Two-phase Stefan problem with nonlinear thermal coefficients and a convective boundary condition

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Abstract. We consider a non-linear two-phase unidimensional Stefan problem, which consists on a solidification process, for a semi-infinite material $x > 0$, with phase change temperature T_1, an initial temperature $T_2 > T_1$ and a convective boundary condition imposed at the fixed face $x = 0$ characterized by a heat transfer coefficient $h > 0$. We assume that the volumetric heat capacity and the thermal conductivity are particular nonlinear functions of the temperature in both solid and liquid phases and they verify a Storm-type relation. A certain inequality on the coefficient h is established in order to get an instantaneous phase change process. We determine sufficient conditions on the parameters of the problem in order to prove the existence and uniqueness of a parametric explicit solution for the Stefan problem.

Keywords. Stefan problem,free boundary problem, phase-change process, similarity solution, Kirchoff transformation.

1. Introduction

The study of phase change processes has occupied scientists of the early eighteenth century. In 1831, Lame and Clapeyron studied problems related to the solidification of planet Earth [15]. In addition, the mathematical formulation of phase change processes as problems of free borders dates from 18th century, because it owes a lot to the works developed by Stefan in 1889 [24, 25, 26]. Currently, his study remains an active area of research. Stefan's problems are present in a wide variety of situations, both natural and industrial. A review of a long bibliography on moving and free boundary value problems for the

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heat equation can be consulted in [29]. In [11, 12, 30] recent applications and future challenges can be found.

We consider the following two-phase Stefan problem (solidification process) with nonlinear thermal coefficients for a semi-infinite region $x > 0$ with phase change temperature T_1, an initial temperature $T_2 > T_1$ and an imposed a convective condition at the fixed face $x = 0$ [32]

$$C_1(u_1)\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\left[K_1(u_1)\frac{\partial u_1}{\partial x}\right], \qquad 0 < x < y(t)\ , \quad t > 0\ , \tag{1.1}$$

$$C_2(u_2)\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\left[K_2(u_2)\frac{\partial u_2}{\partial x}\right], \qquad x > y(t), \quad t > 0, \tag{1.2}$$

$$y(0) = 0\ , \tag{1.3}$$

$$u_2(x,0) = T_2 \qquad x > 0, \tag{1.4}$$

$$u_1(y(t),t) = u_2(y(t),t) = T_1, \quad t > 0, \tag{1.5}$$

$$K_1(u_1)\frac{\partial u_1}{\partial x} - K_2(u_2)\frac{\partial u_2}{\partial x} = L\,\dot{y}\,(t), \qquad x = y(t), \quad t > 0, \tag{1.6}$$

$$K_1(u_1(0,t))\frac{\partial u_1}{\partial x}(0,t) = \frac{h}{\sqrt{t}}[u_1(0,t) - T_\infty], \qquad t > 0,\ \ h > 0, \tag{1.7}$$

where

$$\begin{cases} x : \text{spatial coordinate} \ \ t : \text{time}, \\ u_i(x,t) : \ \text{temperature distribution for phase i}\ , \\ T_1 : \text{phase-change or freezing temperature}\ , \\ T_2 : \text{initial temperature}\ , \\ T_\infty : \text{temperature of the medium}, \\ L > 0 : \ \text{volumetric latent heat}\ , \\ C_i(u_i) > 0 : \ \text{volumetric heat capacity for phase i}\ , \\ K_i(u_i) > 0 : \ \text{thermal conductivity for phase i}\ , \\ h : \text{heat transfer coefficient}, \\ y(t) : \ \text{free boundary (solid-liquid interface) at time } t\ , \\ i = 1 : \text{solid phase}, i = 2 : \text{liquid phase} \end{cases}$$

with

$$T_\infty < u_1(0,t) < T_1 < T_2. \tag{1.8}$$

One common assumption when modeling phase-change processes is to consider constant thermophysical properties. Nevertheless, it is known that certain materials present properties which seem to obey other laws. Some models including variable latent heat, density, melting temperature or thermal conductivity have been proposed in [2, 8, 13, 17, 21].

