Lie analysis, conserved quantities and solitonic structures of Calogero-Degasperis-Fokas equation

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Abstract

The paper investigates a class of exactly solvable third order nonlinear evolution equation [16]. A list of unknown function $F(u)$ is reported for which considered equation contains the nontrivial Lie point symmetries. Moreover, nonlinear self-adjointness is discussed and it is examined that it is not strictly self-adjoint equation for physical parameter $A \neq 0$ but quasi self-adjoint or more generally nonlinear self-adjoint. Additionally, it is observed that Calogero-Degasperis-Fokas (CDF) equation admits a minimal set of Lie algebra under invariance criteria of Lie groups. These classes are utilized one by one to construct the similarity variables to reduce the dimension of the discussed equation. Additionally, Lie symmetries are used to exhibit the associated conservation laws. Henceforth, Lie symmetry reductions of CDF equation are reported with the help of an optimal system. Meantime, this Lie symmetry method reduces the considered equation into ordinary differential equations. Moreover, well-known (G'/G)-expansion method is used to get the exact solutions. The obtained new periodic and solitary wave solutions can be widely used to provide many attractive complex physical phenomena in the different fields of sciences.

Keywords: Calogero-Degasperis-Fokas equation, self-adjointness, Conservation Laws, (G'/G)–expansion method, hyperbolic function solutions, trigonometric function solutions, rational function

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1 Introduction

The importance of nonlinear evolution equations (NLEEs) in various phenomenons of basic sciences such as physics, biology, chemistry, etc, is not cover for nobody, because the mathematical modelling of most of these phenomenons lead to NLEEs. In this paper, a class of exactly solvable third order nonlinear evolution equation [16] of the form:

$$u_t + u_{xxx} - Au_x^3 - F(u)u_x = 0, \qquad A \in \mathbb{R} \tag{1.1}$$

is considered, where $u(x,t)$ is the amplitude of the relevant wave mode. Eq. (1.1) represents different models with different choices of physical parameter A and unknown function F [16]:
(i) For $F = 0$, Eq. (1.1) becomes

$$u_t + u_{xxx} - Au_x^3 = 0, \qquad A \in \mathbb{R}, \tag{1.2}$$

which represents potential modified Korteweg-de Vries (KdV) equation.

(ii) For $A = 0$ and $F = u$, Eq. (1.1) becomes:

$$u_t + u_{xxx} - uu_x = 0, \tag{1.3}$$

which describes KdV equation

(iii) For $A = 0$ and $F = u^2$, Eq. (1.1) represents modified KdV equation of the form:

$$u_t + u_{xxx} - u^2u_x = 0, \qquad A \in \mathbb{R} \tag{1.4}$$

(iv) If $A = \frac{1}{8}$ and the function F satisfies $F'''(u) = 8AF'(u)$ then Eq. (1.1) Calogero-Degasperis-Fokas (CDF) equation takes the form [10]:

$$u_t + u_{xxx} - \frac{1}{8}u_x^3 - (pe^u + qe^{-u})u_x = 0. \tag{1.5}$$

(v) Authors in [34] used exp-function method to construct generalized solitary and periodic solutions of CDF equation of the form

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{4}\frac{u_xu_{xx}}{u} + \frac{3}{8}\frac{u_x^3}{u^2} + \frac{3}{8}(\alpha u^2 + \frac{\beta}{u^2} + \gamma)u_x = 0.$$

The work on CDF equation is done by many researchers because of its applications and importance in applied sciences, like [37] derived soliton solutions to the CDF equations using the improved tanh method, and the references therein.

For the proper analysis of nonlinear differential equations with unknown functions, it is well known practice to investigate such mathematical models via Lie classification [7]. For such investigation classification of selfadjointness is considered to be equally handy tool [7, 24, 26, 36] to see the problem in depth. Further, calculation of conservation laws play a significant role in the study of NLEEs. The integrability for such mathematical equations depends upon the number of conservation laws. The conservation laws are also helpful in the numerical integration of partial differential equations. Conservation laws are the vital part of Lie theory, which are discovered by Noether in 1918. It is well known that the conservation laws have a significant importance in the study of physical phenomenon. In literature many techniques are available to find the conservation laws, some of them are [8, 25, 27, 35, 44, 45]. One of the motives of this article is to compute the conserved quantities and also discover the conditions of nonlinear self-adjointness.

Consequently, obtaining the solutions of NLEEs is a main subject in all branches of sciences. Among the possible solutions to NLEEs, there are certain special forms of solutions known as solitary wave solutions (solitons) which may depend only on a single combination of variables. The concept of solitary waves solutions goes back to works of *John Scott Russell* in 1834. To find more information about solitons, we refer the interested reader to see [42].

The main structure of obtaining the soliton solutions is the comeback to inverse scattering transform [3] which Ablowitz and Clarkson studied for the integrability of the large field of NLEEs. In the past years, many other powerful and direct methods have been developed to find special solutions of NLEEs, such as the Backlund transformation [11], Hirota bilinear method [17], numerical methods [14] and the Wronskian determinant technique [50]. With the help of the computer software, many algebraic methods were proposed, such as the auxiliary equation method [6, 4], the modified Kudryashov method [18, 30], the first integral method [12, 15], the general projective Riccati equation method [38, 39], the new extended direct algebraic method [40, 23], the sine-Gordon expansion method [41, 48], the sub-equation method [5, 29], the extended sinh-Gordon equation expansion method [9, 46], the Khater method [33, 43], the Jacobi elliptic function expansion method [32, 47] and many more.

