q - p^3q + Apq + Bp^3q - Bp^3q^2 + Qpq^2 - Aq - ABq^2 + ABq \end{cases}$$

If $p = 0$, then $\frac{dp}{d\tau} = 0$, and $\frac{dq}{d\tau} = Aq(B - Bq - 1)$;

Then, we have $q = 0$ and $q = \frac{1}{B}(B-1)$

The singularities of the vector field $\bar{Z}_\eta$ are $(0,0)$ and $\left(0, \frac{B-1}{B}\right)$.

A) The Jacobian matrix in the point $(0,0)$ is

$$D\bar{Z}_\eta(0,0) = \begin{pmatrix} A & 0 \\ 0 & A(B-1) \end{pmatrix}.$$

Then, the point $(0,0)$ is

i) a hyperbolic repeller, if and only if, $B > 1$.

ii) a hyperbolic saddle, if and only if, $B < 1$.

iii) a non-hyperbolic saddle, if and only if, $B = 1$.

B) The Jacobian matrix in the point $\left(0, \frac{B-1}{B}\right)$ is

$$D\bar{Z}_\eta\left(0, \frac{B-1}{B}\right) = \begin{pmatrix} A & 0 \\ 0 & -A(B-1) \end{pmatrix}.$$

Then, the point $\left(0, \frac{B-1}{B}\right)$ is

i) a hyperbolic repeller, if and only if, $B < 1$.

ii) a hyperbolic saddle, if and only if, $B > 1$.

iii) a non-hyperbolic saddle, if and only if, $B = 1$.

Then, by the blowing down, the point $(0,0)$ is a non-hyperbolic saddle point in the system (2). □

Remark 2. In the system (2) the point $(0,0)$ has a similar behavior as the Gause type model with the same functional response[21,25]. Then, there no exists the possibility of extinction of both populations simultaneously. Nevertheless, the predator population has high possibilities of be depleted (Lemma 5).

Theorem 1. Let $W^s(0,0)$ the stable manifold of the non-hyperbolic saddle point $(0,0)$ and $W^u(1,0)$ the unstable manifold of the hyperboli saddle point $(1,0)$, respectively; then, there exists a subset of parameters for which the intersection of $W^s(0,0)$ and $W^u(1,0)$ is not empty, giving rise to the heteroclinic curve γ joining the points $(0,0)$ and $(1,0)$.

Proof. By Lemma 5, the point $(0,0)$ is a non-hyperbolic saddle point and by Lemma 4 the point $(1,0)$ is saddle.

It is clear that the $\alpha - limit$ of $W^s(0,0)$ and the $\omega - limit$ of $W^u(1,0)$ are not at infinity on the direction of $v - axis$. Then, there are points $(u^\star, v^s) \in W^s(0,0)$ and $(u^\star, v^u) \in W^u(1,0)$ where v^s and v^u are functions of the parameters A, B, Q, i.e, $v^s = f_1(A,B,Q)$ and $v^u = f_2(A,B,Q)$.

Assuming $0 < u << 1$ then, $v^s < v^u$; if $0 << u < 1$ then, $v^s > v^u$.

Since the vector field is continuous and suffers smooth changes with respect to the parameter values, the stable manifold $W^s(0,0)$ intersects the unstable manifold $W^u(1,0)$; therefore, there exist $\left(u_s^\star, v_s^\star\right) \in \bar{\Gamma}$ (invariant region), such that $v^\star = v^s = v^u$. This equation defines a surface in the parameter space for which a heteroclinic curve γ exists. □

Remark 3. The stable manifold $W^s(0,0)$, the straight line $u = 1$ and the $u-$axis determine a subregion $\bar{\Lambda}$, which is closed and bounded, that is,

$$\bar{\Lambda} = \left\{(u,v) \in (\mathbb{R}_0^+)^2 / 0 \leq u \leq 1, 0 \leq v \leq v_S \text{ with } (u, v_S) \in W^s(0,0)\right\}$$

is a compact region, where it is possible to apply the Poincaré-Bendixson Theorem there.

Analogously, in system (1) it has a compact region Λ.

