MILD SOLUTIONS OF A FRACTIONAL PDE WITH NOISE

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Abstract. This article focuses on studying mild solutions of an original fractional partial differential equation disturbed by multiplicative white noise. We employ techniques of semi group theory, Hausdorff measure, and Darbo fixed point theorem.
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1. Introduction

We consider an original fractional PDE with multiplicative white noise:

$$ {}^{c}D_{0^{+}}^{\alpha}\left[u(x,t)-\eta(u(x,t))\right]=\Delta u(x,t)+u(x,t)\cdot\nabla u(x,t)+g(u(x,t))W(t), \tag{1.1}$$

with

$$u\left(x,0\right)=u_{0}\left(x\right),\ x\in D \tag{1.2}$$

and

$$u\left(x,t\right)=0,\ x\in\partial D, \tag{1.3}$$

in which $D\subset\mathbb{R}^{d}$ is domain, Δ is Laplacian operator, $u\left(\cdot\right)$ on a Hilbert space $\mathcal{H}$ with $\langle\cdot,\cdot\rangle$, $W\left(t\right)$ is a white noise with filtration $F_{t}-$adapted, where $F_{t}=\sigma\left\{W\left(t\right)\right\}$.

${}^{c}D_{t}^{\alpha}$ symbolizes fractional-order derivative operator in the sense of Caputo for $\alpha\in(0,1)$ is defined as:

$$ {}^{c}D_{t}^{\alpha}u\left(t,x\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\int_{0}^{t}\frac{\partial u\left(\omega,x\right)}{\partial\omega}\frac{d\omega}{\left(t-\omega\right)^{\alpha}}. \tag{1.4}$$

The existence and non-existence results for fractional-stochastic PDEs were discussed in [18]. Chemin et al. [6] studied the global regularity for the large solutions to the NSEs. Miura [21] and Germain [13] focused on the uniqueness of mild solutions to the NSEs. Several writers, see [22, 26, 27, 29], have acquired the presence and uniqueness of the SNSEs equations. Just mention a few works of Navier-Stokes time-fractional equations has become a hot topic of studies because of its important role in simulating anomalous diffusion.

Time-fractional differential equations have resulted in numerous mathematical applications of interest. Specifically, see [9, 28, 31, 32, 33] for modeling anomalous processes of diffusion as it may characterize long storage procedures. In addition, exact solutions of some fractional PDEs were studied by different techniques in [14, 23].

Several authors, see for instance Ezzinbi and al. [10] and Cui et al. [5], acquired the presence and regularity of solutions for some PDEs. Research is particularly interested in the stability of differential equations in stochastic concepts. A few papers were in that direction. The comparison theorem was used in [6] to demonstrate stability of stochastic PDEs for a mild solution. Caraballo and others. [3], Liu [19] and Luo [20] provided three techniques for resolving SPDE with delays based on well-known Gronwall inequality, functional Lyapunov and Razuminkhin theorem.

This paper's primary input is to create a mild solution to the$(1.1)-(1.3)$ issue. Using primarily Holder"s inequality, stochastic analysis, Darbo's fixed-point theorem coupled with Hausdorff's methods of measuring non-compactness, we obtain the presence of moderate issue solutions $(1.1)-(1.3)$.

2. Preliminaries

Assume that $\left(\Omega, F, P, \{F_t\}_{t\geq 0}\right)$ is a filter. Assume the operator $\mathcal{A}$ as infinitesimal generator of a strongly continuous semi group on the Hilbert space $H = L^2(D)$. In particular, let

$$A := -\triangle,\ D(A) = H_0^1(D) \cap H^2(D).$$

It is clear that he operator $A$is a positive self-adjoint operator. Let e_k denote the eigenvectors corresponding to eigenvalues λ_k such that

$$Ae_k = \lambda_k e_k,\ e_k = \sqrt{2} sin(k\pi),\ \lambda_k = \pi^2 k^2,\ k \in \mathbb{N}^+.$$

For $\sigma > 0$, we have the following representation for the fractional power $A^{\frac{\sigma}{2}} = (-\Delta)^{\frac{\sigma}{2}}$ and its domain H^σ which can be defined by

$$\sigma > 0, \quad A^{\frac{\sigma}{2}} e_k = \lambda_k^{\frac{\sigma}{2}} e_k,\ k = 1, 2, \ldots$$

and

$$H^\sigma = D\left(A^{\frac{\sigma}{2}}\right) = \left\{ v \in L^2(D),\ s.t.\ \|v\|_{H^\sigma}^2 = \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} v_k^2 < \infty \right\},$$

where $L^2(D)$ is a Hilbert space with,

$$v_k = \langle v, e_k \rangle, \quad \|H^\sigma v\| = \left\| A^{\frac{\sigma}{2}} v \right\|.$$

B is $B(u,v) = u \cdot \nabla v$ with domain $\mathcal{D}(B) = H_0^1(D)$ and $B(u) = B(u,u)$. By using these notations, it is possible to express the main equation $(1.1)-(1.3)$ as follows:

$$\begin{cases} {}^c D_t^\alpha \left[u(t) - \eta(u(t))\right] = Au(t) + B(u(t)) + g(u(t)) \frac{W(t)}{dt},\ t > 0, \\ \qquad u(0) = u_0, \end{cases} \tag{2.1}$$

in which $\{W_t : t \geq 0\}$ is cylindrical L^2-valued Brownian motion (L^2-valued stochastic approach $W(t)$ so that a trace class operator Q denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots$. So

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k,$$

with $\{\beta_k\}_{k=1}^{\infty}$ is a family of Brownian motions.

