An alternative potential method for mixed steady state elastic oscillation problems

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Abstract

We consider an alternative approach to investigate three-dimensional exterior mixed boundary value problems (BVP) for the steady state oscillation equations of the elasticity theory for isotropic bodies. The unbounded domain occupied by an elastic body, $\Omega^- \subset \mathbb{R}^3$, has a compact boundary surface $S = \partial\Omega^-$, which is divided into two disjoint parts, the Dirichlet part S_D and the Neumann part S_N, where the displacement vector (the Dirichlet type condition) and the stress vector (the Neumann type condition) are prescribed respectively.

Our new approach is based on the classical potential method and has several essential advantages compared with the existing approaches. We look for a solution to the mixed boundary value problem in the form of a linear combination of the single layer and double layer potentials with densities supported on the Dirichlet and Neumann parts of the boundary respectively. This approach reduces the mixed BVP under consideration to a system of boundary integral equations, which contain neither extensions of the Dirichlet or Neumann data nor the Steklov-Poincaré type operator involving the inverse of a special boundary integral operator, which is not available explicitly for arbitrary boundary surface. Moreover, the right-hand sides of the resulting boundary integral equations system are vector-functions coinciding with the given Dirichlet and Neumann data of the problem in question. We show that the corresponding matrix integral operator is bounded and coercive in the appropriate L_2-based Bessel potential spaces. Consequently, the operator is invertible, which implies unconditional unique solvability of the mixed BVP in the class of vector-functions belonging to the Sobolev space $[W^1_{2,loc}(\Omega^-)]^3$ and satisfying the Sommerfeld-Kupradze radiation conditions at infinity. We also show that the obtained matrix boundary integral operator is invertible in the L_p-based Besov spaces and prove that under appropriate boundary data a solution to the mixed BVP possesses C^α-Hölder continuity property in the closed domain $\overline{\Omega^-}$ with $\alpha = \frac{1}{2} - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number.

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1 Introduction

We present a new approach for studying a three-dimensional mixed boundary value problem (BVP) for the steady state oscillation equations of the elasticity theory. The unbounded domain occupied by an isotropic elastic body, $\Omega^- \subset \mathbb{R}^3$, has a compact connected boundary surface $S = \partial\Omega^-$, which is divided into two disjoint parts, the Dirichlet part S_D and the Neumann part S_N, where the displacement vector (the Dirichlet type condition) and the stress vector (the Neumann type condition) are prescribed respectively.

Investigations related to the Dirichlet and Neumann BVPs for the Helmholtz equation and for the system of steady state elastic oscillations have a long history and detailed bibliographical information can be found in the monographs [9], [10], [20].

In the references [32]-[33], the direct boundary integral equations method is applied to analyse mixed BVPs for the scalar and vector Helmholtz equations in the L_2-based Sobolev and Bessel potential spaces and with the help of the Wiener-Hopf method the asymptotic behaviour of solutions is established at the collision curve, where the different boundary conditions collide. It should be mentioned that in the case of the direct boundary integral equations method, which is based on the use of the general integral representation formula of a solution vector-function via its Cauchy data, the so-called third Green formula, the right-hand sides of the resulting system of integral equations contain potential type integrals with densities being extensions to the whole boundary of the corresponding Dirichlet and Neumann data of the mixed problem that are originally given on disjoint adjacent proper parts of the boundary. Therefore, the solutions of the system of integral equations essentially depend on the extension operator.

There is another approach for investigation of mixed BVPs, which is based on the standard potential method and which requires extension of the Dirichlet or Neumann data to the whole boundary. This approach uses a representation of solutions by either the single layer potential or by a special combination of the single and double layer potentials with densities of complex form containing the inverses of special boundary integral operators, which are not available explicitly for arbitrary surfaces, in general (see, e.g., [8], [26] and the references therein). In this case, the resulting boundary integral equations contain the so-called Steklov-Poincaré type operators, which in turn, involve the above mentioned inverses of the boundary integral operators. The right-hand side vector-functions of the boundary integral equations again contain potential type integral terms with the densities being extensions of the Dirichlet and Neumann data. Evidently, when extensions of the Dirichlet or Neumann boundary data are used, then the resulting pseudodifferential equations and the corresponding solutions essentially depend on the extension operators. The similar approach is used in [4] for investigating of the mixed impedance problem for the steady state elastic oscillation equations. In spite of the disadvantages described above, this approach turned out to be very useful to derive theoretical qualitative properties related to continuity and asymptotic behaviour of solutions to the mixed BVPs near the collision curve, where the Dirichlet and Neumann boundary conditions collide.

In contrast to the above described direct and indirect boundary integral equations methods, our alternative approach has several essential advantages from the practical point of view. We prove that a solution to the mixed boundary value problem can be represented in the form of a linear combination of the single layer and double layer potentials with sought for densities supported on the Dirichlet and Neumann parts of the boundary respectively. Therefore, on the one hand, our new approach does not require extension of the given Dirichlet or Neumann boundary data to the whole boundary and, on the other hand, the representation of a solution and the corresponding boundary integral equations contain neither the inverse of some boundary integral operators nor the Steklov-Poincaré type operators. Moreover, the right-hand side vector-functions of our boundary integral equations are vector-function coinciding with the original Dirichlet and

Neumann boundary data of the mixed BVP under consideration. These facts will play an essential role in creation of efficient and cheap algorithms for numerical solutions of the mixed BVPs.

It should be mentioned that similar approaches for interior and exterior mixed BVPs are used in [28], [29] and [30] for the Laplace equation, the Helmholtz equation, and the Lamé system. In the case of the steady state elastic oscillation equations, the arguments used in the above references are not directly applicable and we need appropriate modifications to justify our alternative approach.

The paper is organized as follows. In Section 2, we introduce appropriate function spaces, formulate the exterior mixed boundary value problem for the steady state elastic oscillation equations and prove the corresponding uniqueness theorem. In Section 3, we describe mapping properties of the layer potentials and the boundary integral operators generated by them in the Bessel potential and Besov spaces. In Section 4, we show that the matrix integral operator obtained by the alternative approach is bounded and coercive in the appropriate L_2-based Bessel potential spaces. Consequently, the operator is invertible, which implies unconditional unique solvability of the mixed BVP in the class of vector-functions belonging to the Sobolev space $[W^1_{2,loc}(\Omega^-)]^3$ and satisfying the Sommerfeld-Kupradze radiation conditions at infinity. Section 5 is devoted to the extension of the alternative potential method for the exterior mixed BVP to the space of vector-functions belonging to the L_p-based Besov spaces $\big[B^s_{p,2,loc}(\Omega^-)\big]^3$ with $\frac{1}{2} \leqslant s < \frac{1}{2} + \frac{1}{p}$, $p > 4$, and satisfying the Sommerfeld-Kupradze radiation conditions at infinity. Here we derive the almost best regularity results for solutions to the mixed BVPs. In particular, we prove that the obtained matrix operator is invertible in the L_p-based Besov spaces and show that under appropriate boundary data a solution to the mixed BVP possesses C^α-Hölder continuity property in the closed domain $\overline{\Omega^-}$ with $\alpha = \frac{1}{2} - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number. Finally, in Section 6, we justify our new approach for the mixed BVP in the space of vector-functions belonging to the Bessel potential space $\big[H^1_{p,loc}(\Omega^-)\big]^3$ with $\frac{4}{3} < p < 4$, and satisfying the Sommerfeld-Kupradze radiation conditions.

For the readers convenience, in Appendices 1, 2, and 3, we collect some auxiliary material needed for our analysis in the main text.

2 Formulation of the mixed BVP and uniqueness theorem

Let $\Omega^+ \subset \mathbb{R}^3$ be a three-dimensional bounded domain with smooth connected boundary $\partial\Omega^+ = S$, and let $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$ be a connected unbounded complement. Further, let the boundary $\partial\Omega^+ = \partial\Omega^- = S$ be divided into two connected disjoint parts, S_D and S_N, $\overline{S}_D \cup \overline{S}_N = S$, $S_D \cap S_N = \varnothing$. For simplicity, throughout the paper we assume that $S \in C^\infty$ and $\ell = \partial S_D = \partial S_N \in C^\infty$ if not otherwise stated. In particular, some of the results obtained in the article are valid when the S, S_D, and S_N are Lipschitz surfaces, and these cases will always be singled out separately.

By L_p, $L_{p,loc}$, W_p^r, $W_{p,loc}^r$, H_p^s, $H_{p,loc}^s$, $B_{p,q}^s$, and $B_{p,q,loc}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov spaces of complex-valued functions of real variables, respectively (see, e.g., [5], [21], [34], [35]). The following relations $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, hold for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Further, let us introduce the spaces:

$$\widetilde{H}_p^s(S_1) := \{f \,:\, f \in H_p^s(S), \ \mathrm{supp}\, f \subset \overline{S_1}\},$$

$$\widetilde{B}_{p,q}^s(S_1) := \{f \,:\, f \in B_{p,q}^s(S), \ \mathrm{supp}\, f \subset \overline{S_1}\},$$

$$H_p^s(S_1) := \{r_{S_1} f \,:\, f \in H_p^s(S)\},$$

$$B_{p,q}^s(S_1) := \{r_{S_1} f \,:\, f \in B_{p,q}^s(S)\},$$

where $S_1 \in \{S_D, S_N\}$, r_{S_1} is the restriction operator onto S_1. The norms in these spaces are determined by the standard natural way:

$$\|u\|_{\widetilde{H}_p^s(S_1)} = \|u\|_{H_p^s(S)}, \qquad \|u\|_{\widetilde{B}_{p,q}^s(S_1)} = \|u\|_{B_{p,q}^s(S)},$$
$$\|u\|_{H_p^s(S_1)} = \inf \|v\|_{H_p^s(S)}, \quad v \in H_p^s(S), \quad r_{S_1} v = u,$$
$$\|u\|_{B_{p,q}^s(S_1)} = \inf \|v\|_{B_{p,q}^s(S)}, \quad v \in B_{p,q}^s(S), \quad r_{S_1} v = u.$$

Remark 2.1 *Let a function f be defined on an open proper submanifold S_1 of a closed manifold S without boundary. Let $f \in B_{p,q}^s(S_1)$ and $\widetilde{f}$ be the extension of f by zero to $S \setminus S_1$. If the extension preserves the space, that is, if $\widetilde{f} \in \widetilde{B}_{p,q}^s(S_1)$, then we write $f \in \widetilde{B}_{p,q}^s(S_1)$ instead of $f \in r_{S_1} \widetilde{B}_{p,q}^s(S_1)$, when it does not lead to misunderstanding.*

Note that $\widetilde{B}_{p,q}^s(S_1)$ and $B_{p',q'}^{-s}(S_1)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ are dual spaces. Similarly, $\widetilde{H}_p^s(S_1)$ and $H_{p'}^{-s}(S_1)$ are dual spaces as well (for details see [23], [24], [34], [35]).