3.1 | System with a unique positive equilibrium

The nature of the equilibrium point (H,H) with $H < 1$ will be established, considering the obtained relation for $Q = f(H, A)$. The vector field $Y_\eta(u,v)$ or system (2) takes the form:

$$Y_\theta(u,v) : \begin{cases} \dfrac{du}{d\tau} = \left((1-u)\left(u^3 + A\right) - \frac{1}{H}(1-H)\left(H^3 + A\right) v \right) u^2 \\ \dfrac{dv}{d\tau} = B(u-v)\left(u^3 + A\right) v \end{cases} \tag{7}$$

with $\theta = (A, H, B) \in (]0,1[)^2 \times \mathbb{R}$. The Jacobian matrix is:

$$DY_\theta(H,H) = \begin{pmatrix} H^2\left(-A + 3H^2 - 4H^3\right) & (1-H)\left(A + H^3\right) H \\ BH\left(A + H^3\right) & -BH\left(A + H^3\right) \end{pmatrix},$$

then,

$$\det DY_\theta(H,H) = BH^2\left(A + H^3\right)\left(A - 2H^3 + 3H^4\right)$$

and the trace is given by:

$$trDY_\theta(H,H) = H^2\left(-A + 3H^2 - 4H^3\right) - BH\left(A + H^3\right).$$

It has that $trDY_\theta(H,H) = 0$ then $B = \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$.

Let $P = \left(trDY_\theta(H,H)\right)^2 - 4\det DY_\theta(H,H)$. The sign of P determines if the equilibrium (H,H) is a focus or a node.

System (7) has the following properties:

Theorem 2. Let (H,H) be the unique positive equilibrium point at the first quadrant; then, (H,H) is

1. an attractor, if and only if, $trDY_\theta(H,H) < 0$; then $B > \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$. Moreover, it is
   (a) an attractor node, if and only if, $P > 0$ and $B > \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$.
   (b) an attractor focus, if and only if, $P < 0$ and $B > \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$.
2. a repeller, if and only if, $trDY_\theta(H,H) > 0$; thus $B < \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$. Moreover, it is
   (a) a repeller focus, surrounded by a limit cycle, if and only if, $P < 0$.
   (b) a repeller node, if and only if, $P > 0$.
3. a weak focus, if and only if, $B = \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$.

Proof. 1. $B > \frac{H\left(-A+3H^2-4H^3\right)}{H^3+A}$, if and only if, $trDY_\theta(H,H) > 0$. Then, the equilibrium (H,H) is an attractor; moreover,

(a) If $P > 0$ then, the point is an attractor node.

(b) If $P < 0$ then, the point is an attractor focus.

2. $B < \frac{H(-A+3H^2-4H^3)}{H^3+A}$, if and only if, $trDY_\theta(H,H) > 0$. Thus, (H,H) is a repeller; moreover,

(a) If $P < 0$ then, it is a repeller focus. By Hopf bifurcation the point (H,H) is surrounded by at least one infinitesimal limit cycle.

(b) If $P > 0$ and $B < \frac{H(-A+3H^2-4H^3)}{H^3+A}$; therefore, (H,H) becomes a repeller node.
In this case, by the Poincaré-Bendixson Theorem[42,43,45], in the subregion $\bar{\Lambda}$ a non-infinitesimal limit cycle appears.

When the parameters change, both limit cycle can increase their amplitude until to coincide with the heteroclinic curve γ. □

Lemma 6. A Hopf bifurcation at equilibrium point (H,H) occurs in the system (7) for the bifurcation value $B = \frac{H(-A+3H^2-4H^3)}{H^3+A}$.

Proof. The proof follows from the above theorem since the determinant is always positive and the trace changes sign. In addition, the transversality condition[44] is verified, since we have that

$$\frac{\partial\left(trDY_\theta(H,H)\right)}{\partial B} = -H\left(A+H^3\right) < 0.$$

□

Theorem 3. The singularity (H,H) of vector field $Y_\eta(u,v)$ is at least a two order weak focus, if and only if, $B = \frac{H(-A+3H^2-4H^3)}{H^3+A}$.

Proof. Setting $u = U + H$ and $v = V + H$ then the new system translated to origin of coordinates system is

$$Z_\eta(U,V): \begin{cases} \frac{dU}{d\tau} = \left((1-U-H)\left((U+H)^3+A\right) - \frac{1}{H}(1-H)\left(H^3+A\right)(V+H)\right)(U+H)^2 \\ \frac{dV}{d\tau} = B(V+H)\ (U-V)((U+H)^3+A) \end{cases}$$

and the Jacobian matrix of system $Z_\eta(U,V)$ at the point $(0,0)$ is

$$DZ_\eta(0,0) = \begin{pmatrix} H^2\left(-A+3H^2-4H^3\right) & -H(1-H)\left(A+H^3\right) \\ BH\left(A+H^3\right) & -BH\left(A+H^3\right) \end{pmatrix}.$$

