# Global dynamics of an nonautonomous SEIRS epidemic model with vaccination and nonlinear incidence

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**Abstract:** In this paper, a class of nonautonomous SEIRS epidemic models with vaccination and nonlinear incidence is investigated. Under some quite weak assumptions, a couple of new threshold values in the form of integral, i.e., $R_1$, $R_1^*$, $R_2$ and $R_2^*$ on the extinction and permanence of disease for the model are established. As special cases of our model, the autonomous, periodic and almost periodic circumstances are discussed respectively. The nearly necessary and sufficient criteria of threshold on the extinction and permanence of disease for above cases are obtained as well. Numerical examples and simulations are presented to illustrate the analytic results.

**Keywords:** Nonautonomous SEIRS epidemic model; vaccination; nonlinear incidence; permanence; extinction.

## 1. Introduction

As well known, mathematical models reflected in population dynamics of diseases have played a increasingly critical role in the theory of epidemiology. Therein, the susceptible-exposed-infected-removed compartmental models (SEIR models for short) with incubation

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of disease are especially classical for their accurate partition of the total population into four compartments, i.e., susceptible ($S$), exposed ($E$), infected ($I$) and removed ($R$), respectively. Recently, quite a number of significant works about SEIR models have been achieved. The key issues are: calculation of basic reproduction number, stability of disease-free equilibrium, extinction of disease, existence and stability of endemic equilibrium, persistence of disease, and existence and stability of bifurcation phenomena (see, e.g., [2, 3, 4, 5, 12, 14, 16, 17, 18, 19, 21] and references cited therein) etc. In addition, we see that the SEIR models have been successfully applied to investigate and forecast the dynamical behaviors of many fatal infectious disease in reality, e.g., Ebola, Zika, measles (see [13, 15, 20, 22, 23, 24, 25, 29, 40]) etc. For some diseases, e.g., influenza, when the removed individual is not perpetual immune for the disease, then the removed will afresh return to the susceptible as immune system loses its efficacy. Therefore, SEIRS epidemic model is further developed to characterize such diseases. Presently, we see that there is quite a few works on the autonomous SEIRS epidemic models (See [26, 27, 28, 31, 33, 36, 37, 38, 41, 42] and the references cited therein).

Obviously, the nonautonomous phenomena are very prevalent in our real life, e.g., the seasonal alternations, climatic variation etc. As results of these variable factors, many parameters in above epidemic models, e.g., the recruitment of susceptible, transmission of the disease, natural birth and death rates, mortality rate due to disease, remove rate etc., are fluctuant as time $t$. Therefore, nonautonomous epidemic models are more consistent with the real environment, and more realistic to model the dynamics of disease. However, up to now, to our best knowledge, there is very few research on the periodic and general nonautonomous SEIRS type epidemic models (See [1, 7, 10, 11, 30, 32, 34, 35, 39]). Particularly, in [7], Zhang and Teng studied a nonautonomous SEIRS epidemic model. By a new technique of analysis, some new sufficient conditions for the permanence and extinction of disease were established. In [30], the authors investigated a periodic SEIRS epidemic model with a time-dependent latent period. The basic reproduction number $R_0$ was calculated, and the threshold type results on the global dynamics in the term of $R_0$ were established. In [34], a non-autonomous SEIRS model with general incidence rate was considered. The sufficient conditions for the extinction and strong persistence of the infectives were obtained, and as some special cases, including autonomous and periodic circumstances, were discussed as well. In [35], the permanence and extinction for a nonautonomous SEIRS epidemic model with bilinear incidence were investigated. A novel and interesting method was introduced, by which the sufficient conditions were established.

In [10], the author studied the following SEIRS epidemic model with periodic vacci-

nation and transmission rates:

$$
\begin{cases}
\dfrac{dS(t)}{dt} = \mu N(1-p) - \beta(t)S(t)I(t) - (\mu + r(t))S(t) + \delta R(t), \\[2mm]
\dfrac{dE(t)}{dt} = \beta(t)S(t)I(t) - (\mu + \sigma)E(t), \\[2mm]
\dfrac{dI(t)}{dt} = \sigma E(t) - (\mu + \gamma)I(t), \\[2mm]
\dfrac{dR(t)}{dt} = \mu N p + r(t)S(t) + \gamma I(t) - (\mu + \delta)R(t),
\end{cases}
\tag{1}
$$

where $\beta(t)$ and $r(t)$ denote the vaccination and transmission rates, respectively, which are continuous positive periodic functions with common period $\omega > 0$. All the other parameters are positive constants. The basic reproduction ratio $R_0$ was calculated by the method given by Wang and Zhao in [6]. The authors showed that the global dynamics of model (1) is completely determined by the basic reproduction ratio $R_0$. That is, when $R_0 < 1$ then the disease-free periodic solution is globally asymptotically stable by the comparison principle of differential equations, and when $R_0 > 1$ then the disease is permanent by the theory of persistence in dynamical systems.

In this paper, we consider the following general nonautonomous SEIRS model with vaccination and nonlinear incidence

$$
\begin{cases}
\dfrac{dS(t)}{dt} = \Lambda(t)(1-p(t)) - \beta(t)f(S(t), I(t)) - (\mu(t) + r(t))S(t) + \delta(t)R(t), \\[2mm]
\dfrac{dE(t)}{dt} = \beta(t)f(S(t), I(t)) - (\mu(t) + \sigma(t))E(t), \\[2mm]
\dfrac{dI(t)}{dt} = \sigma(t)E(t) - (\mu(t) + \gamma(t))I(t), \\[2mm]
\dfrac{dR(t)}{dt} = \Lambda(t)p(t) + r(t)S(t) + \gamma(t)I(t) - (\mu(t) + \delta(t))R(t),
\end{cases}
\tag{2}
$$

where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the susceptible, the exposed, the infectious and the recovered population at time $t$, respectively. $\Lambda(t)$ denotes the recruitment rate of the susceptible at time $t$. $\beta(t)$ is the transmission rate of the disease at time $t$. $p(t)$ is the vaccination rate of all new-born children at time $t$. $r(t)$ is the vaccination rate of the susceptible population at time $t$. $\mu(t)$ is the common per capita birth and death rate at time $t$. $\sigma(t)$, $\gamma(t)$ and $\delta(t)$ are the per capita rates of leaving the latent stage, infected stage and recovered stage at time $t$, respectively. Function $f(S, I)$ is called the incidence rate, which is defined in $R_+^2 = \{(S, I) : S \geq 0, I \geq 0\}$ and is nonnegative and continuous. It is assumed that all parameters $\Lambda(t)$, $\beta(t)$, $p(t)$, $r(t)$ $\mu(t)$, $\delta(t)$, $\sigma(t)$ and $\gamma(t)$ are continuous and nonnegative functions and $0 \leqslant p(t) \leqslant 1$ for all $t \geqslant 0$.

Our purpose in this paper is to investigate the global dynamical behaviors of model (2). By developing the technique of analysis given in [8, 9, 11], we will establish some new

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threshold conditions of integral form for the extinction and permanence of disease of model (2). Furthermore, as the consequences of these results, we will discuss the autonomous, periodic and almost periodic cases of model (2), and establish the nearly necessary and sufficient threshold criteria on the extinction and permanence of disease for these models as well.

This paper is organized as follows. In section 2, we introduce main assumptions for model (2) and some preliminary lemmas which will be used in the statements and proofs of main results. In section 3, the sufficient condition of integral form for the extinction of disease for model (2) is stated and proved. In section 4, the sufficient condition of integral form for the permanence of disease for model (2) is stated and proved. In section 5, the autonomous, periodic and almost periodic cases of model (2) is discussed and the threshold conditions for the extinction and permanence of disease are stated and proved. In section 6, the theoretical results are illustrated by some special examples and numerical simulations. Finally, a conclusion is given in section 7.

## 2. Preliminaries

For model (2), we always assume that the following conditions hold.

$(H_1)$ Functions $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ are nonnegative, bounded and continuous on $R_+ = [0, +\infty)$.

$(H_2)$ There exist constants $\omega_i > 0$ $(i = 1, 2, 3)$ such that

$$\liminf_{t \to +\infty} \int_t^{t+\omega_1} \beta(s)ds > 0, \quad \liminf_{t \to +\infty} \int_t^{t+\omega_2} \mu(s)ds > 0, \quad \liminf_{t \to +\infty} \int_t^{t+\omega_3} \Lambda(s)ds > 0.$$

$(H_3)$ Function $f(S, I)$ is continuously differentiable for $(S, I) \in R_+^2$ and nondecreasing for $S \geq 0$, $\frac{f(S,I)}{I}$ is nonincreasing for $I > 0$, $f(S, 0) = f(0, I) \equiv 0$ for $S \geq 0$ and $I \geq 0$, respectively.

Let $f(t)$ be a continuous and nonnegative function defined on $R_+$. We define $f^+ = \sup_{t \geq 0} f(t)$ and $f^- = \inf_{t \geq 0} f(t)$. Furthermore, if $f(t)$ also is periodic with period $\omega > 0$, we denote by $\bar{f}$ the average value of $f(t)$, that is $\bar{f} = \frac{1}{\omega} \int_0^\omega f(t)dt$, and if $f(t)$ also is almost periodic, we denote by $m(f)$ the average value of $f(t)$, that is $m(f) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s)ds$ (see [18]) .

**Remark 1.** *When model (2) degenerates into $\omega-$periodic model, that is, all coefficients $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ are $\omega-$ periodic, then $(H_2)$ degenerates into the following form*

$$\bar{\beta} > 0, \quad \bar{\mu} > 0, \quad \bar{\Lambda} > 0.$$

**Remark 2.** *When model (2) degenerates into almost periodic model, that is, all coefficients $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ are almost periodic, then $(H_2)$*

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degenerates into the following form

$$m(\beta) > 0, \quad m(\mu) > 0, \quad m(\Lambda) > 0.$$

**Remark 3.** When $f(S,I) = SI$ (bilinear incidence), $f(S,I) = \frac{SI}{(1+\alpha_1 S)(1+\alpha_2 I)}$ (saturated incidence), $f(S,I) = \frac{SI}{1+\alpha_1 S + \alpha_2 I}$ (Beddington-DeAngelis incidence), and $f(S,I) = \frac{SI}{1+\alpha_1 I^2}$ (non-monotonous incidence), where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are constants, then assumption $(H_3)$ is satisfied.