Therefore, for functions $f \in B^{-s}_{p',q'}(S_1)$ and $g \in \widetilde{B}^{s}_{p,q}(S_1)$ (resp., $f \in H^{-s}_{p'}(S_1)$ and $g \in \widetilde{H}^{s}_{p}(S_1)$) the duality relation $\langle f, g\rangle_{S_1}$ is well defined and it generalizes the classical L_2-inner product,

$$\langle f, g\rangle_{S_1} = \overline{\langle g, f\rangle}_{S_1} = \int\limits_{S_1} f(x)\,\overline{g(x)}\,dS_1 \quad \textit{for} \quad f, g \in L_2(S_1),$$

where the overbar denotes complex conjugation.

Now we formulate the exterior mixed boundary value problem for the steady state elastic oscillation equation [20]: Find a vector-function $u = (u_1, u_2, u_3)^\top \in \big[H^1_{2,loc}(\Omega^-)\big]^3$ satisfying

(i) the partial differential equation of steady state elastic oscillations

$$A(\partial, \omega)\,u(x) := \mu\,\Delta\,u(x) + (\lambda + \mu)\mathrm{grad}\mathrm{div} u(x) + \varrho\,\omega^2\,u(x) = 0 \quad \text{in} \;\; \Omega^-; \tag{2.1}$$

(ii) the mixed boundary conditions

$$r_{S_D}\{u\}^- = f \quad \text{on} \;\; S_D, \tag{2.2}$$

$$r_{S_N}\{T(\partial, n)u\}^- = F \quad \text{on} \;\; S_N \tag{2.3}$$

with

$$f = (f_1, f_2, f_3)^\top \in \big[H^{\frac{1}{2}}_2(S_D)\big]^3, \quad F = (F_1, F_2, F_3)^\top \in \big[H^{-\frac{1}{2}}_2(S_N)\big]^3; \tag{2.4}$$

(iii) the Sommerfeld-Kupradze radiation conditions at infinity, that is, u is representable as a sum of two metaharmonic vectors (see [20]), the so called longitudinal $u^{(1)} = u^{(p)}$ and transverse parts $u^{(2)} = u^{(s)}$,

$$\begin{aligned}
&u = u^{(1)} + u^{(2)} \quad \text{with} \;\; \Delta\,u^{(1)} + k_1^2\,u^{(1)} = 0, \quad \Delta\,u^{(2)} + k_2^2\,u^{(2)} = 0\,,\\
&k_1 \equiv k_p = \omega\,\sqrt{\frac{\varrho}{\lambda + 2\,\mu}}\,, \quad k_2 \equiv k_s = \omega\,\sqrt{\frac{\varrho}{\mu}}\,,\\
&\mu > 0, \quad 3\lambda + 2\mu > 0,
\end{aligned}$$

and for sufficiently large $r = |x|$

$$\frac{\partial u^{(1)}(x)}{\partial r} - i\,k_1\,u^{(1)}(x) = o\,(r^{-1})\,, \quad \frac{\partial u^{(2)}(x)}{\partial r} - i\,k_2\,u^{(2)}(x) = o\,(r^{-1})\,; \tag{2.5}$$

here $u = (u_1, u_2, u_3)^\top$ is a complex-valued displacement vector, λ and μ are the Lamé constants, ϱ is the density of the elastic material, and $\omega \in \mathbb{R}$ is the frequency parameter, $A(\partial, \omega)$ is the matrix differential operator

$$A(\partial, \omega) := A(\partial) + \varrho\,\omega^2\,I_3, \quad A(\partial) := [\mu\,\delta_{kj}\,\Delta + (\lambda + \mu)\partial_k\,\partial_j]_{3\times 3}\,, \quad I_3 = [\,\delta_{kj}\,]_{3\times 3}\,,$$

while $T(\partial,n)$ and $T(\partial,n)u$ denote the stress operator and the stress vector respectively,

$$T(\partial,n):=[T_{kj}(\partial,n)]_{3\times 3},\ \ T_{kj}(\partial,n)=\lambda\,n_k\,\partial_j+\mu\,n_j\,\partial_k+\mu\,\delta_{kj}\,\partial_n,$$

$$[T(\partial,n)u]_k=\sigma_{kj}\,n_j,\ \ \sigma_{kj}=[\lambda\,\delta_{kj}\,\mathrm{div}\,u+2\,\mu\,e_{kj}(u)]\,n_j,\ \ e_{kj}(u)=2^{-1}\,(\partial_k u_j+\partial_j u_k),$$

where Δ is the Laplace operator, $I_3=[\delta_{kj}]$ is the unit matrix, δ_{kj} is the Kronecker delta, $\partial_k=\partial_{x_k}=\partial/\partial x_k$ denotes partial differentiation with respect to the variable x_k, n is the unit outward normal vector to S and $\partial_n=\partial/\partial n$ denotes the normal derivative, $e_{kj}=e_{kj}(u)$ and $\sigma_{kj}=\sigma_{kj}(u)$ denote the strain and stress tensors, respectively.

Here and in what follows the summation over repeated indices is meant from 1 to 3, unless stated otherwise, and the symbol $U^\top$ denotes the transpose of U. The symbols $\{\,\cdot\,\}^+$ and $\{\,\cdot\,\}^-$ denote the interior and exterior one-sided limits on $S=\partial\Omega^\pm$ from $\Omega^\pm$ respectively.

Note that the radiation conditions (2.5) automatically yield the following decay conditions at infinity (for details see [20], [36])

$$u^{(l)}(x)=\mathcal{O}(r^{-1})\,,\quad \partial_j u^{(l)}(x)-i\,k_l\,\frac{x_j}{r}u^{(l)}(x)=\mathcal{O}(r^{-2})\,,\quad l=1,2,\ \ j=1,2,3.$$

Recall that for sufficiently regular vector-functions $u,v\in[C^2(\overline{\Omega^+})]^3$ and $C^{1,\,\alpha}$–smooth domains we have the following Green formula [20]

$$\int\limits_S\{Tu\}^+\cdot\{v\}^+\,dS=\int\limits_{\Omega^+}A(\partial,\omega)u\cdot v\,dx+\int\limits_{\Omega^+}[\,E(u,\overline{v})-\varrho\,\omega^2\,u\cdot v\,]dx, \tag{2.6}$$

where the central dot denotes the scalar product in $\mathbb{C}^3$ and

$$\begin{aligned} E(u,v)\ \ &=\ \ \frac{3\,\lambda+2\,\mu}{3}\,\mathrm{div}u\,\mathrm{div}v+\frac{\mu}{2}\sum_{k\neq j}(\partial_j u_k+\partial_k u_j)\,(\partial_j v_k+\partial_k v_j)+\\ &\qquad +\frac{\mu}{3}\sum_{k,j}(\partial_k u_k-\partial_j u_j)\,(\partial_k v_k-\partial_j v_j). \end{aligned}$$

It is evident that $E(u,\overline{u})\geq 0$, with the equality holding only for a rigid displacement vectors, that is, for vectors of the form $\chi(x)=[a\times x]+b$, where a and b are constant three-dimensional complex-valued vectors and the symbol $\times$ denotes the cross product (see, e.g., [20]).

By the standard limiting procedure, the above Green formula can be generalized to Lipschitz domains and to vector-functions from the Sobolev-Slobodetskii, Bessel potential and Besov spaces. In particular, we can extend Green's formula (2.6) to vector-functions $u\in[W^1_p(\Omega^+)]^3$ with $A(\partial)u\in$

$[L_p(\Omega^+)]^3$ and $v \in [W^1_{p'}(\Omega^+)]^3$ with $1/p + 1/p' = 1$, $1 < p < \infty$ (see, e.g., [2])

$$\langle \{Tu\}^+ \, , \, \{\overline{v}\}^+ \rangle_S = \int\limits_{\Omega^+} A(\partial, \omega)u \cdot v \, dx + \int\limits_{\Omega^+} [\, E(u, \overline{v}) - \varrho\, \omega^2 \, u \cdot v \,]dx, \tag{2.7}$$

where the symbol $\langle \cdot \, , \, \cdot \rangle_S$ denotes duality brackets between the adjoint spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^3$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^3$. Due to the embedding $\{v\}^+ \in [B^{\frac{1}{p}}_{p',p'}(S)]^3$, this relation defines the generalized boundary trace functional $\{Tu\}^+ \in [B^{-\frac{1}{p}}_{p,p}(S)]^3$ associated with the stress vector.

For a vector $u \in [W^1_{p,loc}(\Omega^-)]^3$ with $A(\partial)u \in [L_{p,loc}(\Omega^-)]^3$ the generalized boundary trace functional $\{Tu\}^- \in [B^{-\frac{1}{p}}_{p,p}(S)]^3$ of the stress vector is determined quite similarly by formula

$$\langle \{Tu\}^- \, , \, \{\overline{v}\}^- \rangle_S = -\int\limits_{\Omega^-} A(\partial, \omega)u \cdot v \, dx - \int\limits_{\Omega^-} [\, E(u, \overline{v}) - \varrho\, \omega^2 \, u \cdot v \,]dx \tag{2.8}$$

with arbitrary $v \in [W^1_{p',comp}(\Omega^-)]^3$.

Equation (2.1) we understand in the weak sense. However, due to the strong ellipticity of the matrix differential operator $A(\partial, \omega)$, every solution of equation (2.1) is actually $C^\infty-$regular in $\Omega^\pm$, in view of the interior regularity property, and consequently equation (2.1) is satisfied pointwise (see, e.g., [14]).

The Dirichlet type boundary condition (2.2) is understood in the standard trace sense, while for a weak solution u to equation (2.1) the Neumann type condition (2.3) is understood in the functional sense defined with the help of Green's generalized formula (2.8).

The space of solutions of equation (2.1) satisfying the Sommerfeld-Kupradze radiation conditions (2.5) we denote by $SK(\Omega^-)$. We have the following uniqueness theorem.