In 2008, Wang et al. [49] introduced the (G'/G)–expansion method and reached the traveling wave solution. Since it was a simple method, it was widely used to obtain various exact solutions of NLEEs [1, 2, 22, 28, 31, 49]. Its strategy is reducing the integrable NLEEs to a second-order differential equation with constant coefficients and then by using a simple algebraic computation, the solitary wave solutions of NLEEs can be obtained. The concept of integrability of NLEEs is discussed by F. Calogero in [13].

The main objective of the paper in hand is to consider Eq. (1.1) by means of Lie group classification, nonlinear self-adjointness, conserved vectors of CDF Eq. (1.1) and the application of the (G'/G)–expansion method to obtain the exact solutions of a special class of CDF Eq. (1.1).

The outline of the paper in hand is as follows: Preliminaries are presented in Section 2, nonlinear self-adjointness and conservation laws are given in Section 3. Group classification, equivalence transformations are given in Section 4. Section 5 is devoted for the travelling wave solutions of the Calogero-Degasperis-Fokas equation.

2 Preliminaries

Before presenting the main procedure, we recall some operators and definitions to be used from the literature [7, 19, 20, 21, 26, 36]. Here $x = (x^1, x^2, ..., x^n)$ is an independent variable while $u = u(x)$ is a dependent variable and $u_{(r)}$ represents r^{th} derivative of u. The vector space $\mathcal{A}$ is of all differential functions of finite order.

2.1 Nonlinear self-adjointness and conservation laws

Suppose a PDE of the form:

$$E(x, u_{(1)}, u_{(2)}, ..., u_{(r)}) = 0, \quad E \in \mathcal{A}, \tag{2.1}$$

where the adjoint equation of Eq. (2.1) is:

$$E^*(x, u, v, u_{(1)}, u_{(2)}, ..., u_{(r)}, v_{(r)}) = \frac{\delta(\mathcal{L})}{\delta u} = 0 \tag{2.2}$$

with $\mathcal{L} = vE$, where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{r=1}^{\infty}(-1)^r D_{i_1}...D_{i_r}\frac{\partial}{\partial u_{i_1...i_r}}$$

denotes the Euler-Lagrange operator, while

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + ...$$

A conserved vector $T = T^i$ for the system comprised by Eqs. (2.1)-(2.2) is given by [19, 20]

$$\begin{aligned} T^i &= \xi^i\mathcal{L} + W[\frac{\partial \mathcal{L}}{\partial u_i} - D_j(\frac{\partial \mathcal{L}}{\partial u_{ij}}) + D_kD_j(\frac{\partial \mathcal{L}}{\partial u_{ijk}}) - D_lD_kD_j(\frac{\partial \mathcal{L}}{\partial u_{ijkl}}) + \cdots] \\ &+D_j(W)[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k(\frac{\partial \mathcal{L}}{\partial u_{ijk}}) + \cdots] + D_kD_j(W)[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - D_l(\frac{\partial \mathcal{L}}{\partial u_{ijkl}}) + \cdots] + \cdots , \end{aligned} \tag{2.3}$$

here $W = \phi - \xi^j u_j$ is the Lie characteristic function and $\mathcal{L}$ is Lagrangian.

Definition 1. *The Eq. (2.1) will be strictly self-adjoint if its adjoint Eq. (2.2) obtained by the substitution $v = u$ becomes identical to the Eq. (2.1) [21], i.e;*

$$E^*|_{v=u} = \lambda(x, u, ...)E, \qquad \text{for some} \quad \lambda \in \mathcal{A}.$$

Definition 2. *The Eq. (2.1) is said to be quasi self-adjoint if its adjoint Eq. (2.2) obtained by the substitution* $v = \psi(u)$*, for some* $\psi(u) \neq 0$*, is identical with the Eq. (2.1) [20, 21, 26]; that is*

$$E^*|_{v=\psi(u)} = \lambda(x, u, ...)E, \qquad \text{for some} \quad \lambda \in \mathcal{A}.$$

Definition 3. *Eq. (2.1) is called weakly self-adjoint if the equation obtained from adjoint Eq. (2.2) by the substitution* $v = \psi(x, u)$ *with* $\psi_u \neq 0$ *and* $\psi_{x^i} \neq 0$ *for some* x^i*, is identical to the Eq. (2.1) [26]; that is*

$$E^*|_{v=\psi(x,u)} = \lambda(x, u, ...)E, \qquad \text{for some} \quad \lambda \in \mathcal{A}.$$

Authors in [26] generalized all above three definitions in a fourth definition as follows.