By the biological background of model (2), for any solution $(S(t), E(t), I(t), R(t))$ of model (2) the initial condition is given by

$$S(0) = S_0 > 0, \ E(0) = E_0 > 0, \ I(0) = I_0 > 0, \ R(0) = R_0 > 0. \tag{3}$$

Firstly, on the nonnegativity and boundedness of solution $(S(t), E(t), I(t), R(t))$ of model (2) we have the following results.

**Lemma 1.** Assume that $(H_1) - (H_3)$ hold. Then for any solution $(S(t), E(t), I(t), R(t))$ of model (2) with initial condition (3) we have:

(a) $(S(t), E(t), I(t), R(t))$ is positive for all $t \in R_+$.

(b) $\lim_{t\to\infty}(S(t) + E(t) + I(t) + R(t) - z_0(t)) = 0$, where $z_0(t)$ is the solution of the following equation

$$\frac{dz(t)}{dt} = \Lambda(t) - \mu(t)z(t) \tag{4}$$

with initial condition $z(0) = z_0 > 0$.

(c) There exists a constant $M_0 > 0$ which is independent any solution $(S(t), E(t), I(t), R(t))$ of model (2) such that

$$\limsup_{t\to\infty} S(t) < M_0, \ \limsup_{t\to\infty} E(t) < M_0, \ \limsup_{t\to\infty} I(t) < M_0, \ \limsup_{t\to\infty} R(t) < M_0.$$

**Proof:** Let the solution $(S(t), E(t), I(t), R(t))$ is defined on interval $[0, t_0)$, where $t_0 \leq +\infty$. We firstly prove that $(S(t), E(t), I(t), R(t))$ is positive for $t \in [0, t_0)$. Let $m(t) = \min\{S(t), E(t), I(t), R(t)\}$, then $m(t)$ is continuous for $t \in [0, t_0)$, and $m(0) = \min\{S_0, E_0, I_0, R_0\} > 0$. Suppose that there is a $\bar{t} \in (0, t_0)$ such that $m(\bar{t}) = 0$, and $m(t) > 0$ for all $t \in [0, \bar{t})$. Then, we have the following four case: (1) $m(\bar{t}) = S(\bar{t}) = 0$, (2) $m(\bar{t}) = E(\bar{t}) = 0$, (3) $m(\bar{t}) = I(\bar{t}) = 0$, (4) $m(\bar{t}) = R(\bar{t}) = 0$. If case (1) occurs, then from the first equation of model (2) we obtain

$$\frac{dS(t)}{dt} \geq -(\beta(t)\frac{f(S(t), I(t))}{S(t)} + \mu(t) + \gamma(t))S(t), \ t \in [0, \bar{t}).$$

Since $S(t) > 0$ for all $t \in [0, \bar{t})$, $S(\bar{t}) = 0$ and $\lim_{S\to 0} \frac{f(S,I)}{S} = \frac{\partial f(0,I)}{\partial S}$ exists, we obtain that $\frac{f(S(t), I(t))}{S(t)}$ is defined for all $t \in [0, \bar{t}]$ and also is continuous. Consequently,

$$S(t) \geq S_0 \exp(-\int_0^t (\beta(s)\frac{f(S(s), I(s))}{S(s)} + \mu(s) + \gamma(s))ds), \ t \in [0, \bar{t}].$$

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From this, we further obtain $S(\bar{t}) > 0$, which leads to a contradiction. Hence, case (1) does not occur. Similarly, we also can obtain that cases (2), (3) and (4) do not occur. Therefore, $(S(t), E(t), I(t), R(t))$ is positive for all $t \in [0, t_0)$.

Next, we prove that $(S(t), E(t), I(t), R(t))$ is bounded on $[0, t_0)$. Let $N(t) = S(t) + E(t) + I(t) + R(t)$, then from model (2) we have

$$\frac{dN(t)}{dt} = \Lambda(t) - \mu(t)N(t).$$

Consider the auxiliary equation (4), from assumptions $(H_1)$ and $(H_2)$, and [8, 9], we can obtain that for any initial value $z(0) = z_0 \geq 0$, equation (4) has a unique positive solution $z_0(t)$ defined for $t \in [0, \infty)$, $z_0(t)$ also is bounded on $[0, \infty)$, and $z_0(t)$ is globally uniformly attractive. Thus, for any positive solution $z(t)$ of equation (4) one has

$$\lim_{t \to \infty} (z(t) - z_0(t)) = 0. \tag{5}$$

From this, there is a constant $M_0 > 0$, and $M_0$ is independent of any positive solutions of equation (4), such that

$$\limsup_{t \to \infty} z(t) \leq M_0. \tag{6}$$

Therefore, for $N(t) = S(t) + E(t) + I(t) + R(t)$, from $N(t) = z(t)$ with $N(0) = z(0)$ for all $t \in [0, t_0)$ and (6) we obtain that $(S(t), E(t), I(t), R(t))$ is bounded on $[0, t_0)$.

By the continuation theorem, we further have $t_0 = \infty$, i.e., $(S(t), E(t), I(t), R(t))$ is defined on $[0, \infty)$ and also is positive. Furthermore, from (5) and (6) we also have conclusions $(b)$ and $(c)$ of the lemma. This completes the proof.

From conclusion (b) of Lemma 1, without loss of generality, we can assume that for any solution $(S(t), E(t), I(t), R(t))$ of model (2) with initial condition (3), $S(t) + E(t) + I(t) + R(t) \equiv z_0(t)$ for all $t \in R_+$, where $u_0(t)$ is a fixed solution of equation (4) with initial value $z(0) = z_0 > 0$. Thus, model (2) can be equivalent to the following three-dimensional model:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda(t)(1 - p(t)) - \beta(t)f(S(t), I(t)) - (\mu(t) + r(t))S(t) \\ \qquad\qquad + \delta(t)(z_0(t) - S(t) - E(t) - I(t)), \\ \dfrac{dE(t)}{dt} = \beta(t)f(S(t), I(t)) - (\mu(t) + \sigma(t))E(t), \\ \dfrac{dI(t)}{dt} = \sigma(t)E(t) - (\mu(t) + \gamma(t))I(t), \end{cases} \tag{7}$$

and the initial condition (3) for model (7) becomes into

$$S(0) > 0, \quad E(0) > 0, \quad I(0) > 0. \tag{8}$$

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Next, we consider the following linear equation

$$\frac{du(t)}{dt} = \Lambda(t)(1 - p(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))u(t). \tag{9}$$

Directly from [8, 9], we have the result as follows.

**Lemma 2.** *Assume that* $(H_1) - (H_3)$ *hold. Then we have*

*(a) There are constants* $M_1 > m_1 > 0$ *such that for any solution* $u(t)$ *of equation (9) with initial value* $u(0) = u_0 \geq 0$, *one has* $m_1 < \liminf_{t \to \infty} u(t) \leq \limsup_{t \to \infty} u(t) < M_1$;

*(b) Each fixed solution* $u^*(t)$ *of equation (9) with initial value* $u^*(0) > 0$ *is globally uniformly attractive on* $R_+$.

Lemma 2 indicates that model (7) has the disease-free equilibrium state $(u^*(t), 0, 0)$. Further, we easily verify that model (2) has the disease-free equilibrium state $(u^*(t), 0, 0, z_0(t) - u^*(t))$.

Further, we consider linear equation as follows

$$\begin{aligned}
\frac{dv(t)}{dt} = {} & \Lambda(t)(1 - p(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))v(t) \\
& - \varepsilon_1 (\beta^+ \frac{\partial f(M_0, 0)}{\partial I} + 2\delta^+ + \delta^+ \beta^+ \frac{\partial f(M_0, 0)}{\partial I}\omega_2),
\end{aligned} \tag{10}$$

where constant $M_0$ is given in conclusion $(c)$ of Lemma 1. Let $u(t)$ and $v(t)$ be the solutions of equations (9) and (10) with initial values $u(0) = v(0) = v_0 > 0$, respectively. Directly from [8, 9], we have the following result.

**Lemma 3.** *Assume that* $(H_1) - (H_3)$ *hold. Then there is a constant* $B > 0$ *only dependent on* $\mu(t) + r(t) + \delta(t)$ *such that*

$$\sup_{t \geq 0} |u(t) - v(t)| \leq B\varepsilon_1(\beta^+ \frac{\partial f(M_0, 0)}{\partial I} + 2\delta^+ + \delta^+ \beta^+ \frac{\partial f(M_0, 0)}{\partial I}\omega_2).$$

Let $(S(t), E(t), I(t))$ be any solution of model (7) with initial condition (8), and $u^*(t)$ be the fixed solution of equation (9) with initial value $u^*(0) > 0$. For constant $q > 0$ we define

$$G(q, t) = \beta(t)\frac{\partial f(u^*(t), 0)}{\partial I}q + \gamma(t) - (1 + \frac{1}{q})\sigma(t), \quad W(q, t) = qE(t) - I(t).$$

We have the following result.

**Lemma 4.** *Assume that* $(H_1) - (H_3)$ *hold. If there is constant* $q > 0$ *such that* $\limsup_{t \to \infty} G(q, t) < 0$, *then there is a* $\bar{T} > 0$ *such that either* $W(q, t) > 0$ *for all* $t \geq \bar{T}$ *or* $W(q, t) \leq 0$ *for all* $t \geq \bar{T}$.