Therefore,

$$\det DZ_\eta(0,0) = BH^2\left(A+H^3\right)\left(A-2H^3+3H^4\right)$$

and

$$\mathrm{tr}DZ_\eta(0,0) = H^2\left(-A+3H^2-4H^3\right) - BH\left(A+H^3\right).$$

It has, $\det DZ_\eta(0,0) > 0$, if and only if, $A - 2H^3 + 3H^4 > 0$, i.e., $A > H^3(2-3H)$.
The first Lyapunov quantity[42,44] is $\eta_1 = \mathrm{tr}DZ_\eta(0,0) = 0$.
Hence, $H^2\left(-A+3H^2-4H^3\right) - BH\left(A+H^3\right) = 0$, or

$$B = \frac{H(-A+3H^2-4H^3)}{(A+H^3)}$$

with,
$-A + 3H^2 - 4H^3 > 0$, i.e., $A < H^2(3-4H)$.
Thus, $H^3(2-3H) < H^2(3-4H)$

$$(3-4H) - H(2-3H) = 3(1-H)^2.$$

Let $W^2 = BH^2\left(A+H^3\right)\left(A-2H^3+3H^4\right)$ and $BH\left(A+H^3\right) = H^2\left(-A+3H^2-4H^3\right)$.
The matrix change of basis[46] is

$$M = \begin{pmatrix} Z_\eta 11 - \mathrm{tr}DZ_v & -\det DZ_\eta \\ Z_\eta 21 & 0 \end{pmatrix} = \begin{pmatrix} BH\left(A+H^3\right) & -W \\ BH\left(A+H^3\right) & 0 \end{pmatrix},$$

and

$$M^{-1} = \begin{pmatrix} 0 & \frac{1}{BH(A+H^3)} \\ -\frac{1}{W} & \frac{1}{W} \end{pmatrix}.$$

Considering the change of variables given by

$$\begin{pmatrix} U \\ V \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

it has,

$U = BH\left(A + H^3\right)x - Wy$
$V = BH\left(A + H^3\right)x$

or

$x = \frac{1}{BH\left(A+H^3\right)}V$
$y = \frac{-1}{W}(U + V)$

Then, the new system is

$$\bar{Z}_v(x,y) : \begin{cases} \frac{dx}{d\tau} = \frac{1}{BH\left(A+H^3\right)}\frac{dV}{d\tau} \\ \frac{dy}{d\tau} = \frac{1}{W}\left(-\frac{dU}{d\tau} + \frac{dV}{d\tau}\right) \end{cases}$$

After a large algebraic calculations we obtain the intermediate system $\bar{Z}_v(x,y)$

To obtain the normal form[42] we make the time rescaling given by $T = W\tau$, obtaining

$$\tilde{Z}_v(x,y) : \begin{cases} \frac{dx}{dT} = \begin{array}{l} -y - B\left(A + 4H^3\right)xy + \frac{3H^2W}{A+H^3}y^2 - 6B^2H^3\left(A+H^3\right)x^2y + 9BH^2Wxy^2 \\ -\frac{3HW^2}{A+H^3}y^3 - 4B^3H^3\left(A+H^3\right)^2x^3y + 9B^2H^2W\left(A+H^3\right)x^2y^2 \\ -6BHW^2xy^3 + \frac{W^3}{A+H^3}y^4 - B^4H^3\left(A+H^3\right)^3x^4y + 3B^3H^2W\left(A+H^3\right)^2x^3y^2 \\ -3B^2HW^2\left(A+H^3\right)x^2y^3 + BW^3xy^4 \end{array} \\ \\ \frac{dy}{dT} = \begin{array}{l} x + BH\left(A+H^3\right)\left(-2A - 4BH^3 - AB - 2AH + 16H^3 - 26H^4\right)xy \\ + \frac{B^2H^2\left(2A - 7H^3 + 12H^4\right)\left(A+H^3\right)^2}{W}x^2 - HW\left(2A + 3BH^2 - 9H^2 + 14H^3\right)y^2 \\ + \frac{B^3H^2\left(A - 9H^3 + 19H^4\right)\left(A+H^3\right)^3}{W}x^3 + B^2H\left(A+H^3\right)^2\left(-2A - 6BH^3 - AH + 28H^3 - 58H^4\right)x^2y \\ +BW\left(A+H^3\right)\left(A + 9BH^3 + 2AH - 29H^3 + 59H^4\right)xy^2 \\ +W^2\left(A + 3BH^2 - 10H^2 + 20H^3\right)y^3 + \frac{5B^4H^5(3H-1)\left(A+H^3\right)^4}{W}x^4 \\ -4B^3H^4\left(A+H^3\right)^3(B + 15H - 5)x^3y + 3B^2H^3W\left(A+H^3\right)^2(3B + 30H - 10)x^2y^2 \\ -2BH^2W^2(3B + 30H - 10)\left(A+H^3\right)xy^3 + HW^3(B + 15H - 5)y^4 \\ + \frac{B^5H^5(6H-1)\left(A+H^3\right)^5}{W}x^5 - B^4H^4\left(A+H^3\right)^4(B + 30H - 5)x^4y \\ +B^3H^3W\left(A+H^3\right)^3(3B + 60H - 10)x^3y^2 - B^2H^2W^2\left(A+H^3\right)^2(3B + 60H - 10)x^2y^3 \\ +BHW^3(B + 30H - 5)\left(A+H^3\right)xy^4 + W^4(6H - 1)y^5 + \frac{B^6H^6\left(A+H^3\right)^6}{W}x^6 \\ -6B^5H^5\left(A+H^3\right)^5x^5y + 15B^4H^4W\left(A+H^3\right)^4x^4y^2 - 20B^3H^3W^2\left(A+H^3\right)^3x^3y^3 \\ +15B^2H^2W^3\left(A+H^3\right)^2x^2y^4 - 6BHW^4\left(A+H^3\right)xy^5 - W^5y^6 \end{array} \end{cases}$$