Definition 4. *The Eq. (2.1) is said to be nonlinearly self-adjoint if the equation obtained from adjoint Eq. (2.2) by the substitution* $v = \psi(x, u)$ *with* $\psi(x, u) \neq 0$*, is identical to the original Eq. (2.1) [26]; that is*

$$E^*|_{v=\psi(x,u)} = \lambda(x, u, ...)E, \qquad \text{for some} \quad \lambda \in \mathcal{A}.$$

2.2 Description of the (G'/G)–expansion method

In this section, we briefly explain the application of (G'/G)–expansion method for solving certain NLEEs. For a given NLEEs of the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{2.4}$$

where P is a polynomial in its arguments, the transformation $u(x,t) = U(\xi),\ \xi = x - \omega t$, reduces Eq. (2.4) to a nonlinear ordinary differential equation

$$P(U, U', -\omega U', U'', -\omega U'', \omega^2 U'', ...) = 0, \tag{2.5}$$

where $U = U(\xi)$, and prime denotes derivative with respect to ξ. We assume that the solution of Eq. (2.5) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{n=1}^{m} \alpha_n \Big(\tfrac{G'(\xi)}{G(\xi)}\Big)^n + \alpha_0, \quad \alpha_m \neq 0. \tag{2.6}$$

where α_n, for $n = 0, 1, 2, ..., m$, are constants to be determined later and $G(\xi)$ satisfies a second order linear ordinary differential equation:

$$\tfrac{d^2G(\xi)}{d\xi^2} + \lambda \tfrac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \tag{2.7}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (2.7), we have

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \tag{2.8}$$

and it follows, from (2.6) and (2.7), that

$$
\begin{aligned}
U' &= -\sum_{n=1}^{m} n\ \alpha_n\Big((\tfrac{G'}{G})^{n+1} + \lambda(\tfrac{G'}{G})^{n} + \mu(\tfrac{G'}{G})^{n-1}\Big),\\
U'' &= \sum_{n=1}^{m} n\ \alpha_n\Big((n+1)(\tfrac{G'}{G})^{n+2} + (2n+1)\lambda(\tfrac{G'}{G})^{n+1} + n(\lambda^2+2\mu)(\tfrac{G'}{G})^{n}\\
&\qquad + (2n-1)\lambda\mu(\tfrac{G'}{G})^{n-1} + (n-1)\mu^2(\tfrac{G'}{G})^{n-2}\Big),
\end{aligned}
\tag{2.9}
$$

and so on, here the prime denotes the derivative with respective to ξ. To determine u explicitly, we take the following four steps:

Step 1. Determine the integer m by substituting Eq. (2.6) along with Eq. (2.7) into Eq. (2.5), and balancing the highest order nonlinear term(s) and the highest order partial derivative.

Step 2. Substitute Eq. (2.6) to get the value of m determined in *Step 1*, along with Eq. (2.7) into Eq. (2.5) and collect all terms with the same order of (G'/G) together, the left-hand side of Eq. (2.5) is converted into a polynomial in (G'/G). Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $\omega, \lambda, \mu, \alpha_n$ for $n = 0, 1, 2, ..., m$.

Step 3. Solve the system of algebraic equations obtained in *Step 2*, for $\omega, \lambda, \mu, \alpha_0, ..., \alpha_m$ using Maple 12.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions $u(\xi)$ of Eq. (2.5) depending on (G'/G), since the solutions of Eq. (2.7) have been well known for us, so we can obtain exact solutions of Eq. (2.4).

3 Point symmetries and equivalence transformations

3.1 Lie point symmetries

The one parameter Lie group of infinitesimal transformations are:

$$
\begin{aligned}
\tilde{t} &= t + s\tau(t,x,u) + O(s^2),\\
\tilde{x} &= x + s\xi(t,x,u) + O(s^2),\\
\tilde{u} &= u + s\phi(t,x,u) + O(s^2),
\end{aligned}
$$

where s is the group parameter and τ, ξ and ϕ are the infinitesimals of the transformations for the independent variables t, x and dependent variable u respectively.

The associative vector field for Eq. (1.1) is:

$$
X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \phi(t,x,u)\frac{\partial}{\partial u}. \tag{3.1}
$$

After applying Lie's symmetry criteria, one can have

$$X^{[3]}\left\{u_t + u_{xxx} - A{u_x}^3 - F(u)u_x\right\}|_{Eq.\ (1.1)} = 0, \tag{3.2}$$

where $X^{[3]}$ is the 3^{rd} prolongation of (3.1) and defined as:

$$X^{[3]} = X + \phi^t \partial_{u_t} + \phi^x \partial_{u_x} + \phi^{x^2} \partial_{u_{x^2}} + ... + \phi^{x^3} \partial_{u_{x^3}}. \tag{3.3}$$

In (3.3), $\partial_{u_t} = \frac{\partial}{\partial u_t}$ and so on, while

$$\phi^t = D_t(\phi - \tau u_t - \xi u_x) + \xi u_{xt} + \tau u_{t^2},$$
$$\phi^x = D_x(\phi - \tau u_t - \xi u_x) + \xi u_{x^2} + \tau u_{xt},$$

where D_t and D_x are the total derivatives with respect to t and x respectively. The reader is referred to [36] for the definition and properties of Lie point symmetries.

The condition (3.2) yields a system of following linear PDEs with $\tau = \tau(t)$, $\xi = \xi(t, x)$ and $\phi = a$:

$$(i)\ \xi_{xx} = 0, \quad (ii)\ \tau_t - 3\xi_x = 0, \quad (iii)\ -F_u\phi - F\tau_t + F\xi_x - \xi_{xxx} - \xi_t + 3\phi_{xxu} = 0. \tag{3.4}$$

Differentiating Eq. (3.4(iii)) with respect to u and some manipulations yields to the following two cases:

(i) $F_u \neq 0$, (ii) $F_u = 0$.

3.1.1 $F_u \neq 0$

For this case, we get:

$$\left(\frac{F_u}{F_{uu}}\right)_u = 0. \tag{3.5}$$

The solutions of Eq. (3.5) yield the following functions

(i) $F(u) = ae^{bu} + c$, (ii) $F(u) = au + b$.