**Proof.** Firstly, from conclusion $(a)$ of Lemma 2, there are constants $M_2 > m_2 > 0$ such that

$$m_2 < u^*(t) < M_2 \quad \text{for all} \quad t \geq 0.$$

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Next, from assumption $(H_3)$ we have that $\frac{\partial f(S,0)}{\partial I}$ is uniformly continuous for $S \in [0, 2M_2]$. From this, by $\limsup_{t \to \infty} G(q,t) < 0$, there are constants $\varepsilon_0 > 0$ and $T_0 > 0$ such that

$$\beta(t)\frac{\partial f(u^*(t) + \varepsilon_0, 0)}{\partial I}q + \gamma(t) - (1 + \frac{1}{q})\sigma(t) < -\varepsilon_0 \tag{11}$$

for all $t \geq T_0$. Furthermore, from model (7) we have

$$\frac{dS(t)}{dt} \leq \Lambda(t)(1 - p(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))S(t)$$

for all $t \geq 0$. From the comparison principle and conclusion $(b)$ of Lemma 2, there is a constant $T_1 \geq T_0$ such that

$$S(t) < u^*(t) + \varepsilon_0 \quad \text{for all} \quad t \geq T_1.$$

Suppose that there does not exist such $\bar{T}$, then there exist two increasing time sequences $t_n$ and $s_n$ satisfying following properties: $\lim_{n \to \infty} t_n = \infty$, $0 < t_n < s_n$ for any positive integer $n$, $W(q, t_n) = 0$, $\frac{dW(q,t_n)}{dt} \geq 0$, $W(q,t) > 0$ for all $t \in \bigcup_{n=1}^{\infty}(t_n, s_n)$ and $W(q,t) \leq 0$ for all $t \notin \bigcup_{n=1}^{\infty}(t_n, s_n)$. Hence, we have

$$qE(t_n) = I(t_n) \tag{12}$$

and by assumption $(H_3)$ when $t_n > T_1$

$$\begin{aligned}
\frac{dW(q, t_n)}{dt} &= q\frac{dE(t_n)}{dt} - \frac{dI(t_n)}{dt} \\
&= q\{\beta(t_n)f(S(t_n), I(t_n)) - (\mu(t_n) + \sigma(t_n))E(t_n)\} \\
&\quad -\{\sigma(t_n)E(t_n) - (\mu(t_n) + \gamma(t_n))I(t_n)\} \\
&\leq I(t_n)\{\beta(t_n)\frac{\partial f(u^*(t_n) + \varepsilon_0, 0)}{\partial I}q + (\mu(t_n) + \gamma(t_n))\} \\
&\quad -qE(t_n)\{(\mu(t_n) + \sigma(t_n)) + \frac{1}{q}\sigma(t_n)\}
\end{aligned} \tag{13}$$

Substituting (12) into (13) we obtain

$$qE(t_n)\{\beta(t_n)\frac{\partial f(u^*(t_n) + \varepsilon_0, 0)}{\partial I}q + \gamma(t_n) - (1 + \frac{1}{q})\sigma(t_n)\} \geq 0.$$

We have $E(t_n) > 0$ by conclusion $(a)$ of Lemma 1, and hence,

$$\beta(t_n)\frac{\partial f(u^*(t_n) + \varepsilon_0, 0)}{\partial I}q + \gamma(t_n) - (1 + \frac{1}{q})\sigma(t_n) \geq 0$$

for all $t_n > T_1$, which is a contradiction with (11). This completes the proof.

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# 3. Extinction of disease

**Theorem 1.** *Assume that $(H_1) - (H_3)$ hold. If there are constants $\lambda > 0$ and $q > 0$ such that $\limsup_{t \to \infty} G(q,t) < 0$,*

$$R_1(\lambda, q) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \{\beta(s)\frac{\partial f(u^*(s), 0)}{\partial I}q - (\mu(s) + \sigma(s))\}ds < 0 \tag{14}$$

*and*

$$R_1^*(\lambda, q) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \{\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s))\}ds < 0, \tag{15}$$

*then the disease-free equilibrium $(u^*(t), 0, 0)$ in model (7) is globally attractive.*

**Proof.** From Lemma 4, we only need to consider the following two cases:

(*i*) $qE(t) > I(t)$ for all $t \geq \bar{T}$.

(*ii*) $qE(t) \leq I(t)$ for all $t \geq \bar{T}$.

Firstly, we consider case (*i*). From (14), assumptions $(H_1)$ and $(H_2)$, and the uniform continuity of $\frac{\partial f(S,0)}{\partial I}$ for $S \in [0, 2M_2]$, there exist constants $\varepsilon_0 > 0$, $\delta_1 > 0$ and $T_1 > \bar{T}$ such that

$$\int_t^{t+\lambda} [\beta(s)\frac{\partial f(u^*(s) + \varepsilon_0, 0)}{\partial I}q - (\mu(s) + \sigma(s))]ds < -\delta_1 \tag{16}$$

for all $t \geq T_1$. For any solution $(S(t), E(t), I(t))$ of model (7) with initial condition (8), since

$$\frac{dS(t)}{dt} \leq \Lambda(t)(1 - q(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))S(t)$$

for all $t \geq 0$, by the comparison theorem of differential equations and conclusion (*b*) of Lemma 2, there is a $T_2 \geq T_1$ such that $S(t) \leq u^*(t) + \varepsilon_0$ for all $t \geq T_2$.

From assumption $(H_3)$ and the second equation of model (7), we have

$$
\begin{aligned}
\frac{dE(t)}{dt} &\leq \beta(t)\frac{\partial f(u^*(t) + \varepsilon_0, 0)}{\partial I}I(t) - (\mu(t) + \sigma(t))E(t), \\
&\leq E(t)[\beta(t)\frac{\partial f(u^*(t) + \varepsilon_0, 0)}{\partial I}q - (\mu(t) + \sigma(t))]
\end{aligned}
\tag{17}
$$

for all $t \geq T_2$. Integrating (17) for any $t \geq T_2$, then

$$E(t) \leq E(T_2) \exp\left(\int_{T_2}^t [\beta(s)\frac{\partial f(u^*(t) + \varepsilon_0, 0)}{\partial I}q - (\mu(s) + \sigma(s))]ds\right).$$

From (16), we directly obtain $\lim_{t \to +\infty} E(t) = 0$. Then, from $qE(t) > I(t)$ for all $t \geq \bar{T}$, it follows $\lim_{t \to +\infty} I(t) = 0$.

Next, consider case (*ii*). Since we have $E(t) \leq \frac{I(t)}{q}$ for all $t \geq \bar{T}$, from the third equation of model (7) it follows

$$\frac{dI(t)}{dt} \leq I(t)\{\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t))\} \tag{18}$$

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Integrating (18) for any $t \geq \bar{T}$, then

$$I(t) \leqslant I(\bar{T}) \exp \left( \int_{\bar{T}}^{t} \{\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s))\}ds \right).$$

From (15), we can easily obtain $\lim_{t \to +\infty} I(t) = 0$. Then, from $E(t) \leq \frac{I(t)}{q}$ for all $t \geq \bar{T}$, it follows $\lim_{t \to +\infty} E(t) = 0$.

Thus, for any small enough $\varepsilon > 0$, there is a $T > \bar{T}$ such that $E(t) < \varepsilon$ and $I(t) < \varepsilon$ for all $t \geq T$. Then, we have

$$\frac{dS(t)}{dt} \leq \Lambda(t)(1 - q(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))S(t)$$

for all $t \geq 0$, and by assumption $(H_3)$

$$\frac{dS(t)}{dt} \geq \Lambda(t)(1 - q(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \delta(t))S(t) - (\beta(t)\frac{\partial f(M_0, 0)}{\partial I} + 2\delta(t))\varepsilon$$

for all $t \geq T$. Let $u(t)$ be the solution of equation (9) with initial value $u(T) = S(T)$. We have $u(t) \geq S(t)$ for all $t \geq T$. Let $v(t) = u(t) - S(t)$, then we have $u(t) \geq 0$ for all $t \geq 0$ and

$$\begin{aligned}\frac{dv(t)}{dt} &\leq -(\mu(t) + r(t) + \delta(t))v(t) + (\beta(t)\frac{\partial f(M_0, 0)}{\partial I} + 2\delta(t))\varepsilon \\ &\leq -\mu(t)v(t) + (\beta^+\frac{\partial f(M_0, 0)}{\partial I} + 2\delta^+)\varepsilon\end{aligned}$$

for all $t \geq T$. Hence,

$$v(t) \leq v(T)e^{-\int_T^t \mu(s)ds} + \varepsilon(\beta^+\frac{\partial f(M_0, 0)}{\partial I} + 2\delta^+)\int_T^t e^{-\int_s^t \mu(\tau)d\tau}ds$$

for all $t \geq T$. From assumption $(H_2)$, we easily obtain that there exist constants $\alpha > 0$ and $H \geq 0$ such that for any $t_2 \geq t_1 \geq 0$ one has $\int_{t_1}^{t_2} \mu(s)ds \geq \alpha(t_2 - t_1) - H$. Therefore, we further obtain

$$u(t) \geq S(t) \geq u(t) - v(T)e^{-\alpha(t-T)+H} - \frac{\varepsilon(\beta^+\frac{\partial f(M_0, 0)}{\partial I} + 2\delta^+)}{\alpha}e^H$$

for all $t \geq T$. Taking $t \to \infty$, and by the arbitrariness of $\varepsilon$ we can obtain $\lim_{t \to \infty}(S(t) - u(t)) = 0$. Finally, from conclusion $(b)$ of Lemma 2 we further have $\lim_{t \to \infty}(S(t) - u^*(t)) = 0$. This shows that disease-free equilibrium state $(u^*(t), 0, 0)$ is globally attractive. This completes the proof.

## 4. Permanence of disease

**Theorem 2.** Assume that $(H_1) - (H_3)$ hold. If there exist constants $\lambda > 0$ and $q > 0$ such that $\limsup_{t \to \infty} G(q, t) < 0$,

$$R_2(\lambda, q) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \{\beta(s)\frac{\partial f(u^*(s), 0)}{\partial I}q - (\mu(s) + \sigma(s))\}ds > 0 \qquad (19)$$

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*and*

$$R_2^*(\lambda, q) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \{\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s))\}ds > 0, \tag{20}$$

*then infected I in model (7) is permanent.*

In order to prove Theorem 2, we firstly introduce the following lemma.