We assume that the volumetric heat capacity and the thermal conductivity for each phase i $(i = 1, 2)$ are related as follows: [31]

$$C_i(u_i) = \frac{K_i(u_i)c_0}{k_0\left[1 - \frac{1}{k_0}\int_0^{\frac{u_i - T_1}{T_2 - T_1}} K_i(T_1 + (T_2 - T_1)z)dz\right]^2}\ , \ i = 1, 2 \tag{1.9}$$

with the assumption given by

$$\frac{1}{k_0\,(T_2-T_1)}\int_{T_1}^{T_2} K_2\,(z)\,dz<1 \tag{1.10}$$

where k_0, c_0 are scales for the thermal conductivity and volumetric heat capacity respectively. The goal of this paper is to determine which conditions on the parameters of the problem must be satisfied in order to obtain an explicit solution

$$u(x,t)=\begin{cases} u_1(x,t)<T_1, & 0<x<y(t) \\ T_1, & x=y(t) \\ u_2(x,t)>T_1, & x>y(t) \end{cases} \quad . \tag{1.11}$$

and the free boundary $x=y(t),\ t>0$.

We remark that if equation (1.9) is true then, $K_i(u_i)$ and $C_i(u_i), i=1,2,$ verify the Storm relation given by [5, 6, 19, 27]

$$\frac{1}{\sqrt{K_i(u_i)C_i(u_i)}}\frac{d}{du_i}\left(\log\sqrt{\frac{C_i(u_i)}{K_i(u_i)}}\right)=\frac{1}{\sqrt{c_0k_0}(T_2-T_1)}=const. \tag{1.12}$$

Condition (1.12) was originally obtained by [27] in an investigation of heat conduction in simple monoatomic metals. There, the validity of the equation (1.12) was examined for aluminium, silver, sodium, cadium, zinc, copper and lead. A nonlinear heat conduction problem for semi-infinite material $x>0$, with phase change temperature T_1 an initial temperature $T_2>T_1$ and a heat flux of the type $q(t)=\frac{q_0}{\sqrt{t}}$ imposed on the fixed face $x=0$ was considered by [7], where volumetric heat capacity and thermal conductivity were taken to satisfy the relation (1.9). Sufficient conditions on the parameters of the problem were established in order to obtain an instantaneous nonlinear two-phase Stefan problem (solidification process) and the explicit solution was given. Previously, the explicit solution to the corresponding nonlinear heat conduction problem for the initial (liquid) phase was obtained. Several authors have suggested considering Robin-type conditions (convective conditions) since they represent the fact that the heat transfer in the boundary is proportional to the difference between the temperature imposed and the material is presented at its boundary (see for example books [1, 9]). For this reason, in this article we consider a similar phase-change process to that studied in [16]. We are mainly motivated by improving the modelling of the imposed temperature at the fixed boundary by considering a convective boundary condition.

Other free boundary problems with nonlinear thermal coefficients are given in [3, 4, 5, 16, 19, 20, 23].

In Section 2 we consider the nonlinear two phase Stefan problem (1.1) − (1.7) with assumptions (1.9) − (1.10) and we prove that it admits a unique similarity type solution if certain conditions upon data are satisfied. We will study a solidification process, but a completely similar analysis can be done for the case of melting.

2. Explicit solution for the instantaneous two-phase Stefan process with nonlinear thermal coefficients

Let us consider the problem (1.1) – (1.7) with the hypothesis (1.8) – (1.10). In order to obtain an explicit solution of it, we consider the transformations given in [7] and we define new variables and parameters as follows

$$\begin{cases} \xi = x\sqrt{\frac{c_0}{k_0 t_s}}\,, \quad \tau = \frac{t}{t_s}\,, \quad S(\tau) = y(t)\sqrt{\frac{c_0}{k_0 t_s}}\,, \\ v_i(\xi,\tau) = \frac{u_i(x,t)-T_1}{T_2-T_1}\,, \quad \overline{K}_i(v_i) = \frac{K_i(u_i)}{k_0} > 0\,, \quad \overline{T} = \frac{T_\infty - T_1}{T_2-T_1} < 0\,, \\ \overline{C}_i(v_i) = \frac{C_i(u_i)}{c_0} > 0\,, \quad \overline{L} = \frac{1}{Ste} > 0\,, \quad H = \frac{h}{\sqrt{c_0 k_0}} > 0\,, \end{cases} \tag{2.1}$$

where t_s is a scale for the time and

$$Ste = \frac{c_0(T_2 - T_1)}{L} \tag{2.2}$$

is the Stefan number which, in the remainder of this section, is taken up to 1 due to the fact that it covers most of phase change materials [22].

