EXISTENCE OF POSITIVE SOLUTIONS OF SECOND-ORDER DELAYED DIFFERENTIAL SYSTEM ON INFINITE INTERVAL

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ABSTRACT. The present paper is focused on the analysis on the existence of positive solutions of a second order differential system with two delays

$$\begin{cases} x_1''(t) - a_1(t)x_1(t) + m_1(t)f_1(t, x(t), x_\tau(t)) = 0, \quad t > 0, \\ x_2''(t) - a_2(t)x_2(t) + m_2(t)f_2(t, x(t), x_\tau(t)) = 0, \quad t > 0, \\ x_1(t) = 0, \quad -\tau_1 \ \le t \le 0, \text{ and } \lim_{t\to\infty} x_1(t) = 0, \\ x_2(t) = 0, \quad -\tau_2 \ \le t \le 0, \text{ and } \lim_{t\to\infty} x_2(t) = 0 \end{cases}$$

by using two well-known fixed point theorems.

1. Introduction

In this paper, we study the existence of positive solutions for the second-order two-delay differential system on the positive half line:

$$\begin{cases} x_1''(t) - a_1(t)x_1(t) + m_1(t)f_1(t, x(t), x_\tau(t)) = 0, \quad t > 0, \\ x_2''(t) - a_2(t)x_2(t) + m_2(t)f_2(t, x(t), x_\tau(t)) = 0, \quad t > 0, \\ x_1(t) = 0, \quad -\tau_1 \ \le t \le 0, \text{ and } \lim_{t\to\infty} x_1(t) = 0, \\ x_2(t) = 0, \quad -\tau_2 \ \le t \le 0, \text{ and } \lim_{t\to\infty} x_2(t) = 0, \end{cases} \tag{1.1}$$

where $x(t) = (x_1(t), x_2(t))$, $x_\tau(t) = (x_1(t - \tau_1), x_2(t - \tau_2))$, $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $m_i(t) \in L^1(\mathbb{R}^+)$ and $f_i : {\mathbb{R}^+}^5 \to \mathbb{R}^+$ is continuous,(i=1,2).

Functional differential equations with delays have often been put forward as mathematical model to describe the real phenomenon, see [1, 2, 3, 4]. Motivated by its application background, many researchers has been attracted to study of the theory, methodology and application. For example, J.W. Lee and D. O'Regan establish the general existence principle of differential-difference equations with delay type based on the nonlinear alternative (see [5, 6]). Initialed the work [5, 6], in [7, 8], T. Candan applies Krasnosel'skii's fixed point theorem for the sum of a completely continuous and a contraction mapping to obtain some sufficient conditions for the existence of positive ω-periodic solutions for the first (second) order neutral differential equation, such as

$$[u(t) - p(t)u(t - \tau)]' = -Q(t)u(t) + f(t, u(t - \tau))$$

or

$$[u(t) - p(t)u(t - \tau)]'' = Q(t)u(t) - f(t, u(t - \tau));$$

in [9, 10], the authors apply Mönch fixed point theorem, Schauder fixed point theorem and Banach contraction principle to study the existence and uniqueness

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of solutions for the nonlinear functional differential equations on infinite interval

$$\begin{cases} [p(t)u'(t)]' + f(t, u_t, p(t)u'(t)) = 0, \quad t > 0, \\ u(t) = \phi(t), \quad t \in [\tau, 0], \\ \lim_{t\to\infty} p(t)u'(t) = y_\infty, \end{cases}$$

where the delay τ may be bounded or not; in [11, 12], D. Bai and Y. Xu employ fixed-point theory to show how the parameters effect the number of positive solution for a two-delay singular differential system

$$\begin{cases} \varphi(u_1'(t))' + \lambda_1 h_1(t) f_1(u_1(t-\tau_1), u_2(t-\tau_2)) = 0, \quad 0 < t < 1, \\ \varphi(u_2'(t))' + \lambda_2 h_2(t) f_2(u_1(t-\tau_1), u_2(t-\tau_2)) = 0, \quad 0 < t < 1, \\ u_1(t) = 0, \quad -\tau_1 \le t \le 0, \ u_1(1) = 0, \\ u_2(t) = 0, \quad -\tau_2 \le t \le 0, \ u_2(1) = 0. \end{cases}$$

Based on the model described by BVP (1.1), by constructing the suitable controlling function and applying some fixed point theorems in cones, the main goal of the paper is to find some suitable sufficiently conditions, which guarantee that BVP (1.1) has at least one, two or three positive solutions. Compared to the results in [9, 10, 11, 12], our work presented in this paper considers a more general term $f(t, x(t), x_\tau(t))$ and find some new conditions,which differ from those in the majority of papers as we know.

The paper is organized as follows. In Section 2, we will recall the Green's function of the corresponding linear problem and some basic forms of fixed point theorems. In Section 3, we list the important results. In Section 4, some exact examples are given to illustrate our main results.