**Lemma 5.** *Assume that $(H_1) - (H_3)$ hold. If there exist constants $\lambda > 0$ and $q > 0$ such that condition (20) holds and $\limsup_{t \to \infty} G(q, t) < 0$, then $W(q, t) \leq 0$ for all $t \geq \bar{T}$, where $\bar{T}$ is given as in Lemma 4.*

**Proof.** From Lemma 4 we have only two cases: $W(q, t) > 0$ for all $t \geq \bar{T}$ or $W(q, t) \leq 0$ for all $t \geq \bar{T}$. Suppose that $W(q, t) > 0$ for all $t \geq \bar{T}$. Then, we have $E(t) > \frac{I(t)}{q}$ for all $t \geq \bar{T}$. It follows from the third equation of model (7) that

$$\frac{dI(t)}{dt} > I(t)\{\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t))\} = I(t)\{\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t))\} \tag{21}$$

for all $t \geq \bar{T}$. Integrating (21) for $t \geq \bar{T}$ we have

$$I(t) > I(\bar{T}) \exp\left(\int_{\bar{T}}^t \{\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s))\}ds\right) \tag{22}$$

for all $t \geq \bar{T}$. From condition (20), there exist positive constants $\eta > 0$ and $T^* > 0$ such that

$$\int_t^{t+\lambda} \{\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s))\}ds > \eta \tag{23}$$

for all $t \geq T^*$. Since inequality (22) holds for all $t \geq \max\{\bar{T}, T^*\}$, it follows from (23) that $\lim_{t \to +\infty} I(t) = +\infty$. This contradicts with the boundedness of $I$, stated in conclusion $(c)$ of Lemma 1. This completes the proof.

**Proof of Theorem 2.** Firstly, from the uniform continuity of $\frac{\partial f(S,0)}{\partial I}$ for $S \in [0, 2M_2]$ and inequality (19), there are constants $\bar{\varepsilon} > 0$, $\eta_0 > 0$ and $T_0 > 0$ such that $m_2 - \bar{\varepsilon} > 0$ and for all $t \geq T_0$

$$\int_t^{t+\lambda} (\beta(s)(\frac{\partial f(u^*(s) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon})q - (\mu(s) + \sigma(s)))ds \geq \eta_0. \tag{24}$$

Since $\lim_{I \to 0} \frac{f(S,I)}{I} = \frac{\partial f(S,0)}{\partial I}$ uniformly for $S \in [0, 2M_2]$, we have $\lim_{I \to 0} \frac{f(u^*(t) - \bar{\varepsilon}, I)}{I} = \frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I}$ uniformly for $t \geq T_0$. Hence, there is an $I_0 > 0$ such that $\frac{f(u^*(t) - \bar{\varepsilon}, I)}{I} \geq \frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon}$ for all $t \geq T_0$ and $I \in (0, I_0]$.

For any $v_0 \in R_+$ and $t_0 \in R_+$, let $u(t)$ and $v(t)$ be the solutions of equations (9) and (10) with initial value $u(t_0) = v(t_0) = v_0$, respectively. By Lemma 3, there is a constant $B > 0$ which is only dependent on $\mu(t) + r(t) + \delta(t)$ such that

$$|v(t) - u(t)| \leq B(\beta^+ \frac{\partial f(M_0, 0)}{\partial I} + \delta^+(2 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I})\omega_2)\varepsilon_1$$

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for all $t \geq t_0$. Therefore, for the above $\bar{\varepsilon} > 0$ there is a $\varepsilon_1 > 0$ such that

$$|v(t) - u(t)| < \frac{1}{2}\bar{\varepsilon} \quad \text{for all} \quad t \geq t_0. \tag{25}$$

Since $u^*(t)$ is globally uniformly attractive from Lemma 2, then there is a $T_1 > 0$ which is independent of any $t_0 \in R_+$ and $v_0 \in [0, M_0]$ such that

$$|u(t) - u^*(t)| < \frac{1}{2}\bar{\varepsilon} \quad \text{for all} \quad t \geq t_0 + T_1. \tag{26}$$

Further, for above $\varepsilon_1 > 0$ we can choose constants $\varepsilon_2 > 0$ and $T_2 \geq T_0$ with $\varepsilon_2 < \min\{\varepsilon_1, I_0\}$ such that for all $t \geq T_2$

$$\int_t^{t+\omega_2} (\beta(s)\frac{\partial f(M_0, 0)}{\partial I}\varepsilon_2 - (\mu(s) + \sigma(s))\varepsilon_1)ds < -\eta_0. \tag{27}$$

Let $(S(t), E(t), I(t))$ be any solution of model (7) with initial condition (8). By the conclusion $(c)$ of Lemma 1, there is a constant $T_3 > T_2$ such that $S(t) \leq M_0$, $E(t) \leq M_0$ and $I(t) \leq M_0$ for all $t \geq T_3$. By Lemma 5, there is a constant $T_4 > T_3$ such that

$$W(q, t) \leq 0 \quad \text{for all} \quad t \geq T_4. \tag{28}$$

We firstly prove $\limsup_{t \to \infty} I(t) \geq \varepsilon_2$. Suppose that this conclusion is not true, then there is a $T_5 > T_4$ such that $I(t) < \varepsilon_2$ for all $t \geq T_5$. Suppose that $E(t) \geq \varepsilon_1$ for all $t \geq T_5$, then we have

$$
\begin{aligned}
E(t) =& \; E(T_5) + \int_{T_5}^t [\beta(s)f(S(s), I(s)) - (\mu(s) + \sigma(s))E(s)]ds \\
\leqslant& \; E(T_5) + \int_{T_5}^t [\beta(s)\frac{\partial f(M_0, 0)}{\partial I}\epsilon_2 - (\mu(s) + \sigma(s))\epsilon_1]ds
\end{aligned}
$$

for all $t \geqslant T_5$. Thus, from (27), we have $\lim_{t \to \infty} E(t) = -\infty$, which contradicts with conclusion (a) of Lemma 1. Therefore, there exists an $s_1 \geq T_5$ such that $E(s_1) < \varepsilon_1$. Suppose that there exists an $s_2 > s_1$ such that $E(s_2) > \varepsilon_1 + \beta^+\frac{\partial f(M_0, 0)}{\partial I}\omega_2\varepsilon_2$. Then, there exists an $s_3 \in (s_1, s_2)$ such that $E(s_3) = \varepsilon_1$ and $E(t) > \varepsilon_1$ for all $t \in (s_3, s_2]$. Let $n$ be an integer such that $s_2 \in [s_3 + n\omega_2, s_3 + (n+1)\omega_2]$. Then, from (27) we have

$$
\begin{aligned}
\varepsilon_1 + \beta^+ M_0\omega_2\varepsilon_2 <& \; E(s_2) = E(s_3) + \int_{s_3}^{s_2} [\beta(s)f(S(s), I(s)) - (\mu(s) + \sigma(s))E(s)]ds \\
<& \; \varepsilon_1 + (\int_{s_3}^{s_3+n\omega_2} + \int_{s_3+n\omega_2}^{s_2})[\beta(s)\frac{\partial f(M_0, 0)}{\partial I}\varepsilon_2 - (\mu(s) + \sigma(s))\varepsilon_1]ds \\
<& \; \varepsilon_1 + \int_{s_3+n\omega_2}^{s_2} \beta(s)\frac{\partial f(M_0, 0)}{\partial I}\varepsilon_2 ds \\
<& \; \varepsilon_1 + \beta^+\frac{\partial f(M_0, 0)}{\partial I}\varepsilon_2\omega_2
\end{aligned}
$$

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which is a contradiction. Therefore, we have that

$$E(t) \leqslant \varepsilon_1 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I} \omega_2 \varepsilon_2 \tag{29}$$

for all $t \geq s_1$. Since for all $t \geq s_1$

$$\begin{aligned}
\frac{dS(t)}{dt} \geq {}& \Lambda(t)(1 - q(t)) + \delta(t)u_0(t) - (\mu(t) + r(t) + \sigma(t))S(t) \\
& - (\beta^+ \tfrac{\partial f(M_0,0)}{\partial I} + \delta^+ (2 + \beta^+ \tfrac{\partial f(M_0,0)}{\partial I})\omega_2)\varepsilon_1,
\end{aligned}$$

by the comparison theorem we have $S(t) \geq v(t)$ for all $t \geq s_1$, where $v(t)$ is the solution of equation (10) with initial value $v(s_1) = S(s_1)$. Let $v_0 = S(s_1)$, then $v_0 \in [0, M_0]$. From the inequalities (25) and (26) we obtain for all $t \geq s_1 + T_1$

$$S(t) \geq v(t) \geq u(t) - \frac{1}{2}\bar{\varepsilon} \geq u^*(t) - \bar{\varepsilon}. \tag{30}$$

Since $I(t) < \varepsilon_2 < I_0$ for all $t \geq T_5$, from the second equation of model (7) and (28) we obtain for all $t \geq T_6 \triangleq s_1 + T_1$

$$\begin{aligned}
\frac{dE(t)}{dt} \geq {}& \beta(t)\Big(\frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon}\Big)I(t) - (\mu(t) + \sigma(t))E(t) \\
\geq {}& [\beta(t)\Big(\frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon}\Big)q - (\mu(t) + \sigma(t))]E(t).
\end{aligned}$$

Integrating from $T_6$ to any $t > T_6$ we have

$$E(t) \geq E(T_6) \exp\Big(\int_{T_6}^t [\beta(s)\Big(\frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon}\Big)q - (\mu(s) + \sigma(s))]ds\Big).$$

It follows from (24) that $\lim_{t\to\infty} E(t) = +\infty$. This contradicts with the boundedness of $E(t)$ from Lemma 1. Therefore, we finally have $\limsup_{t\to\infty} I(t) > \epsilon_2$.