We obtain the following problem

$$\overline{C_1}(v_1)\frac{\partial v_1}{\partial \tau} = \frac{\partial}{\partial \xi}\left[\overline{K_1}(v_1)\frac{\partial v_1}{\partial \xi}\right], \qquad 0 < \xi < S(\tau), \quad \tau > 0 \tag{2.3}$$

$$\overline{C_2}(v_2)\frac{\partial v_2}{\partial \tau} = \frac{\partial}{\partial \xi}\left[\overline{K_2}(v_2)\frac{\partial v_2}{\partial \xi}\right], \qquad \xi > S(\tau), \quad \tau > 0, \tag{2.4}$$

$$S(0) = 0, \tag{2.5}$$

$$v_2(\xi, 0) = 1, \qquad \xi > 0, \tag{2.6}$$

$$v_1(S(\tau),\tau) = v_2(S(\tau),\tau) = 0, \qquad \tau > 0, \tag{2.7}$$

$$\overline{K_1}(v_1)\frac{\partial v_1}{\partial \xi} - \overline{K_2}(v_2)\frac{\partial v_2}{\partial \xi} = \overline{L}\,\dot{S}(\tau), \qquad \text{on} \quad \xi = S(\tau) \quad \tau > 0, \tag{2.8}$$

$$\overline{K_1}\left(v_1(0,\tau)\right)\frac{\partial v_1}{\partial \xi}(0,\tau) = \frac{H}{\sqrt{\tau}}\left[v_1(0,\tau) - \overline{T}\right], \qquad \tau > 0. \tag{2.9}$$

(2.10)

By considering the Kirchhoff transformation for $i = 1, 2$ given by

$$\eta_i(\xi,\tau) = \mu_i\left(v_i(\xi,\tau)\right) = \int_0^{v_i(\xi,\tau)} \overline{K}_i(z)dz\ , \ \mu_i(\Psi) = \int_0^{\Psi} \overline{K}_i(z)dz\,, \tag{2.11}$$

then we have the following equivalent problem

$$\frac{\partial \eta_1}{\partial \tau} = (1-\eta_1)^2 \frac{\partial^2 \eta_1}{\partial \xi^2}, \qquad 0 < \xi < S(\tau), \quad \tau > 0, \tag{2.12}$$

$$\frac{\partial \eta_2}{\partial \tau} = (1-\eta_2)^2 \frac{\partial^2 \eta_2}{\partial \xi^2}, \qquad \xi > S(\tau), \quad \tau > 0, \tag{2.13}$$

$$S(0) = 0, \tag{2.14}$$

$$\eta_2(\xi, 0) = \int_0^1 \overline{K_2}(z)dz = \theta < 1, \qquad \xi > 0, \tag{2.15}$$

$$\eta_1(S(\tau), \tau) = \eta_2(S(\tau), \tau) = 0, \qquad \tau > 0, \tag{2.16}$$

$$\frac{\partial \eta_1}{\partial \xi} - \frac{\partial \eta_2}{\partial \xi} = \overline{L} \overset{\bullet}{S}(\tau), \qquad \text{on} \quad \xi = S(\tau), \quad \tau > 0, \tag{2.17}$$

$$\frac{\partial \eta_1}{\partial \xi}(0, \tau) = \frac{H}{\sqrt{\tau}} \left[\mu_1^{-1}(\eta_1(0, \tau)) - \overline{T}\right], \qquad \tau > 0, \tag{2.18}$$

where μ_1^{-1} is the inverse function of μ_1 defined by (2.11) .

Remark 1. According to the conditions at the free boundary we have that [28, 31]

$$S(\tau) = \delta\sqrt{\tau}, \ \tau > 0 \tag{2.19}$$

and the flux of η_2 on the free boundary becomes

$$\frac{\partial \eta_2}{\partial \xi}(S(\tau), \tau) = \frac{\gamma}{\sqrt{\tau}}, \ \tau > 0 \tag{2.20}$$

where the positive constants δ and γ must be established.

