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SHELL EQUATIONS IN TERMS OF GÜNTER'S DERIVATIVES, DERIVED BY THE Γ-CONVERGENCE

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Abstract A mixed boundary value problem for the Láme equation in a thin layer $\Omega^h : \mathscr{C} \times [-h, h]$ around a surface $\mathscr{C}$ with the Lipshitz boundary is investigated. The main goal is to find out what happens when the thickness of the layer tends to zero $h \to 0$. To this end we reformulate BVP into an equivalent variational problem and prove that the energy functional has the Γ-limit being the energy functional on the mid-surface $\mathscr{C}$. The corresponding BVP on $\mathscr{C}$, considered as the Γ-limit of the initial BVP, is written in terms of Günter's tangential derivatives on $\mathscr{C}$ and represents a new form of the shell equation. It is shown that the Neumann boundary condition from the initial BVP on the upper and lower surfaces transforms into a right-hand side term of the basic equation of the limit BVP.

Keywords Hypersurface · Günter's derivatives · Láme equation · Γ-Convergence · Shell equation

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Introduction

In the present paper we study a mixed boundary value problem for the Láme equation in a thin layer $\Omega^h := \mathscr{C} \times [-h, h]$ of thickness $2h$ around a smooth mid-hypersurface $\mathscr{C} \subset (R)^3$ written in terms of Gúnter's derivatives and the energy functional associated to it. We show that when thickness of the layer tends to zero $h \to 0$, the corresponding energy functional, scaled properly, converges in the Γ-limit sense to some functional defined on mid-surface $\mathscr{C}$ of the layer, which corresponds to two-dimensional boundary value problem for associated Euler–Lagrange equation in terms of Gúnter's derivatives. The obtained equations together with boundary conditions can be considered as a boundary value problem defined on a shell model.

Different models of shells were investigated already in nineteenth century by a mathematical justification of the well-known two-dimensional linear Kirchhoff-Love theory of plates [17]. This theory was obtained by retaining the linear terms in the in-plane displacement components and only the constant in the normal displacement as the thickness of the plate approaches zero. There exist a number of approaches proposed for modeling linearly elastic shells. Started by the classical work of brother E. and P. Cosserats [3], many authors contributed the development of the shell theories (see [15], [18], [21], [19], [16], [6] and references therein).

In 1960s, the first attempts were made to apply formal asymptotic methods in linearized elasticity [15]. Shortcoming of these attempts was the lack of convergence theorems of the scaled three-dimensional solution to the leading term of its formal expansion as thickness of the layer tends to zero, because in these works the asymptotic method was applied directly to the partial differential equations of the three-dimensional problem.

P.G. Ciarlet and P. Destuynder [7], [8], applied the formal asymptotic method to the variational formulation of the three-dimensional boundary value problems of linearly and nonlinearly elastic plates and justified the linear and nonlinear Kirchhoff-Love plate theories.

One approach to the shell theory is based on the assumption, that the energy density of the shell can be expressed as a function of the deformation gradient of the mid-surface. The natural mathematical setting in which these results can be formulated is the variational or Γ-convergence, introduced by De Giorgi (see [1] for details). This approach was used in works [13], [14] to strictly justify nonlinear plate theory for surfaces first proposed by G. Kirchhoff.

The equations of three-dimensional linearized elasticity have been studied mostly in Cartesian coordinates. The linear shell theory justified in the present paper is based on the natural curvilinear coordinates, defined on the mid-surface $\mathscr{C}$ and extended by the normal vector field of this surface, which "follow the geometry" of the shell in a most natural way. Accordingly, the purpose of the present preliminary section is to provide a thorough derivation and a mathematical treatment of the equations of linearized three-dimensional elasticity in terms of special curvilinear coordinates.

Let $\mathscr{C} \subset \mathbb{R}^3$ be an open surface with the boundary $\Gamma = \partial\mathscr{C}$ in the Euclidean space $\mathbb{R}^3$, represented by a single coordinate function $\theta \ : \ \omega \to \mathscr{C}$ where ω is an open domain in $\mathbb{R}^2$ (the case of multiple coordinate functions is similar and we skip this case for the simplicity.) Let $\boldsymbol{\nu}(\mathscr{X}) = (\nu_1(\mathscr{X}), \nu_2(\mathscr{X}), \nu_3(\mathscr{X}))^\top$, $\mathscr{X} \in \mathscr{C}$, be the normal vector field on $\mathscr{C}$ and $\mathscr{N}(x) = (\mathscr{N}_1(x), \mathscr{N}_2(x), \mathscr{N}_3(x))^\top$ be its extension in the neighbourhood Ω^h of the surface $\mathscr{C}$. It is known that such extension is unique under the assumption that the extension, as the field on the surface itself, is a gradient vector field $\partial_j \mathscr{N}_k = \partial_k \mathscr{N}_j$ for all $j, k = 1, 2, 3$ and is called the proper extension (see [11] for details).

The 3-tuple of tangential vector fields to the surface $\boldsymbol{g}_1 := \partial_1 \Theta, \boldsymbol{g}_2 := \partial_2 \Theta$ (the covariant basis) together with the proper extension $\boldsymbol{g}_3 := \mathscr{N}$ of normal vector field $\boldsymbol{\nu}$ from the surface $\mathscr{C}$ into the neighborhood $\Omega^h := \{\theta(x') + t\mathscr{N}(x') : \ x' \in \omega, \ -1 < t < 1\}$ depends only on the variable $x' \in \mathscr{C}$ and constitute a basis in Ω^h. That means, that arbitrary vector field $\boldsymbol{U} = \sum_{j=1}^{3} U_j \boldsymbol{e}^j$ can also be represented with this basis in "curvilinear coordinates". Along with the covariant basis is used the contravariant basis $\boldsymbol{g}^1, \boldsymbol{g}^2$ which is the bi-orthogonal system to the covariant basis $\langle \boldsymbol{g}_j, \boldsymbol{g}^k \rangle = \delta_{jk}$, where δ_{jk} denotes the Kroneker's symbol, $j, k = 1, 2$ (see, e.g., [5,6]). In the classical geometry the covariant $\{\langle \boldsymbol{g}_i, \boldsymbol{g}_k \rangle\}_{j,k=1,2}$ and contravariant $\left\{\langle \boldsymbol{g}^i, \boldsymbol{g}^k \rangle\right\}_{j,k=1,2}$ metric tensors together with the Christofells symbols $\Gamma^i_{jk} := \langle \boldsymbol{g}^i, \partial_j \boldsymbol{g}_k \rangle$ are the main tools of calculus on hypersurfaces. For example, the covariant derivatives on the surface $\mathscr{C}$ are defined by $v_{i\|j} := \partial_j v_i - \displaystyle\sum_{k=1}^{2} \Gamma^k_{ij} v_k$.

Our calculus on the surface $\mathscr{C}$ is based on a different curvilinear system of coordinates than the covariant and contravariant vector fields used usually by mathematicians and solid mechanics specialists to derive the shell equations (see, e.g., P. Ciarlet [5,6]). Moreover, the system of curvilinear coordinates introduced below, is linearly dependent but, surprisingly, many partial differential equations are recorded in this system in a simple form (see [10], including Laplace-Beltramy and shell equations on a hypersurface (see below).

From now on, if not stated otherwise, we stick to the following notation: terms with repeated indices are implicitly summed from 1 to 3 if indices are Greek $\alpha, \beta, \gamma, \ldots$ and are summed from 1 to 4 if indices are Latin $j, k, l, \ldots$, as shown on the examples:

$$a_\alpha b_\alpha := \sum_{\alpha=1}^{3} a_\alpha b_\alpha, \quad b_\alpha^2 := \sum_{\alpha=1}^{3} b_\alpha^2, \quad c_j d_j := \sum_{j=1}^{4} c_j d_j, \quad c_j^2 := \sum_{j=1}^{4} c_j^2.$$

We consider a deformation of an isotropic layer domain $\Omega^h := \mathscr{C} \times (-h, h)$ of thickness $2h$ around the mid-surface $\mathscr{C}$ which has the nonempty Lipschitz boundary $\partial\mathscr{C}$. The deformation is governed by the Lamé equation, with the classical mixed boundary conditions, a Dirichlet conditions on the lateral surface $\Gamma^h_L := \partial\mathscr{C} \times (-h, h)$ and a Neumann conditions on the upper and lower surfaces $\Gamma^\pm := \mathscr{C} \times \{\pm h\}$:

$$\begin{aligned}
&\mathscr{L}_{\Omega^h} \boldsymbol{U}(x) = \boldsymbol{F}(x), && x \in \Omega^h := \mathscr{C} \times (-h, h), \\
&\boldsymbol{U}^+(t) = \boldsymbol{G}(t), && t \in \Gamma^h_L := \partial\mathscr{C} \times (-h, h), \\
&(\mathfrak{T}(\mathscr{x}, \nabla)\boldsymbol{U})^+(\mathscr{x}, t) = \boldsymbol{H}(\mathscr{x}, \pm h), && (\mathscr{x}, t) \in \Gamma^\pm = \mathscr{C} \times \{\pm h\}.
\end{aligned} \tag{1}$$

Here $\boldsymbol{U}(x)=(U_1(x),U_2(x),U_3(x))^\top$ is the displacement vector, $\mathscr{L}_{\Omega^h}$ is the Lamé differential operator and $(\mathfrak{T}(\mathscr{X},\nabla)$ is the traction operator

$$\begin{aligned}&\mathscr{L}_{\Omega^h}\boldsymbol{U}=-\mu\,\boldsymbol{\Delta U}-(\lambda+\mu)\,\nabla\mathrm{div}\boldsymbol{U},\\&[\mathfrak{T}(\mathscr{X},\nabla)\boldsymbol{U}]_\beta=\lambda\,\nu_\beta\partial_\gamma\boldsymbol{U}_\gamma+\mu\,\nu_\gamma\partial_\beta\boldsymbol{U}_\gamma+\mu\,\partial_{\boldsymbol{\nu}}\boldsymbol{U}_\beta,\quad\beta=1,2,3,\end{aligned}\tag{2}$$

where $\mu>0$ and $2\mu+3\lambda>0$.

The BVP (1) we consider in the following weak classical setting:

$$\boldsymbol{U}\in\mathbb{H}^1(\Omega^h),\quad\boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\quad\boldsymbol{G}\in\mathbb{H}^{1/2}(\Gamma_L^h),\quad\boldsymbol{H}(\cdot,\pm h)\in\mathbb{H}^{-1/2}(\mathscr{C}).\tag{3}$$

For definitions of Bessel potential spaces $\mathbb{H}^s$, $\widetilde{\mathbb{H}}^s$ see, e.g., [20].