Using the Mathematica package[47] we obtain that the second Lyapunov quantity[42,44] is

$$\eta_2 = \frac{AHf(A,B,H)}{B\left(A+H^3\right)^2}$$

where

$$f(A,B,H) = f_3(A,B,H)B^3 + f_2(A,B,H)B^2 + f_1(A,B,H)B + f_0(A,B,H)$$

with

$f_3(A,B,H) = \left((-39H)A^8 + \left(72H^7 + 31H^6 - 63H^4\right)A^4 + \left(96H^8 - 214H^7\right)A^3 + \left(45H^8 - 31H^6\right)A^2 + 37H^9\right),$

$f_2(A,B,H) = \left(\begin{array}{c} \left(102H - 45H^3 - 15\right)A^8 + \left(151H^5 + 71H^4 + 171H^3\right)A^6 + \left(138H^7\right)A^4 + \left(42H^6\right)A^3 \\ + \left(41H^9 - 121H^6\right)A^2 + \left(18H^9 - 156H^{10}\right) \end{array}\right)$

$f_1(A,B,H) = \left(\left(-74H^5\right)A^5 - 56A^8 + \left(-453H^5 - H^4\right)A^4 + \left(979H^2 - 641H^6\right)A^2 + \left(35H^{10}\right)A + \left(97H^{11} + 46H^9\right)\right)$

$f_0(A,B,H) = \left(\begin{array}{c} \left(197 - 38H - 19H^2\right)A^8 + \left(-456H^3 - 97H^2\right)A^7 + \left(657H^5 - 413H^4\right)A^6 + \left(196H^7 + 95H^4\right)A^5 \\ + \left(32H^9\right)A^4 + \left(385H^8 + 147H^7 + 18H^6\right)A^3 \end{array}\right)$

Remebering that $B = g(A,H)$, replacing and collecting respect to A, we obtaiṅ

$f(A,g(A,H),H) = A^{14}f_{14}(H) + A^{13}f_{13}(H) + +A^{12}f_{12}(H) + A^{11}f_{11}(H) + A^{10}f_{10}(H) + A^9f_9(H) + A^8f_8(H) + A^7f_7(H)$
$+A^6f_6(H) + A^5f_5(H) + A^4f_4(H) + A^3f_3(H) + A^2f_2(H) + A^1f_1(H) + f_0(H),$

where,

$f_{14}(H) = 39H^4$
$f_{13}(H) = -3H^2\left(-34H + 15H^3 + 5\right)$
$f_{12}(H) = -H\left(15H^4 + 249H^5 - 468H^6 + 45H^7 - 56\right)$