Now, we linearize the nonlinear differential equations (2.12) and (2.13). Following [14, 27] we define:

$$\begin{cases} \chi_1(\xi, \tau) = \int_0^\xi \frac{1}{(1-\eta_1(z,\tau))} dz \quad , \ 0 < \xi < S(\tau) \ , \\ \chi_2(\xi, \tau) = \int_{S(\tau)}^\xi \frac{1}{(1-\eta_2(z,\tau))} dz \ , \ \xi > S(\tau) \ , \\ \tau = \tau \ , \ \ w_i(\chi_i, \tau) = \eta_i(\xi, \tau) \ , \ \ i = 1, 2 \, . \end{cases} \tag{2.21}$$

Therefore the free boundary is turning into

$$\beta(\tau) = \chi_1(S(\tau), \tau) = \int_0^{S(\tau)} \frac{1}{(1-\eta_1(z, \tau))} dz. \tag{2.22}$$

We have that problem (2.12) – (2.19) is equivalent to

$$\frac{\partial w_1}{\partial \tau} = \frac{\partial^2 w_1}{\partial \chi_1^2} + \tfrac{H}{\sqrt{\tau}}\left[\mu_1^{-1}(w_1(0,\tau)) - \overline{T}\right] \frac{\partial w_1}{\partial \chi_1}, \quad 0 < \chi_1 < \beta(\tau), \tau > 0, \tag{2.23}$$

$$\frac{\partial w_2}{\partial \tau} = \frac{\partial^2 w_2}{\partial \chi_2^2} + \left[\frac{\delta}{2} + \gamma\right] \frac{1}{\sqrt{\tau}} \frac{\partial w_2}{\partial \chi_2}, \quad \chi_2 > 0, \quad \tau > 0, \tag{2.24}$$

$$\beta(0) = 0, \tag{2.25}$$

$$w_2(\chi_2, 0) = \theta, \quad \chi_2 > 0\ , \tag{2.26}$$

$$w_1(\beta(\tau),\tau) = w_2(0,\tau) = 0, \quad \tau > 0, \tag{2.27}$$

$$\frac{\partial w_1}{\partial \chi_1}(\beta(\tau),\tau) - \frac{\partial w_2}{\partial \chi_2}(0,\tau) = \overline{L}\frac{\delta}{2\sqrt{\tau}}, \quad \tau > 0, \tag{2.28}$$

$$\frac{\partial w_1}{\partial \chi_1}(0,\tau) = \frac{H}{\sqrt{\tau}}\left(1 - w_1(0,\tau)\right)\left[\mu_1^{-1}(w_1(0,\tau)) - \overline{T}\right], \quad \tau > 0. \tag{2.29}$$

and from (2.20) the flux at the fixed face $\chi_1 = 0$ is given by

$$\frac{\partial w_2}{\partial \chi_2}(0,\tau) = \frac{\gamma}{\sqrt{\tau}}, \ \tau > 0. \tag{2.30}$$

We propose a solution of similarity type given by

$$f_1(\phi_1) = w_1(\chi_1,\tau), \ \ \phi_1 = \frac{\chi_1}{2\sqrt{\tau}}\ , \qquad 0 < \chi_1 < \beta(\tau), \quad \tau > 0, \tag{2.31}$$

$$f_2(\phi_2) = w_2(\chi_2,\tau), \ \ \phi_2 = \frac{\chi_2}{2\sqrt{\tau}}, \qquad \chi_2 > 0, \quad \tau > 0. \tag{2.32}$$

Problem (2.23) – (2.29) turns into to the following

$$2(\phi_1 + \Lambda)f_1'(\phi_1) + f_1''(\phi_1) = 0, \qquad 0 < \phi_1 < \Lambda - \lambda_1\ , \tag{2.33}$$

$$2(\phi_2 + \lambda_2)f_2'(\phi_2) + f_2''(\phi_2) = 0, \qquad 0 < \phi_2\ , \tag{2.34}$$

$$f_2(+\infty) = \theta, \tag{2.35}$$

$$f_1'(0) = \lambda_1\left(1 - f_1(0)\right), \tag{2.36}$$

$$f_1(\Lambda - \lambda_1) = f_2(0) = 0, \tag{2.37}$$

$$f_1^{'}(\Lambda - \lambda_1) - f_2^{'}(0) = \overline{L}\delta, \tag{2.38}$$

for the unknown functions f_1 and f_2 , and the unknown coefficients $\delta, \gamma, \lambda_1$ where

$$\Lambda = \frac{\delta}{2}\left[1 + \overline{L}\right] + \gamma\ , \tag{2.39}$$

$$\lambda_1 = H\left[\mu_1^{-1}(f_1(0) - \overline{T}\right] > 0, \tag{2.40}$$

$$\lambda_2 = \frac{\delta}{2} + \gamma, \tag{2.41}$$

and (2.30) is equivalent to

$$f_2'(0) = 2\gamma. \tag{2.42}$$

Remark 2. Taking into account (2.19), (2.22) and

$$\beta(\tau) = 2(\Lambda - \lambda_1)\sqrt{\tau} \ , \ \ \tau > 0 \ . \tag{2.43}$$

Then we have $\Lambda > \lambda_1$. Moreover, by (2.39) and (2.41) we have $\Lambda > \lambda_2$.