Now we discuss the different forms of $F(u)$, up to equivalence transformations, which lead to an extension of the principal Lie algebra of Eq. (1.1) for $A \neq 0$.

3.1.2 When $F(u)$ is arbitrary, (Minimal symmetry algebra)

The minimal algebra for the arbitrary case is:

$$\mathbf{X_1} = \frac{\partial}{\partial t}, \qquad \mathbf{X_2} = \frac{\partial}{\partial x}. \tag{3.6}$$

and appeared in all the rest of the considered cases, thus we shall only present the additional algebra(s).

3.1.3 $F(u) = ae^{bu} + c$

For this case the principal algebra extends and additional generator will be:

$$X_3 = -3bt\frac{\partial}{\partial t} + (2ct - x)b\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}. \tag{3.7}$$

3.1.4 $F(u) = (au + b)$

In this case, additional generators will be:

$$X_3 = -at\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{3.8}$$

3.1.5 $F_u = 0$, (Maximal symmetry algebra)

Taking $F(u) = a$ the principal algebra extends and additional generators will be:

$$X_3 = \frac{\partial}{\partial u}, \quad X_4 = t\frac{\partial}{\partial t} + \frac{1}{3}(x - 2at)\frac{\partial}{\partial x}. \tag{3.9}$$

3.2 One dimensional optimal systems

Next task is to classify Lie algebra $L = \{X_i, X_j\}$ into its subalgebras up to conjugacy classes. For this purpose some basic definitions are taken from the literature [36].

1: A subspace $\pounds_i$ of a Lie algebra [36] $\pounds$ is said to be a Lie subalgebra if it is closed under Lie bracket or Lie commutator which is

$$[X_i,\ X_j] = X_j(X_i) - X_i(X_j),$$

where X_i and X_j are Lie point symmetry generators.

Using the definition of subalgebra one can easily conclude that there are infinite number of one dimensional subalgebras of L. It should be noted that $\pounds_i$ and $\pounds_j$ are said to be equivalent conjugacy classes if

$$\pounds_i = Ad\ X_i(\pounds_j),$$

where $X_i \in L$ and

$$Ad\Big[\exp(tX_i)\Big]X_j = X_j - t[X_i,\ X_j] + \frac{t^2}{2}\Big[X_i, [X_i,\ X_j]\Big] - \cdots .$$

As Lie algebra $L = \{\mathbf{X_1}, \mathbf{X_2}\}$ holds:

$$[\mathbf{X_1}, \mathbf{X_2}] = 0, \tag{3.10}$$

where $[,]$ is called Lie bracket which can be defined as follows:

$$[X_i,\ X_j] = X_j(X_i) - X_i(X_j),$$

while X_i and X_j are Lie point symmetry generators.

For $\mathbf{X_1}$ and $\mathbf{X_2}$ which satisfy (3.10), the one dimensional optimal system [36] is:

$$< \mathbf{X_1} >,\ \ < \mathbf{X_2} >,\ \ < \mathbf{X_1} + \omega \mathbf{X_2} > .$$

3.3 Reduction by using optimal systems

In this section, all possible similarity reductions via similarity variables are computed for Eq. (1.1) for arbitrary choice of F.

3.3.1 $\pounds_1 =< \mathbf{X_1} >$

For this case, one can have

$$\xi = t, \quad u = V(\xi),$$

which gives constant solution.

3.3.2 $\pounds_2 =< \mathbf{X_2} >$

For this class, one can obtain

$$\xi = x, \quad u = v(\xi), \tag{3.11}$$

where V is the solution of

$$v^{'''} - A(v^{'})^3 - F(v)v^{'} = 0. \tag{3.12}$$

where $v^{'} = \frac{dv}{d\xi}$ and so on.

3.3.3 $\pounds_3 =< \mathbf{X_1} + \omega \mathbf{X_2} >$

For this class, one can easily get

$$\xi = x - \omega t, \quad u = v(\xi),$$

where v is the solution of

$$-\omega v^{'} + v^{'''} - A(v^{'})^3 - F(v)v^{'} = 0. \tag{3.13}$$

4 Nonlinear Self-Adjointness and Conserved Vectors

4.1 The nonlinear self-adjointness classification

To investigate whether Eq. (1.1) for arbitrary $F(u)$, is strictly self-adjoint Def. (1), quasi self-adjoint Def. (2), weak self-adjoint Def. (3) or nonlinear self-adjoint Def. (4), we have it Lagrangian

$$L = v[u_t + u_{xxx} - A{u_x}^3 - F(u)u_x] = 0, \tag{4.1}$$

and adjoint equation of Eq. (1.1) is calculated as

$$E^* = -v_t - v_{xxx} + 3Av_x{u_x}^2 + 6Avu_xu_{xx} + v_xF(u). \tag{4.2}$$

Result 1. Eq. (1.1) is not strictly self-adjoint for $A \neq 0$ but strictly self-adjointness exists for Eq. (1.1) by keeping $A = 0$.

Result 2. Eq. (1.1) is a quasi self-adjoint or more generally a nonlinearly self-adjoint equation for

$$v = \psi(u) = pe^{\sqrt{2A}u} + qe^{-\sqrt{2A}u}, \quad \text{with} \;\; p, q \in \mathbb{R}. \tag{4.3}$$

4.2 Conserved Vectors

The conserved vectors for above symmetry generators are calculated next. Here, we will omit the term $\xi^i\mathcal{L}$ from (2.3) as it provides a trivial conserved vector of the equation.