Theorem 2.2 *The mixed boundary value problem* (2.1)-(2.3) *possesses at most one solution in the space* $\big[H^1_{2,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$.

Proof. The proof is much the same as that of Theorem 2.4 in [4]. ■

3 Properties of layer potentials and boundary operators

To make the paper self-contained, here we describe basic mapping properties of the layer potentials associated with the steady state elastic oscillation operator $A(\partial, \omega)$.

Denote by $\Gamma(x,\omega)$ and $\Gamma(x)$ the matrices of fundamental solutions of the differential operators $A(\partial,\omega)$ and $A(\partial,0)=A(\partial)$,

$$A(\partial,\omega)\Gamma(x,\omega)=I_3\,\delta(x),\quad A(\partial)\Gamma(x)=I_3\,\delta(x),$$

where $\delta(x)$ is Dirac's delta functional. The matrices $\Gamma(x,\omega)$ and $\Gamma(x)$, Kupradze's matrix and Kelvin's matrix, are constructed explicitly in terms of elementary functions (see [20, Ch. 2, §1], [22, Ch. VIII, §130])

$$\begin{aligned}
&\Gamma(x,\omega)=[\Gamma_{kj}(x,\omega)]_{3\times3}, \qquad \Gamma_{kj}(x,\omega)=\sum_{l=1}^{2}\left(\delta_{kj}\alpha_l+\beta_l\,\partial_k\,\partial_j\right)\frac{e^{ik_l\,|x|}}{|x|}\,,\\
&\alpha_l=-\frac{\delta_{2l}}{4\,\pi\,\mu},\quad \beta_l=\frac{(-1)^{l+1}}{4\,\pi\,\varrho\,\omega^2},\quad i=\sqrt{-1},\\
&\Gamma(x)=[\Gamma_{kj}(x)]_{3\times3}, \qquad \Gamma_{kj}(x)=\frac{\delta_{kj}\,\lambda'}{|x|}+\frac{\mu'\,x_k\,x_j}{|x|^3},\\
&\lambda'=-\frac{\lambda+3\,\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)},\quad \mu'=-\frac{\lambda+\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)}.
\end{aligned}$$

The real-valued fundamental matrix of statics $\Gamma(x)$ is the principal singular homogeneous part of the complex-valued fundamental matrix of oscillations $\Gamma(x,\omega)$ and the following relations hold:

$$\begin{aligned}
&\Gamma(x,\omega)=\Gamma(-x,\omega)=[\Gamma(x,\omega)]^\top,\quad \Gamma(x)=\Gamma(-x)=[\Gamma(x)]^\top,\\
&|\Gamma_{kj}(x,\omega)|\le c_1(\lambda,\mu)\,|x|^{-1},\\
&|\Gamma_{kj}(x,\omega)-\Gamma_{kj}(x)|\le|\omega|\,c_2(\lambda,\mu),\\
&|\partial_l\Gamma_{kj}(x,\omega)-\partial_l\Gamma_{kj}(x)|\le|\omega|^2\,c_3(\lambda,\mu),\\
&|\partial_m\partial_l\Gamma_{kj}(x,\omega)-\partial_m\partial_l\Gamma_{kj}(x)|\le c_4(\lambda,\mu,\omega)\,|x|^{-1},\\
&|\partial_m\partial_l\partial_q\Gamma_{kj}(x,\omega)-\partial_m\partial_l\partial_q\Gamma_{kj}(x)|\le c_5(\lambda,\mu,\omega)\,|x|^{-2},\quad k,j,l,m,q=1,2,3,
\end{aligned}\tag{3.1}$$

where $c_1(\lambda,\mu)$, $c_2(\lambda,\mu)$, $c_3(\lambda,\mu)$, $c_4(\lambda,\mu,\omega)$ and $c_5(\lambda,\mu,\omega)$ are positive numbers depending upon the material constants λ and μ, and upon the frequency parameter ω.

Note that the functions $\partial_m\partial_l\partial_q\big[\Gamma_{kj}(x-y,\omega)-\Gamma_{kj}(x-y)\big]$ with $x,y\in S$ are singular kernels satisfying the Tricomi condition and, consequently, the corresponding integral operators on S are Calderon-Zygmund type singular integral operators, that is, pseudodifferential operators of zero order [15].

The entries of the matrices $\Gamma(x,\omega)$ and $\Gamma(x)$ are analytic functions of real variables $x =$

(x_1, x_2, x_3) in $\mathbb{R}^3 \setminus \{0\}$ and the columns of the matrix $\Gamma(x, \omega)$ satisfy the Sommerfeld-Kupradze radiation conditions at infinity.

For a solution $u \in \big[W_p^1(\Omega^+)\big]^3$ to the equation $A(\partial, \omega)u = 0$ in Ω^+ we have the following integral representation formula (see [4], [20, Ch. 3, §2.1], [23])

$$W_\omega(\{u\}^+)(x) - V_\omega(\{Tu\}^+)(x) = \begin{cases} u(x) & \text{in} \quad \Omega^+, \\ 0 & \text{in} \quad \Omega^-, \end{cases}$$

where V_ω and W_ω are the single and double layer potentials,

$$V_\omega(g)(x) := \int_S \Gamma(x - y, \omega)\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S\,, \tag{3.2}$$

$$W_\omega(h)(x) := \int_S [T(\partial_y, n(y))\Gamma(x - y, \omega)]^\top\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S\,, \tag{3.3}$$

with densities $g = (g_1, g_2, g_3)^\top$ and $h = (h_1, h_2, h_3)^\top$.

Similar representation formula holds for a solution $u \in \big[W^1_{p,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$ of the equation $A(\partial, \omega)u = 0$ in Ω^- (see [4], [20, Ch. 3, §2.4], [27])

$$-W_\omega(\{u\}^-)(x) + V_\omega(\{Tu\}^-)(x) = \begin{cases} 0 & \text{in} \quad \Omega^+, \\ u(x) & \text{in} \quad \Omega^-. \end{cases}$$

Further, we introduce the boundary integral operators generated by the single and double layer potentials:

$$(\mathcal{H}_\omega\, g)(x) := \int_S \Gamma(x - y, \omega)\, g(y)\, dS_y\,, \quad x \in S,$$

$$(\mathcal{K}^*_\omega\, g)(x) := \int_S \big[\, T(\partial_x, n(x))\Gamma(x - y, \omega)\,\big]\; g(y)\, dS_y, \quad x \in S,$$

$$(\mathcal{K}_\omega\, h)(x) := \int_S \big[\, T(\partial_y, n(y))\,\Gamma(x - y, \omega)\,\big]^\top\; h(y)\, dS_y, \quad x \in S,$$

$$(\mathcal{L}_\omega h)(x) := \big\{\, T(\partial_x, n(x)) W_\omega(h)(x)\,\big\}^\pm, \quad x \in S.$$

The boundary operators $\mathcal{H}_\omega$ and $\mathcal{L}_\omega$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}_\omega$ and $\overline{\mathcal{K}^*_\omega}$ are mutually adjoint singular integral operators, i.e., pseudodifferential operators of order 0 (for details see [1], [2], [15], [18], [19], [20]). Actually, the operator $\mathcal{L}_\omega$ is a singular integro-differential operator (compare, [16], [17], [20]).

The potentials constructed by the fundamental matrix $\Gamma(x-y)$ and the corresponding boundary operators, we denote by the same symbols as above but equipped with the subscript 0, that is, $V_0(g)$ $W_0(h)$, $\mathcal{H}_0 g$, $\mathcal{K}_0^* g$, $\mathcal{K}_0 h$, $\mathcal{L}_0 h$.

The above introduced single and double layer potentials and the boundary integral operators have the following mapping properties. Proofs can be found in the references [2], [7], [8], [11], [15], [18], [19], [23], [25].

Theorem 3.1 *Let S be $C^\infty-$smooth and $1<p<\infty$, $1\le q\le\infty$, and $s\in\mathbb{R}$. The operators*

$$\begin{aligned}
V_\omega \ &: \ \big[B^s_{p,p}(S)\big]^3 \to \big[H_p^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \quad \Big(\big[B^s_{p,p}(S)\big]^3 \to \big[H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),\\
V_\omega \ &: \ \big[B^s_{p,q}(S)\big]^3 \to \big[B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \quad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),\\
W_\omega \ &: \ \big[B^s_{p,p}(S)\big]^3 \to \big[H_p^{s+\frac{1}{p}}(\Omega^+)\big]^3 \quad \Big(\big[B^s_{p,p}(S)\big]^3 \to \big[H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),\\
W_\omega \ &: \ \big[B^s_{p,q}(S)\big]^3 \to \big[B_{p,q}^{s+\frac{1}{p}}(\Omega^+)\big]^3 \quad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B_{p,q,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),
\end{aligned}$$

are continuous.

If S is a Lipschitz surface, then the operators

$$\begin{aligned}
V_\omega \ &: \ \big[H_2^{-\frac{1}{2}}(S)\big]^3 \to \big[H_2^1(\Omega^+)\big]^3 \quad \Big(\big[H_2^{-\frac{1}{2}}(S)\big]^3 \to \big[H^1_{2,loc}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),\\
W_\omega \ &: \ \big[H_2^{\frac{1}{2}}(S)\big]^3 \to \big[H_2^1(\Omega^+)\big]^3 \quad \Big(\big[H_2^{\frac{1}{2}}(S)\big]^3 \to \big[H^1_{2,loc}(\Omega^-)\big]^3\cap SK(\Omega^-)\Big),
\end{aligned}$$

are continuous.

Theorem 3.2 *Let S be $C^\infty-$smooth, $1<p<\infty$, $1\le q\le\infty$, and*

$$g\in\big[B_{p,q}^{-\frac{1}{p}}(S)\big]^3,\quad h\in\big[B_{p,q}^{1-\frac{1}{p}}(S)\big]^3.$$

Then

$$\begin{aligned}
&\big\{V_\omega(g)\big\}^+=\big\{V_\omega(g)\big\}^-=\mathcal{H}_\omega\, g \quad on \quad S,\\
&\big\{T(\partial,n)V_\omega(g)\big\}^\pm=\big[\mp\tfrac{1}{2}I_3+\mathcal{K}^*_\omega\big]\,g \quad on \quad S,\\
&\big\{W_\omega(h)\big\}^\pm=\big[\pm\tfrac{1}{2}I_3+\mathcal{K}_\omega\big]\,h \quad on \quad S,\\
&\big\{\,T(\partial,n)W_\omega(h)\,\big\}^+=\big\{\,T(\partial,n)W_\omega(h)\,\big\}^-=:\mathcal{L}_\omega h \quad on \quad S.
\end{aligned}$$

The same relations hold true for a Lipschitz boundary S and for $p=q=2$.

