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Theoretical Prospects of the Solutions of Fractional Order Weakly Singular Volterra Integro Differential Equations and their Approximations with Convergence Analysis

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In this work, we study a weakly singular Volterra integro differential equation with Caputo type fractional derivative. First, we derive a sufficient condition for the existence and uniqueness of the solution of this problem based on the maximum norm. It is observed that the condition depends on the domain of definition of the problem. Thereafter, we show that this condition will be independent of the domain of definition based on an equivalent weighted maximum norm. In addition, we have also provided a procedure to extend the existence and uniqueness of the solution in its domain of definition by partitioning it. We also derive a sufficient condition under which the model problem will provide an analytic solution. Next, we introduce a operator based parameterized method to generate an approximate solution of this problem. Convergence analysis of this approach is established here. Next, we have optimized this solution based on least square method. For this, residual minimization is used to obtain the optimal values of the auxiliary parameter. In addition, we have also provided an error bound based on this technique. Several numerical examples are produced to clarify the effective behavior of the convergence of the present method. Comparison of the standard method and optimized method based on residual minimization signify the better accuracy of modified one. In addition, we also consider an equivalent form of weakly singular integro differential equation of a Heat transfer problem to show the effectiveness of the present approach.

Keywords: Integral equation, Weakly singular integro differential equation, Parameterized approximation, Optimized solution, Perturbation analysis, Convergence analysis, Caputo fractional derivative.

1. Introduction

Fractional order weakly singular integro differential equations are very frequent in several modeling. For e.g., fractal wave equation in shallow water surface leads to a time fractional model [9]. In addition, if the water wave scattering happens by a thin vertical barrier, immersed in deep water, it produces an integral equation with weakly singular kernel [2]. Therefore, a consequent behavior of two models leads to a fractional order weakly singular integro differential equations. It is observed that the fractional and integral models also appear in flight trajectory movement [6]. One can also see mathematical models available in [22, 19].

There are several notions of fractional derivatives existed in the literature, out of which the Riemann-Lioville [7, 22] and Caputo fractional derivatives [25, 7] are mainly popular in recent days. A coupled system of Riemann-Lioville type fractional differential equations are studied in [27] to obtain a sufficient condition for existence of positive solutions. Under certain conditions depending on the domain, it is observed in [25], that the uniqueness of the solution can be extended in infinite dimensional Banach spaces for fractional order boundary value problems in Caputo sense with $\alpha \in (0, 1)$. A discussion on integro differential equations of integer order and qualitative properties of Volterra integral equations of second kind with some special type of weakly singular kernels can be observed in [26]. But the existence and uniqueness of a general fractional order integro-differential equations with weakly singular kernels in any specified domain are very little in literature, in our knowledge.

The approximate solutions of integral equations are also one of the interesting topic in recent researches. In literature, approximations of a special type Abel integral equations appear in [15] by two-step Laplace decomposition algorithm, in [12] by using fractional order Legendre functions and pseudospectral method and in [16] based on asymptotic behavior of the solution. On the other hand, few specific class of fractional order Volterra integro-differential equations are also considered for numerical

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analysis, based on Wavelet methods in [24], using fast iterative refinement method in [10] and using spline collocation method in [21]. In recent days, semi analytical methods became popular over numerical methods in order to avoid of finding suitable discretization [23] of a general model. Very often, the semi analytical methods provide exact solution of the model. In this context, the popular semi analytical approaches include- methods based on series solution, Laplace transform, variational iteration, to solve fractional equations and integro-differential equations. Recently, homotopy based perturbation method [18, 8] became noticeable due to its simplicity and reliability of solving differential equations. Later, it was modified in [8] for solving mixed type integral equations. Convergence analysis of approximation techniques for integral equations can be also observed in few articles, say [3, 6, 11]. Recently, few numerical methods are proposed to solve fractional integral equation with weakly singular kernel by considering the particular case of $0 < \alpha < 1$ [13]. However, in our knowledge, the approximate solutions of fractional order weakly singular integro differential equations are very not known till today.

In the present work, our aim is to develop the existence and uniqueness result for the fractional order Volterra integro differential equations with weakly singular kernel and provide an approximate solution for this model based on Homotopy perturbation method (HPM) and modified it by least square based Homotopy perturbation method (LSHPM) by choosing the parameters optimally. Note that the present model generalizes the aforementioned problems which include the Abel type problems where the order of the weakly singular kernel is fixed.

We address the present work as follows. In Section 2, few preliminaries of fractional calculus are provided which is required to analyze our model in Section 3. In addition, this section provides few sufficient conditions for existence and uniqueness of the solution. Here, we develop two different theorems to extend the sufficient conditions independent of the domain. Section 4 describes the HPM and explains the convergence analysis of the method. We use the functions involved in this solution, to construct another approximate solution, where the coefficients are optimized by least square method on residual error. This can be pointed as least square homotopy perturbation method. In addition, we also provide an error bound of this modified solution in L^2 norm. Computational experiments are carried out in Section 5 to show the effectivity of the present method. It shows that the optimized LSHPM provides better accuracy compared to HPM with very few number of terms in the approximation.

Notations: Through out the paper, we use $\mathbb{N}$, $\mathbb{R}$ as the set of all natural numbers and real numbers. We define $\overline{\Omega}$ as the closure of the domain Ω. In addition, $C(\Omega)$ defines the set of all continuous functions on Ω and $C^n(\Omega) = \{f(x)|f^n(x)$ exists and $f^n(x)$ is continuous for $x \in \Omega\}$. For the analysis, we use the supremum norm for a function $f(x)$ as $\|f(x)\|_\infty = \|f(x)\| = \max_{x\in\Omega}|f(x)|$ in a domain Ω. The L^2 norm for a function $f(x)$, defined in $\overline{\Omega}$ is defined by $\|f\|_2 = \int_{\overline{\Omega}} f^2 dx$. We also use the symbol $O(p)$ to denote the order of p. For $x \geq 0$, $\gamma(\alpha, x)$ is denoted as incomplete gamma function of order $\alpha > 0$ and is defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt.$$

2. Preliminaries

In this section, we introduce some preliminaries of fractional calculus which will be used through out the work. For more details, one can see [22, 19].

Definition 1 *Riemann-Liouville (R-L) Fractional Integral:- For a locally integrable function f defined on $[0, T]$, the R-L fractional integral of order $\alpha > 0$, is defined by*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x > 0, \tag{1}$$

where Γ defines the Gamma function.

Definition 2 *Liouville Caputo Fractional Derivative:- The fractional derivative of order $\alpha > 0$ for a function f, where f and its derivatives upto order $n-1$ are absolutely continuous, is defined by*

$$D^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, & \text{if } n-1 < \alpha < n, \\ \dfrac{d^n f(x)}{dx^n}, & \alpha = n, \ n \in \mathbb{N}. \end{cases} \tag{2}$$

Riemann-Liouville fractional integral and Liouville-Caputo fractional derivative satisfy the following relations.
For a sufficiently function f and $n-1 < \alpha < n$, $n \in \mathbb{N}$, we have

$$D^\alpha J^\alpha f(x) = f(x), \quad \text{and} \quad D^\alpha f(x) = J^{n-\alpha} D^n f(x), \tag{3}$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0), \quad x > 0. \tag{4}$$

In addition, if $\alpha > 0, \beta > 0$ and $f(x)$ is continuous, then the following result holds

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) = J^\beta J^\alpha f(x). \tag{5}$$

Definition 3 *Mittag-Leffler functions [19]:- The Mittag-Leffler function $E_\alpha(y)$ is denoted as*

$$E_\alpha(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+1)},\qquad y\in\mathbb{R},\quad \alpha>0. \tag{6}$$

3. Fractional Order Volterra Integro Differential Equation with Weakly Singular Kernel

We consider the following fractional order weakly singular Volterra integro differential equation on $\Omega=(0,T]$ with $n-1<\alpha<n$, $n\in\mathbb{N}$, for the analysis:

$$\begin{cases} D^\alpha u(x)=f(x)+\displaystyle\int_0^x (x-t)^{-\mu}K(x,t)u(t)dt, & 0\le\mu<1,\quad x\in\Omega,\\ u^i(0)=\gamma_i,\ i=0,1,\dots,n-1,\ n\in\mathbb{N}. \end{cases} \tag{7}$$

Here, f is considered as a sufficiently smooth function and the kernel $K(x,t)\in C(\overline{\Omega}\times\overline{\Omega})$ is considered to satisfy $|K(x,t)|\le M$ for some constant $M(>0)$ with $K(x,x)\ne 0$ for $x\in\overline{\Omega}$. By means of weakly singular kernel, we note that $(x-t)^{-\mu}K(x,t)$ is a continuous function in $\overline{\Omega}\times\overline{\Omega}$, possibly except for $x=t$. Here, μ defines the order of the weakly singular kernel. We are interested to find an approximation of the solution u defined on $u\in C(\overline{\Omega})$ with suitably smooth given data.

