On a mixed Kernel Hilbert-type integral inequality and its operator expressions with norm

Qiong Liu

College of science, Shao yang university,Shao yang China 422000

Abstract: By using some real analysis techniques, We study the structural characteristics of a multi-parameter Hilbert-type integral inequality with the hybrid kernel, and obtain some equivalent conditions for this inequality. We also consider the operator expression of the equivalent inequalities. The conclusions not only integrates some results of references, but also finds some new Hilbert-type integral inequalities with simple form by choosing suitable parameter values.

Keywords: Hilbert-type integral inequality; hybrid kernel; structural characteristics; operator expression.

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1 Introduction

If $f(x), g(y) \geq 0$, satisfies $0 < \int_0^\infty f^2(x)dx < \infty, 0 < \int_0^\infty g^2(y)dy < \infty$, then the famous Hilbert's integral inequality (cf. [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \Big\{ \int_0^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty g^2(y)dy \Big\}^{\frac{1}{2}} \tag{1.1}$$

holds, and the constant π is the best possible. Besides, we have the following some basic Hilbert integral inequalities(cf. [1,2]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \Big\{ \int_0^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty g^2(y)dy \Big\}^{\frac{1}{2}}, \tag{1.2}$$

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}|}{x+y} f(x)g(y) dxdy < c_0 \Big\{ \int_0^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty g^2(y)dy \Big\}^{\frac{1}{2}}, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}|}{\max\{x,y\}} f(x)g(y) dxdy < 8 \Big\{ \int_0^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty g^2(y)dy \Big\}^{\frac{1}{2}}, \tag{1.4}$$

where these constants $4, c_0 = 8catalan = 7.327724754^+$, and 8 are the best possible of (1.2), (1.3) and (1.4), the constant '*catalan*' is called catalan number.

In 2004, (1.1) and (1.2) are extended introducing the independent parameter $\lambda(>0)$ by Yang, as follows (cf. [3,4])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{2},\frac{\lambda}{2}) \Big\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty y^{1-\lambda} g^2(y)dy \Big\}^{\frac{1}{2}}, \tag{1.5}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < \frac{4}{\lambda} \Big\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty y^{1-\lambda} g^2(y)dy \Big\}^{\frac{1}{2}}, \tag{1.6}$$

where these constants $B(\frac{\lambda}{2},\frac{\lambda}{2})$ are the best possible of (1.5), (1.6), $B(u,v) = \int_0^1 (1-t)^{u-1} t^{v-1} dt \, (u,v>0)$ is called Beta function.

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In 2014, Liu et al. obtained a Hilbert-type integral inequality with independent parameter $\lambda > 0$, as follows(cf. [5])

$$\int_0^\infty \int_0^\infty \min\{x^\lambda, y^\lambda\} f(x)g(y)dxdy < \frac{4}{\lambda}\Big\{\int_0^\infty x^{1+\lambda} f^2(x)dx\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty y^{1+\lambda} g^2(y)dy\Big\}^{\frac{1}{2}}, \tag{1.7}$$

where these constant $\frac{4}{\lambda}$ is the best possible.

The research of Hilbert-type integral inequalities with hybrid kernels is one of the important contents. The so-called hybrid kernel research is to combine some basic kernels into new integral kernels and do the corresponding research works, which began in 2008 and has yielded a lot of results(see [6-9]). In this paper, the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are introduced, and the basic kernels $k_1(x,y) = \frac{1}{x+y}, k_2(x,y) = \frac{|\ln \frac{y}{x}|}{x+y}, k_3(x,y) = \max\{x,y\}, k_4(x,y) = \min\{x,y\}$ are parametric combined to a mixed kernel as $k(x,y) := \frac{|\ln \frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}$. By using the way of weight function and some techniques of real analysis, the structural properties of a Hilbert-type integral inequality with the above mixed kernel $k(x,y)$ are considered. The obtained results not only formulas (1.1) to (1.7) are integrated, but also some new and beautiful Hilbert-type integral inequalities are obtained by selecting the parameter values that meet the conditions. At the same time, the operator expression of the obtained inequalities with norm are discussed.