Define $L_0^2 := L^2\left(Q^{\frac{1}{2}}(H), H\right)$ be a Hilbert-Schmidt space of operators from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\phi\|_{L_0^2} = \left\|\phi Q^{\frac{1}{2}}\right\|_{H^\sigma} = \left(\sum_{n=1}^{\infty} \phi Q^{\frac{1}{2}} e_n\right)^{\frac{1}{2}},$$

i.e.,

$$L_0^2 = \left\{\phi \in L(H) : \sum_{n=1}^{\infty} \left\|\lambda_n^{\frac{1}{2}} \phi Q^{\frac{1}{2}} e_n\right\|^2 < \infty\right\},$$

where $L(H) := \{\phi : \phi : H \mapsto H$ is a linear bounded operator$\}$.

Let K be a Banach space. Then,

$$\|v\|_{L^p(\Omega,K)} = \left(E\|v\|_K^p\right)^{\frac{1}{p}}, \ \forall v \in L^p(\Omega, F, P, K), \text{ for every } 2 \leq p.$$

Lemma 2.1. *([32]) Suppose $T(t) = e^{-tA}$ is a semi group generated by an operator A. For each $\mu > 0$, we have a constant C_μ depending on μ so that*

$$\|AT(t)\|_{L^p} \leq C_\mu t^{-\mu}, \ t > 0.$$

Following Lemma will be introduced to assess the stochastic integrals containing the inequality of the Burkhoder-Davis-Gundy.

Lemma 2.2. ([17]) *For all $2 \leq p$ and $T > t_2 > t_1 \geq 0$, and $u : [0,T] \times \Omega \to L_0^2$ holds for*

$$E\left[\left(\int_0^T \|u(\omega)\|_{L_0^2}^2 d\omega\right)^{\frac{p}{2}}\right] < \infty.$$

The identity

$$E\left[\left\|\int_{t_1}^{t_2} u(\omega)\, dW(\omega)\right\|^p\right] < C(p)\, E\left[\left(\int_{t_1}^{t_2} \|u(\omega)\|_{L_0^2}^2 d\omega\right)^{\frac{p}{2}}\right].$$

holds true under these assumptions.

Inspired by the time-fractional differential equations definition of the mild solution (see [28]), we present the definition:

Definition 2.3. *An F_t-adapted stochastic approach, $u(t)$, is known as mild solution for (2.1) if*

$$u(t) = E_\alpha(t) u_0 + h(u(t)) + \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-\omega) B(u(\omega))\, d\omega$$
$$+ \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-\omega) g(u(\omega))\, dW(\omega), \qquad (2.2)$$

where the $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are stated as:

$$E_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta)\, d\theta,$$

and

$$E_{\alpha,\alpha}(t)=\int_0^\infty \alpha\theta\zeta_\alpha(\theta)\,T(t^\alpha\theta)\,d\theta,$$

where $T(t)=e^{-tA}0$ *is a semi group produced by an operator* $-A$*. In this definition,* $\zeta_\alpha(\theta)$ *is Mainardi's Wright function:*

$$\zeta_\alpha(\theta)=\sum_{m=0}^{\infty}\frac{(-1)^m\theta^m}{m!\Gamma(1-\alpha(1+m))}.$$

Lemma 2.4. ([4]) *For any* $\alpha\in(0,1)$ *and* $-1<\nu<\infty$*, it is clear that*

$$\zeta_\alpha(\theta)\geq 0\,,\ \int_0^\infty \theta^\nu\zeta_\alpha(\theta)\,d\theta=\frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)},\tag{2.3}$$

for every $\theta\geq 0$.

$\{E_\alpha(t)\}$ and $\{E_{\alpha,\alpha}(t)\}$ in (2.8) for $t\geq 0$ possesses the conditions as follows:

Lemma 2.5. *Let* $E_\alpha(t)$ *and* $E_{\alpha,\alpha}(t)$ *be bounded-linear operators. For* $2>\nu\geq 0,\quad \alpha\in(0,1)$*, there is a positive constant* C *for which the inequalities*

$$\|E_\alpha(t)\,y\|_{H^\nu}\leq Ct^{-\frac{\alpha\nu}{2}}\|y\|\,,\quad \|E_{\alpha,\alpha}(t)\,y\|_{H^\nu}\leq Ct^{-\frac{\alpha\nu}{2}}\|y\|\,.\tag{2.4}$$

hold.

Proof. Let $T>0$ and $2>\nu\geq 0$. The Lemmas 2.1, 2.4 imply that

$$\begin{aligned}\|E_\alpha(t)\,\chi\|_{H^\nu}&\leq\int_0^\infty \zeta_\alpha(\theta)\,\|A_\nu T(t^\alpha\theta)\,\chi\|\,d\theta\\&\leq\int_0^\infty C_\nu t^{-\frac{\alpha\nu}{2}}\theta^{-\nu}\zeta_\alpha(\theta)\,\|\chi\|\,d\theta\\&=\frac{C_\nu\Gamma(1-\nu)}{\Gamma(1-\alpha\nu)}t^{-\frac{\alpha\nu}{2}}\|\chi\|\,,\quad \chi\in L^2(D)\,,\end{aligned}$$

and

$$\begin{aligned}\|E_{\alpha,\alpha}(t)\,\chi\|_{H^\nu}&\leq\int_0^\infty \alpha\theta\zeta_\alpha(\theta)\,\|A_\nu T(t^\alpha\theta)\,\chi\|\,d\theta\\&\leq\int_0^\infty C_\nu\alpha t^{-\frac{\alpha\nu}{2}}\theta^{1-\nu}\zeta_\alpha(\theta)\,\|\chi\|\,d\theta\\&=\frac{C_\nu\alpha\Gamma(2-\nu)}{\Gamma(1-\alpha\nu)}t^{-\frac{\alpha\nu}{2}}\|\chi\|\,,\quad \chi\in L^2(D)\,,\end{aligned}$$

so, $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are bounded operators and linear. The proof is completed. □