Now, we convert the fractional differential equation to a equivalent integral equation to obtain the existence and uniqueness of the solution of (7), for analysis [7, Lemma 6.2]. Applying J^α on both sides of (7), and using (3) and (4), we obtain

$$u(x)=\sum_{i=0}^{n-1}\frac{x^i}{i!}\gamma_i+J^\alpha\left[f(x)+\int_0^x(x-t)^{-\mu}K(x,t)u(t)dt\right]. \tag{8}$$

Let us write (8) as $\mathcal{B}u(x)=u(x)$, where $\mathcal{B}$ is defined by

$$\mathcal{B}u(x)=h(x)+J^\alpha\left[\int_0^x(x-t)^{-\mu}K(x,t)u(t)dt\right], \tag{9}$$

where

$$h(x)=\sum_{i=0}^{n-1}\frac{x^i}{i!}\gamma_i+J^\alpha f(x)=g(x)+J^\alpha f(x). \tag{10}$$

Now onwards, we denote $\|.\|_\infty$ as $\|.\|$. The following existence and uniqueness theorem provides a sufficient condition for the solution of (7) for suitably smooth data.

Theorem 4 *Under sufficient smoothness on the data, the solution of (7) will exist and unique in $\overline{\Omega}$, if the condition $0\le\dfrac{MT^{1+\alpha-\mu}\Gamma(1-\mu)}{\Gamma(\alpha+2-\mu)}=\beta<1$ is satisfied.*

Proof: Let us first consider $\mathcal{B}:C(\overline{\Omega})\to C(\overline{\Omega})$ such that

$$\mathcal{B}u(x)=h(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left[\int_0^s(s-t)^{-\mu}K(s,t)u(t)dt\right]ds.$$

Assume $u_1,u_2\in C(\overline{\Omega})$. Then,

$$\begin{aligned}&|\mathcal{B}u_1(x)-\mathcal{B}u_2(x)|\\ &=\left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left[\int_0^s(s-r)^{-\mu}K(s,r)(u_1(r)-u_2(r))dr\right]ds\right|\\ &\le\frac{M}{\Gamma(\alpha)}\|u_1-u_2\|\int_0^x(x-s)^{\alpha-1}\left[\int_0^s(s-r)^{-\mu}dr\right]ds\\ &\le\frac{M\Gamma(1-\mu)T^{1+\alpha-\mu}}{\Gamma(\alpha+2-\mu)}\|u_1-u_2\|.\end{aligned}$$

Since $(C(\overline{\Omega}),\|.\|_\infty)$ is a Banach space, therefore, by Banach contraction principle, we can conclude that (7) has a unique solution in $C(\overline{\Omega})$, when

$$0\le\frac{MT^{1+\alpha-\mu}\Gamma(1-\mu)}{\Gamma(\alpha+2-\mu)}=\beta\ (\text{say})<1. \tag{11}$$

Remark 1 *For any choice of $K(x,t)$ whose bound satisfies $M<\dfrac{\Gamma(\alpha+2-\mu)}{T^{1+\alpha-\mu}\Gamma(1-\mu)}$ for all $x\in[0,T]$, the unique solution of the model problem (7) will be defined in whole interval $[0,T]$.*

Now, we discuss the existence and uniqueness of the solution of (7) by considering weighted maximum norm (see, for e.g., [1]) which is defined by $\|u\|_w = \max_{x\in[0,T]} |e^{-x}u(x)|$. Since $e^{-T}\|u\| \leq \|u\|_w \leq \|u\|$, the norm $\|.\|_w$ is equivalent to maximum norm. In addition, $C(\overline{\Omega})$ is a Banach space with respect to $\|.\|_w$, see [1]. If we show that the operator $\mathcal{B}$ in (9), is a contraction on $(C(\overline{\Omega}), \|.\|_w)$, we can conclude that equation (7) has a unique solution $u(x)$ on $[0, T]$.

Theorem 5 *Under sufficient smoothness on the data, the solution of (7) will exist and unique in $\overline{\Omega}$, if the condition $M < \frac{1}{\Gamma(1-\mu)}$ is satisfied.*

Proof: Assume $u, v \in C(\overline{\Omega})$. Then, from (9), we have

$$|\mathcal{B}u(x) - \mathcal{B}v(x)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}K(s,r)(u(r)-v(r))dr\right]ds\right|.$$

Therefore

$$\begin{aligned}
&e^{-x}|\mathcal{B}u(x) - \mathcal{B}v(x)| \\
&= e^{-x}\left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}K(s,r)(u(r)-v(r))dr\right]ds\right| \\
&\leq \frac{Me^{-x}}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}e^{-r}|u(r)-v(r)|e^r dr\right]ds \\
&\leq \frac{Me^{-x}\|u-v\|_w}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}e^r dr\right]ds \\
&= Me^{-x}\Gamma(1-\mu)\|u-v\|_w\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\frac{1}{\Gamma(1-\mu)}\int_0^s (s-r)^{1-\mu-1}e^r dr\right)ds\right) \\
&= Me^{-x}\Gamma(1-\mu)\|u-v\|_w\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(J^{1-\mu}(e^s))ds\right) \\
&= Me^{-x}\Gamma(1-\mu)\|u-v\|_w J^{\alpha}(J^{1-\mu}(e^x)) \\
&= \frac{Me^{-x}\Gamma(1-\mu)\|u-v\|_w}{\Gamma(1+\alpha-\mu)}\int_0^x (x-s)^{\alpha-\mu}e^s ds.
\end{aligned}$$

By change of variable $x - s = y$, we have

$$\begin{aligned}
e^{-x}|\mathcal{B}u(x) - \mathcal{B}v(x)| &\leq \frac{M\Gamma(1-\mu)\|u-v\|_w}{\Gamma(1+\alpha-\mu)}\int_0^x y^{\alpha-\mu}e^{-y}dy \\
&= M\Gamma(1-\mu)\|u-v\|_w\frac{1}{\Gamma(1+\alpha-\mu)}\int_0^x y^{1+\alpha-\mu-1}e^{-y}dy \\
&= M\Gamma(1-\mu)\frac{\gamma(1+\alpha-\mu, x)}{\Gamma(1+\alpha-\mu)}\|u-v\|_w.
\end{aligned}$$

Since, $\dfrac{\gamma(1+\alpha-\mu, x)}{\Gamma(1+\alpha-\mu)} \leq 1$ for $0 < \mu < 1$, $\alpha > 0$ and for all $x \in \overline{\Omega}$, (see [17]) we have

$$e^{-x}|\mathcal{B}u(x) - \mathcal{B}v(x)| \leq M\Gamma(1-\mu)\|u-v\|_w.$$

This implies

$$\|\mathcal{B}u(x) - \mathcal{B}v(x)\|_w \leq M\Gamma(1-\mu)\|u-v\|_w.$$

Since $(C(\overline{\Omega}), \|.\|_w)$ is a Banach space, therefore, by Banach contraction principle, we can conclude that (7) has a unique solution in $C(\overline{\Omega})$, when

$$M < \frac{1}{\Gamma(1-\mu)}, \tag{12}$$

which is the desired result.

In this above theorem, we discuss existence and uniqueness of the solution for our model problem on $[0, T]$ with respect to weighted maximum norm. Now, we consider this result with respect to maximum norm by dividing the whole domain into a partition. We use the idea of the proof [20, Theorem 3.2] for Volterra integral equations of second kind. First, we discuss a technical lemma which will be used to prove the existence and uniqueness of u on $[0, T]$.

Lemma 1 *Consider $q(x,t) = \dfrac{(x-t)^{\alpha-\mu}}{\Gamma(1+\alpha-\mu)}$, where $\alpha > 0$ and $0 \leq \mu < 1$. This function satisfies the following properties:*
(I). For $x \geq 0$, $q(x,t)$ is an absolute integrable function with respect to t on $[0, x]$ and

$$\lim_{\delta\to 0^+}\int_x^{x+\delta}|q(x+\delta, t)|dt = 0.$$

(II). Let $g(x)$ be a continuous function defined on $[0, T]$ and $0 \leq \zeta_1 \leq \zeta_2 \leq x$, then

$$\int_{\zeta_1}^{\zeta_2} q(x,t)K(x,t)g(t)dt$$

and

$$\int_0^x q(x,t)K(x,t)g(t)dt$$

are continuous on $[0, T]$.
(III). There exists a partition, based on μ in the domain $[0, T]$, say $0 = T_0 < T_1 < T_2 < \cdots < T_N = T$, so that for all $x \in [T_j, T]$ with $j = 0, 1, 2, \cdots, N-1$, we have

$$\widehat{M} \int_{T_j}^{\min(x, T_{j+1})} |q(x,t)|dt \leq \theta < 1,$$

for some constant $\theta < 1$, where $\widehat{M} = M\Gamma(1-\mu)$ and M is defined earlier.