The solution to problem (2.33) – (2.42) is given by

$$\begin{cases} f_1(\phi_1) = \dfrac{erf(\phi_1 + \lambda_1) - erf(\Lambda)}{G(\lambda_1) - erf(\Lambda)} \ , \ \ 0 < \phi_1 < \Lambda - \lambda_1 \\ \\ f_2(\phi_2) = \theta \dfrac{erf(\phi_2 + \lambda_2) - erf(\lambda_2)}{erfc(\lambda_2)} \ , \ \ 0 < \phi_2 \end{cases} \tag{2.44}$$

where

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp(-w^2)dw \ , \tag{2.45}$$

$$G\,(x) = erf(x) + \frac{1}{\sqrt{\pi}}R(x), \quad x > 0, \quad p > 0, \tag{2.46}$$

with

$$R(x) = \frac{\exp(-x^{2)}}{x}, \quad x > 0 \tag{2.47}$$

and the positive unknown coefficients Λ, λ_1 and λ_2 must satisfy the following system of equations

$$\begin{cases} (i) \ \dfrac{\theta}{\sqrt{\pi}} \dfrac{\exp(-\lambda_2^2)}{erfc(\lambda_2)} = \dfrac{\lambda_2\left(1 + \overline{L}\right) - \Lambda}{\overline{L}} \\ \\ (ii) \ \dfrac{\exp(-\Lambda^2)}{\sqrt{\pi}\,[G(\lambda_1) - erf(\Lambda)]} = \dfrac{\lambda_2 + (\overline{L} - 1)\Lambda}{\overline{L}} \\ \\ (iii) \ \dfrac{erf(\lambda_1) - erf(\Lambda)}{G(\lambda_1) - erf(\Lambda)} = \mu_1 \left(\dfrac{\lambda_1}{H} + \overline{T}\right) \end{cases} \tag{2.48}$$

Now, we give preliminaries results to prove the existence and uniqueness of the solution to (2.48).

Lemma **1.** *Let* $G = G(x)$ *be the function defined in* (2.46)*. It satisfies the following properties:*

$$G(0) = +\infty, \quad G(+\infty) = 1, \quad G'(x) < 0 \qquad \forall x > 0. \tag{2.49}$$

Proof. It was proved in [6]. □

Lemma **2.** *Let $P=P(x)$ defined by*

$$P(x)=(1+\overline{L})x-\frac{\overline{L}\theta}{\sqrt{\pi}}F(x)=x\left(1+\overline{L}-\frac{\overline{L}\theta}{Q(x)}\right), \tag{2.50}$$

where

$$F(x)=\frac{\exp(-x^2)}{\operatorname{erfc}(x)} \tag{2.51}$$

and

$$Q(x)=\sqrt{\pi}\,x\,\exp(x^2)\operatorname{erfc}(x) \tag{2.52}$$

It satisfies the following properties:

$$P\left(x^*\right)=0,\quad \textit{where}\quad x^*=Q^{-1}\left(\tfrac{\overline{L}\theta}{\overline{L}+1}\right),\qquad P(+\infty)=+\infty \tag{2.53}$$

$$P'(x)=1+\overline{L}-\tfrac{\overline{L}\theta}{Q(x)}+x\tfrac{\overline{L}\theta}{Q^2(x)}Q'(x),\quad P'(x)>0\quad \textit{if}\quad x>x^* \tag{2.54}$$

$$P(x)>x\quad \textit{if and only if}\quad x>\overline{x}=Q^{-1}(\theta),\quad P(\overline{x})=\overline{x}. \tag{2.55}$$

Proof. The proof is immediate.