Let us consider the following subspace of $\mathbb{H}^1(\Omega^h)$:

$$\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma_L^h):=\left\{\boldsymbol{V}\in\mathbb{H}^1(\Omega^h)\ :\ \boldsymbol{V}^+(t)=0\quad\text{for all}\quad t\in\Gamma_L^h\right\}.\tag{4}$$

Theorem 1 *The BVP* (1) *in the weak classical setting* (3) *has a unique solution.*

Proof: The Láme operator $\mathscr{L}_{\Omega^h}$ is strictly positive on the subspace $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma_L^h)$

$$\langle\mathscr{L}_{\Omega^h}\boldsymbol{V},\boldsymbol{V}\rangle\geqslant M\|\boldsymbol{V}\|^2\qquad\forall\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma_L^h)\tag{5}$$

and the proof follows easily from the Lax-Milgram Lemma (see a similar proof, for example, in [12]). □

To find what happens with the BVP (1), (3) when $h\to0$, we first reformulate this BVP into the equivalent variational problem: Find the vector $\boldsymbol{U}$ which minimizes the energy functional $\mathscr{E}_{\Omega^h}(\boldsymbol{U})$ (see (5)) under the same constraints (3). It is proved that if the weak limits

$$\lim_{h\to0}\boldsymbol{F}(\mathscr{X},h\tau)=\boldsymbol{F}(\mathscr{X}),\qquad\lim_{h\to0}\frac{1}{2h}\left[\boldsymbol{H}(\mathscr{X},+h)-\boldsymbol{H}(\mathscr{X},-h)\right]=\boldsymbol{H}^{(1)}(\mathscr{X}),$$
$$\boldsymbol{F},\boldsymbol{H}^{(1)}\in\mathbb{L}_2(\mathscr{C})$$

exist in $\mathbb{L}_2(\Omega^h)$ and $\mathbb{L}_2(\mathscr{C})$, respectively, than the Γ-limit of the energy functional exists $\lim_{h\to0}\mathscr{E}_{\Omega^h}(\boldsymbol{U})=\mathscr{E}^3_{\mathscr{C}}(\overline{U})$ (cf. (9)) and the equivalent BVP on the surface $\mathscr{C}$, using the Einstein's convention, is written as follows;

$$\begin{cases}\mu\left[\Delta_{\mathscr{C}}\overline{U}_\alpha+\mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta-\mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)\right]\\[1ex]+\dfrac{4\lambda\mu}{\lambda+2\mu}\left[\mathscr{D}_\alpha\mathscr{D}_\beta\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\alpha\mathscr{D}_\beta\overline{U}_\beta\right]=\dfrac{1}{2}F_\alpha+H^{(1)}_\alpha\quad\text{on}\quad\mathscr{C},\\[1ex]\overline{U}_\alpha(t)=0\qquad\qquad\qquad\qquad\text{on}\quad\Gamma=\partial\mathscr{C},\qquad\alpha=1,2,3.\end{cases}\tag{6}$$

In (6) $\boldsymbol{\nu}:=(\nu_1,\nu_2,\nu_3)^\top$ is the unit normal vector filed on $\mathscr{C}$, $\mathscr{H}_{\mathscr{C}}$ is the mean curvature of $\mathscr{C}$, $\mathscr{D}_\alpha:=\partial_\alpha-\nu_\alpha\partial_{\boldsymbol{\nu}}$, $\alpha=1,2,3$ are the Günter's tangential derivatives on $\mathscr{C}$ (see § 1) and $\overline{\boldsymbol{U}}:=(U_1(\mathscr{X},0),U_2(\mathscr{X},0),U_3(\mathscr{X},0))^\top$, $\mathscr{X}\in\mathscr{C}$ is the trace of the displacement vector field $\boldsymbol{U}(\mathscr{X},t):=(U_1(\mathscr{X},t),U_2(\mathscr{X},t),U_3(\mathscr{X},t))^\top$, $(\mathscr{X},t)\in\Omega^h:=\mathscr{C}\times(-h,h)$ on the mid-surface $\mathscr{C}$ (see Theorem 4).

The BVP (6) represents a new 2D equation of shell in terms of Günter's tangential derivatives on the mid-surface $\mathscr{C}$.

1 Curvilinear coordinates

We commence with the definition of new system of coordinates: the system of 4-vectors

$$\boldsymbol{d}^j := \boldsymbol{e}^j - \mathscr{N}_j \mathscr{N}, \qquad j = 1,2,3 \quad \text{and} \quad \boldsymbol{d}^4 := \mathscr{N}, \tag{1}$$

where $\boldsymbol{e}^1 = (1,0,0)^\top, \boldsymbol{e}^2 = (0,1,0)^\top, \boldsymbol{e}^3 = (0,0,1)^\top$ is the Cartesian basis in $\mathbb{R}^3$; the first 3 vectors $\boldsymbol{d}^1, \boldsymbol{d}^2, \boldsymbol{d}^3$ are projections of the Cartesian vectors and are tangential to the surface $\mathscr{C}$, while the last one $\boldsymbol{d}^4 = \mathscr{N}$ is orthogonal to it and, thus, to $\boldsymbol{d}^1, \boldsymbol{d}^2, \boldsymbol{d}^3$. The system is linearly dependent, but full and any vector field $\boldsymbol{U} = U_\alpha \boldsymbol{e}^\alpha$ in Ω_h can be written in the following form:

$$\boldsymbol{U} = U_\alpha \boldsymbol{e}^\alpha = U_j^0 \boldsymbol{d}^j = \boldsymbol{U}^0 = \boldsymbol{U}_0 + U_4^0 \mathscr{N}, \tag{2}$$

$$\boldsymbol{U}_0 := \boldsymbol{U} - \langle \mathscr{N}, \boldsymbol{U} \rangle \mathscr{N}, \qquad U_4^0 := \langle \mathscr{N}, \boldsymbol{U} \rangle = \mathscr{N}_\alpha U_\alpha \tag{3}$$

and the vector $\boldsymbol{U}_0 := (U_1^0, U_2^0, U_3^0)^\top$ is chosen to be tangential to the surface $\langle \mathscr{N}, \boldsymbol{U}_0 \rangle = 0$.

Since the proper extension depends only on the surface variable $\mathscr{N}(\mathscr{x}, t) = \mathscr{N}(\mathscr{x})$ (see [11]), the same is true for the entire basis $\boldsymbol{d}^j(\mathscr{x}, t) = \boldsymbol{d}^j(\mathscr{x})$, $j = 1,2,3,4$.

Note that

$$\mathscr{N}_4 := \langle \mathscr{N}, \mathscr{N} \rangle = 1. \tag{4}$$

Although the system $\left\{\boldsymbol{d}^j\right\}_{j=1}^4$ is linearly dependent, the following holds.

Lemma 1 *The representation* (2) *is unique:*

$$If \quad \boldsymbol{U}^0 = U_j^0 \boldsymbol{d}^j = 0 \quad then \quad U_1^0 = U_2^0 = U_3^0 = U_4^0 = 0. \tag{5}$$

The scalar product and, consequently, the distance between two vectors in Cartesian and new coordinate systems coincide:

$$\langle \boldsymbol{U}^0, \boldsymbol{V}^0 \rangle = U_j^0 V_j^0 = U_\alpha V_\alpha = \langle \boldsymbol{U}, \boldsymbol{V} \rangle, \qquad \|\boldsymbol{U}^0 - \boldsymbol{V}^0\| = \|\boldsymbol{U} - \boldsymbol{V}\| \tag{6}$$

for arbitrary vectors $\boldsymbol{U} = (U_1, U_2, U_3)^\top, \boldsymbol{V} = (V_1, V_2, V_3)^\top \in \mathbb{R}^3$.

Proof: If condition (5) holds $\boldsymbol{U}^0 = 0$, then $U_4 = \langle \boldsymbol{U}, \mathscr{N} \rangle = \langle \boldsymbol{U}^0, \mathscr{N} \rangle = 0$. But then $U_j^0 = U_j - \langle \boldsymbol{U}^0, \mathscr{N} \rangle \mathscr{N}_j = U_j$, $j = 1,2,3$ and, therefore,

$$\begin{aligned} 0 = \boldsymbol{U}^0 = U_\alpha^0 \boldsymbol{d}^\alpha &= U_\alpha \boldsymbol{e}^\alpha - \mathscr{N}_\alpha U_\alpha \mathscr{N} = U_\alpha \boldsymbol{e}^\alpha - U_\alpha \mathscr{N}_\alpha \mathscr{N} \\ &= U_\alpha \boldsymbol{e}^\alpha - \langle \boldsymbol{U}, \mathscr{N} \rangle \mathscr{N} = U_\alpha \boldsymbol{e}^\alpha \end{aligned}$$

which implies $U_j^0 = U_j = 0$, $j = 1,2,3$, because the Cartesian basis $\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3$ is linearly independent.

Let us prove the first equality in (6):

$$\begin{aligned}\langle \boldsymbol{U}^0, \boldsymbol{V}^0\rangle = U^0_j V^0_j &= (U_\alpha - \mathscr{N}_\alpha\langle \mathscr{N}, \boldsymbol{U}\rangle)(V_\alpha - \mathscr{N}_\alpha\langle \mathscr{N}, \boldsymbol{V}\rangle) + \langle \mathscr{N}, \boldsymbol{U}\rangle\langle \mathscr{N}, \boldsymbol{V}\rangle\\ &= \big[U_\alpha V_\alpha - \langle \mathscr{N}, \boldsymbol{V}\rangle U_\alpha \mathscr{N}_\alpha - \langle \mathscr{N}, \boldsymbol{U}\rangle V_\alpha \mathscr{N}_\alpha + \langle \mathscr{N}, \boldsymbol{U}\rangle\langle \mathscr{N}, \boldsymbol{V}\rangle \mathscr{N}_\alpha^2\big]\\ &= U_\alpha V_\alpha = \langle \boldsymbol{U}, \boldsymbol{V}\rangle\end{aligned}$$

and the equality is proved.

The second equality in (6) is a simple consequence of the first one because

$$\|\boldsymbol{U}^0 - \boldsymbol{V}^0\| = \sqrt{\langle \boldsymbol{U}^0 - \boldsymbol{V}^0, \boldsymbol{U}^0 - \boldsymbol{V}^0\rangle} = \sqrt{\langle \boldsymbol{U} - \boldsymbol{V}, \boldsymbol{U} - \boldsymbol{V}\rangle} = \|\boldsymbol{U} - \boldsymbol{V}\|. \quad \square$$

The Günter's derivatives

$$\mathscr{D}_\alpha\varphi := \partial_\alpha\varphi - \nu_\alpha\partial_\nu\varphi, \qquad \alpha = 1, 2, 3 \tag{7}$$

represent tangential differential operators on the surface $\mathscr{C}$ (orthogonal projections of the coordinate derivatives $\partial_1, \partial_2, \partial_3$ and have extensions

$$\mathscr{D}_\alpha\varphi := \partial_\alpha\varphi - \mathscr{N}_\alpha\partial_{\mathscr{N}}\varphi$$

in the neighbourhood of the surface $\mathscr{C}$. The system $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$ is, obviously, linearly dependent, but full: any tangential linear differential operator on the surface $\boldsymbol{A}(D)$ is written in the following form:

$$\boldsymbol{A}(D) = a_\alpha(\mathscr{X})\partial_\alpha = a_\alpha(\mathscr{X})\mathscr{D}_\alpha, \qquad \text{provided} \quad a_\alpha(x)\nu_\alpha(\mathscr{X}) \equiv 0, \quad \mathscr{X} \in \mathscr{C}.$$

In particular

$$\partial_{\boldsymbol{U}} = U_\alpha\partial_\alpha = U^0_j\mathscr{D}_j.$$

The adjoint operator to $\mathscr{D}_j,\ j = 1, 2, 3$, is

$$\mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi + 2\nu_j\mathscr{H}_{\mathscr{C}}\varphi\,, \qquad \varphi \in \mathbb{C}^1(\mathscr{C})\,, \tag{8}$$

where

$$\mathscr{H}_{\mathscr{C}}(\mathscr{X}) := \frac{1}{2}\mathscr{D}_\alpha\nu_\alpha(\mathscr{X}) = \frac{1}{2}\mathscr{D}_\alpha\mathscr{N}_\alpha(\mathscr{X})\,, \qquad \mathscr{X} \in \mathscr{C} \tag{9}$$

is the mean curvature of the surface $\mathscr{C}$.