Theorem 3.3 (i) *Let S be $C^\infty-$smooth and $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. The operators*

$$\begin{aligned}
\mathcal{H}_\omega \ &: \ \big[H_p^s(S)\big]^3 \to \big[H_p^{s+1}(S)\big]^3 \quad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^{s+1}_{p,q}(S)\big]^3\Big),\\
\pm\tfrac{1}{2}I_3+\mathcal{K}_\omega,\ \pm\tfrac{1}{2}I_3+\mathcal{K}^*_\omega \ &: \ \big[H_p^s(S)\big]^3 \to \big[H_p^s(S)\big]^3 \quad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^s_{p,q}(S)\big]^3\Big),\\
\mathcal{L}_\omega \ &: \ \big[H_p^{s+1}(S)\big]^3 \to \big[H_p^s(S)\big]^3 \quad \Big(\big[B^{s+1}_{p,q}(S)\big]^3 \to \big[B^s_{p,q}(S)\big]^3\Big),
\end{aligned}$$

are continuous Fredholm operators with zero index for all $\omega \in \mathbb{R}$.

(ii) *If* S *is Lipschitz, then the operators*

$$\begin{aligned}
\mathcal{H}_\omega &: \ \big[\,H_2^{-\frac{1}{2}}(S)\,\big]^3 \to \big[\,H_2^{\frac{1}{2}}(S)\,\big]^3, \\
\pm\tfrac{1}{2}I_3 + \mathcal{K}^*_\omega &: \ \big[\,H_2^{-\frac{1}{2}}(S)\,\big]^3 \to \big[\,H_2^{-\frac{1}{2}}(S)\,\big]^3, \\
\pm\tfrac{1}{2}I_3 + \mathcal{K}_\omega &: \ \big[\,H_2^{\frac{1}{2}}(S)\,\big]^3 \to \big[\,H_2^{\frac{1}{2}}(S)\,\big]^3, \\
\mathcal{L}_\omega &: \ \big[\,H_2^{\frac{1}{2}}(S)\,\big]^3 \to \big[\,H_2^{-\frac{1}{2}}(S)\,\big]^3,
\end{aligned}$$

are continuous Fredholm operators with zero index for all $\omega \in \mathbb{R}$.

(iii) *The following operator equalities hold in appropriate function spaces:*

$$\mathcal{K}_\omega\,\mathcal{H}_\omega = \mathcal{H}_\omega\,\mathcal{K}^*_\omega, \ \ \mathcal{L}_\omega\,\mathcal{K}_\omega = \mathcal{K}^*_\omega\,\mathcal{L}_\omega,$$
$$\mathcal{L}_\omega\,\mathcal{H}_\omega = -\tfrac{1}{4}I_3 + \big[\,\mathcal{K}^*_\omega\,\big]^2, \ \ \mathcal{H}_\omega\,\mathcal{L}_\omega = -\tfrac{1}{4}I_3 + \big[\,\mathcal{K}_\omega\,\big]^2.$$

Remark 3.4 *The operators* $-\mathcal{H}_0$ *and* $\mathcal{L}_0$ *are strongly elliptic self-adjoint pseudodifferential operators of order* -1 *and* $+1$*, respectively, having positive definite principal homogeneous symbol matrices (for details see [8, Chapters 4 and 6], [11], [12], [15], [16], [17], [19], [26]).*

The entries of the kernel matrices of the integral operators $\mathcal{H}_0$, $\mathcal{K}^*_0$, $\mathcal{K}_0$, *and* $\mathcal{L}_0$ *are real-valued matrix functions. In particular, this implies that* $\mathcal{K}_0$ *and* $\mathcal{K}^*_0$ *are mutually adjoint integral operators. Therefore, for complex-valued vector-functions* $\varphi_1, \varphi_2 \in \big[H_2^{-\frac{1}{2}}(S)\big]^3$ *and* $\psi_1, \psi_2 \in \big[H_2^{\frac{1}{2}}(S)\big]^3$ *the following relations hold*

$$\begin{aligned}
&\big\langle \mathcal{H}_0\varphi_1,\,\varphi_2\big\rangle_S = \big\langle \varphi_1,\,\mathcal{H}_0\varphi_2\big\rangle_S, \quad \big\langle \mathcal{L}_0\psi_1,\,\psi_2\big\rangle_S = \big\langle \psi_1,\,\mathcal{L}_0\psi_2\big\rangle_S, \\
&\big\langle \mathcal{K}^*_0\varphi_1,\,\psi_1\big\rangle_S = \big\langle \varphi_1,\,\mathcal{K}_0\psi_1\big\rangle_S, \quad \big\langle \mathcal{K}_0\psi_1,\,\varphi_1\big\rangle_S = \big\langle \psi_1,\,\mathcal{K}^*_0\varphi_1\big\rangle_S.
\end{aligned} \tag{3.4}$$

Moreover, there are positive constants δ_1 *and* δ_2 *such that the following inequalities hold*

$$\big\langle -\mathcal{H}_0\varphi,\,\varphi\big\rangle_S \geqslant \delta_1\,\|\varphi\|^2_{H_2^{-\frac{1}{2}}(S)} \quad \text{for all} \ \ \varphi \in \big[H_2^{-\frac{1}{2}}(S)\big]^3, \tag{3.5}$$

$$\big\langle \mathcal{L}_0\psi,\,\psi\big\rangle_{S_1} \geqslant \delta_2\,\|\psi\|^2_{\widetilde{H}_2^{\frac{1}{2}}(S_1)} \quad \text{for all} \ \ \psi \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_1)\big]^3, \tag{3.6}$$

where $S_1 \in \{S_D,\,S_N\}$.

The above relations remain valid if S, S_D, *and* S_N *are Lipschitz surfaces.*

Theorem 8.1 stated in Appendix 2 and inequalities (3.5)-(3.6) lead to the following assertion for the boundary operators $\mathcal{H}_0$ and $\mathcal{L}_0$.

Theorem 3.5 *Let $S_1 \in \{S_D, S_N\}$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and $-\frac{1}{2} < s - \frac{1}{p} < \frac{1}{2}$. Then the pseudodifferential operators*

$$
\begin{aligned}
& r_{S_1}\mathcal{H}_0 : \big[\widetilde{H}_p^{s-1}(S_1)\big]^3 \to \big[H_p^s(S_1)\big]^3, && r_{S_1}\mathcal{H}_0 : \big[\widetilde{B}_{p,q}^{s-1}(S_1)\big]^3 \to \big[B_{p,q}^s(S_1)\big]^3, \\
& r_{S_1}\mathcal{L}_0 : \big[\widetilde{H}_p^{s}(S_1)\big]^3 \to \big[H_p^{s-1}(S_1)\big]^3, && r_{S_1}\mathcal{L}_0 : \big[\widetilde{B}_{p,q}^{s}(S_1)\big]^3 \to \big[B_{p,q}^{s-1}(S_1)\big]^3,
\end{aligned}
$$

are invertible.

4 Existence results

We look for a solution to the above formulated mixed boundary value problem (2.1)-(2.3) in the form of the linear combination of single and double layer potentials

$$
u(x) = -V_\omega(\varphi)(x) + W_\omega(\psi)(x) + i\,c\,\omega\, V_\omega(\psi)(x), \quad x \in \Omega^-, \tag{4.1}
$$

with unknown densities

$$
\varphi \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3, \qquad \psi \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^3. \tag{4.2}
$$

Here c is a real number different from zero and $\omega \in \mathbb{R}$ is a frequency parameter.
Evidently, $u \in \big[C^\infty(\Omega^-)\big]^3 \cap \big[H^1_{2,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$ due to Theorem 3.1 and equation (2.1) is automatically satisfied in the classical sense in Ω^-.

In view of the jump relations of the layer potentials and inclusions (4.2), the mixed boundary conditions (2.2) and (2.3) lead to the following integral equations with respect to the unknown vector-functions φ and ψ:

$$
\begin{aligned}
& -\mathcal{H}_\omega\varphi + \mathcal{K}_\omega\psi + i\,c\,\omega\,\mathcal{H}_\omega\psi = f \quad \text{on} \quad S_D, \\
& -\mathcal{K}^*_\omega\varphi + \mathcal{L}_\omega\psi + i\,c\,\omega\,\big(\tfrac{1}{2}\,I_3 + \mathcal{K}^*_\omega\big)\psi = F \quad \text{on} \quad S_N.
\end{aligned} \tag{4.3}
$$

Let us introduce the notation:

$$
\mathbf{A}_\omega := \begin{bmatrix} r_{S_D}(-\mathcal{H}_\omega) & r_{S_D}(\mathcal{K}_\omega + i\,c\,\omega\,\mathcal{H}_\omega) \\ r_{S_N}(-\mathcal{K}^*_\omega) & r_{S_N}\Big(\mathcal{L}_\omega + i\,c\,\omega\big(\tfrac{1}{2}\,I_3 + \mathcal{K}^*_\omega\big)\Big) \end{bmatrix}_{6\times 6}, \tag{4.4}
$$

$$
X := \begin{bmatrix} \varphi \\ \psi \end{bmatrix}_{6\times 1}, \quad G := \begin{bmatrix} f \\ F \end{bmatrix}_{6\times 1},
$$

where $r_{_{S_D}}$ and $r_{_{S_N}}$ are the restriction operators onto S_D and S_N respectively.