4.2.1 Conserved quantities for minimal symmetry algebra

In this section, we will compute the conserved vectors via minimal set of Lie point symmetries given in Eq. (3.6) for arbitrary $F(u)$ and $v = \psi(u)$ from Eq. (4.3).

(I) For this case, components of conserved vector T_1 for $\mathbf{X_1}$ are:

$$T_1^t = -\psi u_t,$$

$$T_1^x = 3A\psi u_t u_x^2 + \psi u_t F(u) - u_t(\psi_u u_{xx} + \psi_{uu}u_x^2) + u_{tx}\psi_u u_x - u_{txx}\psi, \tag{4.4}$$

where ψ is given in Eq. (4.3).

(II) The conserved vectors for $\mathbf{X_2}$ are:

$$T_2^t = -\psi u_x,$$

$$T_2^x = 3A\psi u_x^3 + \psi u_x F(u) - u_x(\psi_u u_{xx} + \psi_{uu}u_x^2) + u_{xx}\psi_u u_x - u_{xxx}\psi. \tag{4.5}$$

where ψ is same as presented in Eq. (4.3).

5 Application of (G'/G)–expansion method to CDF equation

In this section, we will apply the (G'/G)–expansion method to find the solitary wave solutions of a special class of CDF Eq. (1.1). By setting $A = \frac{1}{8}$ and $F(u) = pe^u + qe^{-u}$ in Eq. (1.1), we get Eq. (3.13) of the form

$$-\omega v^{'} + v^{'''} - A(v^{'})^3 - (pe^v + qe^{-v})v^{'} = 0. \tag{5.1}$$

where $pq \neq 0$.
By using the transformation $v(\xi) = \ln V(\xi)$ in Eq. (5.1), we get the following equation:

$$(-\omega V' + V''')V^2 - 3VV'V'' + \tfrac{15}{8}(V')^3 - VV'(pV^2 + q) = 0. \tag{5.2}$$

According to *Step 1*, we put $3m + 3 = 4m + 1$, hence $m = 2$. Therefor, we assume that the solution of Eq. (5.2) can be expressed by a polynomial in (G'/G) as follows:

$$V = \alpha_2(\tfrac{G'}{G})^2 + \alpha_1(\tfrac{G'}{G}) + \alpha_0, \qquad \alpha_2 \neq 0, \tag{5.3}$$

where $G(\xi)$ satisfies a second order linear ordinary differential Eq. (2.7) and $\alpha_2, \alpha_1, \alpha_0$, are unknown to be determined later.

Substituting Eq. (5.3) into Eq. (5.2) and collecting all terms with the same order of (G'/G) together, the left-hand side of Eq. (5.2) is converted into a polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for $\lambda, \mu, \omega, \alpha_0, \alpha_1$ and α_2. Solving the system of algebraic equations with the aid of Maple 12, we obtain the following three general results.

5.1 Case 1

The first set of obtained results is

$$\left\{\mu = \pm\tfrac{2}{3}\sqrt{pq},\ \omega = \mp\tfrac{2}{3}\sqrt{pq} - \tfrac{1}{2}\lambda^2,\ \alpha_0 = 0,\ \alpha_1 = 0,\ \alpha_2 = \tfrac{3}{2p}\right\} \tag{5.4}$$

where λ is an arbitrary constant. Therefore, substituting the above case in Eq. (5.3), we get

$$V(\xi) = \tfrac{3}{2p}(\tfrac{G'}{G})^2, \tag{5.5}$$

where $\xi = x \pm \frac{2}{3}\sqrt{qp}\ t + \frac{1}{2}\lambda^2 t$. Substituting the general solutions (2.8) into Eq. (5.5), we can obtain the following three types of solitary wave solutions of of Eq. (1.5) with respect to the value of $\mathcal{D}_1 = \lambda^2 - 4\mu$.

Firstly, assume that $\mathcal{D}_1 = \lambda^2 - 4\mu = \frac{1}{6}\left(9\lambda^2 \mp 24\sqrt{pq}\right) > 0$, then using the relationship (5.5) and relationship $v(\xi) = \ln V(\xi)$, we obtain hyperbolic function solution $v_{\mathcal{H}}$, of time dependent CDF Eq. (1.5) as follows:

$$v_{\pm\mathcal{H}}(\xi) = \ln\left(\tfrac{3}{2p}(\tfrac{G'}{G})^2_{\pm}\right), \tag{5.6}$$

where

$$(\frac{G'}{G})_{\pm} = \frac{1}{6}\frac{\sqrt{9\lambda^2\mp 24\sqrt{pq}}\Big(C_1\sinh(\frac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi)+C_2\cosh(\frac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi)\Big)}{C_2\sinh(\frac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi)+C_1\cosh(\frac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi)} - \frac{1}{2}\lambda, \tag{5.7}$$

and $\xi = x \pm \frac{2}{3}\sqrt{qp}\ t + \frac{1}{2}\lambda^2 t$, and C_1, C_2, λ are arbitrary constants. It is easy to see that the hyperbolic solution (5.7) can be rewritten at $C_1^2 > C_2^2$, as follows

$$u_{\pm\mathcal{H}}(x,t) = \ln\left(\frac{1}{24p}\Big(\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \tanh(\tfrac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi + \eta_{\mathcal{H}}) - 3\lambda\Big)^2\right), \tag{5.8a}$$

while at $C_1^2 < C_2^2$, one can obtain

$$u_{\pm\mathcal{H}}(x,t) = \ln\left(\frac{1}{24p}\Big(\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \coth(\tfrac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \xi + \eta_{\mathcal{H}}) - 3\lambda\Big)^2\right), \tag{5.8b}$$

where $\xi = x \pm \frac{2}{3}\sqrt{qp}\ t + \frac{1}{2}\lambda^2 t$, and $\lambda,\ \eta_{\mathcal{H}} = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$, are arbitrary constants.