Definition 1 For a function $\varphi \in \mathbb{W}^1(\Omega^h)$ we define the extended gradient

$$\nabla_{\Omega^h}\varphi = \Big\{\mathscr{D}_1\varphi, \mathscr{D}_2\varphi, \mathscr{D}_3\varphi, \mathscr{D}_4\varphi\Big\}^\top, \qquad \mathscr{D}_4\varphi := \partial_{\mathscr{N}}\varphi \tag{10}$$

and, for a vector field $\boldsymbol{U} = U_\alpha\boldsymbol{e}^\alpha = U^0_j\boldsymbol{d}^j \in \mathbb{W}^1(\Omega^h)$–the extended divergence

$$\mathrm{div}_{\Omega^h}\boldsymbol{U} := \mathscr{D}_jU^0_j + 2\mathscr{H}_{\mathscr{C}}U^0_4 = -\nabla^*_{\Omega^h}\boldsymbol{U}\,, \tag{11}$$

where $\nabla^*_{\Omega^h}$ denotes the formally adjoint operator to the gradient ∇_{Ω^h}, $\mathscr{H}_{\mathscr{C}}$ is the mean curvature (cf.(9) and

$$\mathscr{D}_4U^0_4 := \partial_{\mathscr{N}}U^0_4 = \langle \mathscr{N}, \partial_{\mathscr{N}}\boldsymbol{U}\rangle = (\mathscr{D}_4\boldsymbol{U})^0_4.$$

Caution: While defining the extended divergence in (11) we have to use only the representation $\boldsymbol{U} = U_j^0 \boldsymbol{d}^j$ (cf. (2)), because any other representation differs from the indicated one by the vector $c\mathscr{N}$, where $c(\mathscr{x})$ is an arbitrary function. Then the extended divergences will differ by the summand $\mathrm{div}_{\Omega^h}\big(c(\mathscr{x})\mathscr{N}(\mathscr{x})\big) = \partial_{\mathscr{N}} c(\mathscr{x}) + 2c(\mathscr{x})\mathscr{H}_{\mathscr{C}}(\mathscr{x})$

Lemma 2 *The classical gradient $\nabla\varphi := \Big\{\partial_1\varphi, \partial_2\varphi, \partial_3\varphi\Big\}^\top$, written in the full system of vectors $\big\{\boldsymbol{d}^j\big\}_{j=1}^4$ in* (1) *coincides with the extended gradient $\nabla\varphi = \nabla_{\Omega^h}\varphi$ in* (10).

The classical divergence $\mathrm{div}\,\boldsymbol{U} := \partial_\alpha U_\alpha$ of a vector field $\boldsymbol{U} := U_\alpha \boldsymbol{e}^\alpha$, written in the full system (1)*, coincides with the extended divergence:* $\mathrm{div}\,\boldsymbol{U} = \mathrm{div}_{\Omega^h}\boldsymbol{U}^0$ *in* (11).

The gradient and the negative divergence are adjoint operators $\nabla^*_{\Omega^h} = -\mathrm{div}_{\Omega^h}$ with respect to the scalar product induced from the ambient Euclidean space $\mathbb{R}^n$.

The classical Laplace operator in the domain Ω_h

$$\varDelta_{\Omega^h}\varphi(x) := (\mathrm{div}_{\Omega^h}\nabla_{\Omega^h}\varphi)(x) = -(\nabla^*_{\Omega^h}\Big(\nabla_{\Omega^h}\varphi\Big)(x) \qquad x\in\Omega^h,$$

written in the full system (1)*, acquires the following form*

$$\varDelta_{\Omega^h}\varphi = \mathscr{D}_j^2\varphi + 2\mathscr{H}_{\mathscr{C}}\mathscr{D}_4\varphi\,, \qquad \varphi\in\mathbb{W}^2(\Omega^h)\,. \tag{12}$$

Proof: Formulae (10) for the extended gradient follows from the choice of the new coordinate vectors (1):

$$\begin{aligned}\nabla\varphi := \Big\{\partial_1\varphi, \partial_2\varphi, \partial_3\varphi\Big\}^\top &= \boldsymbol{e}^\alpha\partial_\alpha\varphi = \boldsymbol{e}^\alpha(\mathscr{D}_\alpha\varphi + \mathscr{N}_\alpha\mathscr{D}_4\varphi)\\ =\ \boldsymbol{e}^\alpha\mathscr{D}_\alpha\varphi + \mathscr{N}\mathscr{D}_4\varphi &= \boldsymbol{d}^j\mathscr{D}_j\varphi = \nabla_{\Omega^h}\varphi\end{aligned}$$

since

$$\partial_\alpha = \mathscr{D}_\alpha + \mathscr{N}_\alpha\mathscr{D}_4, \qquad \boldsymbol{e}^\alpha\mathscr{D}_\alpha\varphi = \boldsymbol{d}^\alpha\mathscr{D}_\alpha\varphi. \tag{13}$$

By applying

$$\begin{gathered}\mathscr{D}_4 = \partial_{\mathscr{N}}, \quad \mathscr{N}_\alpha U_\alpha = U_4^0, \quad \mathscr{N}_\alpha\mathscr{D}_\alpha = 0, \quad \mathscr{D}_\alpha\mathscr{N}_\alpha = 2\mathscr{H}_{\mathscr{C}},\\ \mathscr{D}_4\mathscr{N}_\alpha = \partial_{\mathscr{N}}\mathscr{N}_\alpha = 0, \qquad \alpha = 1,2,3,\end{gathered} \tag{14}$$

we prove formulae (11) for the extended divergence:

$$\begin{aligned}\mathrm{div}\,\boldsymbol{U} &= \partial_\alpha U_\alpha = (\mathscr{D}_\alpha + \mathscr{N}_\alpha\mathscr{D}_4)(U_\alpha^0 + \mathscr{N}_\alpha U_4^0)\\ &= \mathscr{D}_\alpha U_\alpha^0 + \mathscr{N}_\alpha\mathscr{D}_4 U_\alpha + (\mathscr{D}_\alpha\mathscr{N}_\alpha)U_4^0 + \mathscr{N}_\alpha\mathscr{D}_\alpha U_4^0\\ &= \mathscr{D}_\alpha U_\alpha^0 + \mathscr{D}_4\left[\mathscr{N}_\alpha U_\alpha\right] + \mathscr{H}_{\mathscr{C}} U_4^0\\ &= \mathscr{D}_j U_j^0 + \mathscr{H}_{\mathscr{C}} U_4^0 = \mathrm{div}_{\Omega^h}\boldsymbol{U}^0\,.\end{aligned}$$

Formulae (6), (10) and (11) combined with the classical equality $\nabla^* = -\mathrm{div}$ ensures the equality

$$(\nabla_{\Omega^h}\boldsymbol{U}^0, \boldsymbol{V}^0) = (\nabla\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{U}, -\mathrm{div}\boldsymbol{V}) = (\boldsymbol{U}^0, -\mathrm{div}_{\Omega^h}\boldsymbol{V}^0),$$

which proves that $\nabla^*_{\Omega^h} = -\mathrm{div}_{\Omega^h}$ (cf (11)). The latter can also be verified by direct calculations.

Formula (12) is proved by applying (10) and (11):

$$\begin{aligned}\Delta\varphi &:= \mathrm{div}_{\Omega^h}\nabla_{\Omega^h}\varphi = \mathscr{D}_j^2\varphi + 2\mathscr{H}_{\mathscr{C}}\langle\mathscr{N},\nabla_{\Omega^h}\varphi\rangle = \mathscr{D}_j^2\varphi + 2\mathscr{H}_{\mathscr{C}}\langle\mathscr{N},\nabla\varphi\rangle\\ &= \mathscr{D}_j^2\varphi + 2\mathscr{H}_{\mathscr{C}}\mathscr{D}_4\varphi = \Delta_{\Omega^h}\varphi.\end{aligned}$$

□

Lemma 3 *A matrix-operator $\boldsymbol{A} = [\boldsymbol{A}_{\alpha\beta}]_{3\times3}$ written in curvilinear coordinates* (1)-(3) *acquires the form:*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \langle\boldsymbol{A}_{1,\cdot},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \langle\boldsymbol{A}_{2,\cdot},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \langle\boldsymbol{A}_{3,\cdot},\boldsymbol{\nu}\rangle\\ \langle\boldsymbol{A}_{\cdot,1},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}_{\cdot,2},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}_{\cdot,3},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}\boldsymbol{\nu},\boldsymbol{\nu}\rangle\end{bmatrix} \tag{15}$$

$$\boldsymbol{A}_{\alpha,\cdot} := (\boldsymbol{A}_{\alpha,1},\boldsymbol{A}_{\alpha,2},\boldsymbol{A}_{\alpha,3})^\top,\quad \boldsymbol{A}_{\cdot,\alpha} := (\boldsymbol{A}_{1,\alpha},\boldsymbol{A}_{2,\alpha},\boldsymbol{A}_{3,\alpha})^\top,\quad \alpha=1,2,3,$$

Proof: Indeed, using the representations (2) amd (1) of a vector function $\boldsymbol{U} = (U_1,U_2,U_3)$ and of the coordinate vectors $\boldsymbol{e}^\alpha$ we verify the claimed equality (15) by direct calculations:

$$\begin{aligned}\boldsymbol{A}\boldsymbol{U} &= \boldsymbol{A}_{\alpha\beta}U_\beta\boldsymbol{e}^\alpha = \boldsymbol{A}_{\alpha\beta}(U^0_\beta + \nu_\beta U^0_4)(\boldsymbol{d}^\alpha + \nu_\alpha\boldsymbol{d}^4)\\ &= \boldsymbol{A}_{\alpha\beta}U^0_\beta\boldsymbol{d}^\alpha + \boldsymbol{A}_{\alpha\beta}\nu_\alpha U^0_\beta\boldsymbol{d}^4 + \boldsymbol{A}_{\alpha\beta}\nu_\beta U^0_4\boldsymbol{d}^\alpha + \boldsymbol{A}_{\alpha\beta}\nu_\alpha\nu_\beta U^0_4\boldsymbol{d}^4\\ &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \langle\boldsymbol{A}_{1,\cdot},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \langle\boldsymbol{A}_{2,\cdot},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \langle\boldsymbol{A}_{3,\cdot},\boldsymbol{\nu}\rangle\\ \langle\boldsymbol{A}_{\cdot,1},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}_{\cdot,2},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}_{\cdot,3},\boldsymbol{\nu}\rangle & \langle\boldsymbol{A}\boldsymbol{\nu},\boldsymbol{\nu}\rangle\end{bmatrix}\begin{bmatrix}U^0_1\\U^0_2\\U^0_3\\U^0_4\end{bmatrix}.\end{aligned}$$