The simultaneous equations (4.3) can be rewritten then in vector-matrix form

$$\mathbf{A}_\omega X = G. \tag{4.5}$$

Further, let

$$\widetilde{\mathbb{H}}_2 := \big[\widetilde{H}_2^{-\frac12}(S_D)\big]^3 \times \big[\widetilde{H}_2^{\frac12}(S_N)\big]^3, \quad \mathbb{H}_2 := \big[H_2^{\frac12}(S_D)\big]^3 \times \big[H_2^{-\frac12}(S_N)\big]^3.$$

Obviously, $\widetilde{\mathbb{H}}_2$ and $\mathbb{H}_2$ are mutually adjoint spaces and they are equipped with the norms:

$$\|X\|^2_{\widetilde{\mathbb{H}}_2} := \|\varphi\|^2_{[\widetilde{H}_2^{-\frac12}(S_D)]^3} + \|\psi\|^2_{[\widetilde{H}_2^{\frac12}(S_N)]^3} \quad \text{for} \quad X = \big(\varphi,\, \psi\big)^\top \in \widetilde{\mathbb{H}}_2,$$

$$\|G\|^2_{\mathbb{H}_2} := \|f\|^2_{[H_2^{\frac12}(S_D)]^3} + \|F\|^2_{[H_2^{-\frac12}(S_N)]^3} \quad \text{for} \quad G = \big(f,\, F\big)^\top \in \mathbb{H}_2.$$

In what follows, we show that equation (4.5) is solvable in the space $\widetilde{\mathbb{H}}_2$ for arbitrary right-hand side vector-function $G \in \mathbb{H}_2$. By Theorem 3.3, the operator $\mathbf{A}_\omega$ has the mapping property

$$\mathbf{A}_\omega \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2\ . \tag{4.6}$$

Let us prove that (4.6) is an isomorphism. Denote by $\mathbf{A}_0$ the operator defined by (4.4) for $\omega = 0$,

$$\mathbf{A}_0 := \begin{bmatrix} r_{_{S_D}}(-\mathcal{H}_0) & r_{_{S_D}}(\mathcal{K}_0) \\ r_{_{S_N}}(-\mathcal{K}_0^*) & r_{_{S_N}}\Big(\mathcal{L}_0\Big) \end{bmatrix}_{6\times 6}. \tag{4.7}$$

Due to relations (3.1), $\mathcal{H}_\omega - \mathcal{H}_0$ is a pseudodifferential operators of order -3, $\mathcal{K}_\omega - \mathcal{K}_0$ and $\mathcal{K}^*_\omega - \mathcal{K}^*_0$ are pseudodifferential operators of order -2, while $\mathcal{L}_\omega - \mathcal{L}_0$ is a pseudodifferential operator of order -1. Therefore we have the following mapping properties for arbitrary $\omega \in \mathbb{R}$, $s \in R$, $1 < p < \infty$, and $1 \leqslant q \leqslant \infty$:

$$\mathcal{H}_\omega - \mathcal{H}_0 \,:\, \big[H_2^{-\frac12}(S)\big]^3 \to \big[H_2^{\frac52}(S)\big]^3 \qquad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^{s+3}_{p,q}(S)\big]^3\Big), \tag{4.8}$$

$$\mathcal{K}_\omega - \mathcal{K}_0 \,:\, \big[H_2^{\frac12}(S)\big]^3 \to \big[H_2^{\frac52}(S)\big]^3 \qquad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^{s+2}_{p,q}(S)\big]^3\Big), \tag{4.9}$$

$$\mathcal{K}^*_\omega - \mathcal{K}^*_0 \,:\, \big[H_2^{-\frac12}(S)\big]^3 \to \big[H_2^{\frac32}(S)\big]^3 \qquad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^{s+2}_{p,q}(S)\big]^3\Big), \tag{4.10}$$

$$\mathcal{L}_\omega - \mathcal{L}_0 \,:\, \big[H_2^{\frac12}(S)\big]^3 \to \big[H_2^{\frac32}(S)\big]^3 \qquad \Big(\big[B^s_{p,q}(S)\big]^3 \to \big[B^{s+1}_{p,q}(S)\big]^3\Big). \tag{4.11}$$

Consequently, the operator

$$\mathbf{A}_\omega - \mathbf{A}_0 = \begin{bmatrix} -r_{_{S_D}}(\mathcal{H}_\omega - \mathcal{H}_0) & r_{_{S_D}}(\mathcal{K}_\omega - \mathcal{K}_0 + i\,c\,\omega\,\mathcal{H}_\omega) \\ -r_{_{S_N}}(\mathcal{K}^*_\omega - \mathcal{K}^*_0) & r_{_{S_N}}\Big(\mathcal{L}_\omega - \mathcal{L}_0 + i\,c\,\omega\big(\tfrac12\, I_3 + \mathcal{K}^*_\omega\big)\Big) \end{bmatrix}_{6\times 6} \tag{4.12}$$

possesses the mapping property

$$\mathbf{A}_\omega - \mathbf{A}_0 \,:\, \widetilde{\mathbb{H}}_2 \to \big[H_2^{\frac{3}{2}}(S_D)\big]^3 \times \big[H_2^{\frac{1}{2}}(S_N)\big]^3 \subset \mathbb{H}_2.$$

By the Rellich-Kondrashov compact embedding theorems (see [2], [3]), we infer that the inclusion $\big[H_2^{\frac{3}{2}}(S_D)\big]^3 \times \big[H_2^{\frac{1}{2}}(S_N)\big]^3 \subset \mathbb{H}_2$ is compact, which implies that the operator $\mathbf{A}_\omega$ in (4.6) is a compact perturbation of the operator $\mathbf{A}_0 \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2$.

Now we show that the matrix integral operator $\mathbf{A}_0$ generates a bounded and coercive sesquilinear form. Indeed, let $X' = \big(\varphi',\, \psi'\big)^\top$ and $X'' = \big(\varphi'',\, \psi''\big)^\top$ be arbitrary elements of the space $\widetilde{\mathbb{H}}_2$. Then $\mathbf{A}_0 X' \in \mathbb{H}_2$ and the following duality relation is well-defined

$$\big\langle \mathbf{A}_0 X',\, X'' \big\rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)} := \big\langle -\mathcal{H}_0\varphi',\, \varphi''\big\rangle_{S_D} + \big\langle \mathcal{K}_0\psi',\, \varphi''\big\rangle_{S_D} - \big\langle \mathcal{K}_0^*\varphi', \psi''\big\rangle_{S_N} + \big\langle \mathcal{L}_0\psi', \psi''\big\rangle_{S_N}.$$

Here the symbol $\big\langle \cdot\,,\, \cdot \big\rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)}$ denotes the duality between the mutually adjoint spaces $\mathbb{H}_2$ and $\widetilde{\mathbb{H}}_2$.

With the help of Theorem 3.3, Remark 3.4, inequalities (3.5), (3.6), and relations (3.4), it can be easily shown that the operator

$$\mathbf{A}_0 \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2 \tag{4.13}$$

generates a bounded and strongly coercive bilinear form. Indeed, for arbitrary $X' = \big(\varphi',\, \psi'\big)^\top \in \widetilde{\mathbb{H}}_2$ and $X'' = \big(\varphi'',\, \psi''\big)^\top \in \widetilde{\mathbb{H}}_2$ the following relations hold:

$$\begin{aligned}
\Big|\big\langle \mathbf{A}_0 X',\, X'' \big\rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)}\Big| &\leqslant C_1\Big(\|\varphi''\|_{[\widetilde{H}_2^{-\frac{1}{2}}(S_D)]^3}\,\|\varphi'\|_{[\widetilde{H}_2^{-\frac{1}{2}}(S_D)]^3} + \\
&\quad + \|\varphi''\|_{[\widetilde{H}_2^{-\frac{1}{2}}(S_D)]^3}\,\|\psi'\|_{[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^3} + \\
&\quad + \|\varphi'\|_{[\widetilde{H}^{-\frac{1}{2}}(S_D)]^3}\,\|\psi''\|_{[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^3} + \\
&\quad + \|\psi'\|_{[\widetilde{H}^{\frac{1}{2}}(S_N)]^3}\,\|\psi''\|_{[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^3}\Big) \leqslant \\
&\leqslant C_2\,\|X'\|_{\widetilde{\mathbb{H}}_2}\,\|X''\|_{\widetilde{\mathbb{H}}_2}, \\
\mathrm{Re}\Big[\big\langle \mathbf{A}_0 X',\, X' \big\rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)}\Big] &= \mathrm{Re}\Big[\big\langle -\mathcal{H}_0\varphi',\, \varphi'\big\rangle_{S_D} + \big\langle \mathcal{L}_0\psi', \psi'\big\rangle_{S_N} + \\
&\quad + \big\langle \mathcal{K}_0\psi',\, \varphi'\big\rangle_{S_D} - \big\langle \mathcal{K}_0^*\varphi', \psi'\big\rangle_{S_N}\Big] = \\
&= \mathrm{Re}\Big[\big\langle -\mathcal{H}_0\varphi',\, \varphi'\big\rangle_{S_D} + \big\langle \mathcal{L}_0\psi', \psi'\big\rangle_{S_N}\Big] \geqslant \\
&\geqslant \delta_3\|X'\|^2_{\widetilde{\mathbb{H}}_2},
\end{aligned}$$

where C_1, C_2, and $\delta_3 = \min\{\delta_1, \delta_2\} > 0$ are positive constants (see (3.5)-(3.6)). Here we employed the equalities

$$\big\langle \mathcal{K}_0\psi',\, \varphi' \big\rangle_{S_D} = \big\langle \mathcal{K}_0\psi',\, \varphi' \big\rangle_{S} = \big\langle \psi',\, \mathcal{K}_0^*\varphi' \big\rangle_{S} = \big\langle \psi',\, \mathcal{K}_0^*\varphi' \big\rangle_{S_N} = \overline{\big\langle \mathcal{K}_0^*\varphi',\, \psi' \big\rangle}_{S_N},$$

which follow from relations (3.4) and embeddings $\varphi' \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3$ and $\psi' \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^3$.

By the well-known Lax-Milgram theorem operator (4.13) is invertible and consequently operator (4.6) is Fredholm with zero index for arbitrary $\omega \in \mathbb{R}$ (see, e.g., [23, Ch. 2]).