Proof: The idea of the proof is similar to [7, Lemma 6.10].

Theorem 6 *Under sufficient smoothness on the data and the assumptions given in Lemma 1, the solution of (7) will exist and unique on $\overline{\Omega}$.*

Proof: From (8) and (10), we have

$$u(x) = h(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} K(s,t)u(t)dt \right] ds. \tag{13}$$

We start with the partition $0 = T_0 < T_1 < T_2 < \cdots < T_N = T$, given in (III) of Lemma 1. First we show the existence of a unique solution of (7) in $[T_0, T_1]$ by adopting the technique given in [7, 20]. Then, we extend the unique solution in whole interval $[0, T]$ by induction. Let us define

$$u_m(x) := h(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} K(s,t)u_{m-1}(t)dt \right] ds, \tag{14}$$

for $m = 1, 2, 3, \cdots$, and $u_0(x) := h(x)$. It is easy to see that the above functions are continuous on $[T_0, T_1]$ by Lemma 1 (II) (see also the procedure below, given in (16)). Let us define $\Phi_m(x) := u_m(x) - u_{m-1}(x)$ for $m = 1, 2, 3, \cdots$, and $\Phi_0(x) := u_0(x) = h(x)$. Therefore, $u_m(x)$ can be written as $u_m(x) = \sum_{l=0}^{m} \Phi_l(x)$. Again, for $m = 2, 3, 4, \cdots$, we have

$$\Phi_m(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} K(s,t)(u_{m-1}(t) - u_{m-2}(t))dt \right] ds. \tag{15}$$

For a partition $[T_0, T_1]$ with $x \in [T_0, T_1]$, we obtain

$$\begin{aligned}
|\Phi_m(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} K(s,t)(u_{m-1}(t) - u_{m-2}(t))dt \right] ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} |K(s,t)||(u_{m-1}(t) - u_{m-2}(t))|dt \right] ds \\
&\leq \frac{M\|\Phi_{m-1}\|}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{-\mu} dt \right] ds \\
&= \frac{\widehat{M}\|\Phi_{m-1}\|}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\frac{1}{\Gamma(1-\mu)} \int_0^s (s-t)^{1-\mu-1} dt \right] ds \\
&= \frac{\widehat{M}\|\Phi_{m-1}\|}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (J^{1-\mu}(s^0)) ds \\
&= \widehat{M}\|\Phi_{m-1}\| J^{\alpha}(J^{1-\mu}(x^0)) \\
&= \|\Phi_{m-1}\| \frac{\widehat{M}}{\Gamma(1+\alpha-\mu)} \int_0^x (x-t)^{\alpha-\mu} dt,
\end{aligned} \tag{16}$$

where the last inequality follows from (5). Using Lemma 1 (III) on $[T_0, T_1]$, and taking norm on left side of the above equation, we get the following form

$$\|\Phi_m\| \leq \theta \|\Phi_{m-1}\|,$$

for $m = 2, 3, 4, \ldots,$ and for some positive constant $\theta < 1$. In addition, for $m = 1$, we have

$$\begin{aligned} |\Phi_1(x)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)\Phi_0(t)dt\right]ds\right| \\ &\leq \|\Phi_0\|\frac{M}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}dt\right]ds \\ &= \|\Phi_0\|\frac{\widehat{M}}{\Gamma(1+\alpha-\mu)}\int_0^x (x-t)^{\alpha-\mu}dt \\ &\leq \theta\|\Phi_0\|. \end{aligned}$$

This implies $\|\Phi_m\| \leq \theta\|\Phi_{m-1}\|$ holds true for $m = 1$. Therefore, for $m = 1, 2, 3, \ldots.,$ we obtain

$$\|\Phi_m\| \leq \theta^m\|\Phi_0\|.$$

Since $\theta < 1$, the series $\sum_{l=0}^{\infty}\Phi_l(x)$ is convergent on $[T_0, T_1]$ and the convergence is uniform. Therefore, mth partial sum of $\sum_{l=0}^{\infty}\Phi_l(x)$, i.e., $\{u_m(x)\}$ is uniformly convergent and converges to a function, say $v(x)$, on $[T_0, T_1]$. In addition, continuity of $u_m(x)$ implies $v(x)$ is continuous on $[T_0, T_1]$. Since, $\{u_m(x)\}$ is uniformly convergent, therefore, for $m \to \infty$, we have from (14)

$$\begin{aligned} v(x) &= \lim_{m\to\infty} u_m(x) \\ &= \lim_{m\to\infty}\left(h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)u_{m-1}(t)dt\right]ds\right) \\ &= h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)\lim_{m\to\infty}(u_{m-1}(t))dt\right]ds \\ &= h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)v(t)dt\right]ds. \end{aligned}$$

This implies v is a solution of (13) and hence there exists a solution for the model equation (7) on $[T_0, T_1]$. For uniqueness of the solution, let us consider $u(x)$ as another solution on $[T_0, T_1]$. Then, for $u, v \in C[T_0, T_1]$,

$$\begin{aligned} |u(x) - v(x)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)(u(t)-v(t))dt\right]ds\right| \\ &\leq M\|u-v\|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}dt\right]ds \\ &= \|u-v\|\frac{\widehat{M}}{\Gamma(1+\alpha-\mu)}\int_0^x (x-t)^{\alpha-\mu}dt \\ &\leq \theta\|u-v\|. \end{aligned}$$

Here we have followed the procedure, used in (16). Taking norm on left side of the above equation, we get

$$\|u-v\| \leq \theta\|u-v\|.$$

Since, $\theta < 1$, this implies $u \equiv v$. Hence, the solution of (7) exists and unique on $[T_0, T_1]$. Now, we show that the existence and uniqueness of the solution in $[0, T]$. For this, let us assume that the result holds true on $[T_{j-1}, T_j]$ for $j < N$. We prove that this is also the case on $[T_j, T_{j+1}]$. For $x \in [T_j, T_{j+1}]$, the solution can be written as

$$u(x) = h_j(x) + \frac{1}{\Gamma(\alpha)}\int_{T_j}^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)u(t)dt\right]ds, \tag{17}$$

where

$$h_j(x) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^{T_j} (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)u(t)dt\right]ds. \tag{18}$$

Note that the function $h_j(x)$ is a known function as the solution is known in the interval $[0, T_j]$. In addition, the solution is continuous on $[0, T_j]$ and $h(x)$ is also continuous on $[0, T_j]$. Therefore, Lemma 1 (II) follows that $h_j(x)$ is a continuous function in this domain. Hence, as in (14), we define the following function on $[T_j, T_{j+1}]$

$$u_m^j(x) := h_j(x) + \frac{1}{\Gamma(\alpha)}\int_{T_j}^x (x-s)^{\alpha-1}\left[\int_0^s (s-t)^{-\mu}K(s,t)u_{m-1}^j(t)dt\right]ds, \tag{19}$$

for $m = 1, 2, 3, \cdots$, and $u_0^j(x) := h_j(x)$ and $\Phi_m^j(x) := u_m^j(x) - u_{m-1}^j(x)$ for $m = 1, 2, 3, \cdots$, and $\Phi_0^j(x) := h_j(x)$. Therefore, $u_m^j(x)$ can be written as $u_m^j(x) = \sum_{l=0}^{m}\Phi_l^j(x)$. Hence, for $x \in [T_j, T_{j+1}]$, by proceeding same as before, we obtain a uniformly convergent series

$\sum_{l=0}^{\infty}\Phi_l^j(x)$ whose mth partial sum will converge and leads to a unique continuous solution $\hat{u}(x)$ where $\hat{u}(x)=\lim_{m\to\infty}u_m^j(x)$ on $[T_j,T_{j+1}]$. Now, from (19) and Lemma 1 (I), we have

$$\lim_{x\to T_j+}\hat{u}(x)=h_j(T_j).$$

Since $u(x)$ is defined on $[0,T_j]$, we have from (13)

$$\lim_{x\to T_j-}u(x)=h_j(T_j).$$

This shows that the solution is continuous at T_j. Therefore, by continuous continuation, the solution exists and unique in whole domain $[0,T]$. Hence, the proof is complete.

From above theorem, we can obtain a continuous solution throughout the interval of definition. Now we produce the following result under which an analytic solution can be obtained.