Now, when $\mathcal{D}_1 = \lambda^2 - 4\mu = \frac{1}{6}\sqrt{9\lambda^2\mp 24\sqrt{pq}} < 0$, using the relationship $v(\xi) = \ln V(\xi)$, we obtain trigonometric function solution $U_{\mathcal{T}}$, of Eq. (**??**) as follows:

$$v_{\pm\mathcal{T}}(\xi) = \ln\left(\tfrac{3}{2p}(\tfrac{G'}{G})^2_{\pm}\right), \tag{5.9}$$

where

$$(\frac{G'}{G})_{\pm} = \frac{1}{6}\frac{\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\Big(-C_1\sin(\frac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi)+C_2\cos(\frac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi)\Big)}{C_2\sin(\frac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi)+C_1\cos(\frac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi)} - \frac{1}{2}\lambda, \tag{5.10}$$

and $\xi = x \pm \frac{2}{3}\sqrt{qp}\ t + \frac{1}{2}\lambda^2 t$, and C_1, C_2, λ are arbitrary constants. Similarity, it is easy to see that the trigonometric solution (5.9) can be rewritten at $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows

$$u_{\pm\mathcal{T}}(x,t) = \ln\Big(\frac{1}{24p}\big(\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \tan(\tfrac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi + \eta_{\mathcal{T}}) - 3\lambda\big)^2\Big) \tag{5.11a}$$

and

$$u_{\pm\mathcal{T}}(x,t) = \ln\Big(\frac{1}{24p}\big(\sqrt{9\lambda^2\mp 24\sqrt{pq}}\ \cot(\tfrac{1}{6}\sqrt{\pm 24\sqrt{pq}-9\lambda^2}\ \xi + \eta_{\mathcal{T}}) - 3\lambda\big)^2\Big) \tag{5.11b}$$

respectively, where $\xi = x \pm \frac{2}{3}\sqrt{qp}\ t + \frac{1}{2}\lambda^2 t$, and $\lambda,\ \eta_{\mathcal{T}} = \tan^{-1}\left(\frac{C_1}{C_2}\right)$, are arbitrary constants.

Finally, when $\mathcal{D}_1 = \lambda^2 - 4\mu = 0$, we obtain the following solution

$$u(x,t)_{\pm} = \ln\left(\frac{3}{2p}\Big(\frac{C_2}{C_1 + C_2(x \pm 2\sqrt{pq}\ t)} - \frac{1}{3}\sqrt{6\sqrt{pq}}\Big)^2\right), \tag{5.12}$$

where C_1 and C_2 are arbitrary constants.

The graphics of positive kind of hyperbolic solution (5.8b), trigonometric solution (5.11b) and rational solution (5.12) for $p = q = \frac{1}{2}, \lambda = 1, C_1 = \sqrt{2}$ and $C_2 = \sqrt{3}$ are shown in Figure 1.

5.2 Case 2

The second set of obtained results is

$$\left\{\lambda = \tfrac{\alpha_1^2p^2+9\mu\pm6\sqrt{pq}}{3\alpha_1 p},\ \omega = \tfrac{1}{18}\big(18\mu - \alpha_1^2p^2 \pm 24\sqrt{pq}\big) - \tfrac{81\mu^2\pm108\mu\sqrt{pq}+36pq}{18\alpha_1^2p^2},\ \alpha_0 = \tfrac{1}{6}\alpha_1^2 p,\ \alpha_2 = \tfrac{3}{2p}\right\} \tag{5.13}$$

where μ and $\alpha_1 \neq 0$ are arbitrary constants. Therefore, substitute the above case in (5.3), and using relationship $v(\xi) = \ln V(\xi)$, we get

$$v_{\pm}(\xi) = \ln\left(\tfrac{3}{2p}(\tfrac{G'}{G})_{\pm}^2 + \alpha_1(\tfrac{G'}{G})_{\pm} + \tfrac{1}{6}\alpha_1^2 p\right), \tag{5.14}$$

where

$$\xi = x + \tfrac{1}{8}(\alpha_1^2p^2 - 18\mu \mp 24\sqrt{pq})t + \tfrac{81\mu^2\pm108\mu\sqrt{pq}+36pq}{\alpha_1^2p^2}t. \tag{5.15}$$

According to second set of results (5.13), we have

$$\mathcal{D}_2 = \lambda^2 - 4\mu = \tfrac{1}{6}\left((\alpha_1^2p^2 - 18\mu \pm 12\sqrt{pq}) + \tfrac{81\mu^2\pm108\mu\sqrt{pq}+36pq}{\alpha_1^2p^2}\right). \tag{5.16}$$