Now, we show that the null space of operator (4.6) is trivial, which implies invertibility of the operator for arbitrary frequency parameter $\omega \in \mathbb{R}$. We proceed as follows. Let a pair $(\varphi, \psi) \in \widetilde{\mathbb{H}}_2$ be a solution to the homogeneous system (4.3) with $f = 0$ and $F = 0$. Then the function u represented by formula (4.1) solves the homogeneous exterior mixed BVP (2.1)-(2.3). Due to the uniqueness Theorem 2.2, u vanishes in Ω^-,

$$u(x) = -V_\omega(\varphi)(x) + W_\omega(\psi)(x) + i\,c\,\omega\, V_\omega(\psi)(x) = 0, \quad x \in \Omega^-. \tag{4.14}$$

Let us extend the vector-function $u(x)$ in Ω^+ by the same representation formula (4.1). Using the jump relations for the layer potentials we find

$$\{u\}^+ - \{u\}^- = \psi, \quad \{Tu\}^+ - \{Tu\}^- = \varphi - i\,c\,\omega\,\psi \quad \text{on } S, \tag{4.15}$$

whence by (4.14) we deduce

$$\begin{aligned} &\{u\}^+ = \psi \quad \text{on } S, \\ &\{Tu\}^+ = \varphi - i\,c\,\omega\,\psi \quad \text{on } S. \end{aligned} \tag{4.16}$$

Since $\varphi \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3$ and $\psi \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^3$, from (4.16) we conclude that the vector-function u solves the mixed interior BVP for the steady state oscillation equation in Ω^+ with the Dirichlet type condition on S_D and the Winkler type condition on S_N,

$$\begin{aligned} &A(\partial,\omega)\,u(x) = 0 \quad \text{in } \Omega^+, \\ &\{u\}^+ = 0 \quad \text{on } S_D, \\ &\{Tu\}^+ + i\,c\,\omega\,\{u\}^+ = 0 \quad \text{on } S_N. \end{aligned} \tag{4.17}$$

By Theorem 7.1 in Appendix 1, $u(x) = 0$ in Ω^+. Therefore

$$-V_\omega(\varphi)(x) + W_\omega(\psi)(x) + i\,c\,\omega\, V_\omega(\psi)(x) = 0, \quad x \in \Omega^- \cup \Omega^+,$$

which, along with (4.14) and (4.15), implies $\varphi = 0$ and $\psi = 0$ on S, that is, the null space of the Fredholm operator (4.6) is trivial and consequently it is invertible.

Thus, we have proved the following assertions.

Theorem 4.1 (i) *Operator* (4.6) *is invertible for arbitrary frequency parameter* $\omega \in \mathbb{R}$.
(ii) *The system of integral equations* (4.3) *is uniquely solvable for arbitrary right-hand side vector-functions satisfying inclusions* (2.4) *and the solution pair* (φ, ψ) *meets conditions* (4.2).

From this theorem directly follows the existence theorem for the mixed BVP under consideration.

Theorem 4.2 *Let* f *and* F *be arbitrary vector-functions satisfying inclusions* (2.4). *Then the exterior mixed BVP* (2.1)-(2.3) *is uniquely solvable in the space* $\big[H^1_{2,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$ *and the solution can be represented as a linear combination of single and double layer potentials by formula* (4.1), *where the densities* $\varphi \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3$ *and* $\psi \in \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^3$ *are unique solutions to the system of integral equations* (4.3).

Remark 4.3 *Theorem 4.1 and Theorem 4.2 remain valid if the surfaces* S, S_D, *and* S_N *are Lipschitz. The point is that, for Lipschitz case, the operator* (4.13) *is invertible again. In spite of the fact that the high order smoothing relations* (4.8)-(4.11) *are not true for Lipschitz surfaces, in general, it can be easily shown that the operator* (4.6) *is again a compact perturbation of* (4.13).

5 Regularity results

Using the arguments employed in [30] with appropriate modifications, we can prove the following regularity results for solutions to the mixed BVP of steady state elastic oscillation equations.

Theorem 5.1 *Let the data of the mixed boundary value problem* (2.1)-(2.3) *satisfy the conditions*

$$f \in \big[B^s_{p,2}(S_D)\big]^3, \quad F \in \big[B^{s-1}_{p,2}(S_N)\big]^3 \ \text{ with } \ \frac{1}{2} \leqslant s < \frac{1}{2} + \frac{1}{p} \quad \text{and} \quad p > 4. \tag{5.1}$$

(i) *The system of integral equations* (4.3) *is uniquely solvable and the solution pair* (φ, ψ) *meets the inclusions*

$$\begin{aligned} &\varphi \in \big[\widetilde{B}^{s-1}_{p,2}(S_D)\big]^3 \subset \big[\widetilde{B}^{-\frac{1}{2}}_{2,2}(S_D)\big]^3 = \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3, \\ &\psi \in \big[\widetilde{B}^{s}_{p,2}(S_N)\big]^3 \subset \big[\widetilde{B}^{\frac{1}{2}}_{2,2}(S_N)\big]^3 = \big[\widetilde{H}_2^{\frac{1}{2}}(S_N)\big]^3. \end{aligned} \tag{5.2}$$

Moreover,

$$\psi \in \big[C^{\,t}(S)\big]^3 \quad \textit{with} \quad t = s - \frac{2}{p} \in \Big[\frac{1}{2} - \frac{2}{p},\ \frac{1}{2} - \frac{1}{p}\Big). \tag{5.3}$$

(ii) *The unique solution* u *to the mixed boundary value problem* (2.1)-(2.3) *belongs to the class* $\big[H^1_{2,loc}(\Omega^-)\big]^3 \cap \big[B^{s+\frac{1}{p}}_{p,2,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$ *and it can be represented as the linear combination of the single and double layer potentials* (4.1) *with densities* φ *and* ψ *being solutions to the system of integral equations* (4.3) *belonging to the spaces* (5.2).
Moreover, the solution u *possesses the following Hölder continuity property*

$$u \in \big[C^{\,t}(\overline{\Omega^-}\,)\big]^3 \quad \textit{with} \quad t = s - \frac{2}{p} \in \Big[\frac{1}{2} - \frac{2}{p},\ \frac{1}{2} - \frac{1}{p}\Big). \tag{5.4}$$

Proof. Let conditions (5.1) be satisfied. In view of the embedding properties of the Besov spaces we have (see Appendix 3)

$$\begin{gathered} f \in \big[B^s_{p,2}(S_D)\big]^3 \subset \big[B^{\frac{1}{2}}_{2,2}(S_D)\big]^3 = \big[H^{\frac{1}{2}}_2(S_D)\big]^3, \\ F \in \big[B^{s-1}_{p,2}(S_N)\big]^3 \subset \big[B^{-\frac{1}{2}}_{2,2}(S_N)\big]^3 = \big[H^{-\frac{1}{2}}_2(S_N)\big]^3, \\ \text{for} \quad s \geqslant \frac{1}{2}, \quad s \geqslant \frac{2}{p} - \frac{1}{2}, \quad p > 1. \end{gathered}$$

Therefore, by Theorem 4.1 system (4.3) is uniquely solvable and

$$\varphi \in \big[\widetilde{B}^{-\frac{1}{2}}_{2,2}(S_D)\big]^3 = \big[\widetilde{H}^{-\frac{1}{2}}_2(S_D)\big]^3, \quad \psi \in \big[\widetilde{B}^{\frac{1}{2}}_{2,2}(S_N)\big]^3 = \big[\widetilde{H}^{\frac{1}{2}}_2(S_N)\big]^3. \tag{5.5}$$

Rewrite system (4.3) in the form

$$-\,r_{_{S_D}}\mathcal{H}_0\varphi + r_{_{S_D}}\mathcal{K}_0\psi = f + f^{(1)} \quad \text{on} \quad S_D, \tag{5.6}$$

$$-r_{_{S_N}}\mathcal{K}^*_0\varphi + r_{_{S_N}}\mathcal{L}_0\psi = F + F^{(1)} \quad \text{on} \quad S_N, \tag{5.7}$$

where

$$\begin{aligned} f^{(1)} &= -r_{_{S_D}}\big[\big(\mathcal{H}_0 - \mathcal{H}_\omega\big)\varphi + \big(\mathcal{K}_\omega - \mathcal{K}_0\big)\psi + i\,c\,\omega\,\mathcal{H}_\omega\psi\big], \\ F^{(1)} &= r_{_{S_N}}\Big[\big(\mathcal{L}_0 - \mathcal{L}_\omega\big)\psi + \big(\mathcal{K}^*_\omega - \mathcal{K}^*_0\big)\varphi - i\,c\,\omega\Big(\frac{1}{2}\,\psi + \mathcal{K}^*_\omega\psi\Big)\Big]. \end{aligned}$$

Taking into account inclusions (5.5) and the mapping properties of the operators $\mathcal{H}_\omega$, $\mathcal{K}^*_\omega$, $\mathcal{K}_\omega - \mathcal{K}_0$, $\mathcal{K}^*_\omega - \mathcal{K}^*_0$, $\mathcal{H}_0 - \mathcal{H}_\omega$, and $\mathcal{L}_0 - \mathcal{L}_\omega$, for the vector-functions $f^{(1)}$ and $F^{(1)}$ we conclude (see Theorem 3.3 and (4.8)-(4.11)):

$$f^{(1)} \in \big[H^{\frac{3}{2}}_2(S_D)\big]^3 = \big[B^{\frac{3}{2}}_{2,2}(S_D)\big]^3, \tag{5.8}$$

$$F^{(1)} \in \big[H^{\frac{1}{2}}_2(S_N)\big]^3 = \big[B^{\frac{1}{2}}_{2,2}(S_N)\big]^3. \tag{5.9}$$

In turn, (5.8) and (5.9) imply the following inclusions (see Appendix 3)

$$f^{(1)} \in \big[H_2^{\frac{3}{2}}(S_D)\big]^3 = \big[B_{2,2}^{\frac{3}{2}}(S_D)\big]^3 \subset \big[B_{p,2}^{l}(S_D)\big]^3, \tag{5.10}$$

$$F^{(1)} \in \big[H_2^{\frac{1}{2}}(S_N)\big]^3 = \big[B_{2,2}^{\frac{1}{2}}(S_N)\big]^3 \subset \big[B_{p,2}^{l-1}(S_N)\big]^3, \tag{5.11}$$

$$\text{for} \quad l \leqslant \frac{2}{p} + \frac{1}{2}\,, \;\; l \leqslant \frac{3}{2}\,, \;\; p > 1.$$

Since $B_{p,2}^{l}(S_D) \subset B_{p,2}^{s}(S_D)$ and $B_{p,2}^{l-1}(S_N) \subset B_{p,2}^{s-1}(S_N)$ for $s \leqslant l$ and $p > 1$, in view of (5.10) and (5.11) for the right-hand side functions in (5.6) and (5.7) we deduce

$$f + f^{(1)} \in \big[B_{p,2}^{s}(S_D)\big]^3, \quad F + F^{(1)} \in \big[B_{p,2}^{s-1}(S_N)\big]^3,$$

if the following simultaneous inequalities are satisfied:

$$\begin{aligned} & s \leqslant l\,, \quad p > 1, \\ & l \leqslant \frac{2}{p} + \frac{1}{2}\,, \quad l \leqslant \frac{3}{2}\,, \\ & s \geqslant \frac{2}{p} - \frac{1}{2}\,, \quad s \geqslant \frac{1}{2}. \end{aligned}$$

Rewrite equations (5.6) and (5.7) in matrix form

$$\mathbf{A}_0\, X = \Phi, \tag{5.12}$$

where $\mathbf{A}_0$ is given by (4.7), $X = \big(\varphi, \psi\big)^\top$, and

$$\Phi := \big(f + f^{(1)},\, F + F^{(1)}\big)^\top \in \big[B_{p,2}^{s}(S_D)\big]^3 \times \big[B_{p,2}^{s-1}(S_N)\big]^3. \tag{5.13}$$

In the reference [30, Theorem 5], it is proved that equation (5.12) is uniquely solvable for arbitrary right-hand side vector-function Φ satisfying condition (5.13) if the parameters s and p satisfy the inequalities

$$a(p) \leqslant s < \frac{1}{2} + \frac{1}{p}$$

with

$$a(p) = \begin{cases} \dfrac{2}{p} - \dfrac{1}{2} & \text{for} \quad 1 < p \leqslant 2, \\ \dfrac{1}{2} & \text{for} \quad p \geqslant 2, \end{cases}$$

and for the solution pair $X = \big(\varphi, \psi\big)^\top$ the following inclusions hold

$$\varphi \in \big[\widetilde{B}_{p,2}^{s-1}(S_D)\big]^3, \quad \psi \in \big[\widetilde{B}_{p,2}^{s}(S_N)\big]^3. \tag{5.14}$$

Thus, the vector-functions φ and ψ satisfy inclusions (5.5) and (5.14) which proves the first part of the item (i) of the theorem.