Next, we prove that there is a constant $l > 0$, which is independent of any solution $(S(t), E(t), I(t))$ of model (7), such that

$$\liminf_{t\to\infty} I(t) \geqslant l. \tag{31}$$

From inequalities (24), (27) and $(H_2)$, we can obtain that there exist constants $P > 0$ and $\eta_2 > 0$ such that

$$\int_t^{t+P} [\beta(s)\frac{\partial f(M_0, 0)}{\partial I}\epsilon_2 - (\mu(s) + \sigma(s))\epsilon_1]ds < -M_0 \tag{32}$$

$$\int_t^{t+P} [\beta(s)\Big(\frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\epsilon}\Big)q - (\mu(s) + \sigma(s))]ds > \eta_2 \tag{33}$$

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and

$$\int_t^{t+P} \beta(s)ds > \eta_2 \tag{34}$$

for all $t \geq 0$. Let $K > 0$ be an integer satisfying

$$\varepsilon_1 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I}\omega_2\varepsilon_2 \leq e^{-(\mu^+ + \sigma^+)P}\eta_2 \frac{f(m, M_0)}{M_0}v_2 e^{K\eta_2} \tag{35}$$

where $v_2 = \varepsilon_2 e^{-(\mu^+ + \gamma^+)(2P + T_1)}$ and $m = m_2 - \bar{\varepsilon}$. Since $\limsup_{t \to +\infty} I(t) > \varepsilon_2$, there are only two possibilities as follows:

(i) $\liminf_{t \to \infty} I(t) \geq \varepsilon_2$.

(ii) $I(t)$ oscillates about $\varepsilon_2$ for large $t$.

In case (i), we directly have inequality (31) with $l = \varepsilon_2$. Consider case (ii). Let $t_1, t_2 \geq T_4$ with $t_1 < t_2$ such that $I(t_1) = I(t_2) = \varepsilon_2$ and $I(t) < \varepsilon_2$ for all $t \in (t_1, t_2)$.

Suppose that $t_2 - t_1 \geq C + 2P + T_1$ with $C = PK$. If $E(t) \geq \varepsilon_1$ for all $t \in (t_1, t_1 + P)$, then from (32) we have

$$E(t_1 + P) \leq E(t_1) + \int_{t_1}^{t_1+P} [\beta(s)\frac{\partial f(M_0, 0)}{\partial I}\varepsilon_2 - (\mu(s) + \sigma(s))\varepsilon_1]ds < M_0 - M_0 = 0,$$

which is a contradiction. Therefore, there exists an $s_4 \in [t_1, t_1 + P)$ such that $E(s_4) < \varepsilon_1$. Then, similar to the proof of (29), we can show that

$$E(t) \leq \varepsilon_1 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I}\omega_2\varepsilon_2 < (1 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I}\omega_2)\varepsilon_1 \quad \text{for all} \quad t \in (s_4, t_2]. \tag{36}$$

Since

$$\frac{dS(t)}{dt} \geq \Lambda(t)(1 - q(t)) + \delta(t)z_0(t) - (\mu(t) + r(t) + \sigma(t))S(t)$$
$$-(\beta^+ \frac{\partial f(M_0, 0)}{\partial I} + \delta^+(2 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I})\omega_2)\varepsilon_1.$$

Similarly to above proof of (30) we can obtain

$$S(t) \geq v(t) \geq u(t) - \frac{1}{2}\bar{\varepsilon} \geq u^*(t) - \bar{\varepsilon} \tag{37}$$

for all $t \in [t_1 + P + T_1, t_2]$. From third equation of model (7) we can directly obtain

$$I(t) \geq \varepsilon_2 e^{-(\mu^+ + r^+)(2P + T_1)} \triangleq v_2 \quad \text{for all} \quad t \in [t_1, t_1 + 2P + T_1].$$

Since from (37) for any $t \in [t_1 + P + T_1, t_1 + 2P + T_1]$

$$\frac{dE(t)}{dt} \geq \beta(t)\frac{f(u^*(t) - \bar{\varepsilon}, I(t))}{I(t)}I(t) - (\mu(t) + \sigma(t))E(t)$$
$$\geq \beta(t)\frac{f(m, M_0)}{M_0}v_2 - (\mu^+ + \sigma^+)E(t),$$

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we have from (34)

$$
\begin{aligned}
E(t_1 + 2P + T_1) \geq\ & e^{-(\mu^+ + \sigma^+)(t_1 + 2P + T_1)} \big( E(t_1 + P + T_1) e^{(\mu^+ + \sigma^+)(t_1 + P + T_1)} \\
& + \int_{t_1 + P + T_1}^{t_1 + 2P + T_1} \beta(s) \frac{f(m, M_0)}{M_0} v_2 e^{(\mu^+ + \sigma^+)s} ds \big) \\
\geq\ & e^{-(\mu^+ + \sigma^+)(t_1 + 2P + T_1)} \int_{t_1 + P + T_1}^{t_1 + 2P + T_1} \beta(s) \frac{f(m, M_0)}{M_0} v_2 e^{(\mu^+ + \sigma^+)s} ds \\
\geq\ & e^{-(\mu^+ + \sigma^+)P} \eta_2 \frac{f(m, M_0)}{M_0} v_2.
\end{aligned} \tag{38}
$$

Now, from (28) and (36) we further have

$$
\begin{aligned}
\frac{dE(t)}{dt} =\ & \beta(t) f(S(t), I(t)) - (\mu(t) + \sigma(t)) E(t), \\
\geq\ & \beta(t) \big( \frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon} \big) I(t) - (\mu(t) + \sigma(t)) E(t), \\
\geq\ & E(t) [\beta(t) \big( \frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon} \big) q - (\mu(t) + \sigma(t))]
\end{aligned} \tag{39}
$$

for all $t \in (t_1 + 2P + T_1, t_2]$. Let $t^* = t_1 + 2P + T_1 + C$. Then, from (33), (38) and (39), we have

$$
\begin{aligned}
E(t^*) \geq\ & E(t_1 + 2P + T_1) \exp\big( \int_{t_1 + 2P + T_1}^{t^*} [\beta(s) \big( \frac{\partial f(u^*(t) - \bar{\varepsilon}, 0)}{\partial I} - \bar{\varepsilon} \big) q - (\mu(s) + \sigma(s))] ds \big) \\
>\ & e^{-(\mu^+ + \sigma^+)P} \eta_2 \frac{f(m, M_0)}{M_0} v_2 e^{K\eta_2}.
\end{aligned}
$$

Thus, from (36), we have

$$
\varepsilon_1 + \beta^+ \frac{\partial f(M_0, 0)}{\partial I} \omega_2 \varepsilon_2 > e^{-(\mu^+ + \sigma^+)P} \eta_2 \frac{f(m, M_0)}{M_0} v_2 e^{K\eta_2},
$$

which contradicts with (35). Therefore, we finally have $t_2 - t_1 \leq t_1 + 2P + T_1 + C$. Then, from third equation of model (7) we directly have

$$
I(t) \geq \varepsilon_2 e^{-(\mu^+ + \sigma^+)(2P + T_1 + C)} \triangleq l \quad \text{for all} \quad t \in [t_1, t_2],
$$

which implies $\liminf_{t \to +\infty} I(t) \geq l$. This completes the proof.

From the above results and the equivalence of model (2) and model (7), we can directly have the following corollary on the extinction and permanence of disease for model (2).

**Corollary 1.** *Assume that* $(H_1) - (H_3)$ *hold and there is a constant* $q > 0$ *such that* $\limsup_{t \to \infty} G(q, t) < 0$.

*(a) If there exists constant* $\lambda > 0$ *such that conditions (14) and (15) hold, then the disease-free equilibrium* $(u^*(t), 0, 0, z_0(t) - u^*(t))$ *in model (2) is globally attractive.*

*(b) If there exists constant* $\lambda > 0$ *such that conditions (19) and (20) hold, then infected* $I$ *in model (2) is permanent.*

# 5. Some corollaries

In this section, we consider some special cases of model (2) as the applications of the main results established in the above.

Firstly, we assume that all coefficients in model (2) are positive constants. Thus, model (2) is autonomous. We see that equation (4) is autonomous, and hence solution $z_0(t)$ can be chosen by $z_0(t) = \frac{\Lambda}{\mu} \triangleq z_0$. Then, equation (9) also is autonomous. The solution $u^*(t)$ can be chosen by $u^*(t) = \frac{\Lambda(1-p)\mu+\delta\Lambda}{\mu(\mu+r+\delta)} \triangleq u^*$. Thus, the basic reproduction number for autonomous model (2) is defined by

$$R_0 = \frac{\sigma\beta}{(\mu+\gamma)(\mu+\sigma)} \frac{\partial f(u^*,0)}{\partial I}.$$

For constant $q > 0$, we define

$$R_1(q) = \beta\frac{\partial f(u^*,0)}{\partial I}q - (\mu+\sigma), \quad R_1^*(q) = \sigma\frac{1}{q} - (\mu+\gamma)$$

and

$$G(q) = \beta\frac{\partial f(u^*,0)}{\partial I}q + \gamma - (1+\frac{1}{q})\sigma.$$

**Corollary 2.** *Suppose that* $(H_3)$ *holds, and all parameters* $\Lambda(t)$, $\beta(t)$, $p(t)$, $r(t)$ $\mu(t)$, $\delta(t)$, $\sigma(t)$ *and* $\gamma(t)$ *in model (2) are positive constants. Then we have*

*(1) If* $R_0 < 1$, *then the disease-free equilibrium* $(u^*, 0, 0, z_0 - u^*)$ *of model (2) is globally attractive.*

*(2) If* $R_0 > 1$, *then infected* $I$ *in model (2) is permanent.*

**Proof.** For autonomous model (2), we have $R_1(\lambda, q) = R_2(\lambda, q) = R_1(q)$, $R_1^*(\lambda, q) = R_2^*(\lambda, q) = R_1^*(q)$ and $G(t, q) = G(q)$ in Theorem 1 and Theorem 2. We only need to prove the following conclusions.