Lemma 2.6. *Let* t *be a positive real number. Then,*
(i) *For* $0<\alpha<1$ *and* $0\leq\nu<2$ *and* $0\leq t_1<t_2\leq T$*, there is a* $C>0$ *for which*

$$\|(E_\alpha(t_2)-E_\alpha(t_1))\,\chi\|_{H^\nu}\leq C\,(t_2-t_1)^{\frac{\alpha\nu}{2}}\|\chi\|\,,\tag{2.5}$$

and

$$\|(E_{\alpha,\alpha}(t_2)-E_{\alpha,\alpha}(t_1))\,\chi\|_{H^\nu}\leq C\,(t_2-t_1)^{\frac{\alpha\nu}{2}}\|\chi\|\,.\tag{2.6}$$

(ii) $E_\alpha(t)$ *and* $E_{\alpha,\alpha}(t)$ *are strongly continuous operators.*

Proof. It is not hard to show that for any $0 < T_0 \le t_1 < t_2 \le T$,

$$\int_{t_1}^{t_2} \frac{dT(t^\alpha\theta)}{dt}dt = T(t_2^\alpha\theta) - T(t_1^\theta) = \int_{t_1}^{t_2} \alpha t^{\alpha-1}\theta A T(t^\alpha\theta)\,dt,$$

and by (2.3) and Lemma 2.1, we have

$$\begin{aligned}
\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} &= \|A_\nu(E_\alpha(t_2) - E_\alpha(t_1))\chi\| \\
&= \left\| \int_0^\infty \zeta_\alpha(\theta) A_\nu\left(T(t_2^\alpha\theta) - T(t_1^\theta)\right)\chi d\theta \right\| \\
&\le \int_0^\infty \alpha\theta\zeta_\alpha(\theta)\int_{t_1}^{t_2} t^{\alpha-1}\|A_{2+\nu}T(t^\alpha\theta)\chi\|_{L^2}\,dtd\theta \\
&\le \int_0^\infty C_\nu\alpha\theta^{-\frac{\nu}{2}}\zeta_\alpha(\theta)\left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1}dt\right)\|\chi\|\,d\theta \\
&= \frac{2C_\nu\Gamma\left(1-\frac{\nu}{2}\right)}{\nu\Gamma\left(1-\frac{\alpha\nu}{2}\right)}\left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}}\right)\|\chi\| \\
&\le \frac{2C_\nu\Gamma\left(1-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu}\Gamma\left(1-\frac{\alpha\nu}{2}\right)}(t_2-t_1)^{\frac{\alpha\nu}{2}}\|\chi\|,\ \chi\in L^2(D).
\end{aligned}$$

Also

$$\begin{aligned}
\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\|_{H^\nu} &= \|A_\nu(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\| \\
&= \left\| \int_0^\infty \alpha\theta\zeta_\alpha(\theta) A_\nu\left(T(t_2^\alpha\theta) - T(t_1^\theta)\right)\chi d\theta \right\| \\
&\le \int_0^\infty \alpha^2\theta^2\zeta_\alpha(\theta)\int_{t_1}^{t_2} t^{\alpha-1}\|A_{2+\nu}T(t^\alpha\theta)\chi\|_{L^2}\,dtd\theta \\
&\le \int_0^\infty C_\nu\alpha^2\theta^{1-\frac{\nu}{2}}\zeta_\alpha(\theta)\left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1}dt\right)\|\chi\|\,d\theta \\
&= \frac{2\alpha C_\nu\Gamma\left(2-\frac{\nu}{2}\right)}{\nu\Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)}\left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}}\right)\|\chi\| \\
&\le \frac{2C_\nu\Gamma\left(2-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu}\Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)}(t_2-t_1)^{\frac{\alpha\nu}{2}}\|\chi\|,\ \chi\in L^2(D).
\end{aligned}$$

We have

$$\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \to 0,$$

and

$$\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\|_{H^\nu} \to 0,$$

as $t_1 \to t_2$ which implies the strongly continuity of operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$.

□

Definition 2.7. *Hausdorff measure of non-compactness of a bounded nonempty set Γ of $\widetilde{Y}$ is denoted by $\chi(\Gamma)$ and is defined as*

$$\chi_{\widetilde{Y}}(\Gamma) = inf\left\{\epsilon > 0 \,:\, \Gamma \subset S + \epsilon\Gamma_{\widetilde{Y}},\, S \subset \widetilde{Y},\, S \text{ is finite}\right\}.$$

We recall here the some basic facts about $\chi_{\widetilde{Y}}(\cdot)$.

Lemma 2.8. ([2]) *Suppoe that Y is a Hilbert space and $\Gamma,\, C \subseteq \widetilde{Y}$ are bounded. Then:*
(1) Γ *is pre-compact if and only if* $\chi_{\widetilde{Y}}(\Gamma) = 0$,

(2) $\chi_{\widetilde{Y}}(\Gamma) = \chi_{\widetilde{Y}}\left(\overline{\Gamma}\right) = \chi_{\widetilde{Y}}(conv\Gamma)$, *in which $\overline{\Gamma}$ is closure, $conv\Gamma$ is convex hull of* Γ.