2. Preliminaries

Throughout this paper, we assume that

(A_1) $a_i(t) : [0, \infty) \to (0, \infty)$ is continuous, periodic and bounded. Let

$$H = \max_{i=1,2} \sup_{t\in[0,\infty)} \sqrt{a_i(t)}, \ h = \min_{i=1,2} \inf_{t\in[0,\infty)} \sqrt{a_i(t)} > 0.$$

(A_2) there exists a $k \in [h^2, H^2]$ such that

$$\lim_{t\to\infty} e^{-\rho t} \int_0^t e^{\rho s}[a_i(s) - k]ds \text{ exists, for any } \rho \in \mathbb{R}.$$

(F) $f_i : {\mathbb{R}^+}^5 \to \mathbb{R}^+$ is continuous.

(M) $m_i(t) \in L^1(\mathbb{R}^+)$, and $\mathbb{D} = \mathbb{D}_1 \bigcup \mathbb{D}_2$ is a non-empty compact set satisfying $\mathbb{D} \subset [\tau, +\infty)$, where

$$\tau = \max\{\tau_1, \tau_2\},$$

$$\mathbb{D}_i = \operatorname{supp}\{t \in \mathbb{R}^+ : m_i(t) \ne 0\} \text{ is a non-empty compact set.}$$

Lemma 2.1.[13] Assume that (A_1) holds. Then the linear boundary value problem

$$\begin{cases} x''(t) - a_i(t)x(t) + h(t) = 0, \quad t > 0, \\ x(0) = 0, \text{ and } \lim_{t\to\infty} x(t) = 0 \end{cases} \tag{2.1}$$

has a unique solution $x(t) = \int_0^{+\infty} G_i(t,s)h(s)ds$, where

$$G_i(t,s) = \begin{cases} \phi_{i1}(t)\phi_{i2}(s), \ s \geq t, \\ \phi_{i1}(s)\phi_{i2}(t), \ t \geq s. \end{cases}$$

Lemma 2.2.[13] Assume that (A_1) holds. Then

(i) $0 \leq G_i(t,s) < \frac{1}{2h}$, for all $(t,s) \in [0,\infty) \times [0,\infty)$;

(ii) for any given $\theta \in (1,\infty)$, we have

$$G_i(t,s)\phi_{i2}^{\theta}(t) \leq \frac{H}{h}\phi_{i2}(s)G_i(s,s), \text{ for } (t,s) \in (0,\infty) \times (0,\infty);$$

(iii) for any $t, \ s \in (0,\infty)$, we have

$$G_i(t,s) \geq q_i(t)\phi_{i2}(s)G_i(s,s),$$

where

$$q_i(t) = \min_{t\in(0,+\infty)}\{2h\phi_{i1}(t), \phi_{i2}(t)\}.$$

Let

$$E_i = \{x \in C[-\tau_i,\infty) : x(t) = 0, \ \forall\, t \in [-\tau_i, 0] \text{ and} \lim_{t\to\infty} x(t) = 0\}$$

be a Banach space with norm $|\cdot|_i$, which defined by

$$\begin{aligned} |x|_i &= \sup_{-\tau_i \leq t < \infty} \left[\phi_{i2}^{\theta}(t) \cdot |x(t)|\right] \\ &= \sup_{\mathbb{D}} \left[\phi_{i2}^{\theta}(t) \cdot |x(t)|\right]. \end{aligned}$$

Then $E = E_1 \times E_2$ is a Banach space with norm $\|x\| = |x_1|_i + |x_2|_i$, for $x(t) = (x_1(t), x_2(t)) \in E$. Let $\mathbb{D}^* = \mathbb{D} \cup \mathbb{D}_{\tau_1} \cup \mathbb{D}_{\tau_2}$, where $\mathbb{D}_{\tau_i} = \{s : s = t - \tau_i, \ t \in \mathbb{D}\}$. Define a cone $K \subset E$ by

$$K = \{x \in E : x(t) \geq 0, \min_{t\in\mathbb{D}^*}[x_1(t) + x_2(t)] \geq \delta\|x\|\},$$

where

$$\delta = \frac{h}{H} \min_{i=1,2} \min_{t\in\mathbb{D}^*} q_i(t) \in (0,1).$$

Also, for $r > 0$, define K_r and ∂K_r by

$$K_r = \{x(t) \in K : \|x\| < r\}, \quad \partial K_r = \{x(t) \in K : \|x\| = r\}.$$

Define an operator $\mathfrak{T}$ by $\mathfrak{T}(x)(t) = (\mathfrak{T}_1(x)(t), \mathfrak{T}_2(x)(t))$, where

$$\mathfrak{T}_i(x)(t) = \begin{cases} \int_0^{\infty} G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds, & t > 0, \\ 0, & -\tau_i \leq t \leq 0. \end{cases}$$

Now solutions of (1.1) can be rewritten as fixed points of $\mathfrak{T}$ in Banach space E.