Lemma **3.** *Let $M=M(x)$ be the function given by*

$$M(x)=\operatorname{erf}(P(x))+\frac{\overline{L}R(P(x))}{\sqrt{\pi}[\frac{x}{P(x)}+\overline{L}-1]},\quad x>\overline{x} \tag{2.56}$$

It satisfies:

$$M(\overline{x})=G(\overline{x}),\quad M(+\infty)=1,\quad M'(x)<0,\quad M(x)>1,\quad x>\overline{x}. \tag{2.57}$$

Proof. We first prove that $M'(x)<0$.
By using

$$R'(x)=-\frac{2x^2+1}{x}R(x)$$

we can write

$$M'(x)=\frac{R\left(P(x)\right)}{\sqrt{\pi}}\left\{2P(x)P'(x)\left[1-\frac{\overline{L}}{\frac{x}{P(x)}+\overline{L}-1}\right]-\frac{\overline{L}\left[P'(x)(\overline{L}-1)+1\right]}{\left[\frac{x}{P(x)}+\overline{L}-1\right]^2P(x)}\right\}$$

Taking into account Lemma 2 we have $P'(x)>0$ and $\frac{\overline{L}}{\frac{x}{P(x)}+\overline{L}-1}>1$. Furthermore, since $\overline{L}>1$ we obtain $M'(x)<0$. Finally, it is easy to check the others properties.

Lemma **4.** *Let $V=V(x)$ be the function given by*

$$V(x)=G^{-1}\left(M(x)\right),\quad x>\overline{x} \tag{2.58}$$

It satisfies the following properties:

$$V(\overline{x})=\overline{x},\quad V(+\infty)=+\infty,\quad V'(x)>0\qquad \forall x>\overline{x}. \tag{2.59}$$

Proof. It follows from Lemmas 1 and Lemma 3.

Lemma **5.** *If*

$$H > \frac{Q^{-1}(\theta)}{-\overline{T}} \tag{2.60}$$

then the function $U = U(x)$ *defined by*

$$U(x) = \mu_1 \left(\frac{V(x)}{H} + \overline{T} \right), \qquad x > \overline{x} \tag{2.61}$$

satisfies the following properties:

$$U(\overline{x}) = \mu_1 \left(\frac{\overline{x}}{H} + \overline{T} \right), \quad U'(x) > 0, \tag{2.62}$$

$$U(\hat{x}) = 0, \ \ \textit{with} \ \ \hat{x} = V^{-1}(-H\overline{T}), \tag{2.63}$$

$$U(x) \leq 0 \quad \textit{if} \quad x \in [\overline{x}, \hat{x}]. \tag{2.64}$$

Proof. It is easy to see that U is an increasing function because

$$U'(x) = \mu_1' \left(\frac{V(x)}{H} + \overline{T} \right) \frac{V'(x)}{H} > 0$$

From definition (2.11) we have (2.63). Taking into account (2.60) we get $U(\overline{x}) < 0$ and then from (2.62) and (2.63) follows (2.64).

Lemma **6.** *If data verifies* (2.60) *then, functions* $Y = Y(x)$ *and* $W = W(x)$ *defined by*

$$Y(x) = \operatorname{erf}(V(x)), \quad x \in [\overline{x}, \hat{x}] \tag{2.65}$$

and

$$W(x) = U(x)\left[M(x) - \operatorname{erf}(P(x))\right] + \operatorname{erf}(P(x)), \quad x \in [\overline{x}, \hat{x}] \tag{2.66}$$

satisfy the following properties:

$$Y(\overline{x}) = \operatorname{erf}(\overline{x}), \tag{2.67}$$

$$Y(\hat{x}) = \operatorname{erf}(-H\overline{T}), \tag{2.68}$$

$$Y'(x) > 0, \qquad \forall x \in [\overline{x}, \hat{x}] \tag{2.69}$$

$$W(\overline{x}) = \operatorname{erf}(\overline{x}) + U(\overline{x}) \frac{R(\overline{x})}{\sqrt{\pi}}, \tag{2.70}$$

$$W(\hat{x}) = \operatorname{erf}(P(\hat{x})), \tag{2.71}$$

$$W'(x) > 0, \qquad \forall x \in [\overline{x}, \hat{x}]. \tag{2.72}$$

Taking into account the previous Lemmas we can enunciate the following result:

Theorem 1. *If* (2.60) *holds, then for* $\lambda_2 \in [\overline{x}, \hat{x}]$ *the system of equations* (2.48) *is equivalent to*

$$\begin{cases} (i)\ \Lambda = P(\lambda_2), \\ (ii)\ \lambda_1 = V(\lambda_2), \\ (iii)\ W(\lambda_2) = Y(\lambda_2). \end{cases} \tag{2.73}$$

Proof. The solution to the problem (2.48) must satisfy $\Lambda > \lambda_2$ and $\Lambda > \lambda_1$. Then from (2.48)(i) we write Λ as function of λ_2 and we have (2.73)(i) for $\lambda_2 > \overline{x}$.