Similar as previous case, in the first step, assume that $\mathcal{D}_2 = \lambda^2 - 4\mu > 0$, then the expression $(\frac{G'}{G})_{\pm}$ with respect to second set of results (5.13) will be

$$(\frac{G'}{G})_{\pm} = \frac{\sqrt{\mathcal{D}_2}\big(C_1\sinh(\sqrt{\mathcal{D}_2}\ \xi) + C_2\cosh(\sqrt{\mathcal{D}_2}\ \xi)\big)}{C_2\sinh(\sqrt{\mathcal{D}_2}\ \xi) + C_1\cosh(\sqrt{\mathcal{D}_2}\ \xi)} - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}, \tag{5.17}$$

where ξ and $\mathcal{D}_2$ are mentioned in (5.15) and (5.16) respectively, and $C_1,\ C_2,\ \mu$ and $\alpha_1 \neq 0$ are arbitrary constants. It is easy to see that the hyperbolic solution (5.14) can be rewritten at $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows

$$\begin{aligned} u_{\pm\mathcal{H}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{\mathcal{D}_2}\ \tanh(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{\mathcal{D}_2}\ \tanh(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg) \end{aligned} \tag{5.18a}$$

and

$$\begin{aligned} u_{\pm\mathcal{H}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{\mathcal{D}_2}\ \coth(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{\mathcal{D}_2}\ \coth(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg), \end{aligned} \tag{5.18b}$$

where ξ and $\mathcal{D}_2$ are mentioned in (5.15) and (5.16) respectively, $\mu,\ \alpha_1 \neq 0$ and $\eta_{\mathcal{H}} = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$ are arbitrary constants.

Next, when $\mathcal{D}_2 = \lambda^2 - 4\mu < 0$, then the expression $(\frac{G'}{G})_\pm$ will be

$$(\frac{G'}{G})_\pm = \frac{\sqrt{-\mathcal{D}_2}\big(-C_1\sin(\sqrt{-\mathcal{D}_2}\;\xi) + C_2\cos(\sqrt{-\mathcal{D}_2}\;\xi)\big)}{C_2\sin(\sqrt{-\mathcal{D}_2}\;\xi) + C_1\cos(\sqrt{-\mathcal{D}_2}\;\xi)} - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}, \tag{5.19}$$

where ξ and $\mathcal{D}_2$ are mentioned in (5.15) and (5.16) respectively, $C_1,\ C_2,\ \mu$ and $\alpha_1 \neq 0$ are arbitrary constants.

Similarly, the trigonometric solution (5.14) can be written for $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as

$$\begin{aligned} u_{\pm\mathcal{T}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{-\mathcal{D}_2}\ \tan(\tfrac{1}{2}\sqrt{-\mathcal{D}_2}\;\xi + \eta_{\mathcal{T}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{-\mathcal{D}_2}\ \tan(\tfrac{1}{2}\sqrt{-\mathcal{D}_2}\;\xi + \eta_{\mathcal{T}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg) \end{aligned} \tag{5.20a}$$

and

$$\begin{aligned} u_{\pm\mathcal{T}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{-\mathcal{D}_2}\ \cot(\tfrac{1}{2}\sqrt{-\mathcal{D}_2}\;\xi + \eta_{\mathcal{T}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{-\mathcal{D}_2}\ \cot(\tfrac{1}{2}\sqrt{-\mathcal{D}_2}\;\xi + \eta_{\mathcal{T}}) - \frac{\alpha_1^2p^2 + 9\mu \pm 6\sqrt{pq}}{6\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg), \end{aligned} \tag{5.20b}$$

respectively. Here ξ and $\mathcal{D}_2$ are mentioned in (5.15) and (5.16), $\mu, \alpha_1 \neq 0$ and $\eta_{\mathcal{T}} = \tan^{-1}\left(\frac{C_1}{C_2}\right)$, are arbitrary constants.

Eventually, when $\mathcal{D}_1 = \lambda^2 - 4\mu = 0$, then we obtain the following solution

$$\begin{aligned} u_\pm(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\frac{C_2}{C_1 + C_2(x - 2\sqrt{pq}\,t)} - \frac{1}{3}\alpha_1 p \pm \sqrt{-6\sqrt{pq}}\Big)^2 \\ &+ \alpha_1\Big(\frac{C_2}{C_1 + C_2(x - 2\sqrt{pq}\,t)} - \frac{1}{3}\alpha_1 p \pm \sqrt{-6\sqrt{pq}}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg), \end{aligned} \tag{5.21}$$

where $C_1,\ C_2$ and $\alpha_1 \neq 0$ are arbitrary constants.

Again as previous case, for $p = q = \frac{1}{2},\ C_1 = \sqrt{2},\ C_2 = \sqrt{3}$, the graphics of positive kind of hyperbolic solution (5.18b) for $\mu = 1$ are shown in Figure 2(a), trigonometric solution (5.20b) and rational solution (5.21) for $\mu = \frac{1}{2}$ are shown in Figure 2(b) and Figure 2(c) respectively.