(i) There exists a positive constant $q > 0$ such that $R_1(q) < 0$, $R_1^*(q) < 0$ and $G(q) < 0$ if and only if $R_0 < 1$.

(ii) There exists a positive constant $q > 0$ such that $R_1(q) > 0$, $R_1^*(q) > 0$ and $G(q) < 0$ if and only if $R_0 > 1$.

We only prove conclusion (i). Conclusion (ii) can be proved in a similar manner. Suppose that there exists a constant $q > 0$ such that $R_1(q) < 0$, $R_1^*(q) < 0$ and $G(q) < 0$. Then, it follows from $R_1(q) < 0$ and $R_1^*(q) < 0$ that

$$\frac{\sigma}{\mu+\gamma} < q < \frac{\mu+\sigma}{\beta\frac{\partial f(u^*,0)}{\partial I}}. \tag{40}$$

Hence, we obtain $R_0 < 1$. Conversely, we assume $R_0 < 1$. Since

$$G(\frac{\sigma}{\mu+\gamma}) = \beta\frac{\partial f(u^*,0)}{\partial I}\frac{\sigma}{\mu+\gamma} + \gamma - (1+\frac{\mu+r}{\sigma})\sigma = (\mu+\sigma)(R_0-1) < 1,$$

there exists a constant $q > 0$ being close enough to $\frac{\sigma}{\mu+\gamma}$ such that (40) hold and $G(q) < 0$. Hence, for such $q$ we have $R_1(q) < 0$, $R_1^*(q) < 0$ and $G(q) < 0$. Thus, conclusion (i) holds. This completes the proof.

**Remark 3.** Corollary 1 shows that the results on the extinction and permanence of the disease established in this paper for nonautonomous model (2) cover the threshold-type results in the autonomous case.

Secondly, we consider the periodic case of model (2), that is, all coefficients $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ are nonnegative periodic functions with the common period $\omega > 0$. Firstly, for equation (4), when $\Lambda(t)$ and $\mu(t)$ are $\omega$-periodic functions, then from [8, 9], $z_0(t)$ can be chosen by the unique $\omega$-periodic solution of equation (4). Furthermore, it can be proved that for $\omega$-periodic equation (9) there is a globally uniformly attractive $\omega$-periodic solution $u^*(t)$. Thus, function $G(t, q) = \beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q + r(t) - (1 + \frac{1}{q})\sigma(t)$ is $\omega$-periodic. We see that conditions (14) and (15) given in Theorem 1 are equipollent to $\lambda = \omega$ and

$$R_1(\omega, q) = \int_0^\omega (\beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q - (\mu(t) + \sigma(t)))dt < 0,$$

$$R_1^*(\omega, q) = \int_0^\omega (\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t)))dt < 0,$$

and conditions (19) and (20) in Theorem 2 are equipollent to $R_2(\omega, q) = R_1(\omega, q) > 0$ and $R_2^*(\omega, q) = R_1^*(\omega, q) > 0$. We have the following result.

**Corollary 3.** *Suppose that $(H_3)$ holds, and all parameters $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ in model (2) are $\omega-$periodic continuous functions. Then we have*

*(i) The disease-free periodic equilibrium $(u^*(t), 0, 0, z_0(t) - u^*(t))$ of model (2) is globally attractive if there exist $q > 0$ such that*

$$\frac{\overline{\sigma}}{\overline{\mu + \gamma}} < q < \frac{\overline{\mu + \sigma}}{\overline{\beta\frac{\partial f(u^*(t),0)}{\partial I}}} \tag{41}$$

*and $G(q, t) < 0$ for all $t \in [0, \omega]$.*

*(ii) The infected $I$ in model (2) is permanent if there exists $q > 0$ such that*

$$\frac{\overline{\sigma}}{\overline{\mu + \gamma}} > q > \frac{\overline{\mu + \sigma}}{\overline{\beta\frac{\partial f(u^*(t),0)}{\partial I}}} \tag{42}$$

*and $G(q, t) < 0$ for all $t \in [0, \omega]$.*

**Proof.** In fact, for conclusion (i) we directly obtain $R_1(\omega, q) < 0$ and $R_1^*(\omega, q) < 0$, and for conclusion (ii) then $R_2(\omega, q) > 0$ and $R_2^*(\omega, q) > 0$. Therefore, by Theorem 1 and Theorem 2, Corollary 2 is true. This completes the proof.

**Remark 4.** Model (1) is the special case of model (2) with $\Lambda(t) = \mu N$, $f(S, I) = SI$, and the parameters $p(t)$, $\mu(t)$, $\delta(t)$, $\sigma(t)$ and $\gamma(t)$ are positive constants. Only the parameters $\beta(t)$ and $r(t)$ are $\omega$-periodic continuous function. As a consequence of Corollary 3 we have the following result.

(i) The disease-free periodic equilibrium $(u^*(t), 0, 0, N - u^*(t))$ of model (1) is globally attractive if there exists constant $q > 0$ such that

$$\frac{\sigma}{\mu + \gamma} < q < \frac{\mu + \sigma}{\beta u^*}.$$

and $G(q, t) < 0$ for all $t \in [0, \omega]$.

(ii) The infected $I$ in model (1) is permanent if there exists constant $q > 0$ such that

$$\frac{\sigma}{\mu + \gamma} > q > \frac{\mu + \sigma}{\beta u^*}.$$

and $G(q, t) < 0$ for all $t \in [0, \omega]$.

**Remark 5.** Comparing Corollary 2 and Corollary 3, we can propose the following open problem. For $\omega$-periodic model (2), we define

$$R_0 = \frac{\overline{\sigma \beta \frac{\partial f(u^*(t), 0)}{\partial I}}}{(\mu + \gamma)(\mu + \sigma)}.$$

It is clear that if condition (41) holds, then we have $R_0 < 1$, and if condition (42) holds, then we have $R_0 > 1$. However, conversely, from $R_0 < 1$ (or $R_0 > 1$) we not always can obtain that there exists a constant $q > 0$ such that (41) holds (or (42) holds) and $G(q, t) < 0$ for all $t \in [0, \omega]$. Therefore, an important and interesting open problem is whether only when $R_0 < 1$ (or $R_0 > 1$) we can exactly obtain the global attractivity of the disease-free periodic equilibrium $(u^*(t), 0, 0, z_0(t) - u^*(t))$ (or permanence of infected $I$) in model (2).

**Remark 6.** Furthermore, as a well extension of periodic model (2), we can consider the asymptotic periodic model (2). That is, all coefficients in model (2) are asymptotic periodic continuous functions with common period $\omega > 0$. Thus, there exist $\omega$-periodic continuous functions $\Lambda^*(t)$, $p^*(t)$, $\beta^*(t)$, $\mu^*(t)$, $r^*(t)$, $\delta^*(t)$, $\sigma^*(t)$ and $\gamma^*(t)$ such that $\lim_{t \to \infty}(\Lambda(t) - \Lambda^*(t)) = 0$, $\lim_{t \to \infty}(p(t) - p^*(t)) = 0$, $\lim_{t \to \infty}(\beta(t) - \beta^*(t)) = 0$, $\lim_{t \to \infty}(\mu(t) - \mu^*(t)) = 0$, $\lim_{t \to \infty}(r(t) - r^*(t)) = 0$, $\lim_{t \to \infty}(\delta(t) - \delta^*(t)) = 0$, $\lim_{t \to \infty}(\sigma(t) - \sigma^*(t)) = 0$ and $\lim_{t \to \infty}(\gamma(t) - \gamma^*(t)) = 0$. Assume that $z_0(t)$ is the unique $\omega$-periodic solution of equation

$$\frac{dz(t)}{dt} = \Lambda^*(t) - \mu^*(t)z(t),$$

and $u^*(t)$ is the unique $\omega$-periodic solution of equation

$$\frac{du(t)}{dt} = \Lambda^*(t)(1 - p^*(t)) + \delta^*(t)z_0(t) - (\mu^*(t) + r^*(t) + \delta^*(t))u(t).$$

Furthermore, define

$$G^*(q,t) = \beta^*(t)\frac{\partial f(u^*(t),0)}{\partial I}q + r^*(t) - (1 + \frac{1}{q})\sigma^*(t).$$

As an extension of Corollary 3, we have the following conclusions.

(i) The disease-free periodic equilibrium $(u^*(t), 0, 0, z_0(t) - u^*(t))$ of model (2) is globally attractive if there exist $q > 0$ such that

$$\frac{\overline{\sigma^*}}{\underline{\mu^* + \gamma^*}} < q < \frac{\overline{\mu^* + \sigma^*}}{\underline{\beta^*\frac{\partial f(u^*(t),0)}{\partial I}}}$$

and $G^*(q,t) < 0$ for all $t \in [0, \omega]$.

(ii) The infected $I$ in model (2) is permanent if there exists $q > 0$ such that

$$\frac{\overline{\sigma^*}}{\underline{\mu^* + \gamma^*}} > q > \frac{\overline{\mu^* + \sigma^*}}{\underline{\beta^*\frac{\partial f(u^*(t),0)}{\partial I}}}$$

and $G^*(q,t) < 0$ for all $t \in [0, \omega]$.

Finally, we consider the almost periodic case of model (2). Firstly, for equation (4), when $\Lambda(t)$ and $\mu(t)$ are almost periodic functions, then from [8, 9], $z_0(t)$ can be chosen by the unique almost periodic solution of equation (4). Furthermore, it can be proved that for almost periodic equation (9) there is a globally uniformly attractive almost periodic solution $u^*(t)$. Thus, function $G(q,t) = \beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q + r(t) - (1 + \frac{1}{q})\sigma(t)$ is also almost periodic. We have the following result.