(3) $\chi_{\widetilde{Y}}(\Gamma) \subseteq \chi_{\widetilde{Y}}(C)$ *for* $\Gamma \subset C$,
(4) $\chi_{\widetilde{Y}}(\Gamma + C) \leq \chi_{\widetilde{Y}}(\Gamma) + \chi_{\widetilde{Y}}(C)$ *in which* $\Gamma + C = \{a + b :\, a \in \Gamma,\, b \in C\}$,
(5) $\chi_{\widetilde{Y}}(\Gamma \cup C) = max\left\{\chi_{\widetilde{Y}}(\Gamma), \chi_{\widetilde{Y}}(C)\right\}$,
(6) $\chi_{\widetilde{Y}}(\lambda\Gamma) \leq |\lambda|\, \chi_{\widetilde{Y}}(\Gamma)$ *where λ is a real number.*
(7) *For a Lipschitz continuous map $D(\Phi) \subseteq Y \mapsto B$, the inequality*

$$\chi_B(\Phi\Gamma) \leq k\chi_{\widetilde{Y}}(\Gamma)$$

holds for a bounded set $\Gamma \subseteq D(\Phi)$ in which B is a Banach space.

Definition 2.9. ([25]) *The map : $V : \widetilde{Y} \to Y$ is called a $\chi_{\widetilde{Y}}-$contraction if*

$$\chi_{\widetilde{Y}}(\Phi(\Gamma)) \leq k\chi_{\widetilde{Y}}(\Gamma)$$

for $0 \leq k < 1$, where $\Gamma \subseteq V$ is bounded and closed and $\widetilde{Y}$ is a Banach space.

Lemma 2.10. *(([1])) Let $V \subseteq Y$ be a closed and convex subset of a Hilbert space Y with $0 \in V$. Then, the continuous transformation: $V : Y \to Y$ is a χ_Y-contraction. Furthermore, if $\{u \in V \,:\, u = \lambda\Phi(u)\}$ is bounded where $0 < \lambda < 1$, Φ has at least one fixed point in V.*

Next we present some outcomes on the presence of moderate (2.1) issue alternatives. We create the following hypotheses in order to achieve this:
(H_1) Suppose that the operator A is infinitesimal generator of $\{T(t)\}_{t\geq 0}$ on H. We are also assuming compactness of operator $E_\alpha(t)$.
(H_2) The map $h : \Omega \times H \to L_0^2$ holds the conditions of global growth and Lipshitz continuity:

$$\|h(v)\|_{L_0^2} \leq C\|u\|,\; \|h(u) - h(v)\|_{L_0^2} \leq C\|u - v\|,$$

for each $v, u \in H$.

(H_3) For the initial condition u_0 which is a random variable with measure F_0, the inequality

$$\|u_0\|_{L^p(\Omega, H^\nu)} < \infty, \text{ for any } 0 \leq \nu < \alpha < 2.$$

holds.
(H_4) Assume that $g : L_0^2 \to L_0^2$ is a continuous function. Then,

$$E\|g(u_1(t)) - g(u_2(t))\|_{L_0^2}^p \leq L_g\|u_1(t) - u_2(t)\|_{L_0^2}^p,$$

where $L_g > 0$, and $t \in [0, T]$, $u_1,\, u_2 \in L_0^2$.

and

$$E\left\|g\left(u\left(t\right)\right)\right\|_{L_0^2}^p \leq L_g E\left\|u\left(t\right)\right\|_{L_0^2}^p,\ t\in\left[0,T\right],\ u\in L_0^2.$$

(H_5) The map $Z : L^2(D) \to H^{-1}(D)$ satisfies

$$\left\|Z\left(u\right)\right\|_{H^{-1}} \leq Z_c\left\|u\right\|^2,$$

and

$$\left\|Z\left(u\right)-Z\left(v\right)\right\|_{H^{-1}} \leq Z_c\left(\left\|u\right\|+\left\|v\right\|\right)\left\|u-v\right\|,$$

where $Z_c > 0$, and $u, v \in L^2(D)$.

Lemma 2.11. *Let Φ_1 be the operator expressed by for each $u\in K$*

$$\Phi_1\left(u\right)=\int_0^t\left(t-s\right)^{\alpha-1}E_{\alpha,\alpha}\left(t-s\right)B\left(u\left(s\right)\right)ds. \tag{2.7}$$

Under assumptions (H_1) and (H_2), the operator Φ_1 is continuous and maps K into itself.