Lemma 2.3.[14] Let $\mathbf{BC}(\mathbb{R}^+) = \{u \text{ is a bounded and continuous function}\}$. Then the subset $\Omega \subset \mathbf{BC}(\mathbb{R}^+)$ is compact, if the function $u \in \Omega$ is equicontinuous in each compact interval of $\mathbb{R}^+$ and that for all $u \in \Omega$, we have

$$|u(t)| \leq \zeta(t), \ \ \forall\, t \in \mathbb{R}^+,$$

where $\zeta \in \mathbf{BC}(\mathbb{R}^+)$ verifies

$$\lim_{|t|\to+\infty} \zeta(t) = 0.$$

Lemma 2.4. Assume that (A_1), (A_2), (F) and (M) hold. Then $\mathfrak{T}(K) \subseteq K$ and $\mathfrak{T} : K \to K$ is completely continuous.

Proof. Initially, we show that $\mathfrak{T}(K) \subseteq K$. For any $(x_1, x_2) \in K$, by lemma 2.2, we have

$$\begin{aligned}
\min_{t\in\mathbb{D}^*} \mathfrak{T}_i(x)(t) &\geq \min_{t\in\mathbb{D}^*} \int_0^\infty q_i(t)G_i(s,s)\phi_{i2}(s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\
&= \frac{h}{H}\min_{t\in\mathbb{D}^*} q_i(t)\int_0^\infty \frac{H}{h}\phi_{i2}(s)G_i(s,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\
&\geq \delta \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_0^\infty G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\
&\geq \delta|\mathfrak{T}_i(x)(t)|_i,
\end{aligned}$$

which implies that

$$\min_{t\in\mathbb{D}^*}\big[\mathfrak{T}_1(x)(t)+\mathfrak{T}_2(x)(t)\big] \geq \delta\big(|\mathfrak{T}_1(x)(t)|_1 + |\mathfrak{T}_2(x)(t)|_2\big).$$

Moreover, we claim that $\mathfrak{T} : K \to K$ is continuous. For any $x_n \to x$ as $n \to \infty$, by Lebesgues dominated convergence theorem and letting $n \to \infty$, it is standard to verify that $\|\mathfrak{T}(x_n) - \mathfrak{T}(x)\| \to 0$.

Ultimately, we prove that the operator $\mathfrak{T}$ is compact on K. Let Ω be a subset of K, which is bounded. Then there exists a $C > 0$ such that $\|x\| \leq C$, for each $x \in \Omega$. Since the derivative of $G_i(t,s)$ is bounded in compact interval, the operator $\mathfrak{T}_i(\Omega)$ are equicontinuous on each compact interval. Furthermore, for any $x \in \Omega$, we have

$$|\mathfrak{T}_i(x)(t)| \leq \max_{t\in\mathbb{D}_i,\|x\|\leq C} f_i(t,x,x_\tau)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)ds = \zeta(t).$$

Then from [13] it follows that $\lim_{|t|\to+\infty}\zeta(t) = 0$, namely, $\mathfrak{T}_{\mathrm{i}}$ is completely continuous on K.$\diamond$

At the end of this section, we introduce two crucial lemmas as follows

Lemma 2.5.[15] Let K be a subcone of the Banach space E. Assume that $\mathbb{F} : \overline{K_r} \to K$ be a completely continuous operator satisfying $\mathbb{F}u \neq u$, for $u \in \partial K_r$.

(i) If $\|\mathbb{F}u\| \leq \|u\|$, $u \in K_r$, then $i(\mathbb{F}, K_r, K) = 1$.

(ii) If $\|\mathbb{F}u\| \geq \|u\|$, $u \in K_r$, then $i(\mathbb{F}, K_r, K) = 0$.

Lemma 2.5.[15] Assume that the map $\alpha : K \to \mathbb{R}^+$ is continuous and concave such that $\alpha(u) \leq \|u\|$ for $u \in \overline{K}$. For given constants $0 < d_1 < d_2$, define a convex set by

$$K(\alpha, d_1, d_2) = \{u \in K : d_1 \leq \alpha(u), \|u\| \leq d_2\}.$$

In addition, suppose that $\mathbb{F} : \overline{K_{c_4}} \to \overline{K_{c_4}}$ is completely continuous, and there exist $0 < c_1 < c_2 < c_3 \leq c_4$ such that

(i) $\{u \in K(\alpha, c_2, c_3) : \alpha(u) > c_2\} \neq \emptyset$ and $\alpha(\mathbb{F}u) > c_2$ for $u \in K(\alpha, c_2, c_3)$;

(ii) $\|\mathbb{F}u\| < c_1$ for $\|u\| \leq c_1$;

(iii) $\alpha(\mathbb{F}u) > c_2$ for $x \in K(\alpha, c_2, c_4)$ with $\|\mathbb{F}u\| > c_3$.