□

The Lamé operator

$$\begin{aligned}\mathscr{L}\boldsymbol{U} &= -\mu\,\boldsymbol{\Delta}\boldsymbol{U} - (\lambda+\mu)\,\nabla\mathrm{div}\boldsymbol{U} = -\left[\mu\delta_{\alpha\beta}\partial_k^2 + (\lambda+\mu)\partial_\alpha\partial_\beta\right]_{3\times3}\boldsymbol{U}\\ &= -\left[c_{\alpha\gamma\beta\omega}\partial_\gamma\partial_\omega\right]_{3\times3}\boldsymbol{U},\qquad c_{\alpha\gamma\beta\omega} = \lambda\delta_{\alpha\gamma}\delta_{\beta\omega} + \mu(\delta_{\alpha\beta}\delta_{\gamma\omega} + \delta_{\alpha\omega}\delta_{\beta\gamma})\end{aligned} \tag{16}$$

is formally self-adjoint differential operator of the second order and, written in the full system (1), acquires the form

$$\mathscr{L}_{\Omega^h}\boldsymbol{U}^0 = -\mu\,\boldsymbol{\Delta}_{\Omega^h}\boldsymbol{U}^0 - (\lambda+\mu)\,\nabla_{\Omega^h}\mathrm{div}_{\Omega^h}\boldsymbol{U}^0. \tag{17}$$

To reformulate the BVP (1) in curvilinear coordinates we also need to represent the traction operator (cf. (2))

$$\begin{aligned}\mathfrak{T}(x,\partial)\boldsymbol{U} = (\mathfrak{T}_{\alpha\beta}(x,\partial)\boldsymbol{U}_\beta)\,e^\alpha = (\{\lambda\nu_\alpha\partial_\beta + \mu\nu_\beta\partial_\alpha + \delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\}\,U_\beta)\,\boldsymbol{e}^\alpha,\\ \boldsymbol{U} = (U_1,U_2,U_3)^\top = U_\alpha\boldsymbol{e}^\alpha\end{aligned} \tag{18}$$

in Gunter's derivatives. For this we apply the representations (1) amd (7) of the coordinate vectors $\boldsymbol{e}^\alpha$ and the differential operators ∂_α, take into account the equalities

$$\partial_{\boldsymbol{\nu}} = \mathscr{D}_4,\qquad \nu_\alpha\boldsymbol{e}^\alpha = \boldsymbol{\nu} = \boldsymbol{d}^4,\qquad \nu_\alpha\boldsymbol{d}^\alpha = 0,\qquad \nu_\alpha^2 = 1.$$

and find the claimed representation of the traction operator $\mathfrak{T}(x,\partial)$ by direct calculations:

$$\begin{aligned}\mathfrak{T}(\mathscr{x},\mathscr{D}) &= \boldsymbol{e}^\alpha\otimes\boldsymbol{e}^\beta\left\{\lambda\nu_\alpha\partial_\beta+\mu\nu_\beta\partial_\alpha+\delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\right\}\\ &= \lambda\boldsymbol{d}^4\otimes(\boldsymbol{d}^\beta+\nu_\beta\boldsymbol{d}^4)(\mathscr{D}_\beta+\nu_\beta\mathscr{D}_4)\\ &\quad+\mu(\boldsymbol{d}^\beta+\nu_\alpha\boldsymbol{d}^4)\otimes(\boldsymbol{d}^\beta+\nu_\beta\boldsymbol{d}^4)\mathscr{D}_4\\ &\quad+\mu(\boldsymbol{d}^\beta+\nu_\beta\boldsymbol{d}^4)\otimes\boldsymbol{d}^4(\mathscr{D}_\beta+\nu_\beta\mathscr{D}_4)\\ &= \begin{bmatrix}\mu\mathscr{D}_4 & 0 & 0 & \mu\mathscr{D}_1\\ 0 & \mu\mathscr{D}_4 & 0 & \mu\mathscr{D}_2\\ 0 & 0 & \mu\mathscr{D}_4 & \mu\mathscr{D}_3\\ \lambda\mathscr{D}_1 & \lambda\mathscr{D}_2 & \lambda\mathscr{D}_3 & (\lambda+2\mu)\mathscr{D}_4\end{bmatrix}.\end{aligned} \tag{19}$$

2 Variational reformulation of the problem

To apply the method of Γ-convergence we have to reformulate the BVP (1) in an equivalent varational problem for the energy functional. For this we need to consider the BVP with the vanishing Dirichlet condition on the lateral surface

$$\begin{aligned}&\mathscr{L}_{\Omega^h}\boldsymbol{U}_0(x)=\boldsymbol{F}_0(x), && x\in\Omega^h:=\mathscr{C}\times(-h,h),\\ &\boldsymbol{U}_0^+(t)=0, && t\in\Gamma^h_L:=\partial\mathscr{C}\times(-h,h),\\ &(\mathfrak{T}(\mathscr{x},\nabla)\boldsymbol{U}_0)^+(\mathscr{x},\pm h)=\boldsymbol{H}_0(\mathscr{x},\pm h), && \mathscr{x}\in\mathscr{C}.\end{aligned} \tag{1}$$

It is possible to rewrite the BVP (1) in an equivalent BVP (1). Indeed, consider the BVP

$$\begin{aligned}&\mathscr{L}_{\Omega^h}\boldsymbol{V}(x)=0, && x\in\Omega^h:=\mathscr{C}\times(-h,h),\\ &\boldsymbol{V}^+(t)=\boldsymbol{G}(t), && t\in\Gamma^h_L,\\ &(\mathfrak{T}((\mathscr{x},\nabla)\boldsymbol{V})^+(\mathscr{x},\pm h)=0, && (\mathscr{x},\pm h)\in\Gamma^\pm=\mathscr{C}\times\{\pm h\},\end{aligned} \tag{2}$$

which has a unique solution $\boldsymbol{V}\in\mathbb{W}^1(\Omega^h)$ (see Theorem 1) and note, that the difference $\boldsymbol{U}_0:=\boldsymbol{U}-\boldsymbol{V}$ of solutions to BVPs (1) and (2) is a solution to the BVP (3), where $\boldsymbol{F}_0(\mathscr{x})=\boldsymbol{F}(\mathscr{x})-\mathscr{L}_{\Omega^h}\boldsymbol{V}(\mathscr{x})$, $\boldsymbol{H}_0(\mathscr{x},\pm h):=\boldsymbol{H}(\mathscr{x},\pm h)-(\mathfrak{T}((\mathscr{x},\nabla)\boldsymbol{V})^+(\mathscr{x},\pm h)$. Vice versa, solution to the BVP (1) is recovered as the sum of solutions $\boldsymbol{U}=\boldsymbol{U}_0+\boldsymbol{V}$ of the BVPs (3) and (2).

Thus, in the BVP (1) we can assume, without restricting generality, that $\boldsymbol{G}=0$ and consider the BVP

$$\begin{aligned}&\mathscr{L}_{\Omega^h}\boldsymbol{U}(x)=\boldsymbol{F}(x), && x\in\Omega^h:=\mathscr{C}\times(-h,h),\\ &\boldsymbol{U}^+(t)=0, && t\in\Gamma^h_L:=\partial\mathscr{C}\times(-h,h),\\ &(\mathfrak{T}(\mathscr{x},\nabla)\boldsymbol{U})^+(\mathscr{x},\pm h)=\boldsymbol{H}(\mathscr{x},\pm h), && \mathscr{x}\in\mathscr{C}.\end{aligned} \tag{3}$$

Theorem 2 *The problem* (3) *with the constraints*

$$\boldsymbol{U}\in\mathbb{H}^1(\Omega^h,\Gamma^h_L),\qquad \boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\qquad \boldsymbol{H}(\cdot,\pm h)\in\mathbb{H}^{-1/2}(\mathscr{C}) \tag{4}$$

is reformulated into the following equivalent variational problem: Under the same constraints (4) *look for a displacement vector-function $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$, which is a stationary point of the following functional*

$$\begin{aligned}
&\mathscr{E}_{\Omega^h}(\boldsymbol{U}):=\frac{1}{2}\int_{\Omega^h}\Big[\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\beta\boldsymbol{U}_\alpha+\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\alpha\boldsymbol{U}_\beta+\lambda\partial_\alpha\boldsymbol{U}_\alpha\cdot\partial_\gamma\boldsymbol{U}_\gamma\\
&+2\boldsymbol{F}_\beta\cdot\boldsymbol{U}_\beta\Big]dx+\int_{\mathscr{C}}\big[\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{U}^+(\mathscr{x},+h)\rangle-\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{U}^+(\mathscr{x},-h)\rangle\big]\,d\sigma\\
&=\frac{1}{2}\int_{-h}^{h}\int_{\mathscr{C}}\Big[\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\beta\boldsymbol{U}_\alpha+\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\alpha\boldsymbol{U}_\beta+\lambda\partial_\alpha\boldsymbol{U}_\alpha\cdot\partial_\gamma\boldsymbol{U}_\gamma+2\boldsymbol{F}_\beta\cdot\boldsymbol{U}_\beta\\
&+\frac{1}{h}\big[\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{U}^+(\mathscr{x},+h)\rangle-\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{U}^+(\mathscr{x},-h)\rangle\big]\Big]d\sigma dt,
\end{aligned}\tag{5}$$

Remark 1 The integral on $\mathscr{C}$ in (5) is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C})$ and $\mathbb{H}^{-1/2}(\mathscr{C})$ because $\boldsymbol{H}(\cdot,\pm h)\in\mathbb{H}^{-1/2}(\mathscr{C}_N)$ and the condition $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ implies the inclusion $\boldsymbol{U}^+(\cdot,\pm h)\in\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$.

Prior proving Theorem 2 we prove the following auxiliary lemma.

Lemma 4 *In the representation of the energy functional $\mathscr{E}_{\Omega^h}(\boldsymbol{U})$ from* (5)

$$\mathscr{E}_{\Omega^h}(\boldsymbol{U})=\frac{1}{2}\mathscr{Q}(\boldsymbol{U})-\mathscr{F}(\boldsymbol{U}) \tag{6}$$

as the sum of quadratic and linear functionals

$$\begin{aligned}
&\mathscr{Q}(\boldsymbol{U}\!\!\!=)\int_{-h}^{h}\int_{\mathscr{C}}\Big[\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\beta\boldsymbol{U}_\alpha+\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\alpha\boldsymbol{U}_\beta+\lambda\partial_\alpha\boldsymbol{U}_\alpha\cdot\partial_\gamma\boldsymbol{U}_\gamma\Big]d\sigma dt\\
&\mathscr{F}(\boldsymbol{U}\!\!\!=)-\int_{-h}^{h}\int_{\mathscr{C}}\Big[\boldsymbol{F}_\beta\cdot\boldsymbol{U}_\beta+\frac{1}{2h}\big[\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{U}^+(\mathscr{x},+h)\rangle\\
&\qquad-\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{U}^+(\mathscr{x},-h)\rangle\big]\Big]d\sigma dt
\end{aligned}$$

the quadratic part $\mathscr{Q}(\boldsymbol{U})$ is positive definite on the space $\mathbb{H}^1(\Omega^h,\Gamma^h_L)$:

$$\mathscr{Q}(\boldsymbol{U})\geq\|\boldsymbol{U}\big|\mathbb{H}^1(\Omega^h)\|,\qquad \textit{for all}\quad \boldsymbol{U}\in\mathbb{H}^1(\Omega^h,\Gamma^h_L). \tag{7}$$

Proof: Note, that

$$\begin{aligned}
\mathscr{Q}(\boldsymbol{U})&=\int_{-h}^{h}\int_{\mathscr{C}}Q(\nabla\boldsymbol{U}(\mathscr{x},t))d\sigma dt \\
Q(\boldsymbol{F})&=2\mu|\boldsymbol{E}|^2+\lambda(\mathrm{Trace}\,\boldsymbol{E})^2,\quad \boldsymbol{E}=\frac{1}{2}(\boldsymbol{F}+\boldsymbol{F}^\top),
\end{aligned}\tag{8}$$

where $\boldsymbol{E}=[E_{\alpha\beta}]_{3\times3}$ is a 3×3 matrix and $|\boldsymbol{E}|^2=\mathrm{Trace}(\boldsymbol{E}^\top\boldsymbol{E})=\sum\limits_{\alpha,\beta}E^2_{\alpha\beta}$.