In order to obtain (2.73)(ii) we note that (2.48)(ii) is equivalent to

$$G(\lambda_1) = M(\lambda_2). \tag{2.74}$$

Moreover since $\frac{\lambda_2}{P(\lambda_2)} < 1$ we have that $\frac{\overline{L}}{\frac{\lambda_2}{P(\lambda_2)}+\overline{L}-1} > 1$ then

$$M(\lambda_2) > G\left(P(\lambda_2)\right) > 1 \tag{2.75}$$

and from (2.74) and Lemma 1 we can define $\lambda_1 = G^{-1}(M(\lambda_2)) = V(\lambda_2)$. Moreover this inequality implies that $\Lambda > \lambda_1$.

Now, we rewrite (2.48)(iii) as

$$\frac{erf(V(\lambda_2)) - erf(P(\lambda_2))}{M(\lambda_2) - erf(P(\lambda_2))} = U(\lambda_2)$$

or equivalently

$$erf(V(\lambda_2)) - erf(P(\lambda_2)) = U(\lambda_2)\left[M(\lambda_2) - erf(P(\lambda_2))\right].$$

Since $\Lambda > \lambda_1$ the left hand of the above equation is negative, it follows that $U(\lambda_2)$ must be negative and in consequence (2.48)(iii) can be written as (2.73)(iii) for $\overline{x} < \lambda_2 < \hat{x}$. Then the thesis holds.

Lemma **7.** *If data verifies* (2.60) *then, there exists a unique solution* λ_2^0 *to* (2.73)(iii).

Proof. We see at once that $W(\overline{x}) < Y(\overline{x})$. It is clear that from (2.75) follows

$$V(\lambda_2) < P(\lambda_2), \qquad\qquad \overline{x} < \lambda_2 < \hat{x},$$

that is to say, $erf(V(\lambda_2)) < erf(P(\lambda_2))$. In particular, if we take $\lambda_2 = \hat{x}$, we get $Y(\hat{x}) < W(\hat{x})$.

Therefore there exist λ_2^0 solution of (2.73)(iii). The uniqueness follows inmediatly from (2.69) and (2.72).

Theorem 2. *If* (2.60) *holds, then the system of equations* (2.73) *has a unique solution* λ_2^0, $\Lambda^0 = P(\lambda_2^0)$ *and* $\lambda_1^0 = V(\lambda_2^0)$. *Moreover*

$$\begin{cases} f_1(\phi_1) = \dfrac{\operatorname{erf}(\phi_1 + \lambda_1^0) - \operatorname{erf}(\Lambda^0)}{G(\lambda_1^0) - \operatorname{erf}(\Lambda^0)}\ , \quad 0 < \phi_1 < \Lambda^0 - \lambda_1^0 \\ \\ f_2(\phi_2) = \theta \dfrac{\operatorname{erf}(\phi_2 + \lambda_2^0) - \operatorname{erf}(\lambda_2^0)}{\operatorname{erfc}(\lambda_2^0)}\ , \quad 0 < \phi_2 \end{cases} \tag{2.76}$$

is the solution to the problem $(2.33) - (2.42)$.

We are thus led to the following strengthening of Theorem 3

Theorem 3. *If* Ste <1 *and the heat transfer coefficient* h *verifies*

$$h > \left(\frac{T_2-T_1}{T_1-T_\infty}\right)\sqrt{c_0k_0}\;\; Q^{-1}\left(\frac{1}{k_0\left(T_2-T_1\right)}\int\limits_{T_1}^{T_2} K_2\left(z\right)dz\right) \tag{2.77}$$

problem (1.1)$-$(1.7) *with condition* (1.8)$-$(1.10) *has the following parametric explicit solution:*