Then $\mathbb{F}$ has three fixed points u_1, u_2, u_3 satisfying

$$\|u_1\| < c_1,\ c_2 < \alpha(u_2),$$
$$\|u_3\| > c_1 \text{ and } \alpha(u_3) < c_2.$$

3. Main results

For convenience, let $\rho = \min\{\rho_1, \rho_2\}$, where

$$\rho_1 = \min\{\min_{\mathbb{D}} \phi_{12}^{\theta}(t), \min_{\mathbb{D}} \phi_{22}^{\theta}(t)\} > 0,$$

$$\rho_2 = \min\{\min_{\mathbb{D}} \phi_{12}^{\theta}(t-\tau_1),\ \min_{\mathbb{D}} \phi_{22}^{\theta}(t-\tau_2)\} > 0.$$

For $r > 0,\ r_j > 0,\ i = 1,2,3,4$, we introduce the height function

$$\varphi_i(t,r) = \max\{|f_i(t,x,x_\tau)| : 0 \le x_1 + x_2 \le \frac{r}{\rho_1},\ \ 0 \le x_{1\tau_1} + x_{2\tau_2} \le \frac{r}{\rho_2}\},$$

$$\psi_i(t,r_j) = \min\{|f_i(t,x,x_\tau)| : r_1 \le x_1 + x_2 \le r_2,\ \ r_3 \le x_{1\tau_1} + x_{2\tau_2} \le r_4\},\ (j = 1,2,3,4).$$

Theorem 3.1. Assume that (A_1), (A_2), (F) and (M) hold. If there exist $b > a > 0$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s) m_i(s) \psi_i(s, \delta b, \frac{b}{\rho_1}, \delta b, \frac{b}{\rho_2}) ds \ge \frac{b}{2}$$

and

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s) m_i(s) \varphi_i(s,a) ds \le \frac{a}{2},$$

then there exist one positive solutions to problem (1.1).

Proof. Choosing $r = a$, let $K_r = \{x = (x_1, x_2) \in K :\ \|x\| < r\}$. Then for $x = (x_1,x_2) \in \partial K_r$ and $t \in \mathbb{D}$, we have

$$\begin{aligned} r = \|x\| &= \sup_{\mathbb{D}} \Big[\phi_{12}^{\theta}(t) \cdot |x_1(t)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t) \cdot |x_2(t)|\Big] \\ &\ge \min_{\mathbb{D}} \phi_{12}^{\theta}(t) \cdot |x_1(t)| + \min_{\mathbb{D}} \phi_{22}^{\theta}(t) \cdot |x_2(t)| \\ &\ge \rho_1(x_1 + x_2) \end{aligned}$$

and

$$\begin{aligned} r = \|x\| &= \sup_{\mathbb{D}} \Big[\phi_{12}^{\theta}(t) \cdot |x_1(t)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t) \cdot |x_2(t)|\Big] \\ &\ge \sup_{\mathbb{D}} \Big[\phi_{12}^{\theta}(t-\tau_1) \cdot |x_1(t-\tau_1)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t-\tau_2) \cdot |x_2(t-\tau_2)|\Big] \\ &\ge \min_{\mathbb{D}} \phi_{12}^{\theta}(t-\tau_1) \cdot |x_1(t-\tau_1)| + \min_{\mathbb{D}} \phi_{22}^{\theta}(t-\tau_2) \cdot |x_2(t-\tau_2)| \\ &= \min_{\{t: t\in\mathbb{D},\ t\ge\tau_1\}} \phi_{12}^{\theta}(t-\tau_1) \cdot |x_1(t-\tau_1)| + \min_{\{t: t\in\mathbb{D},\ t\ge\tau_2\}} \phi_{22}^{\theta}(t-\tau_2) \cdot |x_2(t-\tau_2)| \\ &\ge \rho_2\Big[x_1(t-\tau_1) + x_2(t-\tau_2)\Big]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |\mathfrak{T}_i x|_i &= \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_0^{\infty} G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\ &= \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\ &\le \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,a)ds \\ &\le \frac{a}{2}, \end{aligned}$$

which implies that

$$\|\mathfrak{T}x\| = |\mathfrak{T}_1 x|_1 + |\mathfrak{T}_2 x|_2 \le a = \|x\|, \text{ for } x\in\partial K_r,$$

namely, $i(\mathfrak{T},K_r,K)=1$.

Choosing $R=b$, let $K_R=\{x=(x_1,x_2)\in K:\ \|x\|<R\}$. Then for $x=(x_1,x_2)\in\partial K_R$ and $t\in\mathbb{D}$, on one hand, we have

$$x_1(t)+x_2(t)\ge\delta\|x\|=\delta R \text{ and } x_1(t-\tau_1)+x_2(t-\tau_2)\ge\delta\|x\|=\delta R.$$

On the other hand, it is clear that

$$x_1(t)+x_2(t)\le\frac{\|x\|}{\rho_1}=\frac{R}{\rho_1} \text{ and } x_1(t-\tau_1)+x_2(t-\tau_2)\le\frac{\|x\|}{\rho_2}=\frac{R}{\rho_2}.$$

So, we have

$$\begin{aligned} |\mathfrak{T}_i x|_i &= \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\ &\ge \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\psi_i(s,\delta b,\frac{b}{\rho_1},\delta b,\frac{b}{\rho_2})ds \\ &\ge \frac{b}{2}, \end{aligned}$$

which implies that

$$\|\mathfrak{T}x\| = |\mathfrak{T}_1 x|_1 + |\mathfrak{T}_2 x|_2 \ge b = \|x\|, \text{ for } x\in\partial K_R,$$

namely, $i(\mathfrak{T},K_R,K)=0$

From the above discussions, we have $i(\mathfrak{T},\overline{K_R}\setminus K_r,K)=-1$, which means that $\mathfrak{T}$ has a fixed point $x=(x_1,x_2)\in\overline{K}_R\backslash K_r$.$\diamond$

Theorem 3.2. Assume that (A$_1$), (A$_2$), (F) and (M) hold. If there exist $b>a>0$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\psi_i(s,\delta a,\frac{a}{\rho_1},\delta a,\frac{a}{\rho_2})ds\ge\frac{a}{2}$$

and

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,b)ds\le\frac{b}{2},$$

then there exist one positive solutions to problem (1.1).