$f_{11}(H) = -\left(360H^8 - 421H^7 - 887H^6 + 561H^5 - 202H^4 + 19H^2 + 38H - 197\right)$
$f_{10}(H) = -H^2\left(33H - 110H^2 + 57H^3 - 209H^5 - 492H^6 + 1952H^7 - 2221H^8 + 360H^9 + 97\right)$
$f_9(H) = H^4\left(366H - 1174H^2 + 586H^3 - 1803H^4 + 1027H^5 + 2502H^6 - 816H^7 - 413\right)$
$f_8(H) = H^4\left(\begin{array}{c} +95 + H + 453H^2 - 1043H^3 + 1722H^4 - 1862H^5 + 141H^6 + 2681H^7 \\ -4313H^8 + 3920H^9 - 720H^{10} \end{array}\right)$
$f_7(H) = -H^7\left(341H - 2773H^2 + 4116H^3 - 3520H^4 + 3692H^5 - 5848H^6 + 1152H^7 - 285\right)$
$f_6(H) = H^3\left(\begin{array}{c} 18H^3 + 785H^4 - 970H^5 + 1812H^6 + 33H^7 + 1194H^8 - 1521H^9 - 698H^{10} \\ +2009H^{11} + 2312H^{12} - 1040H^{13} - 979 \end{array}\right)$
$f_5(H) = H^6\left(\begin{array}{c} 54H^3 + 2443H^4 - 2540H^5 + 3468H^6 + 839H^7 + 2908H^8 - 15\,400H^9 \\ +17\,380H^{10} - 4608H^{11} - 1958 \end{array}\right)$
$f_4(H) = -H^5\left(\begin{array}{c} 3916H + 2902H^4 - 2518H^5 - 335H^7 + 67H^8 - 4357H^9 + 6750H^{10} \\ -6864H^{11} + 8456H^{12} - 10\,624H^{13} + 4608H^{14} - 2937 \end{array}\right)$
$f_3(H) = -H^8\left(\begin{array}{c} 7832H + 4935H^4 - 7937H^5 + 996H^6 + 677H^7 + 7623H^8 - 27\,382H^9 \\ +41\,184H^{10} - 27\,520H^{11} + 6144H^{12} - 5874 \end{array}\right)$
$f_2(H) = -H^{11}\left(\begin{array}{c} 3778H + 184H^2 - 291H^3 + 4237H^4 - 7873H^5 + 1721H^6 + 5628H^7 \\ -5496H^8 + 2224H^9 - 2937 \end{array}\right)$
$f_1(H) = -H^{15}(4H-3)\left(-540H + 818H^2 + 35H^4 + 146\right)$
$f_0(H) = -H^{18}(4H-3)\left(-1428H + 1313H^2 + 433\right).$

Asigning some values to the parameter A, we can prove that $f(A, g(A,H), H)$ and $f(A,B,H)$ change of sign; then, the equilibrium $(0,0)$ of vector field $\bar{Z}_v$ is a weak focus of order two, i.e., it is surrounded by at least two limit cycles..

So, in the system (2) the equilibrium (H,H) is surrounded by at least two limit cycles. □

3.2 | System with positive equilibria

Case 2a Remebering that

$$a_0 = L^2 + (1-H)L - H(1-H) > 0 \text{ and } \Delta = (1-H+L)^2 - 4\left(L^2 + (1-H)L - H(1-H)\right) = 0,$$

then there exists a unique solution $u_* = \frac{1}{2}(1-H+L)$ of the equation (6), due to u_1 and u_2 conicide.

Then, the system (2) has two positive equilibria, (H,H) and (u_*, u_*).

The evaluation of the Jacobian matrix is the same for both points, i.e.,

$$DY_\eta(u,u) = \begin{pmatrix} u^2\left(-4u^3 + 3u^2 - A\right) & -Qu^2 \\ Bu\left(u^3 + A\right) & -Bu\left(u^3 + A\right) \end{pmatrix},$$

obtaining that

$$\det DY_\eta(u,u) = Bu^3\left(A + u^3\right)\left(A + Q - 3u^2 + 4u^3\right)$$

which sign depends of the factor

$$T = A + Q - 3u^2 + 4u^3.$$

Furthermore,

$$\mathrm{tr}DY_\eta(u,u) = u^2\left(-4u^3 + 3u^2 - A\right) - Bu\left(u^3 + A\right).$$

When the three equilibrium points of system (3.1) coincide it has

$$u_* = \frac{1}{2}(1-H+L) = H.$$

Then, $H = \frac{1+L}{3}$; replacing u_* in Q it obtains

$$Q = \frac{\left(27A + (1+L)^3\right)(2-L)}{27(L+1)} > 0, \text{ with } L < 2.$$

Lemma 7. When $H = \frac{1+L}{3}$, the equilibrium point $\left(\frac{1+L}{3}, \frac{1+L}{3}\right)$ is the unique positive equilibrium point, which is

i) an atractor node, if and only if, $B > \frac{\left((5-4L)(L+1)^2 - 27A\right)(L+1)}{3\left(27A + (L+1)^3\right)}$,

ii) a repeller node, if and only if, $B < \frac{\left((5-4L)(L+1)^2 - 27A\right)(L+1)}{3\left(27A + (L+1)^3\right)}$,

iii) a cusp point, if and only if, $B = \frac{\left((5-4L)(L+1)^2 - 27A\right)(L+1)}{3\left(27A + (L+1)^3\right)}$.