**Corollary 4.** *Suppose that $(H_3)$ holds, and all parameters $\Lambda(t)$, $\mu(t)$, $p(t)$, $\beta(t)$, $\gamma(t)$, $r(t)$, $\sigma(t)$ and $\delta(t)$ in model (2) are almost periodic continuous functions. Then, we have*

*(i) The disease-free almost periodic equilibrium $(u^*(t), 0, 0, z_0(t) - u^*(t))$ of model (2) is globally attractive if there exists $q > 0$ such that*

$$\frac{m(\sigma)}{m(\mu + \gamma)} < q < \frac{m(\mu + \sigma)}{m(\beta\frac{\partial f(u^*(t),0)}{\partial I})} \tag{43}$$

*and $\limsup_{t\to\infty} G(q,t) < 0$.*

*(ii) The infected $I$ in model (2) is permanent if there exists $q > 0$ such that*

$$\frac{m(\sigma)}{m(\mu + \gamma)} > q > \frac{m(\mu + \sigma)}{m(\beta\frac{\partial f(u^*(t),0)}{\partial I})}$$

*and $\limsup_{t\to\infty} G(q,t) < 0$.*

**Proof.** We only prove conclusion (i). Conclusion (ii) can be proved in a similar manner. From condition (43) and the properties of almost periodic functions (see [8]), we obtain that there are constants $\lambda > 0$ and $\eta > 0$ such that for all $t \geq 0$

$$\int_t^{t+\lambda} (\beta(s)\frac{\partial f(u^*(s),0)}{\partial I}q - (\mu(s) + \sigma(s)))ds < -\eta$$

and

$$\int_t^{t+\lambda} (\sigma(s)\frac{1}{q} - (\mu(s) + \gamma(s)))ds < -\eta.$$

Hence, we further have

$$R_1(\lambda, q) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \{\beta(v)\frac{\partial f(u^*(v),0)}{\partial I}q - (\mu(v) + \sigma(v))\}dv < 0$$

and

$$R_1^*(\lambda, q) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \{\sigma(v)\frac{1}{q} - (\mu(v) + \gamma(v))\}dv < 0.$$

Therefore, by Theorem 1, conclusion (i) in Corollary 4 is true. This completes the proof.

**Remark 7.** For the almost periodic model (2), we define

$$R_0 = \frac{m(\sigma)m(\beta\frac{\partial f(u^*(t),0)}{\partial I})}{m(\mu + \gamma)m(\mu + \sigma)}.$$

Similarly to Remark 5, we also can propose the following open problem. That is, whether only when $R_0 < 1$ (or $R_0 > 1$) we can exactly obtain the global attractivity of the disease-free almost periodic equilibrium $(u^*(t), 0, 0, z_0(t) - u^*(t))$ (or permanence of infected $I$) in model (2).

**Remark 8.** Similarly to Remark 6, as an extension of almost periodic model (2), we can consider the asymptotic almost periodic model (2). We can establish the similar conclusions as in Remark 6.

## 6. Numerical examples

In order to illustrate the validity of our theoretical results, we give some examples and numerical simulations in this section.

**Example 1.** Take $\Lambda(t) = 20 + 0.1\sin(t)$, $\mu(t) = 1.2 + 0.1\sin(t)$, $p(t) = 0.2 + 0.1\sin(t)$, $\beta(t) = 0.25 + 0.2\sin(t)$, $r(t) = 0.2 + 0.1\sin(t)$, $\delta(t) = 0.2 + 0.1\sin(t)$, $\sigma(t) = 0.8 + 0.1\sin(t)$, $\gamma(t) = 0.8 + 0.1\sin(t)$ and $f(S, I) = \frac{SI}{1+2I^2}$ in model (2), which is a $2\pi$-periodic model. It is easy to verify that assumptions $(H_1) - (H_3)$ hold. Solving equation (4), we obtain

$$
\begin{aligned}
z_0(t) = {}& \exp(-1.2t + 0.1\cos(t) + 0.1)\int_0^t (20x + 0.1\sin(x))\exp(1.2x - 0.1\cos(x) - 0.1)dx \\
& + \frac{z_0\exp(0.1\cos(t) - 1.2t + 0.1)}{\exp(0.2)},
\end{aligned}
$$

see Fig.1 (a), by which we can see that the periodic solution $z_0(t)$ is globally asymptotically stable. Then, from numerical simulation, we can see that equation (9) has a $2\pi$-periodic solution $u^*(t)$ (see Fig.1 (b)) which is globally asymptotically stable, by which we can see that $10 < u^*(t) < 15$.

Choosing $q = 0.41$, we can see $\frac{\partial f(u^*(t),0)}{\partial I} = u^*(t)$, then we can obtain by the numerical calculation

$$
\begin{aligned}
G(t,q) = {} & \beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q + r(t) - (1 + \tfrac{1}{q})\sigma(t) \\
\leq {} & (0.25 + 0.2\sin(t)) * 15 * 0.41 + 0.2 + 0.1\sin(t) - (1 + 100/41) * (0.8 + 0.1\sin(t)) \\
\approx {} & -1.0137 + 0.9861\sin(t) < 0,
\end{aligned}
$$

$$
\begin{aligned}
R_1(\omega,q) = {} & \int_0^{2\pi} (\beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q - (\mu(t) + \sigma(t)))dt \\
\leq {} & \int_0^{2\pi} ((0.25 + 0.2\sin(t)) * 15 * 0.41 - (1.2 + 0.1\sin(t) + 0.8 + 0.1\sin(t)))dt \\
\approx {} & -2.9060 < 0,
\end{aligned}
$$

$$
\begin{aligned}
R_1^*(\omega,q) = {} & \int_0^{2\pi} (\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t)))dt \\
= {} & \int_0^{2\pi} ((0.8 + 0.1\sin(t)) * 100/41 - (1.2 + 0.1\sin(t) + 0.8 + 0.1\sin(t)))dt \\
\approx {} & -0.3065 < 0.
\end{aligned}
$$

Therefore, all conditions given in Corollary 3 are satisfied. From numerical simulations (see Fig 1. (c)-(f)), we can see that the infected $I(t)$ in model (2) is extinct, and disease-free equilibrium state $(u^*(t), 0, 0, z_0(t) - u^*(t))$ is globally attractive.

**Example 2.** Take $\Lambda(t) = 100 + 0.1\sin(t)$, $\mu(t) = 1.2 + 0.1\sin(t)$, $p(t) = 0.5 + 0.3\sin t$, $\beta(t) = 10 + 0.4\sin t$, $r(t) = 0.2 + 0.1\sin(t)$, $\delta(t) = 0.8 + 0.4\sin t$, $\sigma(t) = 3 + 0.3\sin t$, $\gamma(t) = 0.8 + 0.4\sin t$ and $f(S, I) = \frac{SI}{1+2S+3I}$ in model (2), which is a $2\pi$-periodic model. It is easy to verify that assumptions $(H_1) - (H_3)$ hold. Solving equation (4), we obtain

$$
\begin{aligned}
z_0(t) = {} & \exp(-1.2t + 0.1\cos(t) + 0.1)\int_0^t (100x + 0.1\sin(x))\exp(1.2x - 0.1\cos(x) - 0.1)dx \\
& + \frac{z_0 \exp(0.1\cos(t) - 1.2t + 0.1)}{\exp(0.2)},
\end{aligned}
$$

see Fig.2 (a), by which we can see that the periodic solution $z_0(t)$ is globally asymptotically stable. From numerical simulation, we can see that equation (9) has a $2\pi$-periodic solution $u^*(t)$ which is globally asymptotically stable(see Fig.2 (b)), by which we can see that $32 < u^*(t) < 58$.

Choosing $q = 0.07$, we can see $\frac{\partial f(u^*(t),0)}{\partial I} = \frac{u^*(t)+2u^{*2}(t)}{1+2u^*(t)} = u^*(t)$, then we can obtain by the numerical calculation

$$
\begin{aligned}
G(t,q) = \quad & \beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q + r(t) - (1+\tfrac{1}{q})\sigma(t) \\
< \quad & (10 + 0.4*\sin(t))*58*0.07 + 0.2 + 0.1\sin(t) - (1 + 1/0.07)*(3 + 0.3*\sin(t)) \\
\approx \quad & -5.0571 - 2.8617\sin(t) < 0,
\end{aligned}
$$

$$
\begin{aligned}
R_1(\omega,q) = \quad & \int_0^{2\pi}(\beta(t)\frac{\partial f(u^*(t),0)}{\partial I}q - (\mu(t) + \sigma(t)))dt \\
> \quad & \int_0^{2\pi}((10 + 0.4*\sin(t))*32*0.07 - (1.2 + 0.1*\sin(t) + 3 + 0.3*\sin(t)))dt \\
\approx \quad & 114.3542 > 0,
\end{aligned}
$$

$$
\begin{aligned}
R_1^*(\omega,q) = \quad & \int_0^{2\pi}(\sigma(t)\frac{1}{q} - (\mu(t) + \gamma(t)))dt \\
= \quad & \int_0^{2\pi}((3 + 0.3*\sin(t))*100/7 - (1.2 + 0.1*\sin(t) + 0.8 + 0.4*\sin(t)))dt \\
\approx \quad & 256.7136 > 0.
\end{aligned}
$$

Therefore, all conditions given in Corollary 3 are satisfied. From numerical simulations, we can see that system that the infected $I(t)$ in model (2) is permanent (see Fig.2 (c)-(f)). In addition, model (2) has a positive $2\pi$-periodic solution which may be globally asymptotically stable.

# 7. Conclusions

In this paper, we investigate the global dynamic behaviors of a class of SEIRS epidemic models with vaccination, nonlinear incidence and all coefficients depending on time $t$. Under the certain reasonable assumptions, some new threshold values are obtained to determine the permanence and extinction of disease for model (2). The threshold conditions for the permanence of disease has the integrable form of limit inferior, while the threshold conditions for the extinction of disease has the integrable form of limit superior. Particularly, in Section 5, we prove that when all parameters of model (2) degrade into positive constants the conditions given in Theorems 1 and 2 become the threshold conditions by the basic reproduction number $R_0$.