Further, with the help of Theorem 3.1 along with the representation (4.1) and inclusions (5.5) and (5.14), for the solution u to the mixed BVP we get the following inclusion

$$u = -V_\omega(\varphi) + W_\omega(\psi) + i\,c\,\omega\,V_\omega(\psi) \in \big[H^1_{2,loc}(\Omega^-)\big]^3 \cap \big[B^{s+\frac{1}{p}}_{p,2,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-) \tag{5.15}$$
$$\text{with } \; a(p) \leqslant s < \tfrac{1}{2} + \tfrac{1}{p}, \;\; p > 1.$$

Recall that for arbitrary domain $\Omega \subset \mathbb{R}^3$ with smooth two-dimensional boundary $\partial\Omega$, the following embedding relations hold (see Appendix 3)

$$\begin{gathered} H^s_p(\Omega) \subset \mathcal{C}^t(\overline{\Omega}), \quad B^s_{p,q}(\Omega) \subset \mathcal{C}^t(\overline{\Omega}), \qquad H^r_p(\partial\Omega) \subset \mathcal{C}^\tau(\partial\Omega), \quad B^r_{p,q}(\partial\Omega) \subset \mathcal{C}^\tau(\partial\Omega), \\ 1 < p < \infty, \;\; 1 \leqslant q \leqslant \infty, \;\; t \geqslant 0, \;\; s > t + \frac{3}{p}, \;\; \tau \geqslant 0, \;\; r > \tau + \frac{2}{p}, \end{gathered} \tag{5.16}$$

since $\dim\Omega = 3$ and $\dim\partial\Omega = 2$. If t (resp. τ) is not an integer, then in (5.16) one can take $s = t + \frac{3}{p}$ (resp. $r = \tau + \frac{2}{p}$). Note that here $\mathcal{C}^t(\overline{\Omega})$ and $\mathcal{C}^\tau(\partial\Omega)$ with $t > 0$ and $\tau > 0$ denote the Zygmund spaces, which for non-integers t and τ coincide with the Hölder spaces $C^t(\overline{\Omega})$ and $C^\tau(\partial\Omega)$, respectively.

Now, if $p > 4$ and $\frac{1}{2} \leqslant s < \frac{1}{2} + \frac{1}{p}$, from (5.14) and (5.15) we deduce the following smoothness properties

$$\begin{gathered} \psi \in \big[\widetilde{B}^s_{p,2}(S_N)\big]^3 \subset \big[C^{\,t}(S)\big]^3, \\ u = \, -V_\omega(\varphi) + W_\omega(\psi) + i\,c\,\omega\,V_\omega(\psi) \in \big[B^{s+\frac{1}{p}}_{p,2}(\Omega)\big]^3 \subset \big[C^{\,t}(\overline{\Omega})\big]^3 \end{gathered}$$

with

$$t = s - \frac{2}{p} \in \Big[\frac{1}{2} - \frac{2}{p}, \, \frac{1}{2} - \frac{1}{p}\Big).$$

This completes the proof.

■

With the help of Theorem 5.1 we can prove the following proposition.

Theorem 5.2 *Let the boundary data of the mixed BVP* (2.1)-(2.3) *satisfy the relations*

$$f \in \big[B^{\frac{1}{2}}_{\infty,\,2}(S_D)\big]^3, \qquad F \in \big[B^{-\frac{1}{2}}_{\infty,\,2}(S_N)\big]^3. \tag{5.17}$$

Then the density function ψ and the solution u to the mixed BVP have the following Hölder continuity properties

$$\psi \in \bigcap_{t<\frac{1}{2}} \big[C^{t}(S)\big]^3, \quad u \in \bigcap_{t<\frac{1}{2}} \big[C^{t}(\overline{\Omega})\big]^3.$$

Proof. First of all, we recall the well known embedding relations for the Besov spaces (see, e.g., [35, Sections 2.3.5, 3.3.1], [5, Theorem 6.2.4], [34, Section 4.6])

$$B^{\frac{1}{2}}_{\infty,2}(S_D) \subset B^{\frac{1}{2}}_{p,2}(S_D), \qquad B^{-\frac{1}{2}}_{\infty,2}(S_N) \subset B^{-\frac{1}{2}}_{p,2}(S_N) \quad \text{for all} \quad p > 1. \tag{5.18}$$

Therefore, the items (i) and (ii) of Theorem 5.1 hold for all $p > 4$ in view of (5.17) and (5.18). Now, using the inclusions (5.3) and (5.4) we deduce

$$\psi \in \bigcap_{p>4} \big[C^{\frac{1}{2}-\frac{2}{p}}(S)\big]^3, \qquad u \in \bigcap_{p>4} \big[C^{\frac{1}{2}-\frac{2}{p}}(\overline{\Omega})\big]^3,$$

which completes the proof. ■

6 Solvability and alternative representation of solutions in $\big[H^1_{p,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$

Now we discuss the question: For which values of the parameter p can be applied the above described alternative approach to the exterior mixed BVP in the spaces $\big[H^1_{p,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$. The answer to this question is given by the following assertion.

Theorem 6.1 *Let the boundary data of the exterior mixed BVP* (2.1)-(2.3) *meet the conditions:*

$$f \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3, \quad F \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3 \quad \textit{with} \ \frac{4}{3} < p < 4. \tag{6.1}$$

Then exterior mixed BVP (2.1)-(2.3) *possesses a unique solution $u \in \big[H^1_{p,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$, which can be represented by the linear combination of the single and double layer potentials* (4.1), *where $\varphi \in \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3$ and $\psi \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$ are unique solutions to the system of integral equations* (4.3).

Proof. First we show that the operator

$$\mathbf{A}_\omega \,:\, \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3 \to \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3, \ \ \frac{4}{3} < p < 4, \tag{6.2}$$

where $\mathbf{A}_\omega$ is defined in (4.4), is invertible. Note that continuity of the operator (6.2) follows from Theorem 3.3. Due to Theorem 7(ii) in the reference [30], the operator

$$\mathbf{A}_0 \,:\, \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3 \to \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3, \;\; \frac{4}{3} < p < 4, \tag{6.3}$$

where $\mathbf{A}_0$ is given by (4.7), is invertible. In view of Theorem 3.3 and the mapping properties (4.8)-(4.11), the difference $\mathbf{A}_\omega - \mathbf{A}_0$ is a smoothing operator (see (4.12))

$$\mathbf{A}_\omega - \mathbf{A}_0 \,:\, \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3 \to \big[B^{2-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3.$$

Therefore, by the Rellich-Kondrashov compact embedding theorems, the operator A_ω defined by (6.2) is a compact perturbation of the invertible operator $\mathbf{A}_0$ defined by (6.3) and consequently it is a Fredholm operator with zero index. Further, we show that the null space of the operator A_ω is trivial. To this end, let us assume that a vector-function

$$X = (\varphi, \psi)^\top \in \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$$

is a solution to the homogeneous equation

$$\mathbf{A}_\omega X = 0.$$

This equation implies that the vector-function

$$u(x) = -V_\omega(\varphi)(x) + W_\omega(\psi)(x) + i\,c\,\omega\, V_\omega(\psi)(x), \quad x \in \Omega^-, \tag{6.4}$$

solves the homogeneous mixed BVP (2.1)-(2.3). According to Theorem 3.1 we have the inclusion $u \in \big[H^1_{p,loc}(\Omega^-)\big]^3 \cap SK(\Omega^-)$. Since $\frac{4}{3} < p < 4$, due to the uniqueness theorem for the exterior mixed BVP under consideration (see [4, Theorem 3.3]) we conclude $u(x) = 0$ in Ω^-. If we extend the vector-function u in the domain Ω^+ by the same formula (6.4), with the help of the jump relations for the layer potentials stated in Theorem 3.2 and using the same arguments as in Section 4, we easily find the relations:

$$\{u\}^+ = 0 \;\text{ on }\; S_D, \quad \{Tu\}^+ + i\,c\,\omega\,\{u\}^+ = 0 \;\text{ on }\; S_N,$$

which are understood in the sense of the spaces $\big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3$ and $\big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3$ respectively. Therefore, the vector-function $u \in \big[H^1_p(\Omega^+)\big]^3$ defined by (6.4) solves the homogeneous interior Dirichlet-Winkler type mixed BVP (4.17) in Ω^+. Since $\frac{4}{3} < p < 4$, by the uniqueness theorem for the interior mixed BVP (4.17) in the space $\big[H^1_p(\Omega^+)\big]^3$ (see Theorem 7.2 in Appendix 7, [4, Theorem 3.6]) we have $u(x)=0$ in Ω^+. Thus, the vector-function $u(x)$ given by (6.4) vanishes in $\Omega^+\cup\Omega^-$, which in turn implies $\varphi=0$ and $\psi=0$ on S. Therefore the null space of the operator (6.2) is trivial and consequently it is invertible.