Indeed, the equality (8) holds because

$$Q(\nabla\boldsymbol{U})=\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\beta\boldsymbol{U}_\alpha+\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\alpha\boldsymbol{U}_\beta+\lambda\partial_\alpha\boldsymbol{U}_\alpha\cdot\partial_\gamma\boldsymbol{U}_\gamma$$
$$=\frac{1}{2}\mu(\partial_\alpha\boldsymbol{U}_\beta+\partial_\beta\boldsymbol{U}_\alpha)(\partial_\alpha\boldsymbol{U}_\beta+\partial_\beta\boldsymbol{U}_\alpha)+\lambda(\partial_\alpha\boldsymbol{U}_\alpha)(\partial_\beta\boldsymbol{U}_\beta)$$
$$=2\mu\left|\mathrm{Def}(\boldsymbol{U})\right|^2+\lambda\left(\mathrm{Trace\ Def}(\boldsymbol{U})\right)^2,\qquad \mathrm{Def}(\boldsymbol{U}):=\frac{\nabla\boldsymbol{U}+(\nabla\boldsymbol{U})^\top}{2}. \quad (9)$$

To prove the positive definiteness (7) let us rewrite the expression (9), using the notation $\boldsymbol{E}=\dfrac{\boldsymbol{F}+\boldsymbol{F}^\top}{2}$, as follows

$$\begin{aligned}Q(\boldsymbol{F})\ &=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+2\mu\sum_{\alpha}E^2_{\alpha\alpha}+(\mu+\lambda)\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}-\mu\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}\\ &=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\Big[2\sum_{\alpha}E^2_{\alpha\alpha}-\sum_{\alpha\neq\beta}E_{\alpha\alpha}E_{\beta\beta}\Big]\\ &=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha}-E_{\beta\beta})^2\geq 0, \quad (10)\end{aligned}$$

since $\mu>0,\ \mu+\lambda>\dfrac{2\mu+3\lambda}{3}>0$ (see (2)).

From (10) follows that if $Q_4(\boldsymbol{F})=0$, $\boldsymbol{F}=\nabla\boldsymbol{U}$, then the deformation tensor is zero $\boldsymbol{E}=\mathrm{Def}(\boldsymbol{U}):=\dfrac{\nabla\boldsymbol{U}+(\nabla\boldsymbol{U})^\top}{2}=0$. The latter infers that

$$\begin{aligned}&\boldsymbol{U}(x)=a+\mathscr{B}x=a+b\wedge x\,,\\ &a=(a_1,a_2,a_3)^\top,\ b:=(b_1,b_2,b_3)^\top\in\mathbb{R}^3\,,\quad x\in\Omega^\varepsilon\,,\end{aligned}\qquad \mathscr{B}:=\begin{bmatrix}0&-b_3&b_2\\ b_3&0&-b_1\\ -b_2&b_1&0\end{bmatrix}.$$

Since $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ vanishes on the lateral surface Γ_L. Then the vector $\boldsymbol{U}$ is identically zero (see [4]). □

Proof of Theorem 2: Let $\boldsymbol{U}$ be a solution to the mixed problem (3). By taking the scalar product of the first equation $\mathscr{L}_{\Omega^h}\boldsymbol{U}(x)=\boldsymbol{F}(x)$ in (3) with a function $\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ and applying the Green's formulae we get the following equality:

$$\int\limits_{\Omega^h}\big\langle\boldsymbol{F}(x),\boldsymbol{V}(x)\big\rangle\,dx=\int\limits_{\Omega^h}\big\langle\mathscr{L}_{\Omega^h}\boldsymbol{U}(x),\boldsymbol{V}(x)\big\rangle\,dx$$
$$=-\int_{\Omega^h}\Big[\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\beta\boldsymbol{V}_\alpha+\mu\partial_\beta\boldsymbol{U}_\alpha\cdot\partial_\alpha\boldsymbol{V}_\beta$$
$$+\lambda\partial_\alpha\boldsymbol{U}_\alpha\cdot\partial_\gamma\boldsymbol{V}_\gamma\Big]dx+\int\limits_{\Gamma^h_L}\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(y)\rangle d\sigma$$
$$+\int_{\mathscr{C}}\big[\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(\mathscr{x},+h)\rangle-\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(\mathscr{x},-h)\rangle\big]\,d\sigma.$$

By inserting the boundary conditions from (3) and recalling that the trace $\boldsymbol{V}^+$ of a vector-function $\boldsymbol{V} \in \mathbb{H}^1(\Omega^h, \Gamma_L^h)$ vanishes on Γ_L, we derive that the solution $\boldsymbol{U}$ to the BVP (3) solves the following variational problem for arbitrary trial function $\boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$:

$$\begin{aligned}\int_{\Omega^h} &\Big[\mu\partial_\beta \boldsymbol{U}_\alpha \cdot \partial_\beta \boldsymbol{V}_\alpha + \mu\partial_\beta \boldsymbol{U}_\alpha \cdot \partial_\alpha \boldsymbol{V}_\beta + \lambda\partial_\alpha \boldsymbol{U}_\alpha \cdot \partial_\gamma \boldsymbol{V}_\gamma\Big] dx \\ &= \int_{\mathscr{C}} \Big[\langle \boldsymbol{H}(\mathscr{x}, +h), \boldsymbol{V}^+(\mathscr{x}, +h)\rangle - \langle \boldsymbol{H}(\mathscr{x}, -h), \boldsymbol{V}^+(\mathscr{x}, -h)\rangle\Big] d\sigma \\ &\quad - \int_{\Omega^h} \langle \boldsymbol{F}(x), \boldsymbol{V}(x)\rangle dx. \end{aligned} \tag{11}$$

Next note, that the quadratic form (i.e. when $\boldsymbol{V} = \boldsymbol{U}$) in the left hand side of the equality (11) is positive definite in the space $\widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ and, therefore, defines an equivalent norm in the Hilbert space $\widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$. On the other hand, the functional in the right hand side with a fixed $\boldsymbol{U}$ is bounded in the same space $\widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$. Therefore, by the Riesz theorem on functionals in the Hilbert spaces there exists a unique function $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ which defines the functional in (11).

Now let $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ be the solution to the variational problem (5) and $\boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ is arbitrary. A direct verification shows that

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U} + \boldsymbol{V}) = \mathscr{E}_{\Omega_h}(\boldsymbol{U}) + \big[\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) - \mathscr{F}(\boldsymbol{V})\big] + \frac{1}{2}\mathscr{Q}_4(\boldsymbol{V}, \boldsymbol{V}), \tag{12}$$

where $\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V})$ is the bilinear form (cf. (6))

$$\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) = \int_{-h}^{h} \int_{\mathscr{C}} \Big[\mu\partial_\beta \boldsymbol{U}_\alpha \cdot \partial_\beta \boldsymbol{V}_\alpha + \mu\partial_\beta \boldsymbol{U}_\alpha \cdot \partial_\alpha \boldsymbol{V}_\beta + \lambda\partial_\alpha \boldsymbol{U}_\alpha \cdot \partial_\gamma \boldsymbol{V}_\gamma\Big] d\sigma dt$$

and $\mathscr{F}(\boldsymbol{V})$ is the functional, defined in (6). Then the equality holds

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U}) = \frac{1}{2}\mathscr{Q}(\boldsymbol{U}, \boldsymbol{U}) - \mathscr{F}(\boldsymbol{U}) \tag{13}$$

and, due to the equality (12), Lemma 4

$$\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) - \mathscr{F}(\boldsymbol{V}) = 0 \quad \text{for all} \quad \boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h) \quad \text{implies}$$

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U} + \boldsymbol{V}) - \mathscr{E}_{\Omega_h}(\boldsymbol{U}) = \frac{1}{2}\mathscr{Q}_4(\boldsymbol{V}, \boldsymbol{V}) \geqslant \frac{C}{2}\|\boldsymbol{V}\big|\widetilde{\mathbb{H}}^1(\Omega^h)\|^2 \quad \forall \boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h).$$

Then, obviously, $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ is the minimizer of the functional $\mathscr{E}_{\Omega_h}(\boldsymbol{U})$.

Conversely: Let $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ be the minimizer of $\mathscr{E}_{\Omega_h}(\boldsymbol{V})$ and the vector-function $\boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma_L^h)$ be arbitrary. The inequality (cf. (12))

$$0 \leqslant \mathscr{E}_{\Omega_h}(\boldsymbol{U}+\varepsilon\boldsymbol{V}) - \mathscr{E}_{\Omega_h}(\boldsymbol{U}) = \varepsilon\big\{\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) - \mathscr{F}(\boldsymbol{V})\big\} + \frac{\varepsilon^2}{2}\mathscr{Q}(\boldsymbol{V}, \boldsymbol{V}) \qquad \forall \varepsilon \in \mathbb{R}.$$

implies that $\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) = \mathscr{F}(\boldsymbol{V})$. Indeed, the first summand in the right-hand side of the equality dominates for small ε (positive and negative) and the second is non-negative. If we assume contrary $\mathscr{Q}(\boldsymbol{U}, \boldsymbol{V}) \neq \mathscr{F}(\boldsymbol{V})$ the difference $\mathscr{E}_{\Omega_h}(\boldsymbol{U} + \varepsilon\boldsymbol{V}) - \mathscr{E}_{\Omega_h}(\boldsymbol{U})$ would become negative for certain small ε, which is a contradiction. □

3 Γ-limit of the energy functional and main theorem

Main theorem of the present paper, Theorem 4, will be proved later. We commence with the investigation of the Γ-limit of the energy functional $\mathscr{E}_{\Omega^h}(\boldsymbol{U})$ in (5).