In the case ii) and iii) a non-infinitesimal limit cycles exists.

Proof. Evaluating the factor T it becomes

$T = A + \frac{1}{27}\left(27A + (1+L)^3\right)\frac{2-L}{L+1} + 4\left(\frac{1+L}{3}\right)^3 - 3\left(\frac{1+L}{3}\right)^2$
$= \frac{(L-1)(L+1)^3+27A}{9(L+1)}.$

Thus, when the factor $T = 0$, it has that the numerator of T,
$N = (L-1)(L+1)^3 + 27A = 0$, and
$A = \frac{1}{27}(1-L)(L+1)^2 > 0$; so, $L < 1$.

Then, the point $\left(\frac{1+L}{3}, \frac{1+L}{3}\right)$ is a non-hyperbolic equilibrium.

Assuming $\operatorname{tr}DY_\eta\left(\frac{1+L}{3}, \frac{1+L}{3}\right) = 0$, and replacing $H = \frac{1}{3}(1+L)$ in the expression of B, it has that,
$B = \frac{\left((5-4L)(L+1)^2-27A\right)(L+1)}{3\left(27A+(L+1)^3\right)}$,
with,
$A < \frac{1}{27}(5-4L)(L+1)^2$; so, $L < \frac{5}{4}$.

Considering
i) $\operatorname{tr}DY_\eta\left(\frac{1+L}{3}, \frac{1+L}{3}\right) < 0$, it obtains $B > \frac{\left((5-4L)(L+1)^2-27A\right)(L+1)}{3\left(27A+(L+1)^3\right)}$,
ii) $\operatorname{tr}DY_\eta\left(\frac{1+L}{3}, \frac{1+L}{3}\right) > 0$, it obtains $B < \frac{\left((5-4L)(L+1)^2-27A\right)(L+1)}{3\left(27A+(L+1)^3\right)}$.
iii) For $\operatorname{tr}DY_\eta\left(\frac{1+L}{3}, \frac{1+L}{3}\right) = 0$, the point $\left(\frac{1+L}{3}, \frac{1+L}{3}\right)$ is a cusp point for $L < 1$.

In the case ii and iii), since the point $\left(\frac{1+L}{3}, \frac{1+L}{3}\right)$ is the unique on the invariant region $\bar{\Gamma}$, the trajectories must have an $\omega-limit$; so, by the Poincaré-Bendixson Theorem a non-infinitesimal limit cycle exists. □

Remark 4. 1. Replacing $\left(\frac{1+L}{3}\right)$ in $P(u)$ and $P_1(u)$, it has
$P(u) = \left(\frac{1+L}{3}\right)^4 - \left(\frac{1+L}{3}\right)^3 + \left(A + \frac{1}{27}\left(27A + (1+L)^3\right)\frac{2-L}{L+1}\right)\left(\frac{1+L}{3}\right) - A = 0,$
and
$P_1(u) = \left(\frac{1+L}{3}\right)^3 - \left(1 - \left(\frac{1+L}{3}\right)\right)\left(\frac{1+L}{3}\right)^2 - \left(\frac{1+L}{3}\right)\left(1 - \left(\frac{1+L}{3}\right)\right)\left(\frac{1+L}{3}\right) + \frac{A}{\left(\frac{1+L}{3}\right)}$
$= \frac{1}{9}\frac{L^4+2L^3-2L+(27A-1)}{L+1} = \frac{T}{9(L+1)} = 0.$

2. The dynamics of the system 2 when $u_* \neq H$ will be presented in a future work.

3. The existence of the non-infinitesimal limit cycle, which is not obtained by Hopf bifurcation, it persists below small perturbations on the parameter values. This has a great importance in the model, because in the systems (7) and (2), when there exist two or three positive equilibria, any be their nature, they can be enclosed by a non-infinitesimal limit cycle (See Figures 5-7).