Corollary 3.3. Assume that (A$_1$), (A$_2$), (F) and (M) hold. If there exist $0<a<b<c$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,a)ds\le\frac{a}{2},$$

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\psi_i(s,\delta b,\frac{b}{\rho_1},\delta b,\frac{b}{\rho_2})ds > \frac{b}{2}$$

and

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,c)ds \leq \frac{c}{2},$$

then there exist two positive solutions to problem (1.1).

Corollary 3.4. Assume that (A_1), (A_2), (F) and (M) hold. If there exist $0 < a < b < c$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\psi_i(s,\delta a,\frac{a}{\rho_1},\delta a,\frac{a}{\rho_2})ds \geq \frac{a}{2},$$

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,b)ds < \frac{b}{2}$$

and

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\psi_i(s,\delta c,\frac{c}{\rho_1},\delta c,\frac{c}{\rho_2})ds \geq \frac{c}{2},$$

then there exist two positive solutions to problem (1.1).

Theorem 3.5. Assume that (A_1), (A_2), (F) and (M) hold. If there exist positive constants c_1, c_2 and c_4 with $0 < c_1 < c_2 < \frac{\rho\delta}{8}c_4$, such that

(H_1) $\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,c_1)ds \leq \frac{c_1}{2}$;

(H_2) there exists $i_0 \in \{1,2\}$, such that

$$\sup_{t\in\mathbb{D}} \phi_{i_0 2}^{\theta}(t) \int_{\mathbb{D}_{i_0}} G_{i_0}(t,s)m_{i_0}(s)\psi_{i_0}(s,\frac{c_2}{\rho},\frac{8c_2}{\rho^2\delta},0,\frac{8c_2}{\rho^2\delta})ds \geq \frac{c_2}{\rho\delta};$$

(H_3) $\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,c_4)ds \leq \frac{c_4}{2}$,

then there exist three positive solutions to problem (1.1).

Proof. For $x = (x_1, x_2) \in K$, define

$$\alpha(x) = \min_{t\in\mathbb{D}} \rho(x_1(t) + x_2(t)),$$

then it is clear that α is a nonnegative continuous concave functional on K with $\alpha(x) \leq \|x\|$. In fact, for any $x \in K$,

$$\begin{aligned} \|x\| &= \sup_{\mathbb{D}} \left[\phi_{12}^{\theta}(t)\cdot|x_1(t)|\right] + \sup_{\mathbb{D}} \left[\phi_{22}^{\theta}(t)\cdot|x_2(t)|\right] \\ &\geq \min_{\mathbb{D}} \phi_{12}^{\theta}(t)\cdot|x_1(t)| + \min_{\mathbb{D}} \phi_{22}^{\theta}(t)\cdot|x_2(t)| \\ &\geq \rho_1(x_1 + x_2) \\ &\geq \rho(x_1 + x_2) \geq \alpha(x). \end{aligned}$$

First, we show that $\mathfrak{T} : \overline{K_{c_4}} \to \overline{K_{c_4}}$. For any $x \in \overline{K_{c_4}}$, we have

$$0 \leq x_1(t) + x_2(t) \leq \frac{\|x\|}{\rho_1} \leq \frac{c_4}{\rho_1}$$

and

$$0 \leq x_1(t-\tau_1) + x_2(t-\tau_2) \leq \frac{\|x\|}{\rho_2} \leq \frac{c_4}{\rho_2}.$$

Then from (H_3) it follows that

$$\begin{aligned}
|\mathfrak{T}_i x|_i &= \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)f_i(s,x(s),x_\tau(s))ds \\
&\le \sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i} G_i(t,s)m_i(s)\varphi_i(s,c_4)ds \\
&\le \frac{c_4}{2},
\end{aligned}$$

which implies that

$$\|\mathfrak{T}x\| = |\mathfrak{T}_1 x|_1 + |\mathfrak{T}_2 x|_2 \le c_4.$$

In the similar way, we can verify that $\mathfrak{T} : \overline{K_{c_1}} \to \overline{K_{c_1}}$. So (ii) of Lemma 2.6 holds.