Let us rewrite the kernel $Q(\nabla\boldsymbol{U})$ of the quadratic part $\mathscr{Q}_4(\boldsymbol{U})$ of the energy functional in (6),(8), (10) by using the equalities (13) and (14) as follows:

$$
\begin{aligned}
Q_4(\nabla\boldsymbol{U}) = Q(\nabla\boldsymbol{U}) &= \mu(\mathscr{D}_\beta U_\alpha + \mathscr{N}_\beta\mathscr{D}_4 U_\alpha)^2 + \mu(\mathscr{D}_\beta U_\alpha + \mathscr{N}_\beta\mathscr{D}_4 U_\alpha)(\mathscr{D}_\alpha U_\beta \\
&\quad + \mathscr{N}_\alpha\mathscr{D}_4 U_\beta) + \lambda(\mathscr{D}_\alpha U_\alpha + \mathscr{N}_\alpha\mathscr{D}_4 U_\alpha)(\mathscr{D}_\beta U_\beta + \mathscr{N}_\beta\mathscr{D}_4 U_\beta) \\
&= \mu\big[(\mathscr{D}_\beta U_\alpha)^2 + (\mathscr{D}_4 U_\alpha)^2 + \mathscr{D}_\beta U_\alpha\cdot\mathscr{D}_\alpha U_\beta + \mathscr{N}_\alpha\mathscr{D}_\beta U_\alpha\mathscr{D}_4 U_\beta \\
&\quad + \mathscr{N}_\beta\mathscr{D}_\alpha U_\beta\mathscr{D}_4 U_\alpha + (\mathscr{D}_4 U_4)^2\big] + \lambda(\mathscr{D}_\alpha U_\alpha + \mathscr{D}_4 U_4)(\mathscr{D}_\beta U_\beta + \mathscr{D}_4 U_4) \\
&= \mu\big[(\mathscr{D}_j U_k)^2 + \mathscr{D}_\beta U_\alpha\cdot\mathscr{D}_\alpha U_\beta + 2\mathscr{N}_\beta\mathscr{D}_\alpha U_\beta\mathscr{D}_4 U_\alpha\big] \\
&\quad + \lambda(\mathscr{D}_\alpha U_\alpha + \mathscr{D}_4 U_4)(\mathscr{D}_\beta U_\beta + \mathscr{D}_4 U_4) \\
&= \mu\big[(\mathscr{D}_\alpha U_\beta)^2 + \mathscr{D}_4 U_\beta)^2 + \mathscr{D}_\beta U_\alpha\mathscr{D}_\alpha U_\beta + 2\mathscr{N}_\alpha\mathscr{D}_\beta U_\alpha\mathscr{D}_4 U_\beta\big] + \lambda\big[\mathscr{D}_\alpha U_\alpha\mathscr{D}_\beta U_\beta \\
&\quad + 2\mathscr{D}_\alpha U_\alpha\mathscr{D}_4 U_4\big] + (\lambda+\mu)(\mathscr{D}_4 U_4)^2, \quad \alpha,\beta = 1,2,3,\ j,k = 1,2,3,4, \qquad (1)
\end{aligned}
$$

since $\mathscr{N}_\alpha\mathscr{D}_\alpha = 0$, $\mathscr{N}_\alpha\mathscr{D}_4 U_\alpha = \mathscr{D}_4(\mathscr{N}_\alpha U_\alpha) = \mathscr{D}_4 U_4$ and $U_4 = \mathscr{N}_\alpha U_\alpha$.

Next we perform the scaling of the variable $t = h\tau$, $-1 < \tau < 1$ in the modified kernel $Q_4(\nabla\boldsymbol{U})$ of the quadratic part of energy functional (1), divide by h and study the following kernel in the scaled domain $\Omega^1 = \mathscr{C}\times(1,1)$

$$
\begin{aligned}
Q_4^0\left(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\right) &:= \frac{1}{h}Q_4\left(\nabla\boldsymbol{U}(\mathscr{x},h\tau)\right) \\
&= \mu\left[(\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau))^2 + \left(\frac{1}{h}\frac{\partial U_\alpha(\mathscr{x},h\tau)}{\partial\tau}\right)^2\right. \\
&\quad \left. + \mathscr{D}_\beta U_\alpha(\mathscr{x},h\tau)\cdot\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau) + 2\mathscr{N}_\beta\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau)\frac{1}{h}\frac{\partial U_\alpha(\mathscr{x},h\tau)}{\partial\tau}\right] \\
&\quad + \lambda\left[\mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau)\mathscr{D}_\beta U_\beta(\mathscr{x},h\tau) + 2\mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau)\frac{1}{h}\frac{\partial U_4(\mathscr{x},h\tau)}{\partial\tau}\right] \\
&\quad + (\lambda+\mu)\left(\frac{1}{h}\frac{\partial U_4(\mathscr{x},h\tau)}{\partial\tau}\right)^2, \qquad (2)
\end{aligned}
$$

where

$$
\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau) := U_1(\mathscr{x},h\tau), U_2(\mathscr{x},h\tau), U_3(\mathscr{x},h\tau), U_4(\mathscr{x},h\tau))^\top, \quad U_4 = \mathscr{N}_\alpha U_\alpha (3)
$$

Lemma 5 *The scaled and divided by h energy functional*

$$
\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \frac{1}{h}\mathscr{E}_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \frac{1}{2}\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h) - \mathscr{F}^0(\widetilde{\boldsymbol{U}}^0_h) \qquad (4)
$$

with the quadratic and linear parts

$$\mathscr{Q}_4^0(\widetilde{\boldsymbol{U}}^h)=\int_{-1}^{1}\int_{\mathscr{C}} Q_4^0\left(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\right)d\sigma d\tau$$

$$\mathscr{F}^0(\widetilde{\boldsymbol{U}}^0_h)=\int_{-h}^{h}\int_{\mathscr{C}}\Big[\langle\widetilde{\boldsymbol{F}}^0_h,\boldsymbol{U}^0_h\rangle+\frac{1}{h}\big[\langle\widetilde{\boldsymbol{H}}(\mathscr{x},+h),\widetilde{\boldsymbol{U}}^0(\mathscr{x},+h)\rangle$$

$$-\langle\widetilde{\boldsymbol{H}}^0(\mathscr{x},-h),\widetilde{\boldsymbol{U}}^0(\mathscr{x},-h)\rangle\big]\Big],$$

$$\widetilde{\boldsymbol{F}}^0_h(\mathscr{x},\tau):=(F^0_1(\mathscr{x},h\tau),F^0_2(\mathscr{x},h\tau),F^0_3(\mathscr{x},h\tau),F^0_4(\mathscr{x},h\tau))^\top,$$
$$\widetilde{\boldsymbol{H}}^0_h(\mathscr{x},\tau):=(H^0_1(\mathscr{x},h\tau),H^0_2(\mathscr{x},h\tau),H^0_3(\mathscr{x},h\tau),H^0_4(\mathscr{x},h\tau))^\top,$$
$$H^0_4=\mathscr{N}_\alpha H_\alpha,\qquad F^0_4=\mathscr{N}_\alpha F_\alpha$$

is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$ *(see* (4)*) and is convex*

$$\mathscr{E}^0_{\Omega^h}(\theta\widetilde{\boldsymbol{U}}^h+(1-\theta)\widetilde{\boldsymbol{V}}^h)\leqslant\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h)+(1-\theta)\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{V}}^h),\qquad 0<\theta<1 \quad (5)$$

for arbitrary vector $\widetilde{\boldsymbol{V}}^h(\mathscr{x},\tau):=(V_1(\mathscr{x},h\tau),V_2(\mathscr{x},h\tau),V_3(\mathscr{x},h\tau),V_4(\mathscr{x},h\tau))^\top$, $\widetilde{\boldsymbol{V}}^h\in\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Moreover, if $\widetilde{\boldsymbol{F}}^0_h(\mathscr{x},\tau):=\boldsymbol{F}^0(\mathscr{x},h\tau)$ *are uniformly bounded in* $\mathbb{L}_2(\Omega^1)$

$$\sup_{h<h_0}\|\widetilde{\boldsymbol{F}}^0_h|\mathbb{L}_2(\Omega^1)\|<\infty \quad (6)$$

for some $h_0>0$*, the energy functional has the following quadratic estimate: there exist positive constants* C_1,C_2 *and* C_3*, independent of the parameter* h *such that*

$$C_1\int\limits_{\Omega^1}\left[(\mathscr{D}_\alpha U^0_j(\mathscr{x},h\tau))^2+\left(\frac{1}{h}\frac{\partial U^0_j(\mathscr{x},h\tau)}{\partial\tau}\right)^2\right]dx-C_2\leqslant\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h)$$
$$\leqslant C_3\left\{1+\int\limits_{\Omega^1}\left[(\mathscr{D}_\alpha U^0_j(\mathscr{x},h\tau))^2+\left(\frac{1}{h}\frac{\partial U^0_j(\mathscr{x},h\tau)}{\partial\tau}\right)^2\right]dx\right\} \quad (7)$$

for all $\widetilde{\boldsymbol{U}}^h\in\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Proof: The convexity of the linear part $\mathscr{F}^0(\widetilde{\boldsymbol{U}}^h)$ is trivially obvious and we concentrate on the convexity of the quadratic part $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$. It follows easily from the well known inequality:

$$[\theta a+(1-\theta)b]^2=\theta^2a^2+2\theta(1-\theta)ab+(1-\theta)^2b^2\leqslant\theta^2a^2+\theta(1-\theta)(a^2+b^2)$$
$$+(1-\theta)^2b^2=\theta a^2+(1-\theta)b^2.$$

Thus, the inequality (5) is proved.

For the quadratic part $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$ of the energy functional the upper estimate in the inequality (7) is trivial, while the lower estimate, even with $C_2=0$, follows due to the inequality (7) since the norms of the vector-functions $\boldsymbol{U}(\mathscr{x},ht)$ and $\widetilde{\boldsymbol{U}}^h(\mathscr{x},t)$ are equivalent (cf. Lemma 1).

On the other hand $\mathscr{F}^0(\widetilde{\boldsymbol{U}}^h) \leqslant C_4\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$ (i.e., the quadratic part dominates the linear part) and the estimate (7) for the difference $\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \frac{1}{2}\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h) - \mathscr{F}^0(\widetilde{\boldsymbol{U}}^h)$ follows from the proved estimate for $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$. □

Theorem 3 *Let the weak limits*

$$\lim_{h\to 0}\boldsymbol{F}(\mathscr{x},h\tau) = \boldsymbol{F}(\mathscr{x}), \qquad \lim_{h\to 0}\frac{1}{2h}\left[\boldsymbol{H}(\mathscr{x},+h) - \boldsymbol{H}(\mathscr{x},-h)\right] = \boldsymbol{H}^{(1)}(\mathscr{x}), \tag{8}$$
$$\boldsymbol{F}, \boldsymbol{H}^{(1)} \in \mathbb{L}_2(\mathscr{C})$$

in $\mathbb{L}_2(\Omega^h)$ *and* $\mathbb{L}_2(\mathscr{C})$*, respectively, exist. Then the* Γ*-limit of the energy functional* $\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h)$ *exists*

$$\Gamma - \lim_{h\to 0}\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}}) := \int_{\mathscr{C}} Q_3\left(\overline{\boldsymbol{U}}(\mathscr{x})\right)\, d\sigma, \tag{9}$$

where

$$Q_3(\overline{\boldsymbol{U}}) = \mu\Big[(\mathscr{D}_\alpha\overline{U}_\beta)(\mathscr{D}_\alpha\overline{U}_\beta) + \mathscr{D}_\beta\overline{U}_\alpha\mathscr{D}_\alpha\overline{U}_\beta - \nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big]$$
$$+\frac{2\lambda\mu}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2 + \langle\boldsymbol{F}(\mathscr{x}) + 2\boldsymbol{H}^{(1)}(\mathscr{x}), \overline{U}(\mathscr{x})\rangle \tag{10}$$

and

$$\overline{\boldsymbol{U}}(\mathscr{x}) := (\overline{U}_1(\mathscr{x}), \overline{U}_2(\mathscr{x}), \overline{U}_3(\mathscr{x}))^\top, \quad \overline{U}_\alpha(\mathscr{x}) := U_\alpha(\mathscr{x},0), \quad \alpha = 1,2,3.$$

Proof: Since the Γ-convergence of the linear part $\mathscr{F}^0(\boldsymbol{U}^0_h)$ coincides with the point-wise convergence of a functional, it is trivial part of the proof and we will only consider the Γ-convergence of the quadratic part $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$ (cf. (4)).