Lemma 2 *Let $f(x)$ and $K(x,t)$ are analytic in $\bar{\Omega}$ and $\bar{\Omega}\times\bar{\Omega}$ respectively. Then, the solution $u(x)$ of (7) is analytic if and only if*

$$\int_0^x (x-t)^{\alpha-1}\left(f(t)+\left[\int_0^t (t-r)^{-\mu}K(t,r)g(r)dr\right]dt\right)=0, \tag{20}$$

where $g(x)$ is given in (10).

Proof: The idea of the proof is similar to [7]. From (8), we have

$$u(x)=g(x)+J^{\alpha}\left[f(x)+\int_0^x (x-t)^{-\mu}K(x,t)u(t)dt\right], \tag{21}$$

where $g(x)=\sum_{i=0}^{n-1}\frac{x^i}{i!}\gamma_i$ is analytic in $\bar{\Omega}$.

From (20), we have $J^{\alpha}\left[f(x)+\int_0^x (x-t)^{-\mu}K(x,t)u(t)dt\right]=0$. This implies $u(x)=g(x)$ is an analytic solution of (7), since $g(x)$ is analytic. For the converse, let us assume that $u(x)$ is analytic. We show that this follows the condition (20). Since $f(x)$ is analytic at $x=0$, it can be represented by a power series expansion at $x=0$, i.e.,

$$f(x)=\sum_{j=0}^{\infty}c_jx^j.$$

Let us define a function $r:\bar{\Omega}\times\bar{\Omega}\to\mathbb{R}$ such that $r(x,t)=K(x,t)u(t)$. Since $K(x,t)$ is analytic at $(0,0)$ and $u(t)$ is also analytic, this follows $r(x,t)$ is analytic at $(0,0)$. So, $r(x,t)$ can be represented as

$$r(x,t)=\sum_{k,l=0}^{\infty}c_{kl}x^kt^l.$$

Now, from (21), we have

$$\begin{aligned}
u(x)-g(x)&=J^{\alpha}\left[f(x)+\int_0^x (x-t)^{-\mu}r(x,t)dt\right]\\
&=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\sum_{j=0}^{\infty}c_js^j+\int_0^s (s-r)^{-\mu}\sum_{k,l=0}^{\infty}c_{kl}s^kr^ldr\right]ds\\
&=\sum_{j=0}^{\infty}\frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)}c_jx^{\alpha+j}+\frac{1}{\Gamma(\alpha)}\sum_{k,l=0}^{\infty}c_{kl}\int_0^x (x-s)^{\alpha-1}s^k\left[\int_0^s (s-r)^{-\mu}r^ldr\right]ds\\
&=x^{\alpha}\sum_{j=0}^{\infty}\frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)}c_jx^j+x^{1+\alpha-\mu}\sum_{k,l=0}^{\infty}c_{kl}x^{k+l}\frac{\Gamma(l+1)\Gamma(1-\mu)}{\Gamma(2+l-\mu)}\frac{\Gamma(2-\mu+k+l)}{\Gamma(2+\alpha-\mu+k+l)}.
\end{aligned} \tag{22}$$

At the point $x=0$, the left side of the above equation is analytic. Therefore, right hand side must be analytic. Again, for a fractional number $n-1<\alpha<n$, $n\in\mathbb{N}$ and $0\le\mu<1$, right hand side of the above equation cannot be analytic unless the series terms are identically zero. This implies

$$\int_0^x (x-t)^{\alpha-1}\left(f(t)+\left[\int_0^t (t-r)^{-\mu}K(t,r)u(r)dr\right]\right)dt=0,$$

and hence, $u(x)=g(x)$ is a solution of (7). Combining both of the above results, we obtain the required condition.

Now we introduce the HPM which will be used to approximate the solution of (7). This analytical method attaches a continuous mapping between exact and approximate solutions and uses the traditional perturbation analysis to obtain a suitable approximation of the solution within few iterations. Here we discuss a modified version of HPM based on least square approach, and produce a residual based error analysis. Numerically it is observed that the modified HPM is leading to a better accuracy of the solution, compared to HPM. In particular, numerical experiments suggest that the modified approach also requires less number of iterations compared to HPM.

4. Solution Approximation

To define the procedure of finding an approximated solution, let us reformulate the model (7) to the following form

$$\mathrm{L}u + \mathcal{R}u = f(x), \quad \text{for} \quad x \in \Omega. \tag{23}$$

Here L defines a linear fractional order differential operator, $\mathcal{R}$ defines a linear integral operator, $f(x)$ is a sufficiently smooth known function and L satisfies

$$\mathrm{L}v = 0, \quad \text{for} \quad v = 0. \tag{24}$$

Now we construct a homotopy $\mathrm{H}(W, p) : \mathcal{F} \times [0,1] \to \mathbb{R}$, denoted by $\mathrm{H} := \mathrm{H}(W, p)$ for $W \in \mathcal{F}$ and $p \in [0,1]$. Here $\mathcal{F} \equiv \{W(x,p) | (x,p) \in \Omega \times [0,1]\}$ defines a sufficiently smooth function space. We consider the homotopy H as

$$\mathrm{H}(W, p) = (1-p)[\mathrm{L}(W) - \mathrm{L}(u_0)] + p[\mathrm{L}(W) + \mathcal{R}(W) - f(x)] = 0, \quad \text{for} \quad x \in \Omega. \tag{25}$$

Here, $p \in [0,1]$ is an embedding parameter and $u_0 = u_0(x)$ is an initial chosen approximation of the solution of (23) which can be obtained by using the initial condition in (7) and $\mathrm{L}(u) = f(x)$.

From (25), note that $p = 0$ and $p = 1$ lead to

$$\mathrm{H}(W, p)|_{p=0} = \mathrm{L}(W) - \mathrm{L}(u_0) \text{ and } \mathrm{H}(W, p)|_{p=1} = \mathrm{L}(W) + \mathcal{R}(W) - f(x). \tag{26}$$

Therefore, from (24), we can say that $W(x,0) = u_0(x)$ and $W(x,1) = u(x)$ are the solutions of the equation $\mathrm{H}(W,p)|_{p=0} = 0$ and $\mathrm{H}(W,p)|_{p=1} = 0$ respectively.

Now we can relate this approach in topological sense. In topology, it is possible to continuously deform one continuous function, say $W(x,0)$, to another continuous function $W(x,1)$ by means of homotopy. In the present case, we expect that the initial approximation $u_0(x)$ will deform to the original solution $u(x)$ when p tends from 0 to 1, i.e., $W(x,p)$ approaches to the solution $u(x)$ from the initial approximation u_0, see for e.g., [18].
Now, considering $W(x,p)$ as a smooth function of p, we can write $W(x,p)$ in the following series form of p:

$$W(x,p) = \sum_{j=0}^{\infty} p^j W_j(x). \tag{27}$$

We can obtain $W_j(x)$ by substituting (27) in (25) and equating the powers of p. Therefore, we get the following relation

$$\begin{cases} W_0(x) = u_0(x) = \displaystyle\sum_{i=0}^{n-1} \frac{x^i}{i!}\gamma_i + \mathrm{L}^{-1} f(x), \\ W_j(x) = -\mathrm{L}^{-1}(\mathcal{R}W_{j-1}), \quad j \geq 1. \end{cases} \tag{28}$$

Now, if the series at (27) is uniformly convergent in p, then we can write

$$u(x) = \lim_{p \to 1-} W(x,p) = \sum_{j=0}^{\infty} W_j(x). \tag{29}$$

The approximate solution $\Phi_n(x)$ is defined by $\Phi_n(x) = \sum_{j=0}^{n-1} W_j(x)$.

Now we produce the convergence analysis of the above series in the next subsection.

4.1. Convergence Analysis

Here, we show that the series (27) and (29) are uniformly convergent.

Lemma 3 *Let the function $K(x,t) \in C(\overline{\Omega} \times \overline{\Omega})$ bounded by $M(>0)$ for all $(x,t) \in \overline{\Omega} \times \overline{\Omega}$ and $f(x) \in C(\overline{\Omega})$. In addition, assume that the initial approximation u_0 is continuous on $\overline{\Omega}$ and $0 \leq \beta < 1$ in (11). Then, the series in (29) is uniformly convergent in $\overline{\Omega}$.*

Proof: First, we prove absolute convergence of the series in (29) by producing a convergent geometric series as its upper bound.