In turn, the invertibility of the operator (6.2) implies that the system of integral equations (4.3) is uniquely solvable in the space $\big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3\times\big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$ for arbitrary right-hand side vector-functions f and F satisfying conditions (6.1).

From these results directly follows that the exterior mixed BVP (2.1)-(2.3) with boundary data f and F, satisfying conditions (6.1), possesses a unique solution $u\in\big[H^1_{p,loc}(\Omega^-)\big]^3\cap SK(\Omega^-)$, which can be represented by the linear combination of the single and double layer potentials (4.1), where the density vector-functions $\varphi\in\big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\big]^3$ and $\psi\in\big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$ are unique solutions to the system of integral equations (4.3). This completes the proof. ■

7 Appendix 1: Uniqueness theorem for the mixed Dirichlet-Winkler type BVPs

For arbitrary solution $u\in\big[H^1_p(\Omega^+)\big]^3$ to the homogeneous equation $A(\partial,\omega)u(x)=0$ in Ω^+ we have the general integral Somigliana type representation formulas (see [4], [20], [23])

$$W_\omega(\{u\}^+)(x)-V_\omega(\{Tu\}^+)(x)=\begin{cases}u(x) & \text{in}\quad \Omega^+,\\ 0 & \text{in}\quad \Omega^-,\end{cases}\tag{7.1}$$

where V_ω and W_ω are the single and double layer potentials defined in (3.2) and (3.3).

Theorem 7.1 *The homogeneous interior Dirichlet-Winkler type mixed boundary value problem* (4.17) *possesses only the trivial solution in the space* $\big[H^1_2(\Omega^+)\big]^3=\big[W^1_2(\Omega^+)\big]^3$ *for all* $\omega\in\mathbb{R}$.

Proof. Let $u\in\big[H^1_2(\Omega^+)\big]^3$ be a solution to the homogeneous Dirichlet-Winkler type mixed boundary value problem (4.17). Since $\{u\}^+\in\big[\widetilde{H}^{\frac{1}{2}}_2(S_N)\big]^3$ and $\{Tu\}^++ic\omega\{u\}^+=0$ on S_N, from Green's formula (2.7) we get

$$\int\limits_{\Omega^+}[\,E(u,\overline{u})-\varrho\,\omega^2\,|u|^2]dx+ic\omega\int\limits_{S_N}|\{u\}^+|^2\,dS=0.\tag{7.2}$$

Keeping in mind that $E(u,\overline{u})\geq 0$, by separating the imaginary and real parts from (7.2) we find $\{u\}^+=0$ on S_N, implying $\{u\}^+=0$ on S for $\omega\neq 0$. If $\omega=0$, then from (7.2) we have $E(u,\overline{u})=0$

in Ω^+. Therefore $u(x) = [a \times x] + b$, where a and b are constant three-dimensional complex-valued vectors, and due to the homogeneous Dirichlet type condition on S_D we infer $u(x) = 0$ in Ω^+. Thus, $\{u\}^+ = 0$ on S for all $\omega \in \mathbb{R}$. Consequently, $\{Tu\}^+ = 0$ on S_N and due to formula (7.1) we have

$$u(x) = -V_\omega(\{Tu\}^+)(x) = -\int\limits_{S_D} \Gamma(x-y,\omega)\big\{T(\partial_y, n(y))u(y)\big\}^+ dS_y, \ \ x \in \Omega^+, \tag{7.3}$$

and

$$-V_\omega(\{Tu\}^+)(x) = -\int\limits_{S_D} \Gamma(x-y,\omega)\big\{T(\partial_y, n(y))u(y)\big\}^+ dS_y = 0, \ \ x \in \Omega^-. \tag{7.4}$$

Since the single layer potential $V_\omega(\{Tu\}^+)$ with the density $\{Tu\}^+ \in \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^3$ is analytic vector-function with respect to the real variable x in the connected domain $\mathbb{R}^3 \setminus \overline{S_D}$, from (7.4) it follows that $V_\omega(\{Tu\}^+)(x) = 0$ in $\mathbb{R}^3 \setminus \overline{S_D}$. Therefore, $u(x) = 0$ in Ω^+ in view of (7.3) for all $\omega \in \mathbb{R}$. ■

There holds a more general theorem (see [4, Thorem 3.6]).

Theorem 7.2 *The homogeneous interior Dirichlet-Winkler type mixed boundary value problem* (4.17) *possesses only the trivial solution in the space* $\big[H_p^1(\Omega^+)\big]^3$ *with* $\frac{4}{3} < p < 4$.

Remark 7.3 *If the surfaces* S, S_D, *and* S_N *are Lipschitz, then the representation* (7.1) *and Theorem 7.1 remain true.*

8 Appendix 2: Some results from the theory of pseudodifferential equations on manifolds with boundary

Here we present some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission and crack type problems by the potential and boundary integral equations methods. They can be found in [6], [13], [31].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, non-self-intersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order

$\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\sigma(x,0,\cdots,0,+1)\,]^{-1}[\,\sigma(x,0,\cdots,0,-1)\,], \quad x \in \partial\overline{\mathcal{M}},$$

and introduce the notation

$$\delta_j(x) = \mathrm{Re}\left[\,(2\pi\, i)^{-1}\ln\lambda_j(x)\,\right],\ j=1,\cdots,N.$$

Here the branch in the logarithmic function $\ln\zeta$ is chosen with regard to the inequality $-\pi < \arg\zeta \leq \pi$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$, $j = \overline{1,N}$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. In the particular case, when $\sigma(x,\xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1,\cdots,N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary are characterized by the following theorem.

Theorem 8.1 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\mathrm{Re}\left[\sigma(x,\xi)\,\eta\cdot\eta\right] \geq c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\mathcal{A}\ :\ \left[\widetilde{H}^s_p(\mathcal{M})\right]^N \to \left[H^{s-\nu}_p(\mathcal{M})\right]^N \quad \left(\left[\widetilde{B}^s_{p,q}(\mathcal{M})\right]^N \to \left[B^{s-\nu}_{p,q}(\mathcal{M})\right]^N\right) \tag{8.1}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\leq j\leq N}\delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\leq j\leq N}\delta_j(x). \tag{8.2}$$

Moreover, the null-spaces and indices of the operators (8.1) *are the same* (*for all values of the parameter $q \in [1,+\infty]$*) *provided p and s satisfy the inequality* (8.2).

9 Appendix 3: Embedding properties of the Bessel potential and Besov spaces

For a sufficiently smooth m-dimensional manifold $\mathcal{M}$ ($\mathcal{M} \in C^\infty$, say), the following continuous inclusions hold in the case of the Bessel potential and Besov spaces ([34, Section 4.6], [5, Theorem 6.2.4])

$$H_p^s(\mathcal{M}) \subset H_q^t(\mathcal{M}), \quad B_{p,r}^s(\mathcal{M}) \subset B_{q,r}^t(\mathcal{M}),$$
$$1 < p, q < \infty, \ \ 1 \leqslant r \leqslant \infty, \ \ -\infty < t \leqslant s < \infty, \ \ s - \frac{m}{p} \geqslant t - \frac{m}{q};$$
$$H_p^s(\mathcal{M}) \subset B_{q,p}^t(\mathcal{M}), \quad B_{p,q}^s(\mathcal{M}) \subset H_q^t(\mathcal{M}),$$
$$1 < p, q < \infty, \ \ -\infty < t < s < \infty, \ \ s - \tfrac{m}{p} \geqslant t - \tfrac{m}{q};$$
$$B_{p,\min\{2,p\}}^s(\mathcal{M}) \subset H_p^s(\mathcal{M}) \subset B_{p,\max\{2,p\}}^s(\mathcal{M}),$$
$$B_{p,\infty}^{s+\varepsilon}(\mathcal{M}) \subset B_{p,1}^s(\mathcal{M}) \subset B_{p,q_1}^s(\mathcal{M}) \subset B_{p,q_2}^s(\mathcal{M}) \subset B_{p,\infty}^s(\mathcal{M}) \subset B_{p,1}^{s-\varepsilon}(\mathcal{M}),$$
$$\varepsilon > 0, \ \ -\infty < s < \infty, \ \ 1 < p < \infty, \ \ 1 \leqslant q_1 \leqslant q_2 \leqslant \infty.$$

These inclusions hold true when $\mathcal{M}$ is replaced by a smooth bounded region $\Omega \subset \mathbb{R}^m$.

Evidently, if $\mathcal{M} \in \{S_D, S_N\}$, then $m = 2$, and if

$$s \geqslant \frac{2}{p} - \frac{1}{2}, \quad s \geqslant \frac{1}{2}, \quad p > 1,$$

we have the inclusions

$$B_{p,2}^s(S_D) \subset B_{2,2}^{\frac{1}{2}}(S_D) = H_2^{\frac{1}{2}}(S_D), \ \ B_{p,2}^{s-1}(S_N) \subset B_{2,2}^{-\frac{1}{2}}(S_N) = H_2^{-\frac{1}{2}}(S_N).$$

For a bounded domain $\Omega \subset \mathbb{R}^m$ and for $(m-1)$-dimensional manifold $\mathcal{M}$ ($\mathcal{M} = \partial\Omega$, say), the following embedding relations hold (see, e.g., [34, Section 4.6])

$$H_p^r(\mathcal{M}) \subset \mathcal{C}^\tau(\mathcal{M}), \ \ B_{p,q}^r(\mathcal{M}) \subset \mathcal{C}^\tau(\mathcal{M}), \ \ H_p^s(\Omega) \subset \mathcal{C}^t(\overline{\Omega}), \ \ B_{p,q}^s(\Omega) \subset \mathcal{C}^t(\overline{\Omega}),$$
$$1 < p < \infty, \ \ 1 \leqslant q \leqslant \infty, \ \ \tau \geqslant 0, \ \ r > \tau + \frac{m-1}{p}, \ \ t \geqslant 0, \ \ s > t + \frac{m}{p}. \tag{9.1}$$

If t (resp. τ) is not an integer, then in (9.1) one can take $s = t + \frac{m}{p}$ (resp. $r = \tau + \frac{m-1}{p}$). Note that here $\mathcal{C}^t(\overline{\Omega})$ and $\mathcal{C}^\tau(\mathcal{M})$ with $t > 0$ and $\tau > 0$ denote the Zygmund spaces, which for non-integers t and τ coincide with the Hölder spaces $C^t(\overline{\Omega})$ and $C^\tau(\mathcal{M})$ respectively (see, e.g., [34, Sections 4.5, 4.6]).

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