To check the Γ-convergence of the quadratic part $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$ first we prove the lower estimate for its kernel

$$\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h) \geqslant \mathscr{Q}^3_{\mathscr{C}}(\overline{\boldsymbol{U}}) := \int_{\mathscr{C}} Q^0_3\left(\overline{\boldsymbol{U}}(\mathscr{x})\right)\, d\sigma, \tag{11}$$
$$Q^0_3(\overline{\boldsymbol{U}}) = \frac{\mu}{2}\Big[\left[\mathscr{D}_\alpha\overline{U}_\beta + \mathscr{D}_\beta\overline{U}_\alpha\right]^2 - 2\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big] + \frac{2\lambda\mu}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2.$$

For this let us rewrite Q^0_4 in (2) in the form

$$Q^0_4\left(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\right) = \mu\big[\,[(\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau))]^2 + \xi^2_\alpha$$
$$+\mathscr{D}_\beta U_\alpha(\mathscr{x},h\tau)\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau) + 2\mathscr{N}_\beta\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau)\xi_\alpha\big]$$
$$+\lambda\big[\mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau)\mathscr{D}_\beta U_\beta(\mathscr{x},h\tau) + 2\mathscr{D}_\beta U_\beta(\mathscr{x},h\tau)\xi_4\big] + (\lambda+\mu)\xi^2_4, \tag{12}$$

where the variables

$$\xi_\alpha=\xi_\alpha(\mathscr{x},h\tau):=\frac{1}{h}\frac{\partial U_\alpha(\mathscr{x},h\tau)}{\partial\tau},\quad \alpha=1,2,3,\quad \xi_4=\mathscr{N}_\alpha\xi_\alpha \tag{13}$$

depend on h and we will find minimum of the kernel $Q^0_4\left(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}(\mathscr{x},\tau)\right)$ with respect to the variables ξ_1,ξ_2,ξ_3. To this end we write the condition of minimum, by using the equality $\partial_{\xi_\alpha}\xi_4^2=2\xi_4\partial_{\xi_\alpha}\xi_4=2\mathscr{N}_\alpha\xi_4$ (cf. (13)):

$$\frac{\partial}{\partial\xi_\alpha}Q^0_4(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h)=2\mu(\xi_\alpha+\mathscr{N}_\beta\mathscr{D}_\alpha U_\beta)+2\lambda\mathscr{N}_\alpha\mathscr{D}_\beta U_\beta+2(\lambda+\mu)\mathscr{N}_\alpha\xi_4=0 \tag{14}$$

for $\alpha=1,2,3$. From (14) follows

$$\begin{aligned}0=\mathscr{N}_\alpha\frac{\partial}{\partial\xi_\alpha}Q^0_4(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h)&=2\mu(\mathscr{N}_\alpha\xi_\alpha+\mathscr{N}_\beta\mathscr{N}_\alpha\mathscr{D}_\alpha U_\beta)+2\lambda\mathscr{N}_\alpha\mathscr{N}_\alpha\mathscr{D}_\beta U_\beta\\ +2(\lambda+\mu)\mathscr{N}_\alpha\mathscr{N}_\alpha\xi_4&=2\mu\xi_4+2\lambda\mathscr{D}_\beta U_\beta+2(\lambda+\mu)\xi_4,\end{aligned} \tag{15}$$

Finding ξ_4 from (15) and inserting into (14) we find ξ_α:

$$\xi_4=-\frac{\lambda}{\lambda+2\mu}\mathscr{D}_\beta U_\beta, \tag{16}$$

$$\xi_\alpha=-\mathscr{N}_\beta(\mathscr{D}_\alpha U_\beta)-\frac{\lambda}{\lambda+2\mu}\mathscr{N}_\alpha(\mathscr{D}_\beta U_\beta),\ \ \alpha=1,2,3. \tag{17}$$

From (16), (17) and (12) we find the minimum $Q_3(\overline{\boldsymbol{U}})$:

$$\begin{aligned}Q_3(\overline{\boldsymbol{U}})&=\min_{\xi_1,\xi_2,\xi_3}Q^0_4\left(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h\right)\\ &=\mu\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta\big]^2+\nu_\beta(\mathscr{D}_\alpha\overline{U}_\beta)\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)+\mathscr{D}_\beta\overline{U}_\alpha\mathscr{D}_\alpha\overline{U}_\beta\\ &\quad+\frac{\lambda^2}{(\lambda+2\mu)^2}(\mathscr{D}_\beta\overline{U}_\beta)^2\Big]-2\mu\nu_\beta\Big[\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)+\frac{\lambda}{\lambda+2\mu}\nu_\alpha(\mathscr{D}_\gamma\overline{U}_\gamma)\Big](\mathscr{D}_\alpha\overline{U}_\beta)\\ &\quad+\lambda(\mathscr{D}_\alpha\overline{U}_\alpha)^2-\frac{2\lambda^2}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2+(\lambda+\mu)\frac{\lambda^2}{(\lambda+2\mu)^2}(\mathscr{D}_\alpha\overline{U}_\alpha)^2\\ &=\mu\big[\big[\mathscr{D}_\alpha\overline{U}_\beta\big]^2+(\mathscr{D}_\beta\overline{U}_\alpha)(\mathscr{D}_\alpha\overline{U}_\beta)\big]+\mu\nu_\beta\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\beta)(\mathscr{D}_\alpha\overline{U}_\gamma)\\ &\quad+\frac{\mu\lambda^2}{(\lambda+2\mu)^2}(\mathscr{D}_\alpha\overline{U}_\alpha)^2-2\mu\nu_\beta\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)(\mathscr{D}_\alpha\overline{U}_\beta)\\ &\quad+\lambda(\mathscr{D}_\alpha\overline{U}_\alpha)^2-\frac{2\lambda^2}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2+(\lambda+\mu)\frac{\lambda^2}{(\lambda+2\mu)^2}(\mathscr{D}_\alpha\overline{U}_\alpha)^2\\ &=\mu\Big[(\mathscr{D}_\alpha\overline{U}_\beta)^2+\mathscr{D}_\beta\overline{U}_\alpha\mathscr{D}_\alpha\overline{U}_\beta-\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big]+\frac{2\lambda\mu}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2\\ &=\frac{\mu}{2}\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha\big]^2-2\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big]+\frac{2\lambda\mu}{\lambda+2\mu}(\mathscr{D}_\alpha\overline{U}_\alpha)^2.\end{aligned} \tag{18}$$

Thus, the equality (11) is proved.

To accomplish the proof of the Γ-convergence (9) it remains to build a recovery sequence $\overline{\boldsymbol{U}}^k \in \mathbb{H}^1(\Omega^1, \Gamma^1_L)$

$$\begin{aligned}
&\overline{\boldsymbol{U}}^k(\mathscr{X}, t) = (U^k_1(\mathscr{X}, t), U^k_2(\mathscr{X}, t), U^k_3(\mathscr{X}, t), U^k_4(\mathscr{X}, t))^\top \to (\overline{\boldsymbol{U}}(\mathscr{X}), 0),\\
&\overline{\boldsymbol{U}}(\mathscr{X}) = (\overline{U}_1(\mathscr{X}), \overline{U}_2(\mathscr{X})\overline{U}_3(\mathscr{X}))^\top,\\
&U^k_\alpha(\mathscr{X}, t) := U_\alpha(\mathscr{X}, h_k t), \qquad \overline{U}_\alpha(\mathscr{X}) := U_\alpha(\mathscr{X}, 0)
\end{aligned}$$

($\alpha = 1, 2, 3$), along which the quadratic form attains its minimum

$$\lim_{h_k\to 0} \mathscr{Q}^0_4(\overline{\boldsymbol{U}}^k) = \mathscr{Q}^3_{\mathscr{C}}(\overline{\boldsymbol{U}}(\mathscr{X})). \tag{19}$$

The first three components of the minimizing sequence $U^k_1(\mathscr{X}, t), U^k_2(\mathscr{X}, t), U^k_3(\mathscr{X}, t)$ should be found from the initial value problem, minimizing the quadratic form (cf. (14)-(17)):

$$\frac{1}{h_k}\frac{\partial U^k_\alpha(\mathscr{X}, \tau)}{\partial \tau} = -\nu_\beta \mathscr{D}_\alpha \overline{U}_\beta(\mathscr{X}) - \frac{\lambda}{\lambda + 2\mu}\nu_\alpha \mathscr{D}_\beta \overline{U}_\beta(\mathscr{X}), \quad \alpha = 1, 2, 3, \tag{20}$$

$$U^k_\alpha(\mathscr{X}, 0) = \overline{U}_\alpha(\mathscr{X}), \quad \alpha = 1, 2, 3 \tag{21}$$

and $U^k_4(\mathscr{X}, t) = \mathscr{N}_\alpha(\mathscr{X}) U^k_\alpha(\mathscr{X}, t)$. From (20) and (21) we have for $\alpha = 1, 2, 3$:

$$U^k_\alpha(\mathscr{X}, \tau) = \overline{U}_\alpha(\mathscr{X}) - h_k\tau\Big[\nu_\beta \mathscr{D}_\alpha U^0_\beta(\mathscr{X}) + \frac{\lambda}{\lambda + 2\mu}\nu_\alpha \mathscr{D}_\beta U^0_\beta(\mathscr{X})\Big]. \tag{22}$$

By inserting the obtained solutions into the quadratic form $\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h)$ and by sending $h_k \to 0$ we prove that the limit in (9) is attained. □

Theorem 4 *Let $\boldsymbol{F}, \boldsymbol{H}^{(1)} \in \mathbb{L}_2(\mathscr{C})$. The vector-function $\overline{\boldsymbol{U}} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$ which minimizes the energy functional $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ in* (9)-(10) *is a solution to the following boundary value problem*

$$\begin{cases}
\mu\left[\Delta_{\mathscr{C}}\overline{U}_\alpha + \mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta - 2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta - \mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)\right]\\
+\dfrac{4\lambda\mu}{\lambda + 2\mu}\left[\mathscr{D}_\alpha\mathscr{D}_\beta\overline{U}_\beta - 2\mathscr{H}_{\mathscr{C}}\nu_\alpha\mathscr{D}_\beta\overline{U}_\beta\right] = \dfrac{1}{2}F_\alpha + H^{(1)}_\alpha \quad \text{on} \quad \mathscr{C},\\
\overline{U}_\alpha(t) = 0 \qquad\qquad \text{on} \quad \Gamma = \partial\mathscr{C}, \quad \alpha = 1, 2, 3.
\end{cases} \tag{23}$$

Vice versa: on the solution $\overline{\boldsymbol{U}} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$ to the boundary value problem (23) *under the condition $\boldsymbol{F}, \boldsymbol{H}^{(1)} \in \mathbb{L}_2(\mathscr{C})$, the energy functional $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ in* (9)-(10) *attains the minimum.*

Remark 2 According to the foregoing Theorem 4 the boundary value problem (23) can be considered as the Γ-limit of the BVP (3).