Next, let $c_3 = \frac{8c_2}{\rho\delta}$ and $\widetilde{x} = \left(\frac{c_2}{2\rho} + \frac{3\delta}{16}c_3, \frac{c_2}{2\rho} + \frac{3\delta}{16}c_3\right)$. Since

$$\alpha(\tilde{x}) = \min_{\mathbb{D}} \rho(x_1(t)+x_2(t)) = 2\rho(\frac{c_2}{2\rho} + \frac{3\delta}{16}c_3) > c_2$$

and

$$\begin{aligned}
\|\tilde{x}\| &= \sup_{\mathbb{D}} \Big[\phi_{12}^{\theta}(t)\cdot|x_1(t)| + \phi_{22}^{\theta}(t)\cdot|x_2(t)|\Big] \\
&\le x_1 + x_2 \\
&= 2(\frac{c_2}{2\rho} + \frac{3\delta}{16}c_3) \\
&= \frac{\delta}{2}c_3 < c_3,
\end{aligned}$$

it implies that $\widetilde{x} \in K(\alpha, c_2, c_3)$, namely, the set $\{x \in K(\alpha, c_2, c_3) : \alpha(x) > c_2\}$ is not empty. Then for any $x \in K(\alpha, c_2, c_3)$ and $t \in \mathbb{D}$, we have

$$\begin{aligned}
c_3 \ge \|x\| &= \sup_{\mathbb{D}} \Big[\phi_{12}^{\theta}(t)\cdot|x_1(t)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t)\cdot|x_2(t)|\Big] \\
&\ge \min_{\mathbb{D}} \phi_{12}^{\theta}(t)\cdot|x_1(t)| + \min_{\mathbb{D}} \phi_{22}^{\theta}(t)\cdot|x_2(t)| \\
&\ge \rho_1(x_1+x_2) \\
&\ge \rho(x_1+x_2) \\
&\ge \alpha(x) > c_2
\end{aligned}$$

and

$$\begin{aligned}
c_3 \ge \|x\| &= \sup_{t\in\mathbb{D}} \Big[\phi_{12}^{\theta}(t)\cdot|x_1(t)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t)\cdot|x_2(t)|\Big] \\
&\ge \sup_{t\in\mathbb{D}} \Big[\phi_{12}^{\theta}(t-\tau_1)\cdot|x_1(t-\tau_1)|\Big] + \sup_{\mathbb{D}} \Big[\phi_{22}^{\theta}(t-\tau_2)\cdot|x_2(t-\tau_2)|\Big] \\
&\ge \min_{\mathbb{D}} \phi_{12}^{\theta}(t-\tau_1)\cdot|x_1(t-\tau_1)| + \min_{\mathbb{D}} \phi_{22}^{\theta}(t-\tau_2)\cdot|x_2(t-\tau_2)| \\
&\ge \rho_2\Big[x_1(t-\tau_1) + x_2(t-\tau_2)\Big] \\
&\ge \rho(x_1(t-\tau_1) + x_2(t-\tau_2)) \ge 0.
\end{aligned}$$

Then by (H_2), we have

$$\begin{aligned}
\alpha(\mathfrak{T}x(t)) &= \min_{t\in\mathbb{D}} \rho(\mathfrak{T}_1x(t)+\mathfrak{T}_2x(t)) \\
&\geq \rho\mathfrak{T}_{i_0}x(t) \\
&\geq \rho\delta\|\mathfrak{T}_{i_0}x(t)\| \\
&= \rho\delta\sup_{t\in\mathbb{D}}\phi^{\theta}_{i_0 2}(t)|\mathfrak{T}_{i_0}x(t)| \\
&\geq \rho\delta\sup_{t\in\mathbb{D}}\phi^{\theta}_{i_0 2}(t)\int_{\mathbb{D}_{i_0}} G_{i_0}(t,s)m_{i_0}(s)f_{i_0}(s,x(s),x_\tau(s))ds \\
&\geq \rho\delta\sup_{t\in\mathbb{D}}\phi^{\theta}_{i_0 2}(t)\int_{\mathbb{D}_{i_0}} G_{i_0}(t,s)m_{i_0}(s)\psi_{i_0}(s,\frac{c_2}{\rho},\frac{c_3}{\rho},0,\frac{c_3}{\rho})ds \\
&\geq c_2,
\end{aligned}$$

which implies that (i) of Lemma 2.6 is satisfied.

Ultimately, for any $x\in K(\alpha,c_2,c_4)$ with $\|\mathfrak{T}x\|>c_3$, then we have

$$\begin{aligned}
\alpha(\mathfrak{T}x(t)) &= \min_{t\in\mathbb{D}} \rho(\mathfrak{T}_1x(t)+\mathfrak{T}_2x(t)) \\
&\geq \rho\delta(|\mathfrak{T}_1x|_1+|\mathfrak{T}_2x|_2) \\
&= \rho\delta\|\mathfrak{T}x\| > \rho\delta c_3 \\
&> \frac{\rho\delta c_3}{8} = c_2,
\end{aligned}$$

which implies that (iii) of Lemma 2.6 is satisfied.