Proof. It is obvious that Φ_1 is continuous. Next, we show that $\Phi_1(K)\subset K$. By (H_1) and (H_2) and the equation 2.7, we get

$$\begin{aligned}
E\left\|\left(\Phi_1 u\right)(t)\right\|_{H^\nu}^p &= E\left\|\int_0^t\left(t-\omega\right)^{\alpha-1}A_1E_{\alpha,\alpha}\left(t-\omega\right)A_{\nu-1}Z\left(u\left(s\right)\right)d\omega\right\|_{H^\nu}^p\\
&\leq C_\alpha^p\left(\int_0^t\left(t-\omega\right)^{\frac{p\left(\frac{\alpha-1}{2}\right)}{p-1}}d\omega\right)^{p-1}\int_0^t E\left[\left\|A_{\nu-1}Z\left(u\left(\omega\right)\right)\right\|^p\right]d\omega\\
&\leq C^pC_\alpha\left[\frac{2\left(p-1\right)}{p-2}\right]^{p-1}\left(T\right)^{\frac{p-2}{2}}\int_0^t E\left[\left\|u\left(t\right)\right\|_{H^\nu}^p\right]\\
&= \gamma_1\int_0^t E\left[\left\|u\left(\omega\right)\right\|_{H^\nu}^p\right]d\omega \qquad ,
\end{aligned} \tag{2.8}$$

with $\gamma_1 = C^pC_\alpha\left[\frac{2(p-1)}{p-2}\right]^{p-1}\left(T\right)^{\frac{p-2}{2}}$. This complete the proof. □

Lemma 2.12. *Let K be a Hilbert space. The operator*

$$\Phi_2\left(u\right)=\int_0^t S_\alpha\left(t-\omega\right)g\left(u\left(\omega\right)\right)dW\left(\omega\right)\ \textit{for each } u\in K.$$

is continuous and maps K into itself by the assumptions of (H_1) and (H_2).

Proof. By Lemma 2.2, we obtain

$$E\left\|\left(\Phi_2 u\right)(t)\right\|_{H^\nu}^p = E\left\|\int_0^t\left(t-\omega\right)^{\alpha-1}E_{\alpha,\alpha}\left(t-\omega\right)g\left(u\left(\omega\right)\right)dW\left(\omega\right)\right\|_{H^\nu}^p$$

$$\leq C(p)\,E\left[\left(\int_0^t \left\|(t-\omega)^{\alpha-1}E_{\alpha,\alpha}(t-\omega)\right\|^2 \|A_\nu g(u)\|^2_{L_0^2}\,d\omega\right)^{\frac{p}{2}}\right]$$

$$\leq C(p)\,C_\alpha^p\left(\int_0^t (t-\omega)^{\frac{2p(\alpha-1)}{p-2}}\right)^{\frac{p-2}{2}}\int_0^t E\,\|A_\nu g(u)\|^p_{L_0^2}\,d\omega$$

$$\leq C(p)\,C_\alpha^p\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p-2}{2}}\int_0^t E\,\|A_\nu g(u)\|^p_{L_0^2}\,d\omega$$

$$=\gamma_2\int_0^t E\left[\|u(s)\|^p_{H^\nu}\right]ds, \tag{2.9}$$

where $\gamma_2 = C(p)\,C_\alpha^p C^p\left[\frac{p-2}{p(2\alpha-1)-2}\right]^{\frac{p-2}{2}}$. This implies that $\Phi_2(Z)\subset Z$. □

Lemma 2.13. *Assume that the operator Φ_3 defined in Z, satisfies*

$$(\Phi_3 u)(t) = E_\alpha(t)\,u_0 + g(u(t))\,.$$

By the assumptions (H_1) and (H_4), the operator $\Phi_3 : Y \mapsto Y$ is continuous.

Proof. The continuity in $p-th$ moment of Φ_3 follows from (H_4).
Next, we show that $\Phi_3(Y)\subset Y$. By (2.9) and the assumptions (H_1), (H_5), one gets

$$E\,\|(\Phi_3 u)(t)\|^p_{L_0^2} \leq E\,\|g(u(t))\|^p_{L_0^2} \leq L_h E\,\|u(t)\|^p_{L_0^2}\,.$$

So, we conclude $\Phi_3(Z)\subset Z$. □

Lemma 2.14. *By (H_1) and (H_2), we have*

$$E\left[\|E_\alpha(t)\,u_0\|_{H^\nu}\right] \leq E\left[\|u_0\|_{H^\nu}\right].$$

Proof. The Lemma 2.1 implies that

$$E\left[\|E_\alpha(t)\,u_0\|_{H^\nu}\right] \leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\|A_\nu T(t^\alpha\theta)\,u_0\|^2\right)^{\frac{1}{2}}d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu e^{-t^\alpha\theta A}u_0, e_n\right\rangle^2\right)^{\frac{1}{2}}d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu u_0, e^{-t^\alpha\theta\lambda_n^{\frac{\nu}{2}}}, e_n\right\rangle^2\right)^{\frac{1}{2}}d\theta\right]$$

$$\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\,\|u_0\|_{H^\nu}\,d\theta\right] = E\left[\|u_0\|_{H^\nu}\right].$$

□

Now, we set $T = T_1 + T_2$, where

$$(T_1 u)(t) = E_\alpha(t)\, u_0(s) + g(u(t)),$$

and

$$(T_2 u)(t) = \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-\omega)\, Z(u(\omega))\, d\omega + \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-\omega)\, f(u(\omega))\, dW(\omega),$$

for $t \in [0, T]$.