Proof of Theorem 4: If $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ attains the minimum on the vector-function $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{V}}=(\overline{V}_1,\overline{V}_2,\overline{V}_3)^\top\in\widetilde{\mathbb{H}}^1(\mathscr{C},\Gamma)$ is arbitrary, then

$$\begin{aligned}
&\frac{d}{dt}\mathscr{E}^3_{\mathscr{C}}((\overline{\boldsymbol{U}}+t\overline{\boldsymbol{V}})\Big|_{t=0}\\
&\qquad=\int_{\mathscr{C}}\frac{d}{dt}\Big[Q_3(\overline{\boldsymbol{U}}+t\overline{\boldsymbol{V}})+\langle\boldsymbol{F}(\mathscr{X})+2\boldsymbol{H}^{(1)}(\mathscr{X}),\overline{\boldsymbol{U}}(\mathscr{X})+t\overline{\boldsymbol{V}}(\mathscr{X})\rangle\Big]_{t=0}d\sigma\\
&\qquad=\int_{\mathscr{C}}\Bigg\{2\mu\left[\mathscr{D}_\beta\overline{U}_\alpha\mathscr{D}_\beta\overline{V}_\alpha+\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\beta\overline{V}_\alpha-\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta\mathscr{D}_\gamma\overline{V}_\alpha\right]\\
&\qquad\qquad+\frac{4\lambda\mu}{\lambda+2\mu}\mathscr{D}_\beta\overline{U}_\beta\mathscr{D}_\alpha\overline{V}_\alpha+\langle\boldsymbol{F}+2\boldsymbol{H}^{(1)},\overline{\boldsymbol{V}}\rangle\Bigg\}\,d\sigma.
\end{aligned}$$

Now we apply formula (8) (we remind that $\overline{U}_\alpha(t)=\overline{V}_\alpha(t)=0$ on $\Gamma=\partial\mathscr{C}$) and get

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}^3_{\mathscr{C}}((\overline{\boldsymbol{U}}+t\overline{\boldsymbol{V}})\Big|_{t=0}&=-\int_{\mathscr{C}}\Bigg\{2\mu\big[\mathscr{D}^2_\beta\overline{U}_\alpha-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\beta\overline{U}_\alpha+\mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta\\
&\quad-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta-\mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)+2\mathscr{H}_{\mathscr{C}}\nu_\alpha\nu_\beta\nu_\gamma\mathscr{D}_\gamma\overline{U}_\beta\big]\\
&\quad+\frac{4\lambda\mu}{\lambda+2\mu}\left[\mathscr{D}_\alpha\mathscr{D}_\beta\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\alpha\mathscr{D}_\beta\overline{U}_\beta\right]-F_\alpha-2H^{(1)}_\alpha\Bigg\}\overline{V}_\alpha\,d\sigma\\
&=-\int_{\mathscr{C}}\Bigg\{2\mu\left[\varDelta_{\mathscr{C}}\overline{U}_\alpha+\mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta-\mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)\right]\\
&\quad+\frac{4\lambda\mu}{\lambda+2\mu}\left[\mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\alpha-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\alpha\right]-F_\alpha-2H^{(1)}_\alpha\Bigg\}\overline{V}_\alpha\,d\sigma
\end{aligned}$$

Since $\overline{V}_1,\overline{V}_2,\overline{V}_3$ are arbitrary, from the obtained equality follows that the vector-function $\overline{\boldsymbol{U}}(\mathscr{X})=(\overline{U}_1(\mathscr{X}),\overline{U}_2(\mathscr{X}),\overline{U}_3(\mathscr{X}))^\top$ is a solution to the BVP (23).

The inverse assertion that if $\overline{\boldsymbol{U}}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ is the solution to the boundary value problem (23) under the condition $\boldsymbol{F},\boldsymbol{H}^{(1)}\in\mathbb{L}_2(\mathscr{C})$, then the energy functional $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ in (9)-(10) attains the minimum at $\overline{\boldsymbol{U}}$, is proved as in Theorem 2. □

Remark 3 Note, that when $\mathscr{C}=\Omega\subset\mathbb{R}^2$ is the flat 2-dimensional domain, then the energy functional $\mathscr{E}^3_{\Omega}(\overline{\boldsymbol{U}})$ in (9) coincides with the density of the energy functional $\mathscr{E}_2(\boldsymbol{U})$ of the plane theory of elesticity (see [13], p.16).

Indeed,

$$\mathscr{E}_2(\boldsymbol{U}):=\int_\Omega Q_2(\boldsymbol{U})(x)dx,\qquad \boldsymbol{U}=(U_1,U_2)^\top$$

(see (see [13], p.16), where

$$Q_2(\boldsymbol{U})=2\mu\Big|\frac{\nabla\boldsymbol{U}+\nabla\boldsymbol{U}^\top}{2}\Big|^2+\frac{2\lambda\mu}{\lambda+2\mu}\big(\operatorname{trace}\nabla\boldsymbol{U}\big)^2,\tag{24}$$

$$\nabla\boldsymbol{U}:=\begin{pmatrix}\partial_1U_1&\partial_2U_1\\ \partial_1U_2&\partial_2U_2\end{pmatrix}.$$

From (24)

$$Q_2(\boldsymbol{U}) = 2\mu[(\partial_1 U_1)^2 + (\partial_2 U_2)^2] + \mu(\partial_1 U_2 + \partial_2 U_1)^2 \\ + \frac{2\lambda\mu}{\lambda + 2\mu}(\partial_1 U_1 + \partial_2 U_2)^2. \tag{25}$$

On the other hand, if $\mathscr{C} = \Omega = \Omega \subset \mathbb{R}^2$, then $x_3 = 0$, $\boldsymbol{\nu} = (0, 0, 1)$, $\mathscr{H}_{\mathscr{C}} = 0$, $\mathscr{D}_1 = \partial_1$, $\mathscr{D}_2 = \partial_2$, $\mathscr{D}_3 = 0$, $\overline{U}_1 = U_1$, $\overline{U}_2 = U_2$, $\overline{U}_3 = U_3$ and from (10) we have

$$\begin{aligned} Q_3(\overline{\boldsymbol{U}}) &= \mu\big[(\mathscr{D}_1\overline{U}_1)^2 + (\mathscr{D}_1\overline{U}_2)^2 + (\mathscr{D}_1\overline{U}_3)^2 + (\mathscr{D}_2\overline{U}_1)^2 + (\mathscr{D}_2\overline{U}_2)^2 \\ &\quad +(\mathscr{D}_2\overline{U}_3)^2 + (\mathscr{D}_1\overline{U}_1)^2 + (\mathscr{D}_2\overline{U}_2)^2 + 2(\mathscr{D}_1\overline{U}_2)(\mathscr{D}_2\overline{U}_1) - (\mathscr{D}_1\overline{U}_3)^2 \\ &\quad -(\mathscr{D}_2\overline{U}_3)^2\big] + \frac{2\lambda\mu}{\lambda + 2\mu}(\mathscr{D}_1\overline{U}_1 + \mathscr{D}_2\overline{U}_2)^2 \\ &= 2\mu[(\partial_1\overline{U}_1)^2 + (\partial_2\overline{U}_2)^2] + \mu(\partial_1\overline{U}_2 + \partial_2\overline{U}_1)^2 + \frac{2\lambda\mu}{\lambda + 2\mu}(\partial_1\overline{U}_1 + \partial_2\overline{U}_2)^2 \\ &= 2\mu[(\partial_1 U_1)^2 + (\partial_2 U_2)^2] + \mu(\partial_1 U_2 + \partial_2 U_1)^2 + \frac{2\lambda\mu}{\lambda + 2\mu}(\partial_1 U_1 + \partial_2 U_2)^2 \\ &= Q_2(\boldsymbol{U}) \end{aligned}$$

and we have verified that $Q_3(\overline{\boldsymbol{U}}) = Q_2(\boldsymbol{U})$.

Remark 4 Note, that in the case of a flat surface the energy functional $\mathscr{E}^3_{\Omega}(\overline{\boldsymbol{U}}) = \mathscr{E}_2(\boldsymbol{U})$ is independent of the displacement in the vertical direction $\overline{U}_3 = 0$ (cf. (25)).

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References

1. A. Braides, Gamma-convergence for beginners. University Press, Oxford, 2002.
2. T. Buchukuri, R. Duduchava, G. Tephnadze, Dirichlet problem for Laplace-Beltrami equation on hypersurfaces - FEM approximation. *Transactions of A. Razmadze Mathematical Institute* **170**, 3, 300–307, 2016.
3. E. Cosserat, P. Cosserat, Thiorie des Corps Diformables. Hermann, Paris, 1909.
4. P.G. Ciarlet, *Mathematical Elasticity, Vol. I: Three dimensional elasticity*, Studies in Mathematics and Applications, 20, Elsevier, North-Holland, Amsterdam, 1988.
5. P.G. Ciarlet, *Introduction to Linear Shell Theory*, Series in Applied Mathematics, Vol. 1, Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, North-Holland, Amsterdam, 1998.
6. P.G. Ciarlet, *Mathematical Elasticity, Vol. III: Theory of Shells*, Studies in Mathematics and Applications, 29, Elsevier, North-Holland, Amsterdam, 2000.
7. P.G. Ciarlet, P. Destuynder, A justification of the two-dimensional plate model, J. Micanique 18, 315-344.
8. P.G. Ciarlet, P. Destuynder, A justification of a nonlinear model in plate theory, Comput. Methods Appl. Mech. Engrg. 17/18, 227-258.
9. E. DeGiorgi and T. Franzoni, Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 58 (1975), 842-850.
10. R.Duduchava, D.Mitrea, M.Mitrea, Differential operators and boundary value problems on surfaces. *Mathematis che Nachrichten* **279**, No. 9-10 (2006), 996-1023.
11. R. Duduchava, E. Shargorodsky, G. Tephnadze, Extension of the unit normal vector field from a hypersurface and Eikonal equation. *Georgian Mathematical Journal*. **22**, 5, 355-359, 2015.

12. R. Duduchava, M. Tsaava, T. Tsutsunava, Mixed boundary value problem on hypersurfaces. *International Journal of Differential Equations*, Hindawi Publishing Corporation, Volume 2014, Article ID 245350, 8 pages.
13. G. Friesecke, R.D. James, S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity, *Communications on Pure and Applied Mathematics* bf 55, 11, 2002, 1461-1506.
14. G. Friesecke, R.D. James, M. G. Mora, S. Müller, Derivation of nonlinear bending theory for shells from three dimensional nonlinear elasticity by Gamma onvergence, *C. R. Acad. Sci. Paris.* Sér. I **336** 2003, 697-702.
15. A.L. Goldenveizer, Theory of Elastic Thin Shells, Pergamon Press, Oxford, 1961.
16. W.T. Koiter, On the foundations of the linear theory of thin elastic shells, Proc. Kon. Ned. Akad. Wetensch. B78, 169-195.
17. A.E.H. Love, The Mathematical Theory of Elasticity, Cambridge University Press, 1934.
18. P.M. Naghdi, Foundations of elastic shell theory, in Progress in Solid Mechanics, Vol. 4, pp. 1-90, North-Holland, Amsterdam, 1963.
19. V.V. Novozhilov, Thin Shell Theory, Second Edition, Wolters-Noordhoff, Groningen 1970.
20. H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig 1995.
21. I.N. Vekua, Theory of thin shallow shells of variable thickness, Acad. Nauk Gruzin. S.S.R. Trudy Tbilis. Mat. Inst. Razmadze, 30, 3-103.