5.3 Case 3

The third set of obtained results is

$$\Big\{\mu = 0,\ \lambda = \tfrac{1}{3}\alpha_1 p \pm 2\frac{\sqrt{pq}}{\alpha_1 p},\ \omega = \tfrac{2}{9}\big(\alpha_1^2p^2 \pm 6\sqrt{pq}\big) - \frac{5\alpha_1^4p^3 + 36q}{18\alpha_1^2 p},\ \alpha_0 = \tfrac{1}{6}\alpha_1^2 p,\ \alpha_2 = \tfrac{3}{2p}\Big\} \tag{5.22}$$

where $\alpha_1 \neq 0$ is an arbitrary constant. Therefore, using relationship $v(\xi) = \ln V(\xi)$, we get

$$v_{\pm}(\xi) = \ln\Big(\tfrac{3}{2p}(\tfrac{G'}{G})^2_{\pm} + \alpha_1(\tfrac{G'}{G})_{\pm} + \tfrac{1}{6}\alpha_1^2 p\Big), \tag{5.23}$$

where

$$\xi = x - \tfrac{2}{9}(\pm\alpha_1^2 p^2 + 6\sqrt{pq})t + \tfrac{5\alpha_1^4 p^3 + 36q}{\alpha_1^2 p}t. \tag{5.24}$$

and the parameter $\mathcal{D}_3 = \lambda^2 - 4\mu$ will be

$$\mathcal{D}_3 = \lambda^2 - 4\mu = \tfrac{1}{6}\left(\frac{(\pm\alpha_1^2 p^2 + 6\sqrt{pq})^2}{\alpha_1^2 p^2}\right). \tag{5.25}$$

It is easy to see that the parameter $\mathcal{D}_3$ always will be positive, then the only possible expression $(\frac{G'}{G})_{\pm}$ with respect to third set of results (5.22) in case 3 will be

$$(\frac{G'}{G})_{\pm} = \frac{\sqrt{\mathcal{D}_3}\big(C_1\sinh(\sqrt{\mathcal{D}_3}\ \xi) + C_2\cosh(\sqrt{\mathcal{D}_3}\ \xi)\big)}{C_2\sinh(\sqrt{\mathcal{D}_3}\ \xi) + C_1\cosh(\sqrt{\mathcal{D}_3}\ \xi)} - \frac{1}{6}\alpha_1 p \mp \frac{\sqrt{pq}}{\alpha_1 p}, \tag{5.26}$$

where ξ and $\mathcal{D}_3$ are mentioned in (5.24) and (5.25) respectively, $C_1,\ C_2$ and $\alpha_1 \neq 0$ are arbitrary constants. Therefore, the hyperbolic solution of (5.23) can be rewritten for $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows

$$\begin{aligned} u_{\pm\mathcal{H}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{\mathcal{D}_2}\ \tanh(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{1}{6}\alpha_1 p \mp \frac{\sqrt{pq}}{\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{\mathcal{D}_2}\ \tanh(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{1}{6}\alpha_1 p \mp \frac{\sqrt{pq}}{\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg) \end{aligned} \tag{5.27a}$$

and

$$\begin{aligned} u_{\pm\mathcal{H}}(x,t) = \ln\Bigg(&\frac{3}{2p}\Big(\sqrt{\mathcal{D}_2}\ \coth(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{1}{6}\alpha_1 p \mp \frac{\sqrt{pq}}{\alpha_1 p}\Big)^2 \\ &+ \alpha_1\Big(\sqrt{\mathcal{D}_2}\ \coth(\tfrac{1}{2}\sqrt{\mathcal{D}_2}\ \xi + \eta_{\mathcal{H}}) - \frac{1}{6}\alpha_1 p \mp \frac{\sqrt{pq}}{\alpha_1 p}\Big) + \tfrac{1}{6}\alpha_1^2 p\Bigg), \end{aligned} \tag{5.27b}$$

where ξ and $\mathcal{D}_2$ are mentioned in (5.24) and (5.25) respectively, $\alpha_1 \neq 0$ and $\eta_{\mathcal{H}} = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$ are arbitrary constants.

Finally, from $\mathcal{D}_1 = \lambda^2 - 4\mu = 0$, we get $\alpha_1 = \pm\sqrt{\frac{6\sqrt{pq}}{p^2}}$ then the following solution can be obtain

$$u_{\pm}(x,t) = \ln\left(\frac{3}{2p}\frac{C_2^2}{(C_1 + C_2(x + \frac{2pq}{\sqrt{pq}}\,t))^2} \pm \frac{\sqrt{\frac{6\sqrt{pq}}{p^2}}\,C_2}{C_1 + C_2(x + \frac{2pq}{\sqrt{pq}}\,t)} + \frac{\sqrt{pq}}{p}\right). \tag{5.28}$$

where C_1, C_2 and $\alpha_1 \neq 0$ are arbitrary constants.

Similarly, as previous cases, for $p = q = \frac{1}{2}, \alpha_1 = 1, \lambda = 2, C_1 = \sqrt{2}$ and $C_2 = \sqrt{3}$, the graphics of positive kind of hyperbolic solution (5.27b) and rational solution (5.28) are shown in Figure 3(a)and Figure 3(b) respectively.

6 Conclusions

In this paper, Lie classification for a class of exactly solvable third order nonlinear evolution equation [16] is carried out. The nonlinear self-adjointness is also investigated proving it a quasi self-adjoint equation and conserved vectors are computed for the set of minimal Lie algebra. Further, a Calogero–Degasperis–Fokas equation is introduced and the solitary wave solutions of mentioned equation are obtained via (G'/G)-expansion method. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. With the aid of Maple 12, we have assured the correctness of the obtained solutions by putting them back into the original equation. We hope that obtained solutions will be useful for further studies in applied sciences and engineering.

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