By above assumptions u_0 is continuous in $\overline{\Omega}$, therefore there exist $\delta > 0$, such that $|u_0| \le \delta$ for all $x \in \overline{\Omega}$. Using the stated assumptions, we obtain the following estimates

$$\begin{aligned}
&|W_0(x)| = |u_0(x)| \le \delta,\\
&|W_1(x)| = |J^\alpha \int_0^x |x-t|^\mu K(x,t) W_0(x) dt| \le \frac{\delta M \Gamma(1-\mu) T^{1+\alpha-\mu}}{\Gamma(2+\alpha-\mu)} = \delta\beta,\\
&|W_2(x)| = |J^\alpha \int_0^x |x-t|^\mu K(x,t) W_1(x) dt| \le \frac{\delta M \beta \Gamma(1-\mu) T^{1+\alpha-\mu}}{\Gamma(2+\alpha-\mu)} = \delta\beta^2.
\end{aligned}$$

By induction, the j-th term of the series (29) satisfies

$$|W_j(x)| \le \delta\beta^j.$$

Therefore, for all $x \in \overline{\Omega}$,

$$\sum_{j=0}^{\infty} |W_j(x)| \le \sum_{j=0}^{\infty} \delta\beta^j.$$

For $0 < \beta < 1$, $\sum_{j=0}^{\infty} \delta\beta^j$ is a convergent geometric series. Therefore, by Weierstrass M-test, we conclude that $\sum_{j=0}^{\infty} W_j(x)$ converges uniformly on $\overline{\Omega}$. Using the above Lemma 3, we can obtain the uniform convergence of the series in (27).

Theorem 7 *For all $p \in [0,1]$, the assumptions stated in Lemma 3 implies that the series (27) is uniformly convergent in $\overline{\Omega}$. In particular, an approximate solution of (7) can be obtained from (29).*

Proof: For all $p \in [0,1]$ and for all $x \in \overline{\Omega}$, note that

$$\sum_{j=0}^{\infty} p^j W_j(x) \le \sum_{j=0}^{\infty} |W_j(x)|.$$

Hence, by Lemma 3, the series $\sum_{j=0}^{\infty} |W_j(x)|$ converges uniformly on $\overline{\Omega}$. Therefore, by Weierstrass M-test, we obtain the uniform convergence of (27) on $\overline{\Omega}$. Hence, we can write $\lim_{p\to 1^-} W(x,p) = \sum_{j=0}^{\infty} W_j(x)$. Again, from (26), we have $u(x) = \lim_{p\to 1^-} W(x,p)$. Now, by combining these estimates, we get an approximate solution of (7).

Note 8 *Let $\sum_{j=0}^{\infty} W_j(x)$ be the solution of (7). In practice, we only take a finite number of terms of the series (29), say N terms, to approximate the exact solution by a partial sum of (29). Note that the upper bound of the absolute error, based on this partial sum will be given by $\frac{\delta\beta^N}{1-\beta}$ (see Lemma 3), where β is noted in Theorem 4 and δ is defined in Lemma 3.*

Remark 2 *In addition, if ϵ defines a tolerable error based on the partial sum up to N terms, then we can provide an upper bound of N, which is*

$$N \ge \left\lfloor \frac{\ln(\epsilon(1-\beta)/\delta)}{\ln(\beta)} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ defines the nearby least integer of x, β is mentioned in Theorem 4 and δ is defined in Lemma 3.

4.2. Least square homotopy perturbation method

Now, we discuss a modified version of HPM which can accelerate the convergence compared to the standard HPM. The method is constructed by combing the HPM and least square method and known as least square homotopy perturbation method [4]. Here we observe that the modified HPM provides less residual error compared to standard HPM. In addition, we also obtain an estimate of the approximation error based on the residual error. First, we obtain the approximate solution $\Phi_n(x)$ by taking a finite partial sum of (29). Let us assume that it contains $m+1$ number of linearly independent functions $\{\Psi_{n0}, \Psi_{n1}, \cdots, \Psi_{nm}\}$ in the vector space of continuous functions on $[0,T]$. Now, consider a set $\mathbf{S}_n$ $(n = 1, 2, \cdots)$ which contains the linear combinations of the functions $\{\Psi_{n0}, \Psi_{n1}, \cdots, \Psi_{nm}\}$ such that $\Phi_n(x)$ can be written by a linear combination of these functions and $\mathbf{S}_{n-1} \subseteq \mathbf{S}_n$.

For $n \ge 0$, let us assume $\bar{\Phi}_n(x) = \sum_{k=0}^{m} C_n^k \Psi_{nk}$ be the approximate solution of (7) where C_n^k's are unknown constants. Now, we calculate the optimal value of the constants C_n^k by using least square residual minimization method. We do it by the following algorithm:

(i) First, calculate the residual function by substituting $\bar{\Phi}_n(x)$ at (23):

$$R(x,\bar{\Phi}_n(x)) \equiv \mathbf{L}\bar{\Phi}_n(x) + \mathcal{R}\bar{\Phi}_n(x) - f(x).$$

(ii) Now, Construct a functional

$$J(C_n^k) = \int_0^T R^2(x,\bar{\Phi}_n(x))dx.$$

(iii) Minimize the functional defined in (ii) by computing C_n^k by solving the system of equations $\dfrac{\partial J}{\partial C_n^k} = 0$.

(iv) Use the calculated values C_n^k to obtain the approximate solution $\bar{\Phi}_n(x)$.

Lemma 4 *Let $\bar{\Phi}_n(x)$ be an approximation of (7). Then, it satisfies*

$$\lim_{n\to\infty}\int_0^T R^2(x,\bar{\Phi}_n(x))dx = 0.$$

Proof: *By the construction of $\bar{\Phi}_n(x)$ we have*

$$\int_0^T R^2(x,\bar{\Phi}_n(x))dx \le \int_0^T R^2(x,\Phi_n(x))dx.$$

Since, $\Phi_n(x)$ is a convergent solution of (7), we have

$$\lim_{n\to\infty}\int_0^T R^2(x,\Phi_n(x))dx = 0.$$

Therefore,

$$0 \le \lim_{n\to\infty}\int_0^T R^2(x,\bar{\Phi}_n(x))dx \le \lim_{n\to\infty}\int_0^T R^2(x,\Phi_n(x))dx = 0.$$

This implies that the desired result holds true.

Error Analysis

By Lemma 4, for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > N$, $n \in \mathbb{N}$, it follows that $|(R(x,\bar{\Phi}_n(x))| < \epsilon$. This implies

$$\left| D^\alpha\bar{\Phi}_n(x) - f(x) - \int_0^x (x-t)^{-\mu}K(x,t)\bar{\Phi}_n(t)dt \right| < \epsilon.$$

For $x > t$, the integral operator is monotonically increasing. Hence, from the above equation, we get

$$\left| \bar{\Phi}_n(x) - h(x) - J^\alpha\left[\int_0^x (x-t)^{-\mu}K(x,t)\bar{\Phi}_n(t)dt\right] \right| < J^\alpha(\epsilon) \le \epsilon_1 \text{ say,} \tag{30}$$

where $\epsilon_1 = \dfrac{\epsilon T^\alpha}{\Gamma(\alpha+1)}$. Let $u(x)$ be the solution of the model equation (7). Then from equation (13) and (30), we have for all $x \in [0,T]$

$$\begin{aligned}
&\left| (\bar{\Phi}_n(x) - u(x)) - \left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}K(s,r)(u(r)-\bar{\Phi}_n(r))dr\right]ds \right) \right| < \epsilon_1 \\
&\Rightarrow |u(x) - \bar{\Phi}_n(x)| < \left| \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\int_0^s (s-r)^{-\mu}K(s,r)(u(r)-\bar{\Phi}_n(r))dr\right]ds \right| + \epsilon_1 \\
&\Rightarrow |u(x) - \bar{\Phi}_n(x)| < \epsilon_1 + \frac{M\Gamma(1-\mu)T^{1+\alpha-\mu}}{\Gamma(\alpha+2-\mu)}\|u-\bar{\Phi}_n\| = \epsilon_1 + \beta\|u-\bar{\Phi}_n\|.
\end{aligned}$$

This implies

$$\|u-\bar{\Phi}_n\| < \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)(1-\beta)}.$$

Also, we have

$$\int_0^T |(u(x)-\bar{\Phi}_n(x))|^2 dx \le T\|u-\bar{\Phi}_n\|^2.$$

Error with respect to L^2 norm is given by

$$\|u-\bar{\Phi}_n\|_2 \le \sqrt{T}\|u-\bar{\Phi}_n\| < \frac{\epsilon T^{\alpha+1/2}}{\Gamma(\alpha+1)(1-\beta)}.$$

5. Computational Experiments

In this section, we produce several examples in favor of the convergence of the present method. These examples satisfy the existence and uniqueness result from Theorem 4. Before going to the computation, we provide the key steps of HPM by the following algorithm. This algorithm can be modified for LSHPM appropriately.