Therefore, from Leggett-Williams theorem it follows that problem (1.1) has at least three positive solutions $x^1=(x_1^1,x_2^1)$, $x^2=(x_1^2,x_2^2)$ and $x^3=(x_1^3,x_2^3)$ such that

$$\|x^1\|<c_1,\ c_2<\alpha(x^2),\ \|x^3\|>c_1 \text{ and } \alpha(x^3)<c_2. \diamond$$

4. Examples

Example 4.1. Now we consider the following problem:

$$\left\{\begin{array}{l}
x_1''(t)-(\sin t+3)^2x_1(t)+m_1(t)f_1(t,x(t),x_\tau(t))=0,\quad t>0,\\
x_2''(t)-(\sin t+3)^2x_2(t)+m_2(t)f_2(t,x(t),x_\tau(t))=0,\quad t>0,\\
x_1(t)=0,\quad -1\leq t\leq 0,\ \text{and } \lim_{t\to\infty}x_1(t)=0,\\
x_2(t)=0,\quad -1\leq t\leq 0,\ \text{and } \lim_{t\to\infty}x_2(t)=0,
\end{array}\right. \tag{4.1}$$

where $\tau_1=\tau_2=1$, $\alpha_i,\beta_i>0$,

$$f_i(t,x(t),x_\tau(t))=(\sin^2 t+2)[c_{i1}(x_1+x_2)^{\alpha_i}+c_{i2}(x_1(t-\tau_1)+x_2(t-\tau_2))^{\beta_i}]$$

and

$$m_1(t)=\begin{cases}1,\ t\in\mathbb{D}_1=[2,4],\\ 0,\ t\notin\mathbb{D}_1=[2,4],\end{cases}\qquad m_2(t)=\begin{cases}1,\ t\in\mathbb{D}_2=[5,11],\\ 0,\ t\notin\mathbb{D}_2=[5,11].\end{cases}$$

It is clear that (A_1), (A_2), (F) and (M) hold.

(**I**): Since

$$\begin{aligned}
\varphi_i(t,a) &= \max\{f_i: 0\leq x_1+x_2\leq\frac{a}{\rho_1},\ 0\leq x_{1\tau_1}+x_{2\tau_2}\leq\frac{a}{\rho_2}\} \\
&= (\sin^2 t+2)[c_{i1}(\frac{a}{\rho_1})^{\alpha_i}+c_{i2}(\frac{a}{\rho_2})^{\beta_i}]
\end{aligned}$$

and

$$\begin{aligned}\psi_i(t,\delta b,\frac{b}{\rho_1},\delta b,\frac{b}{\rho_2}) &= \min\{f_i:\delta b\le x_1+x_2\le\frac{b}{\rho_1},\ \ \delta b\le x_{1\tau_1}+x_{2\tau_2}\le\frac{b}{\rho_2}\}\\ &= (\sin^2 t+2)[c_{i1}(\delta b)^{\alpha_i}+c_{i2}(\delta b)^{\beta_i}],\end{aligned}$$

if $\alpha_i>1,\beta_i>1$, then there two positive constants a (small enough) and b (big enough) with $b>a$ such that

$$\sup_{t\in\mathbb{D}}\phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i}G(t,s)(\sin^2 s+2)ds[(\frac{a}{\rho_1})^{\alpha_i}+(\frac{a}{\rho_2})^{\beta_i}]\le\frac{a}{2}$$

and

$$\sup_{t\in\mathbb{D}}\phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i}G(t,s)(\sin^2 s+2)ds\cdot[(\delta b)^{\alpha_i}+(\delta b)^{\beta_i}]\ge\frac{b}{2}.$$

Therefore from Theorem 3.1, it concludes that problem (4.1) has one positive solution.

(**II**): Since

$$\begin{aligned}\varphi_i(t,b) &= \max\{f_i:0\le x_1+x_2\le\frac{b}{\rho_1},\ \ 0\le x_{1\tau_1}+x_{2\tau_2}\le\frac{b}{\rho_2}\}\\ &= (\sin^2 t+2)\cdot[c_{i1}(\frac{b}{\rho_1})^{\alpha_i}+c_{i2}(\frac{b}{\rho_2})^{\beta_i}]\end{aligned}$$

and

$$\begin{aligned}\psi_i(t,\delta a,\frac{a}{\rho_1},\delta a,\frac{a}{\rho_2}) &= \min\{f_i:\delta a\le x_1+x_2\le\frac{a}{\rho_1},\ \ \delta a\le x_{1\tau_1}+x_{2\tau_2}\le\frac{a}{\rho_2}\}\\ &= (\sin^2 t+2)[c_{i1}(\delta a)^{\alpha_i}+c_{i2}(\delta a)^{\beta_i}],\end{aligned}$$

if $\alpha_i<1,\beta_i<1$, then there exists two positive constants a (small enough) and b (big enough) with $b>a$ such that

$$\sup_{t\in\mathbb{D}}\phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i}G(t,s)(\sin^2 s+2)ds\cdot[c_{i1}(\frac{b}{\rho_1})^{\alpha_i}+c_{i2}(\frac{b}{\rho_2})^{\beta_i}]\le\frac{b}{2}$$

and

$$\sup_{t\in\mathbb{D}}\phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i}G(t,s)(\sin^2 s+2)ds\cdot[c_{i1}(\delta a)^{\alpha_i}+c_{i2}(\delta b)^{\beta_i}]\ge\frac{a}{2}.$$

Therefore from Theorem 3.2, it concludes that (4.1) has one positive solution.$\diamond$