$$\begin{cases} u_1(x,t) = T_1 + (T_2-T_1)\mu_1^{-1}\left(\frac{\operatorname{erf}(\frac{\chi_1}{2\sqrt{\tau}}+\lambda_1^0)-\operatorname{erf}(\Lambda^0)}{G(\lambda_1^0)-\operatorname{erf}(\Lambda^0)}\right), \\ \qquad 0<\chi_1<\beta(\tau)=2\left(\Lambda^0-\lambda_1^0\right)\sqrt{\tau},\ \tau>0 \\ u_2(x,t) = T_1 + (T_2-T_1)\mu_2^{-1}\left(\theta\frac{\operatorname{erf}(\frac{\chi_2}{2\sqrt{\tau}}+\lambda_2^0)-\operatorname{erf}(\lambda_2^0)}{\operatorname{erfc}(\lambda_2^0)}\right), \\ \qquad \chi_2>0\,,\tau>0 \end{cases} \tag{2.78}$$

$$\begin{aligned} x &= \sqrt{\tfrac{k_0t_s}{c_0}}\left\{\left(1+\tfrac{\operatorname{erf}(\Lambda^0)}{G(\lambda_1^0)-\operatorname{erf}(\Lambda^0)}\right)\chi_1 \right. \\ &\left. -\tfrac{2\sqrt{\tau}}{G(\lambda_1^0)-\operatorname{erf}(\Lambda^0)}\left[(\tfrac{\chi_1}{2\sqrt{\tau}}+\lambda_1^0)G\left(\tfrac{\chi_1}{2\sqrt{\tau}}+\lambda_1^0\right)-\lambda_1^0G(\lambda_1^0)\right]\right\}, \\ &0<\chi_1<2\left(\Lambda^0-\lambda_1^0\right)\sqrt{\tau}\ ,\quad \tau>0 \\ x &= \sqrt{\tfrac{k_0t_s}{c_0}}\left\{\left(1+\theta_2\tfrac{\operatorname{erf}(\lambda_2^0)}{\operatorname{erfc}(\lambda_2^0)}\right)\chi_2 - \right. \\ &\left. -\tfrac{2\sqrt{\tau}\theta_2}{\operatorname{erfc}(\lambda_2^0)}\left[\left(\tfrac{\chi_2}{2\sqrt{\tau}}+\lambda_2^0\right)G\left(\tfrac{\chi_2}{2\sqrt{\tau}}+\lambda_2^0\right)-\lambda_2^0G(\lambda_2^0)\right]\right.\ ,\ \chi_2>0\ ,\ \tau>0 \\ t &= t_s\tau\ ,\qquad \tau>0 \end{aligned} \tag{2.79}$$

where the free boundary is given by

$$y(t) = 2\mathrm{Ste}\sqrt{\frac{k_0}{c_0}}\left(\Lambda^0-\lambda_1^0\right)\sqrt{t}\quad,\quad t>0 \tag{2.80}$$

Proof. Condition (2.77) is equivalent to (2.60) . Then, if (2.77) holds, from Theorem 2 , problem (2.33)-(2.42) has a unique solution given by (2.76). If we invert transformations (2.31), (2.32), (2.21), (2.11) and (2.1) we obtain the expressions for the solution (2.78)-(2.79). From (2.1), (2.19), (2.22) and (2.43) we have that the free boundary $y(t)$ is given by (2.80) which concludes the proof.

Remark 3. If we get $u_1(0,t)=T^*$ constant, we obtain that the convective boundary condition (1.7) becomes as a heat flux condition of the type:

$$K_1(u_1(0,t))\frac{\partial u_1}{\partial x}(0,t) = \frac{h[T^*-T_\infty]}{\sqrt{t}},\ \ t>0\ \ h>0\,.$$

Then we obtain the same solution as in [7] for this particular case.

3. Conclusions

A nonlinear two-phase unidimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_1, an initial temperature $T_2\,(> T_1)$ and a convective boundary condition imposed on the fixed face $x = 0$ is considered. The volumetric heat capacity and the thermal conductivity are nonlinear functions of the temperature and they verify a particular relation in each phase, these relations imply that the material is of Storm's type.
For $Ste < 1$, sufficient conditions on the parameters of the problem are established in order to obtain a unique explicit solution. If heat transfer coefficient h verifies a certain inequality then there exists an instantaneous phase-change and the corresponding parametric solution is given.

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Conflict of interest

The authors declare that they have no conflict of interest.

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