5.1. Algorithm

Step 1. Fix ϵ as an user chosen desired accuracy. Find N from Remark 2.

Step 2. Find W_j by

$$\begin{cases} W_0(x) = u_0(x), \\ W_j(x) = -\mathbf{L}^{-1}(\mathcal{R}W_{j-1}), \quad j \geq 1. \end{cases}$$

for $j = 0, 1, \cdots, n$ for some positive number $n \leq N$ where $\mathbf{L}$ and $\mathcal{R}$ are defined in (23) and $u_0(x)$ is defined in (28).

Step 3. Consider $\Phi_0(x) = 0$. Now, compute the approximate solution by $\Phi_j(x) = \Phi_{j-1}(x) + W_{j-1}(x)$ for $j = 1, \cdots, n-1$. $\Phi_n(x)$ is our desired approximate solution.

The mathematical model of Heat transfer problem is given by

$$\begin{aligned} &\frac{\partial T(x,t)}{\partial t} = A\frac{\partial^2 T(x,t)}{\partial^2 x}, \quad x \in (0,\infty), \quad t > 0, \\ &T(x,0) = 0, \quad x \in (0,\infty), \end{aligned} \tag{31}$$

with boundary conditions

$$\begin{aligned} &K\frac{\partial T(0,t)}{\partial x} = H[T(0,t) - T_\infty], \\ &\lim_{x\to\infty} T(x,t) = 0, \quad t > 0. \end{aligned} \tag{32}$$

Here, $T(0,t)$ is the surface temperature, T_∞ is the temperature of surrounding medium, H is the convection coefficient. We assume thermal conductivity K and thermal diffusivity A are constant. The boundary condition

$$K\frac{\partial T(0,t)}{\partial x} = H[T(0,t) - T_\infty]$$

represents the heat flux across the surface, which is proportional to the temperature between the surface and surrounding medium.

Equation (31) can be written as an equivalent weakly singular integral form by using Fourier cosine transform. For more details, see [14]. The solution of the equation (31) is given by

$$T(x,t) = -\lambda \int_0^t H(t-s)^{-1/2}[T(0,s) - T_\infty] \times exp\left(\frac{-x^2}{4A(t-s)}\right)ds, \tag{33}$$

where $\lambda = \frac{1}{K}\sqrt{\frac{A}{\pi}}$. If one evaluates the equation (33) by setting $x = 0$, the dimension of (33) would reduce. Therefore, at $x = 0$, this equation can be written as:

$$T(t) = f(t) - \lambda \int_0^t H(t-s)^{-1/2} T(s)ds, \tag{34}$$

$$\text{where } f(t) = \lambda \int_0^t HT_\infty(t-s)^{-1/2}ds.$$

Here, $T(t) = T(0,t)$ is the unknown surface temperature which can be easily obtained from (34). Since, $T(x_i,t)$ only depends on the surface temperature, we can obtain the temperature at any location of the domain from (33). In the following example, we are interested to obtain the surface temperature of the environment by considering an equivalent fractional order weakly singular integral equation.

Example 1 *Consider:*

$$\begin{cases} D^{\frac{3}{4}}T(t) = f(t) - \lambda \int_0^t H(t-s)^{-1/2} T(s)ds, \; 0 < t \leq 1, \\ T(0) = 0, \end{cases} \tag{35}$$

$$\textit{where } f(t) = \lambda \int_0^t HT_\infty(t-s)^{-1/2}ds.$$

In our case, we will obtain the surface temperature for the given value of $H = 1$, $\lambda = 1$ and $T_\infty = 25°C$. The exact solution of Example 1 is unknown. The approximate solution of Example 1 is $\Phi_n(t)$, defined by $\Phi_{n_1}(t) = \sum_{m=0}^{n_1-1} W_m(t)$. Using the initial condition and by equating the coefficient of $O(p^j)$, we obtain the following results

$$O(p^0): D^{3/4}(W_0(t)) = f(t) \quad \Longrightarrow \quad W_0(t) = \frac{25\sqrt{\pi}}{\Gamma(9/4)} t^{5/4}.$$
$$O(p^1): D^{3/4}(W_1(t)) + \int_0^t (t-s)^{-1/2} W_0(s)ds = 0 \quad \Longrightarrow \quad W_1(t) = -\frac{40\sqrt{\pi}}{3} t^{5/2}.$$

Similarly, we can obtain the next terms of the series (29) by the relation

$$D^{3/4}(W_{i+1}(t)) + \int_0^t (t-s)^{-1/2} W_i(s)ds = 0, \text{ for } i = 1, 2, \cdots.$$

Now, we produce the absolute residual error $E^\infty_{n_1}(t)$ to show the effectiveness of our present method. It is defined as follows:

$$E^\infty_{n_1}(t) = |D^{3/4}(\Phi_{n_1}(t)) + \int_0^t (t-s)^{-1/2}\Phi_{n_1}(s)ds - f(t)|.$$

These errors are given at Table 1.

t	$E^\infty_8(t)$	$E^\infty_9(t)$	t	$E^\infty_8(t)$	$E^\infty_9(t)$
0.1	1.24345E-14	3.55271E-15	0.6	1.69893E-06	7.40958E-08
0.2	1.66125E-11	1.81188E-13	0.7	8.57271E-06	4.53334E-07
0.3	1.17317E-09	2.15081E-11	0.8	3.48364E-05	2.17683E-06
0.4	2.40557E-08	6.31999E-10	0.9	1.19987E-04	8.68692E-06
0.5	2.50480E-07	8.69788E-09	1	3.62734E-04	2.99582E-05

Table 1. Absolute residue errors of Example 1

Example 2 *First, consider the Volterra integro-differential equation of arbitrary fractional order α:*

$$\begin{cases} D^\alpha u(x) = \int_0^x u(t)dt, & \alpha \in (0,1), \ \textit{and} \ x \in (0,1], \\ u(0) = 1. \end{cases} \tag{36}$$

By considering $\mathbf{L}$ as the fractional operator at (25), we obtain the following relations by using initial condition and by equating the coefficient of $O(p^j)$

$$O(p^0): \mathbf{L}(W_0(x)) = 0 \quad \Longrightarrow \quad W_0(x) = u(0) = 1.$$
$$O(p^1): \mathbf{L}(W_1(x)) - \int_0^x W_0(t)dt = 0 \quad \Longrightarrow \quad W_1(x) = \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$O(p^m): \mathbf{L}(W_m(x)) - \int_0^x W_{m-1}(t)dt = 0 \quad \Longrightarrow \quad W_m(x) = \frac{x^{m+m\alpha}}{\Gamma(m+1+m\alpha)}.$$

Using the above terms, we can write from (29)

$$u(x) = 1 + \tfrac{x^{1+\alpha}}{\Gamma(2+\alpha)} + \tfrac{x^{2+2\alpha}}{\Gamma(3+2\alpha)} + \tfrac{x^{3+3\alpha}}{\Gamma(4+3\alpha)} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n(1+\alpha)}}{\Gamma(n(1+\alpha)+1)}$$
$$= E_{(1+\alpha)}(x^{1+\alpha}),$$

where $E_\alpha(x)$ is the Mittag-Leffler function [19].

Example 3 *Now, consider the following fractional problem with weakly singular kernel:*

$$\begin{cases} D^{\frac{3}{4}} u(x) = f(x) - \frac{1}{20}\int_0^x (2+3x+5t)(x-t)^{-1/2}u(t)dt, \ 0 < x \le 1, \\ u(0) = 0, \end{cases} \tag{37}$$

$$\textit{where } f(x) = \frac{x^{1/4}}{\Gamma(5/4)} + \frac{\Gamma(9/2)}{\Gamma(15/4)}x^{11/4} + \frac{2x^{3/2}}{15} + \frac{7x^{5/2}}{15} + \frac{7\pi x^4}{256} + \frac{105\pi x^5}{1024}.$$