Example 4.2. Now we consider the following problem:

$$\begin{cases} x_1''(t)-a_1(t)x_1(t)+m_1(t)f_1(t,x(t),x_\tau(t))=0, & t>0,\\ x_2''(t)-a_2(t)x_2(t)+m_2(t)f_2(t,x(t),x_\tau(t))=0, & t>0,\\ x_1(t)=0,\ \ -\tau_1\ \le t\le 0,\ \text{and}\ \lim_{t\to\infty}x_1(t)=0,\\ x_2(t)=0,\ \ -\tau_2\ \le t\le 0,\ \text{and}\ \lim_{t\to\infty}x_2(t)=0,\end{cases}$$

where

$$f_i(t,x(t),x_\tau(t))=\begin{cases}A_i(x_1^2+x_2^2+x_{1\tau_1}^2+x_{2\tau_2}^2)^2, & (x,x_\tau)\in\overline{B_R(0)},\\ B_i(x_1^2+x_2^2+x_{1\tau_1}^2+x_{2\tau_2}^2)^{\frac14}+D_i, & (x,x_\tau)\in\mathbb{R}^4\backslash\overline{B_R(0)},\end{cases}$$

$$\overline{B_R(0)}=\{(x,x_\tau)\in\mathbb{R}^4:x_1^2+x_2^2+x_{1\tau_1}^2+x_{2\tau_2}^2\le R^2\}.$$

Let

$$\theta_i=\sup_{t\in\mathbb{D}}\phi_{i2}^{\theta}(t)\int_{\mathbb{D}_i}G(t,s)m_i(s)ds>0,$$

and the positive constant A_i, B_i, D_i, R satisfy the following relation

$$\begin{cases} B_i R^{\frac{1}{2}} + D_i = A_i R^4, \\ (2)^{\frac{1}{4}} B_i R^{\frac{1}{2}} + D_i = \frac{4R}{\delta\theta_i}. \end{cases}$$

First, let $c_1 < \frac{\rho R}{2}$, then we have

$$x_1^2 + x_2^2 + x_{1\tau_1}^2 + x_{2\tau_2}^2 \le (x_1 + x_2)^2 + (x_{1\tau_1} + x_{2\tau_2})^2 \le \frac{c_1^2}{\rho_1^2} + \frac{c_1^2}{\rho_2^2} \le \frac{2}{\rho^2} c_1^2 < \frac{R^2}{2}.$$

Furthermore, we have

$$|f_i| \le A_i[(x_1 + x_2)^2 + (x_{1\tau_1} + x_{2\tau_2})^2]^2 \le A_i(\frac{c_1^2}{\rho_1^2} + \frac{c_1^2}{\rho_2^2})^2.$$

Thus there exists a sufficiently small $c_1 < \frac{\rho R}{2}$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G(t,s) m_i(s) \varphi_i(s, c_1) ds = \theta_i A_i (\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2})^2 c_1^4 < \frac{c_1}{2},$$

which implies that (H_1) holds.

Next, let $c_4 > \frac{16R}{\delta}$, then we have

$$\begin{aligned} |f_i| &\le A_i R^4 + B_i[(x_1 + x_2)^2 + (x_{1\tau_1} + x_{2\tau_2})^2]^{\frac{1}{4}} + D_i \\ &\le A_i R^4 + B_i(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2})^{\frac{1}{4}} c_4^{\frac{1}{2}} + D_i. \end{aligned}$$

Thus there exists a sufficiently large $c_4 > \frac{16R}{\delta}$ such that

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G(t,s) m_i(s) \varphi_i(s, c_4) ds = \theta_i[A_i R^4 + B_i(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2})^{\frac{1}{4}} c_4^{\frac{1}{2}} + D_i] < \frac{c_4}{2},$$

which implies that (H_3) holds.

Finally, choosing $c_2 = 2\rho R < \frac{\rho\delta}{8} c_4$. Since

$$\frac{c_2}{\rho} \le x_1 + x_2 \le \frac{8c_2}{\rho^2\delta},\ 0 \le x_{1\tau_1} + x_{2\tau_2} \le \frac{8c_2}{\rho^2\delta},$$

we have

$$x_1^2 + x_2^2 + x_{1\tau_1}^2 + x_{2\tau_2}^2 \ge \frac{(x_1 + x_2)^2}{2} + \frac{(x_{1\tau_1} + x_{2\tau_2})^2}{2} \ge \frac{c_2^2}{2\rho^2} = 2R^2 > R^2.$$

Furthermore, we have

$$\sup_{t\in\mathbb{D}} \phi_{i2}^{\theta}(t) \int_{\mathbb{D}_i} G(t,s) m_i(s) \psi_i(s, \frac{c_2}{\rho}, \frac{8c_2}{\rho^2\delta}, 0, \frac{8c_2}{\rho^2\delta}) ds = \theta_i B_i (2R^2)^{\frac{1}{4}} + \theta_i D_i = \frac{4R}{\delta} > \frac{2R}{\delta} = \frac{c_2}{\rho\delta},$$

which implies that (H_2) holds.

Therefore from Theorem 3.5, it concludes that (4.2) has three positive solutions.$\diamond$

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