4. For a wide subset of parameter values, the populations can coexist oscillating their population sizes, although the phenomenon of tri-stability can also exists (Figure 7). In this situation, the system (2) has two attractor focus coexisting with the stable non-infinitesimal stable limit cycle.

Case 2b. Assuming that $a_0 = L^2 + (1-H)L - H(1-H) = 0$, the equation (2.6) has a unique solution $u_{**} = 1 - H + L$. . Moreover, $u = 0$ is a root of multiplicity two.

Then, the system (2) has two positive equilibria, (H, H) and $\left(u_{**}, u_{**}\right)$.

The evaluation of the Jacobian matrix is the same for both points, as the above case.

The sign of $\det DY_\eta(u,u)$ depends of the factor T.

Without loss generality we assume that $u_{**} \leq H$; then, $H \geq \frac{1+L}{2}$.

In particular, when $u_{**} = H$, it obtains $H = \frac{1+L}{2}$.

Lemma 8. When $H = \frac{1+L}{3}$, the equilibrium point $\left(\frac{1+L}{2}, \frac{1+L}{2}\right)$ is the unique positive equilibrium point, which is
i) an atractor node, if and only if, $B > \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3}$,
ii) a repeller node, if and only if, $B < \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3}$,
iii) a cusp point, if and only if, $B = \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3}$.
In the case ii) and iii) a non-infinitesimal limit cycles exists.

Proof. Replacing u_* in Q and factoring it obtains

$Q = \frac{\left(8A+(L+1)^3\right)(1-L)}{8(L+1)} > 0.$

Then, $L < 1$.

Replacing in the factor T and factoring it becomes

$T = \frac{2}{L+1}A - \frac{1}{8}\left(1-3L\right)\left(L+1\right)^2.$

$T = 0$ implies, $A = \frac{1}{16}\left(1-3L\right)\left(L+1\right)^3$; so, $L < \frac{1}{3}$.

Then, the point $\left(\frac{1+L}{2}, \frac{1+L}{2}\right)$ is a non-hyperbolic equilibrium.

Assuming $\operatorname{tr}DY_\eta\left(\frac{1+L}{2}, \frac{1+L}{2}\right) = 0$, and replacing $H = \frac{1+L}{2}$ in the expression of B, it has that,

$B = \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3},$

with

$A < \frac{1}{4}\left(1-2L\right)\left(L+1\right)^2$; so, $L < \frac{1}{2}$.

Considering

i) $\operatorname{tr}DY_\eta\left(\frac{1+L}{2}, \frac{1+L}{2}\right) < 0$, it obtains $B > \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3}$,

ii) $\operatorname{tr}DY_\eta\left(\frac{1+L}{2}, \frac{1+L}{2}\right) > 0$, it obtains $B < \frac{\left((1-2L)(L+1)^2-4A\right)(L+1)}{8A+(L+1)^3}$.

iii) For $\operatorname{tr}DY_\eta\left(\frac{1+L}{3}, \frac{1+L}{3}\right) = 0$, the point $\left(\frac{1+L}{2}, \frac{1+L}{2}\right)$ is a cusp point of codimension two, for $L < \frac{1}{3}$.

Since the point $\left(\frac{1+L}{2}, \frac{1+L}{2}\right)$ is the unique on the invariant region $\bar{\Gamma}$, in the case ii and iii), the trajectories must have an $\omega-limit$; so, newly by the Poincaré-Bendixson Theorem applies and a non-infinitesimal limit cycle exists. □

Remark 5. 1. The properties of the system when $u_{**} < H$ and when exist three equilibrium points will be exposed in a future work. Here, we shown some simulations for this last case.

2. The non-infinitesimal stable limit cycle, which existence is proved in the above lemma can maintain under small perturbations of the parameters, when three positive equilibria exist; but after it disappears for a subset of the parameter values. This implies that both species can coexist for different population sizes, without oscillations on these sizes (Figure 8).

Although we do not analyze the case in which there are 3 positive equilibrium points, we will show some examples with the dynamics of the model in that situation.

4 | SOME NUMERICAL SIMULATIONS

In order to reinforce the obtained results, some computer simulations are also given, presenting different behaviors of system (2.2). The diverse natures of the positive equilibrium point $(H, H+C)$ are shown for different parameters values.

Case 1. There exists a unique positive equilibrium point

HERE figures 2, 3 and 4

Case 2. There exists three positive equilibrium points

HERE figures 5, 6, 7 and 8

5 | CONCLUSIONS

In this work, a bidimensional continuous-time differential equations system was analyzed, which is derived from a Leslie-Gower type predator-prey model by considering a Holling type IV or non-monotonic functional response.