Lemma 2.15. *Assume $(H_2), (H_4)$, (H_5) hold and $0 < \nu < \alpha \leq 2$, $p \geq 2$. Then,*

$$E \left\| E_\alpha(t_2) - E_\alpha(t_1) \right\|_{H^\nu}^p \leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E \left\| u_0 \right\|^p.$$

Proof. We set

$$I_1 = T_1(t_2) - T_1(t_1) = E_\alpha(t_2)\, u_0 - E_\alpha(t_2)\, u_0$$

For any $p \geq 2$, by virtue of Lemma 2.6, one gets

$$\begin{aligned} E\left[\|I_1\|_{H^\nu}^p\right] &= E\left[A \left\| E_\alpha(t_2)\, u_0 - E_\alpha(t_2)\, u_0 \right\|^p\right] \\ &\leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E \left\| u_0 \right\|^p. \end{aligned}$$

It is clear that $\left\|(T_1(t_2) - T_1(t_1))\right\|_Y \to 0$ as $t_1 \to t_2$ which means that the operator F_1 is strongly continuous. □

Lemma 2.16. *Assume $(H_2), (H_4)$, (H_5) hold and $0 < \nu < \alpha \leq 2$, $p \geq 2$, the operator T_2 is uniformly bounded.*

Proof. From Lemma 2.7, using the estimate (2.8), we have

$$\underset{t \in [0,T]}{sup}\; E\left[\|T_2(u(t))\|_{H^\nu}^p\right] \leq \infty,$$

that is the operator T_2 is uniformly bounded. □

Lemma 2.17. *Assume $(H_2), (H_4)$, (H_5) hold and $0 < \nu < \alpha \leq 2$, $p \geq 2$, Then the operator T_2 is equi-continuous.*

Proof. For $T \geq t_2 > t_1 \geq 0$,

$$(T_2 u)(t_2) - (T_2 u)(t_1) = \int_0^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega)\, Z(u(\omega))\, d\omega$$

$$-\int_0^{t_1} (t_1-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_1-\omega)\, Z(u(\omega))\, d\omega + \int_0^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega)\, g(u)\, dW(\omega).$$

$$-\int_0^{t_1} (t_1-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_1-\omega)\, g(u)\, dW(\omega) = J_2 + J_3, \tag{2.10}$$

where

$$J_2 = \int_0^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega)\, Z(u(\omega))\, d\omega - \int_0^{t_1} (t_1-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_1-\omega)\, Z(u)\, d(\omega)$$

$$= \int_0^{t_1} (t_1-\omega)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-\omega) - E_{\alpha,\alpha}(t_1-\omega)\right] Z(u(\omega))\, d\omega$$

$$+ \int_0^{t_1} \left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right] E_{\alpha,\alpha}(t_2-\omega) Z(u(\omega))\, d\omega$$

$$+ \int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega) Z(u(\omega))\, d\omega$$

$$= J_{21} + J_{22} + J_{23}, \tag{2.11}$$

and

$$J_3 = \int_0^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega) f(u(\omega))\, dW\omega - \int_0^{t_1} (t_1-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_1-\omega) f(u)\, dW(\omega)$$

$$= \int_0^{t_1} (t_1-\omega)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-\omega) - E_{\alpha,\alpha}(t_1-\omega)\right] f(u(\omega))\, dW(\omega)$$

$$+ \int_0^{t_1} \left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right] E_{\alpha,\alpha}(t_2-\omega) f(u(\omega))\, dW(\omega)$$

$$+ \int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1} E_{\alpha,\alpha}(t_2-\omega) f(u(\omega))\, dW(\omega)$$

$$= J_{31} + J_{32} + J_{33}. \tag{2.12}$$

For J_{21} in (2.11), by (H_5) and Lemma 2.6, one gets

$$E\left[\|J_{21}\|_{H^\nu}^p\right] = E\left[\left\| \int_0^{t_1} (t_1-\omega)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-\omega) - E_{\alpha,\alpha}(t_1-\omega)\right] Z(u(\omega))\, d\omega \right\|^p\right]$$

$$\leq C_{\alpha\nu}^p (t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}} \left(\int_0^{t_1} (t_1-\omega)^{\frac{p(\alpha-1)}{p-1}} d\omega \right)^{p-1} \int_0^t E\left[\|A_{-1} Z(u(\omega))\|_{H^1}^p\right] d\omega$$

$$\leq C^p C_{\alpha\nu}^p T^{p\alpha} \left(\frac{p-1}{p\alpha-1}\right)^{p-1} \left(\sup_{t\in[0,T]} E\left[\|u(\omega)\|_{H^1}^{2p}\right]\right) (t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}}. \tag{2.13}$$

Using the assumptions (H_5) and Lemma 2.6 and Holder inequality, we have

$$E\left[\|J_{22}\|_{H^\nu}^p\right] = E\left[\left\| \int_0^{t_1} \left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right] \left[A_\nu E_{\alpha,\alpha}(t_2-\omega)\right] Z(u(\omega))\, d\omega \right\|^p\right]$$

$$\leq C_\alpha^p \left(\int_0^{t_1} \left\{ \left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right] \times (t_2-\omega)^{\frac{-\alpha(\nu+1)}{2}} \right\}^{\frac{p}{p-1}} d\omega \right)^{p-1}$$

$$\times \int_0^t E\left[\|A_{-1}B(u(\omega))\|_{H^1}^p\right] d\omega$$

$$\leq C^p C_\alpha^p T \left(\frac{p-1}{p\left(\alpha - \frac{\alpha(\nu+1)}{2}\right)}\right)^{p-1} \left(\underset{t\in[0,T]}{sup}\, E\left[\|u(\omega)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)-2}{2}}, \tag{2.14}$$

and

$$E\left[\|J_{23}\|_{H^\nu}^p\right] = E\left[\left\|\int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-\omega) B(u(\omega))\, d\omega\right\|^p\right]$$

$$\leq C_\alpha^p \left(\int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1-\frac{\alpha(\nu+1)}{2}} d\omega\right)^{p-1} \int_{t_1}^{t_2} E\left[\|A_{-1}B(u(\omega))\|_{H^1}^p\right] d\omega$$

$$\leq C^p C_\alpha^p \left(\frac{p-1}{p\left(\alpha - \frac{\alpha(\nu+1)}{2}\right)-1}\right)^{p-1} \left(\underset{t\in[0,T]}{sup}\, E\left[\|u(\omega)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)}{2}}. \tag{2.15}$$