The exact solution of (37) is $u(x) = x + x^{7/2}$. Now, we use HPM described in Section 4 to solve the above problem. Using (25), the initial condition and equating the coefficient of $O(p^j)$, we obtain the following outcome

$$O(p^0) : \mathrm{L}(W_0(x)) - f(x) = 0$$

$$\Rightarrow W_0(x) = x + x^{7/2} + \frac{\sqrt{\pi}x^{9/4}}{10\Gamma(13/4)} + \frac{7\sqrt{\pi}x^{13/4}}{26\Gamma(13/4)} + \frac{2\pi x^{19/4}}{627\Gamma(9/4)} + \frac{105\pi x^{23/4}}{876\Gamma(15/4)}.$$

$$O(p^1) : \mathrm{L}(W_1(x)) - \frac{1}{20}\int_0^x (2 + 3x + 5t)(x - t)^{-1/2} W_0(t)dt = 0$$

$$\Rightarrow W_1(x) = -\frac{32\sqrt{\pi}}{175\Gamma(6)}x^{7/2} - \frac{32\sqrt{\pi}}{35\Gamma(6)}x^{9/2} - \frac{568\sqrt{\pi}}{495\Gamma(6)}x^{11/2} - \frac{7\pi\Gamma(5/2)}{80\Gamma(7)}x^6 - \frac{17\pi\ \Gamma(5/2)}{32\Gamma(7)}x^7$$
$$-\frac{333\pi\ \Gamma(9/2)}{512\Gamma(8)}x^8 - \frac{\sqrt{\pi}x^{9/4}}{10\Gamma(13/4)} - \frac{7\sqrt{\pi}x^{13/4}}{26\Gamma(13/4)} - \frac{2\pi x^{19/4}}{627\Gamma(9/4)} - \frac{105\pi x^{23/4}}{876\Gamma(15/4)}.$$

Similarly, we obtain the next three terms $W_2(x), W_3(x), W_4(x)$ of (29). The approximate solution $\Phi_{n_1}(x)$ is defined by $\Phi_{n_1}(x) = \sum_{m=0}^{n_1-1} W_j(x)$. Now, we produce the absolute error $Er_{ab}(x)$ and the relative error (%) $Er_{re}(x)$ to show the effectiveness of our present method. These errors are defined as follows [5, 23]:

$$Er_{ab}(x) = |u(x) - \Phi_{n_1}(x)| = |u(x) - \textstyle\sum_{m=0}^{n_1-1} W_m(x)| \text{ and}$$
$$Er_{re}(x) = \frac{\left|u(x) - \sum_{m=0}^{n_1-1} W_m(x)\right|}{|u(x)|} \times 100,$$

by taking the first 5 terms of the series (29), i.e., $n_1 = 5$. Define $Er_{n_1}(x) \equiv Er_{ab}(x)|_{n_1}$. These errors are given at Table 2.

(a) First Subtable

x	$Er_{ab}(x)$	$Er_{re}(x)(\%)$
0.1	3.27044E-15	3.26013E-12
0.2	1.16559E-12	5.72552E-10
0.3	4.56740E-11	1.45094E-08
0.4	6.96185E-10	1.58052E-07
0.5	6.20180E-09	1.05403E-06

(b) Second Subtable

x	$Er_{ab}(x)$	$Er_{re}(x)(\%)$
0.6	3.89153E-08	5.07164E-06
0.7	1.90552E-07	1.93067E-05
0.8	7.75263E-07	6.16293E-05
0.9	2.73167E-06	1.71631E-04
1	8.58020E-06	4.29010E-04

Table 2. Absolute errors and relative errors for Example 3

Now, we produce the absolute residual error $Er_{res}(x)$ is defined as follows

$$Er_{res}(x) = |\mathrm{L}(\Phi_{n_1}(x)) + \mathcal{R}(\Phi_{n_1}(x)) - f(x)|,$$

where L and $\mathcal{R}$ is defined in Section 4. For Example 3, it can be clearly seen from Figure 1 that the absolute residual errors are converging to zero as the number of terms n_1 in the approximate solution increases.

Note that, if we consider the absolute tolerable error $\epsilon = .00001$ at Remark 2, then we need at least 17 terms of the partial sum for the desired accuracy for Example 3. However, Table 2 suggests $n_1 = 5$ is sufficient for Example 3. It will be more beneficial to take the value n_1 which is suggested in Remark 2 in the series of (29). In addition, we also produce the comparison of the exact and approximate solutions in Figure 1 and their errors in Figure 2 to show the convergence behavior.

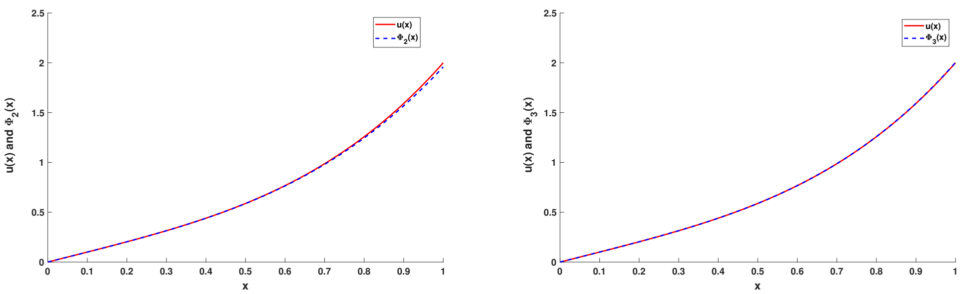

Figure 1. Comparison of exact solution and approximate solution for Example 3

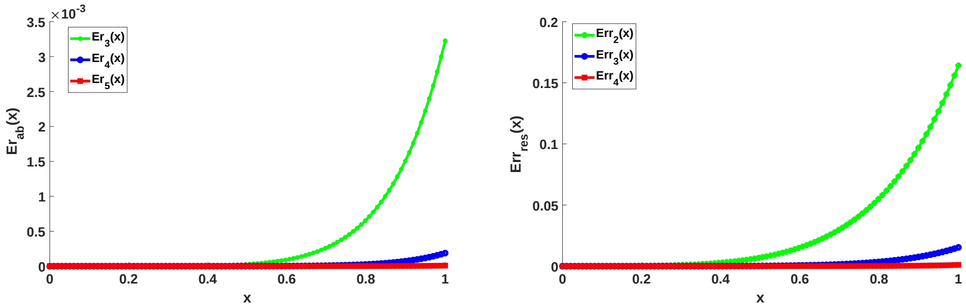

Figure 2. Comparison of absolute maximum errors and absolute residual errors for Example 3 for different values of n_1

Example 4 *Now, consider the following fractional problem with weakly singular kernel:*

$$\begin{cases} D^{\frac{3}{4}}u(x) = 3x + 2x^2 - \dfrac{1}{20}\displaystyle\int_0^x (5x-3t)(x-t)^{-1/2}u(t)dt, \ 0 < x \leq 2, \\ u(0) = 1. \end{cases} \tag{38}$$

The exact solution of (38) is unknown. Now, we use HPM described in Section 4 to solve the above problem. Using (25), the initial condition and equating the coefficient of $O(p^j)$, we obtain the following outcome

$$O(p^0) : \mathbf{L}(W_0(x)) - 3x - 2x^2 = 0 \Rightarrow W_0(x) = 1 + \tfrac{3x^{7/4}}{\Gamma(11/4)} + \tfrac{4x^{11/4}}{\Gamma(15/4)}.$$

Similarly, we obtain the next terms of the series (29) by the relation

$$\mathbf{L}(W_{j+1}(x)) - \frac{1}{20}\int_0^x (5x-3t)(x-t)^{-1/2}W_j(t)dt = 0.$$

The approximate solution $\Phi_n(x)$ is defined by $\Phi_n(x) = \sum_{m=0}^{n-1} W_j(x)$.

We produce the error with respect to L^2 norm over $\overline{\Omega} = [0, 2]$ as

$$E_n^{2,HPM} = \left(\int_0^2 (u(x) - \Phi_n(x))^2 \, dx\right)^{1/2}. \tag{39}$$

Now, we use least square homotopy method defined in Subsection 4.2 to approximate the solution. For $n = 1$, the approximate solution $\Phi_1(x)$ is given by $\Phi_1(x) = 1 + \frac{3x^{7/4}}{\Gamma(11/4)} + \frac{4x^{11/4}}{\Gamma(15/4)}$. This implies that the set S_1 contains the functions $\{1, x^{7/4}, x^{11/4}\}$, which are linearly independent and continuous. Hence, for applying LSHPM, we construct the approximate solution $\bar{\Phi}_1(x)$ as $\bar{\Phi}_1(x) = A_0 + A_1x^{7/4} + A_2x^{11/4}$, where A_0, A_1, A_2 are unknown and obtained by residual minimization and given initial condition. By initial condition, $\bar{\Phi}_1(0) = 1$, we get $A_0 = 1$. Therefore, $\bar{\Phi}_1(x) = 1 + A_1x^{7/4} + A_2x^{11/4}$. The residual error is

$$R(x, \bar{\Phi}_1(x)) = (1.60836A_1 - 3)x + 0.3x^{3/2} + x^2(2.21149A_2 - 2) + 0.137633A_1$$
$$x^{13/4} + 0.11132A_2x^{17/4}.$$

Next, we compute the functional $J(A_1, A_2)$ as

$$J(A_1, A_2) = \int_0^2 R^2(x, \bar{\Phi}_1(x))dx.$$

The optimal value of A_1, A_2 are calculated by minimizing the functional $J(A_1, A_2)$. For minimization, we need to solve the following system of equations

$$\frac{\partial J}{\partial A_1} = 0, \quad \frac{\partial J}{\partial A_2} = 0.$$

By solving the above system of equations, we obtain $A_1 = 2.31021$ and $A_2 = 0.275744$. Therefore, the approximate solution is given by $\bar{\Phi}_1(x) = 1 + 2.31021x^{7/4} + 0.275744x^{11/4}$. Similarly, we can obtain $\bar{\Phi}_2$ and $\bar{\Phi}_3$, $\bar{\Phi}_4$, etc. Now we produce the error with respect to L^2 norm over $\overline{\Omega} = [0, 2]$ as

$$E_n^{2,LSHPM} = \left(\int_0^2 \left(u(x) - \bar{\Phi}_n(x)\right)^2 dx\right)^{1/2}. \tag{40}$$

At Table 3, the comparison of HPM and LSHPM is given for the Example 4, which shows that the solution accuracy by LSHPM method is better than standard HPM. In addition we also plot the errors based on different methods at Figure 3 for this example.