The Leslie-Gower type predation models not are based in a transference mass principle as it happens with the Gause type models. However, the analyzed model illustrates the dynamical complexities that can produce in relatively simple predator-prey interactions. when are described by an ordinary differential equation system.

In order to simplify the calculationsby, a topologically equivalent polynomial system was obtained, by mean a reparameterization and a time rescaling.

Using the method of blowing up[43], we have shown that the point $(0,0)$ is non-heperbolic saddle point, although the modified Leslie-Gower model proposed originally is not defined there. Moreover, it was proved that the equilibrium $(1,0)$ is a hyperbolic saddle for all parameter values. Moreover, it is proved that there exists a heteroclinic curve joining the saddle point $(1, 0)$ and the non-hyperbolic singularity $(0, 0)$.

From an ecological point of view, we can to say that this model is persistent, the populations cannot go to extinction simultaneously, although the predator population can extinct and the prey population can attain its maximun size in the environment.

Also, we proved the boundedness of solutions of the dimensionless system (2), using the extended real line to apply the compactification of Poincaré[45], showing that the modified Leslie-Gower model with a non-monotonic functional response is well posed[2].

Assuming the existence a unique equilibrium point (H, H), we have established the existence of the parameter constraints for which the positive equilibrium point is an attractor or a repeller surrounded by at least one limit cycle, existing a Hopf bifurcation.

Using the method of Lyapunov numbers was proved that there are conditions on the parameter values for which the equilibrium at the interior of the first quadrant is a weak focus of order two, i.e., there exist two concentric limit cycles surrounding that equilibrium, the innermost unstable and the outermost stable.

Taylor in[17] hypothesizes that a non-monotonic functional response should tend to destabilize a system; the results obtained would seem to support this conjecture, because for a wide range of parameter the bistability phenomenon is prevalent.

Considering the exposed simulations, it is further shown that by choosing different values of parameters the model more interesting dynamics are possibles, such as the existence of heteroclinic and homoclinic curves, and a non-infinitesimal limit cycle enclosing the three equilibrium points. Besides, there exists a subset of the parameter space in which three limit cycles coexist: a non-infinitesimal stable limit cycle surrounding three positive equilibria and two infiitesimal unstable limit cycles.

We show that the qualitatively behavior of the model studied in this work is not similar to May-Holling-Tanner[8] because in the model here studied can have up to three equilibrium point at the interior of the first quadrant; but both systems can have two concentric limit cycles; nonetheless, the precise number of such solutions must be proven as in[48].

However, the behavior of the model studied is similar to the model analyzed in[22,23] in which rhe non-monotonic functional response is described by $h(x) = \frac{qx}{x^2+a}$; both systems have the same number of equilibrium points and parameter constraints assuring the existence of two limit cycles. This should imply that the exponents on the non-monotonic functional response could have not importance.

Nevertheless, this question could be resolved by considering the study of models assuming the general form $h(x) = \frac{qx^m}{x^n+a}$, with $n > m \geq 1$, n and $m \in \mathbb{N}$[49].

Although the system has mathematical complexities, we consider that the Holling type IV represents adequately the antipredator behavior named group defense.

Furthermore, the system studied here has clear differences with the system modelling the phenomenon of "herd behavior" studied in[34]. In that model, among other properties, there are solutions that arrive to the vertical axis (y-axis) in a finite time, implying the extinction of the prey population in a short lapse, which can happen locally. This phenomenon cannot be described by the system (2) since $(0, 0)$ is a saddle point, repelling by the direction of the $x - axis$.

In short, our main results can be highlighted as follows:

1. Group defense is an antipredator behavior that produces a rich dynamic for the predator-prey interaction, if a non-monotonic functional response is considered.

2. When exists a unique positive equlibrium point, there are parameter constraints for which the phenomenon of bi-stabilty appears, since coexist a stable limit cycle with an attractor focus.

3. The model is persistent because the point $(0, 0)$ is a non-hyperbolic saddle point.

4. It can possibly the existence up three positive equilibria, all the which can be surrounded by a non-infinitesimal limit cycle, not obtained by Hopf bifurcation.

5. The phenomenon of the multi-stability exists since can have two attractor focus (and a saddle point) enclosed by a stable non-infinitesimal limit cycle.

6. By simulations is shown there is a subset of the parameter values, for which three limit cycles coexist. Two of them are unstable and the third is a non-infinitesimal stable limit cycle.

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