Next, we can show via (2.13) – (2.15) and the Lemma 2.2:

$$E\left[\|J_{31}\|_{H^\nu}^p\right] = E\left[\left\|\int_0^{t_1} (t_1-\omega)^{\alpha-1}\left[E_{\alpha,\alpha}(t_2-\omega) - E_{\alpha,\alpha}(t_1-\omega)\right] f(u(\omega))\, dW\omega\right\|^p\right]$$

$$\leq C(p)\, E\left[\left(\int_0^{t_1} \left\|(t_1-\omega)^{\alpha-1} A_\nu\left[E_{\alpha,\alpha}(t_2-\omega) - E_{\alpha,\alpha}(t_1-\omega)\right]\right\|^2 \|f(u(\omega))\|_{L_0^2}^2 d\omega\right)^{\frac{p}{2}}\right]$$

$$\leq C(p)\, C_{\alpha\nu}^p (t_2-t_1)^{\frac{p\alpha\nu}{2}} \left(\int_0^{t_1} (t_1-\omega)^{\frac{2p(\alpha-1)}{p-2}} d\omega\right)^{\frac{p-2}{2}} \int_0^{t_1} E\,\|f(u(\omega))\|_{L_0^2}^p d\omega$$

$$\leq C^p C_{\alpha\nu}^p T^{\frac{2p\alpha-p-1}{2}} \left(\frac{p-1}{2p\alpha-p-2}\right)^{p-1} \left(\underset{t\in[0,T]}{\omega up}\, E\left[\|u(\omega)\|^p\right]\right)(t_2-t_1)^{\frac{p\alpha\nu}{2}}, \tag{2.16}$$

and

$$E\left[\|J_{32}\|_{H^\nu}^p\right] = E\left[\left\|\int_0^{t_1} \left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-\omega)\right] f(u(\omega))\, dW\omega\right\|^p\right]$$

$$\leq C(p)\, E\left[\left(\int_0^{t_1} \left\|\left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-\omega)\right]\right\|^2 \|f(u(\omega))\|_{L_0^2}^2 d\omega\right)^{\frac{p}{2}}\right]$$

$$\leq C(p)\, C_\alpha^p \left(\int_0^{t_1} \left\{\left[(t_2-\omega)^{\alpha-1} - (t_1-\omega)^{\alpha-1}\right] \times (t_2-\omega)^{\frac{-\alpha\nu}{2}}\right\}^{\frac{2p}{p-2}} d\omega\right)^{\frac{p-2}{2}}$$

$$\times \int_0^t E\left[\|f(u(\omega))\|_{L_0^2}^p\right] d\omega$$

$$\leq C(p) C^p C_\alpha^p T \left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}}$$

$$\times \left(\sup_{t\in[0,T]} E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2(p+2)}{4}}, \tag{2.17}$$

and

$$E\left[\|J_{33}\|_{H^\nu}^p\right] = E\left[\left\|\int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-\omega) Z(u(\omega)) d\omega\right\|^p\right]$$

$$\leq C(p) E\left[\left(\int_0^{t_1} \left\|(t_2-\omega)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2-\omega)\right\|^2 \|f(u(\omega))\|_{L_0^2}^2 d\omega\right)^{\frac{p}{2}}\right]$$

$$\leq C(p) C_\alpha^p \left(\int_{t_1}^{t_2} (t_2-\omega)^{\alpha-1-\frac{\alpha\nu}{2}}\right)^{\frac{p-2}{2}} \times \int_{t_1}^{t_2} E\left[\|f(u(\omega))\|_{L_0^2}^p\right] d\omega$$

$$\leq C(p) C^p C_\alpha^p \left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}}$$

$$\left(\sup_{t\in[0,T]} E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2p}{4}}.$$

By expectation of (2.10) and $(2.13)-(2.18)$, one gets

$$\|(T_2u)(t_2)-(T_2u)(t_1)\|_{L^p(\Omega,H^\nu)} \leq C(t_2-t_1)^\gamma,$$

where $\gamma = min\left\{\frac{\alpha\nu}{2}, \frac{\alpha p(1-\nu)-2}{2p}, \frac{2p\alpha(2-\nu)-2(p+2)}{4p}\right\}$ when $0 < t_2 - t_1 < 1$.
Otherwise, if $t_2 - t_1 \geq 1$, then we set $\gamma = max\left\{\frac{\alpha(\nu+1)}{2}, \frac{\alpha(2-\nu-1)}{2}, \frac{2p\alpha(2-\nu)-2p}{4p}\right\}$. □

Lemma 2.18. *F maps Y into itself by (H_1) and (H_2).*

Proof. Let the nonlinear operator F be given by,

$$(Fu)(t) = E_\alpha(t) u_0 + h(u(t)) + \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(\omega)) d\omega$$

$$+ \int_0^t (t-\omega)^{\alpha-1} E_{\alpha,\alpha}(t-\omega) g(u) dW(\omega).$$

Now, we construct mild solutions of $(1.1)-(1.2)$:

Step 1. Let $\lambda \in (0,1)$, set $\{u \in T : u = \lambda F u\}$ is bounded.
Assume that $u \in Y$ be a solution of $u = \lambda F u$. Then, by $(H_1) - (H_4)$ and applying the similar arguments in Lemmas 2.9and 2.12, we get

$$\begin{aligned} E\left\|u(t)\right\|_{H^\nu}^p &\leq 3^{p-1}\left\|E_\alpha(t)u_0\right\|_{H^\nu}^p + 3^{p-1}\left\|h(u(t))\right\|_{H^\nu}^p + 3^{p-1}E\left\|\Phi_1(u(t))\right\|_{H^\nu}^p \\ &\quad + 3^{p-1}E\left\|\Phi_2(u(t))\right\|_{H^\nu}^p \\ &\leq 3^{p-1}E\left[\left\|u_0\right\|_{H^\nu}\right] + 3^{p-1}(\gamma_1+\gamma_2)\int_0^t E\left[\left\|u\right\|_{H^\nu}^p\right]ds. \\ &\sup_{t\in[0,T]} E\left\|u(t)\right\|_{H^\nu}^p < \infty. \end{aligned}$$

which implies boundedness of $u(\cdot)$.

Step 2. $F : Y \to C([0,T], H^\sigma)$ is continuous. Let $\{u_n(t)\}_{n\geq 0}$ with $u_n \to u\ (n\to\infty)$ in Y. Then there is a number $r > 0$ such that $E\left\|u_n(t)\right\|_{H^\nu}^2 \leq r$ for $t \in [0,T]$ and all n, $u_n \in B(0,Y) = \left\{u \in Y : \sup\limits_{t\in[0,T]} \left\|u\right\|_{H^\sigma}\right\}$ and $u \in B_r(0,Y)$.
By the assumptions (H_2) and similar argument to get (2.8) and (2.9), we have

$$\begin{aligned} E\left\|(Fu_n)(t) - (Fu)(t)\right\|_{H^\nu}^p &\leq 3^{p-1}\left\|h(u_n(t)) - h(u(t))\right\|_{H^\nu}^P + 3^{p-1}E\left\|\Phi_1(u_n(t) - u(t))\right\|_{H^\nu}^p \\ &\quad + 3^{p-1}E\left\|\Phi_2(u_n(t) - u(t))\right\|_{H^\nu}^p \\ &\leq 3^{p-1}\left\|h(u_n(t)) - h(u(t))\right\|_{H^\nu}^p + 3^{p-1}(G\gamma_1 + K\gamma_2)\left(\int_0^t E\left\|u_n - u\right\|_{H^\nu}^p ds\right). \end{aligned}$$

Then, for $t \in [0,T]$,

$$\left\|Fu_n - Fu\right\|_Y^p \longrightarrow 0, \text{ while } n \longrightarrow \infty.$$

Therefore, F is a continuous map.

Step 3. Let us write F as $F = T_1 + T_2$ to prove that F is $\chi-$contraction.

(1) T_1 is a contraction on Y. Let $u,\ v \in Y$. By the Lemma 2.11 :

$$\begin{aligned} E\left\|T_1u - T_1v\right\|_{H^\nu}^p &\leq L_g E\left\|u(\omega) - v(\omega)\right\|_{H^\nu}^p \\ &\leq L_g \sup_{\omega\in[0,T]} E\left\|u(\omega) - v(\omega)\,d\omega\right\|_{H^\nu}^p \\ &\leq L_g\left\|u(\omega) - v(\omega)\,d\omega\right\|_Y^p \end{aligned}$$

Taking supremum over t

$$\left\|T_1u - T_1v\right\|_Y^p \leq L_0\left\|u(\omega) - v(\omega)\right\|_Y^p,$$

where $L_0 = L_g < 1$.
Hence T_1 is a contraction on Y.

(2) T_2 is compact operator. Let $u,\ v \in Y$. It follows from (H_2), (H_5) and Lemma 2.12 that

$$E\left\|T_2u - T_2v\right\|_{H^\nu}^2 \leq 2^{p-1}E\left\|\int_0^t (t-\omega)^{\alpha-1}E_{\alpha,\alpha}(t-\omega)A_\nu\left[g(u(\omega)) - g(v(\omega))\right]dW(\omega)\right\|_{H^\nu}^2$$

$$+2^{p-1}E\left\|\int_0^t (t-\omega)^{\alpha-1}E_{\alpha,\alpha}(t-\omega)A_\nu\left[B(u(\omega))-B(v(\omega))\right]d\omega\right\|_{H^\nu}^p$$

$$\leq(\gamma_1+\gamma_2)E\left(\int_0^t\|u-v\|_{H^\nu}^2\,d\omega\right),$$

which implies

$$\sup_{t\in[0,T]}E\|T_2u-T_2v\|_{H^\nu}^2=(\gamma_1+\gamma_2)\sup_{t\in[0,T]}E\|u-v\|_{H^\nu}^2.$$

Since $0<L=\gamma_1+\gamma_2<1$, then F is contraction maping on Y.

From the Lemma 2.11 and Lemma 2.12, the operator T_1 is relatively compact.

Because T_1 is a compact operator,

$$\chi_K(T_1V)=0$$

for a bounded $V\subset K$. Hence,

$$\chi_K(F)=\chi_K(F_1V+F_2V)\leq\chi_K(F_1V)+\chi_K(F_2V)\leq L\chi_K(V)<\chi_K(V).$$

Therefore, F is a $\chi-$contraction mapping. Using the Lemma 2.10, we get F has at least one fixed point $u^*\in V\subset K$ which is a mild solution of $(1.1)-(1.2)$.

□

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