$x \in [0, 2]$	n	$E_n^{2,HPM}$	$E_n^{2,LSHPM}$
	2	2.58973	6.13537E-03
	3	2.16748E-01	1.51576E-05
	4	1.27902E-02	2.00751E-06

Table 3. L^2 norm based errors for Example 4

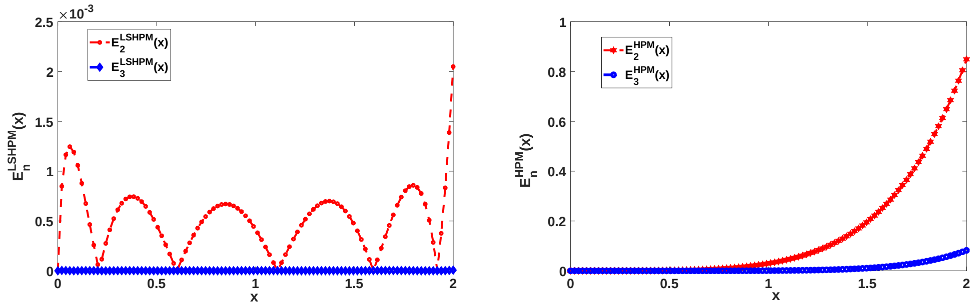

Figure 3. Comparison of errors for Example 4 for different values of n_1

Example 5 *Now, consider the following fractional problem with weakly singular kernel:*

$$\begin{cases} D^{\frac{4}{5}} u(x) = 2x + 3x^{3/2} - \dfrac{1}{15} \displaystyle\int_0^x (x + 2t)(x - t)^{-3/5} u(t) dt, \;\; 0 < x \leq 2, \\ u(0) = 0. \end{cases} \tag{41}$$

The exact solution of (38) is unknown. Now, we use HPM described in Section 4 to solve the above problem. Using (25), the initial condition and equating the coefficient of $O(p^j)$, we obtain the following outcome

$$O(p^0) : \mathrm{L}(W_0(x)) - 2x - 3x^{3/2} = 0 \Rightarrow W_0(x) = \tfrac{25x^{9/5}}{18\Gamma(4/5)} + \tfrac{9\sqrt{\pi}x^{23/10}}{4\Gamma(33/10)}.$$

Similarly, we obtain the next terms of the series (29) by the relation

$$\mathrm{L}(W_{j+1}(x)) - \frac{1}{15} \int_0^x (x + 2t)(x - t)^{-3/5} W_j(t) dt = 0.$$

The approximate solution $\Phi_n(x)$ is defined by $\Phi_n(x) = \sum_{m=0}^{n-1} W_j(x)$.

We compute the error with respect to L^2 norm over $\overline{\Omega} = [0, 2]$ as

$$E_n^{2,HPM} = \left(\int_0^2 (u(x) - \Phi_n(x))^2 \, dx \right)^{1/2}. \tag{42}$$

Now, we use least square homotopy method defined in Subsection 4.2 to approximate the solution. For $n = 1$, the approximate solution $\Phi_1(x)$ is given by $\Phi_1(x) = \dfrac{25x^{9/5}}{18\Gamma(4/5)} + \dfrac{9\sqrt{\pi}x^{23/10}}{4\Gamma(33/10)}$. This implies that the set S_1 contains the functions $\{x^{9/5}, x^{23/10}\}$, which are linearly independent and continuous. Hence, for applying LSHPM, we construct the approximate solution $\bar{\Phi}_1(x)$ as $\bar{\Phi}_1(x) = A_1 x^{9/5} + A_2 x^{23/10}$, where A_1, A_2 are unknown and obtained by residual minimization. Since $\bar{\Phi}_1(0) = 0$, this implies the approximate solution satisfied the initial condition. The residual error is

$$R(x, \bar{\Phi}_1(x)) = (1.67649A_1 - 2)x + x^{3/2}(2.01862A_2 - 3) + 0.28126A_1 x^{16/5}$$
$$+0.26486A_2 x^{37/10}.$$

Next, we compute the functional $J(A_1, A_2)$ as

$$J(A_1, A_2) = \int_0^2 R^2(x, \bar{\Phi}_1(x)) dx.$$

The optimal values of A_1, A_2 are calculated by minimizing the functional $J(A_1, A_2)$. For minimization, we need to solve the following system of equations

$$\frac{\partial J}{\partial A_1} = 0, \quad \frac{\partial J}{\partial A_2} = 0.$$

By solving the above system of equations, we obtain $A_1 = 3.37741$ and $A_2 = -0.760781$. Therefore, the approximate solution is given by $\bar{\Phi}_1(x) = 3.37741x^{9/5} - 0.760781x^{23/10}$. Similarly, we can obtain $\bar{\Phi}_2$ and $\bar{\Phi}_3$, $\bar{\Phi}_4$, etc. Now we produce the error with respect to L^2 norm over $\overline{\Omega} = [0, 2]$ as

$$E_n^{2,LSHPM} = \left(\int_0^2 \left(u(x) - \bar{\Phi}_n(x) \right)^2 dx \right)^{1/2}. \tag{43}$$

At Table 4, the comparison of HPM and LSHPM is given for this example, which shows that the solution accuracy by LSHPM method is better than standard HPM. We also plot the errors at Figure 4 based on different methods.

$x \in [0, 2]$	n	$E_n^{2,HPM}$	$E_n^{2,LSHPM}$
	3	1.281937	7.46274E-04
	4	1.64422E-01	1.03303E-05
	5	1.66611E-02	2.30637E-07

Table 4. L^2 norm based errors for Example 5

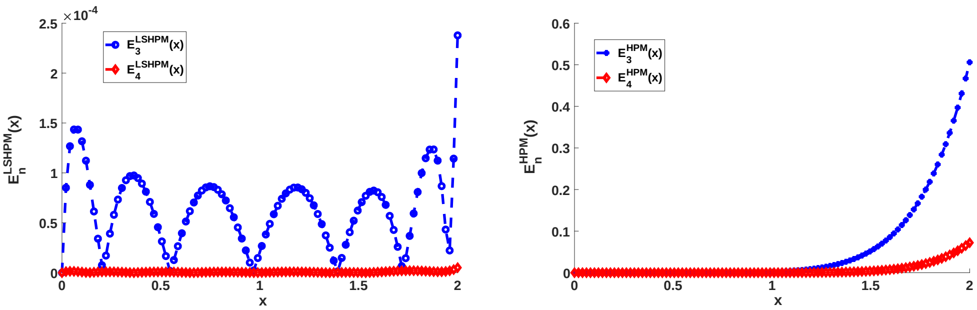

Figure 4. Comparison of errors for Example 5 for different values of n

6. Conclusions

Here, several sufficient conditions for the existence and uniqueness of the solutions of fractional order weakly singular Volterra integro differential equations are derived based on maximum norm and weighted maximum norm. In addition, we have also noted the restricted conditions if one proceeds with the usual approaches. Hence, we provide a suitable approach to extend the existence and uniqueness of the solution which is defined throughout its domain of definition. In addition, an approximate solution of this model is proposed based on HPM. We have theoretically shown that the proposed approach is convergent to the exact solution. This approach can be enhanced by LSHPM, where the constants multiplied with the involving functions in HPM, can be chosen optimally. It turns out that this approach produces comparatively less error than HPM in L^2 norm. We have theoretically estimate an error bound based on LSHPM. Several experiments are produced which observe that the modified optimized method LSHPM is computationally effective than HPM for weakly singular fractional order Volterra integro differential equations.

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