For the special case of model (2) in which only $\beta(t)$ and $r(t)$ in model (2) is given as $\omega$-periodic functions, incidence function $f(S,I) = SI$ and all other parameters are positive constants, then we have model (1). We obtained the new threshold criteria on the global attractivity of the disease-free periodic equilibrium and the permanence of infected $I$ for model (1) which are different from the threshold criteria given in [10]. It is easy to see that the new threshold criteria more easily verify than those in [10]. However, unfortunately,

an additional condition $G(q, t) < 0$ for all $t \in [0, \omega]$ is requested. Therefore, we proposed an open problem in Remark 5.

We also see that in model (2) we do not introduced the disease-related death rate of infected. Therefore, another open problem is to investigate the dynamical behaviors of general nonautonomous SEIRS model with vaccination, disease-related death and nonlinear incidence.

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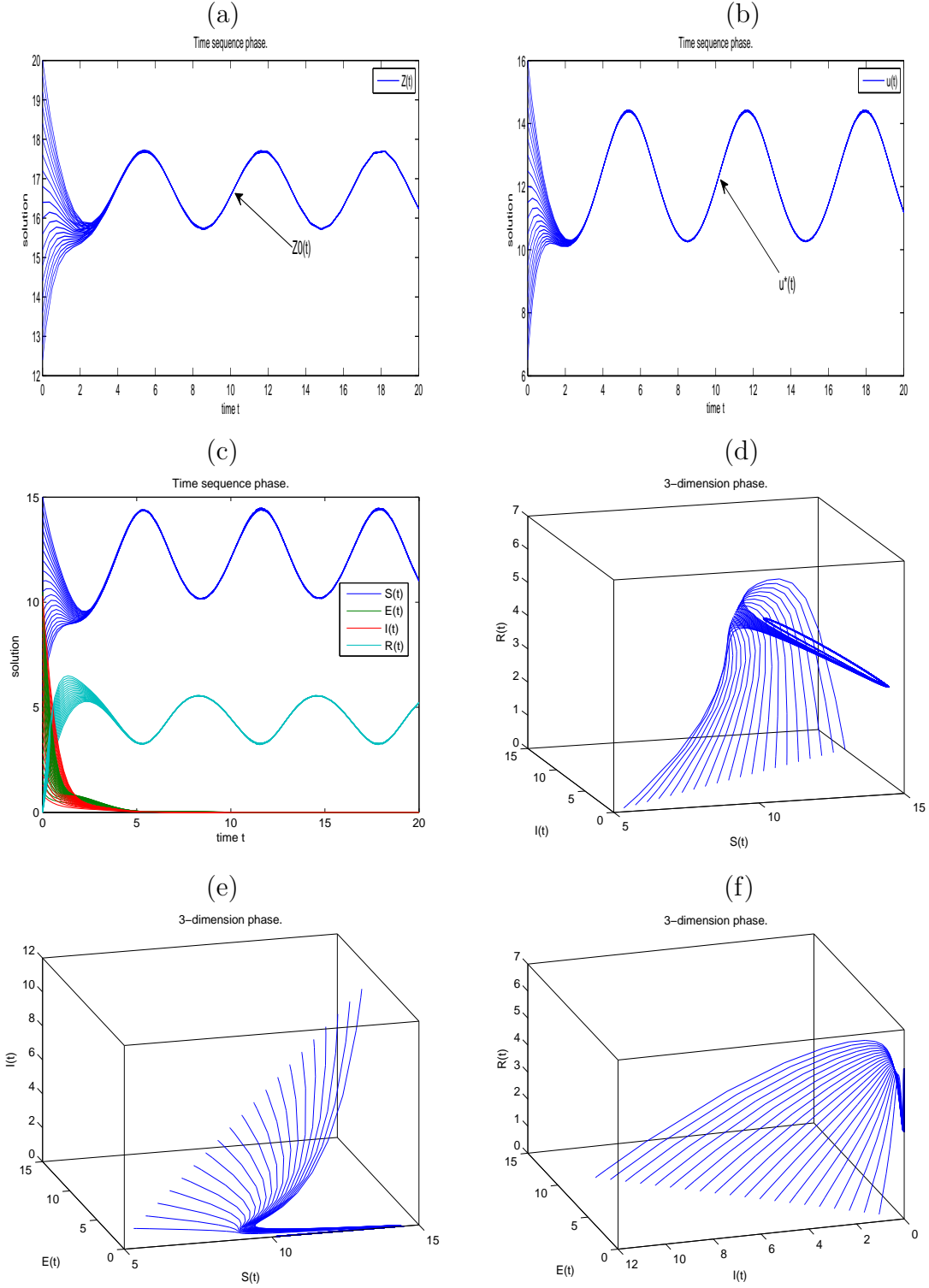
Figure 1: *(a): Time series of solutions $z(t)$ of system (4) in Example 1 with initial values: $z_0 = 12 + 0.4 * i$, $i = 1, 2, \cdots$, respectively. (b):Time series of solutions $u(t)$ of system (9) with initial values: $u_0 = 6 + i * 0.5$, $i = 1, 2, \cdots, 20$, respectively. (c): time series of solutions of $(S(t), E(t), I(t), R(t))$ of system (2), (d)-(f): 3-dimensional phases of system (2) with initial values $(S(0), E(0), I(0), R(0)) = (5 + 0.5 * i, 0.5 + 0.5 * i, 0.3 + 0.5 * i, 0)$, $i = 1, 2, \cdots, 20$, respectively.*
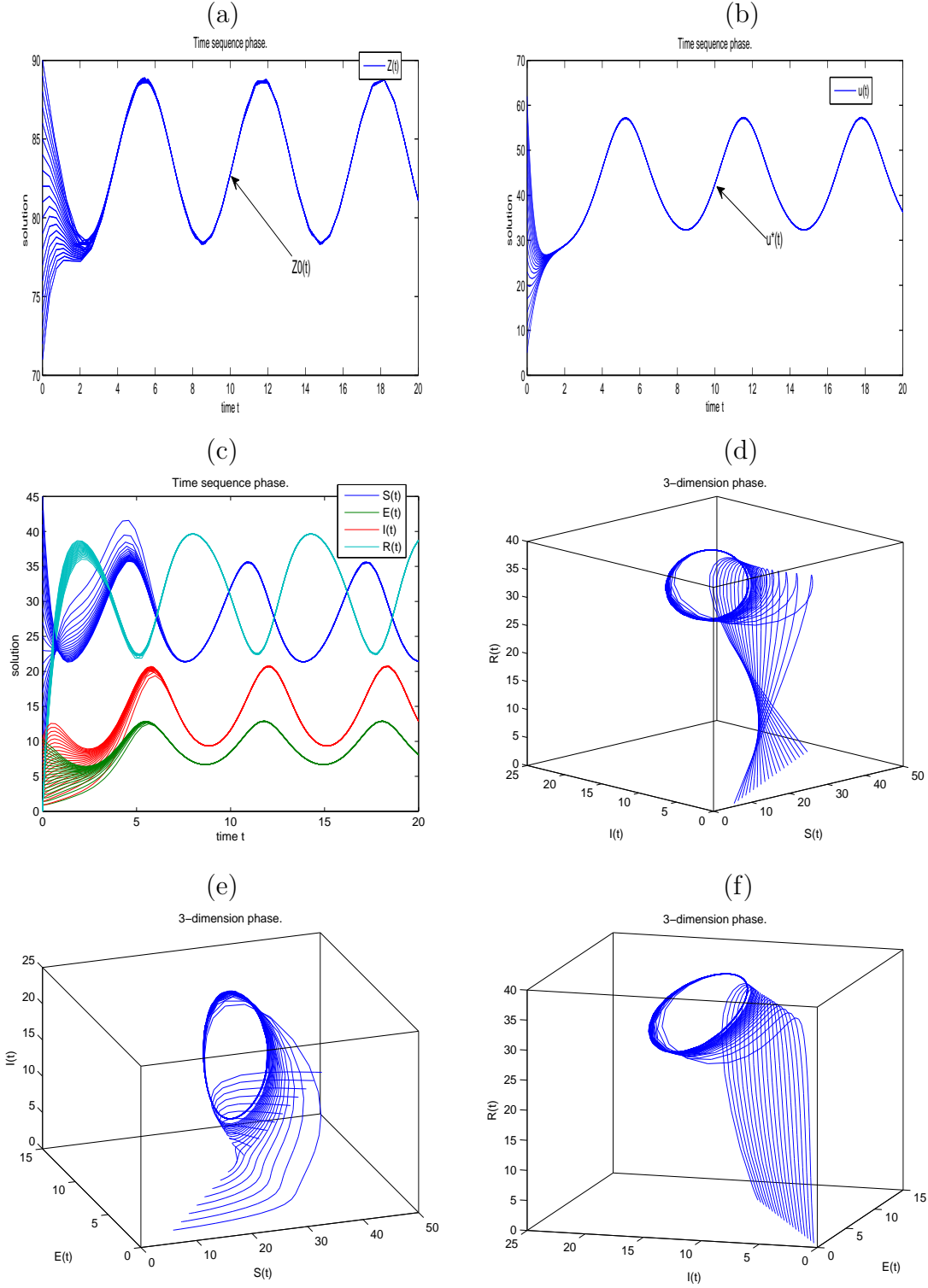
Figure 2: *(a): Time series of solutions $z(t)$ of system (4) in Example 2 with initial values: $z_0 = 70 + i$, $i = 1, 2, \cdots$, respectively. (b):Time series of solutions $u(t)$ of system (9) with initial values: $u_0 = 2 + 3 * i$, $i = 1, 2, \cdots, 20$, respectively. (c): time series of solutions of $(S(t), E(t), I(t), R(t))$ of system (2), (d)-(f): 3-dimensional phases of system (2) with initial values $(S(0), E(0), I(0), R(0)) = (5 + 2 * i, 0.5 + 0.5 * i, 0.3 + 0.5 * i, 0)$, $i = 1, 2, \cdots, 20$, respectively.*