2 Some Lemmas

In this paper, to avoid subsequent repetition, we always assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $\sigma, \sigma_1, \lambda_i (i = 1,2,3,4) \in \mathbb{R}$, which meet the conditions $\lambda_1 > -1, -\lambda_2 < \sigma(\sigma_1) < \lambda_3 + \lambda_4$. $f(x), g(y)$ are two non-negative measurable functions on $\mathbb{R}_+$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty, \quad 0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy < \infty.$$

Let $m > -1, n > -1$, then we have integral formula (cf. [10])

$$\int_0^1 t^m \big(\ln \frac{1}{t}\big)^n dt = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \tag{2.1}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ is called Gamma function

Let $\alpha \in \mathbb{R}, |x| < 1$, we have the following series expansion (cf. [11])

$$(1+x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n, \tag{2.2}$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$.

Lemma 2.1 Calculating the following integral, we have

$$\begin{aligned} K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &:= \int_0^\infty \frac{|\ln t|^{\lambda_1} (\min\{1,t\})^{\lambda_2} t^{\sigma-1}}{(1+t)^{\lambda_3} \big(\max\{1,t\}\big)^{\lambda_4}} dt \\ &= \Gamma(\lambda_1+1) \sum_{n=0}^\infty \frac{\binom{-\lambda_3}{n}}{(n+\lambda_2+\sigma)^{\lambda_1+1} + (n+\lambda_3+\lambda_4-\sigma)^{\lambda_1+1}}. \end{aligned} \tag{2.3}$$

Proof. By (2.1) and (2.2), we find

$$\begin{aligned} K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4) :&= \int_0^\infty \frac{|\ln t|^{\lambda_1} (\min\{1,t\})^{\lambda_2} t^{\sigma-1}}{(1+t)^{\lambda_3} \big(\max\{1,t\}\big)^{\lambda_4}} dt \\ &= \int_0^1 \frac{(\ln \frac{1}{t})^{\lambda_1} t^{\lambda_2+\sigma-1}}{(1+t)^{\lambda_3}} dt + \int_1^\infty \frac{(\ln t)^{\lambda_1} t^{\sigma-\lambda_4-1}}{(1+t)^{\lambda_3}} dt \end{aligned}$$

$$= \int_0^1 \frac{(\ln\frac{1}{t})^{\lambda_1} t^{\lambda_2+\sigma-1}}{(1+t)^{\lambda_3}} dt + \int_0^1 \frac{(\ln\frac{1}{u})^{\lambda_1} u^{\lambda_3+\lambda_4-\sigma-1}}{(1+u)^{\lambda_3}} du$$

$$= \Gamma(\lambda_1+1) \sum_{n=0}^{\infty} \frac{\binom{-\lambda_3}{n}}{(n+\lambda_2+\sigma)^{\lambda_1+1} + (n+\lambda_3+\lambda_4-\sigma)^{\lambda_1+1}}.$$

Commentary 1: If $\sigma+\sigma_1 = \lambda_3+\lambda_4-\lambda_2$, replacing σ in (2.3) with $\sigma_1 = \lambda_3+\lambda_4-\lambda_2-\sigma$, we obtain

$$K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = K(\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4) \tag{2.4}$$

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Commentary 2: letting $\sigma = \sigma_1, \Rightarrow \sigma = \frac{\lambda_3+\lambda_4-\lambda_2}{2}$, we have

$$K(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = 2\Gamma(\lambda_1+1) \sum_{n=0}^{\infty} \frac{\binom{-\lambda_3}{n}}{(n+\frac{\lambda_2+\lambda_3+\lambda_4}{2})^{\lambda_1+1}}. \tag{2.5}$$

Lemma 2.2 Let $\sigma+\sigma_1 = \lambda_3+\lambda_4-\lambda_2$, we defined two weight functions by

$$\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) := \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \frac{y^{\sigma_1-1}}{x^{\frac{p}{q}(\sigma-1)}} dy, x\in(0,\infty),$$

$$\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,y) := \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \frac{x^{\sigma-1}}{y^{\frac{q}{p}(\sigma_1-1)}} dx, y\in(0,\infty)$$

then holds the relations

$$\begin{aligned}\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) &= K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)x^{p(1-\sigma)-1},\\ \omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,y) &= K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)y^{q(1-\sigma_1)-1}.\end{aligned} \tag{2.6}$$

Proof Setting $\frac{y}{x} = t$, by Lemma 2.1 and (2.4), we have

$$\begin{aligned}\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) &= \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \frac{y^{\sigma_1-1}}{x^{\frac{p}{q}(\sigma-1)}} dy\\ &= x^{p(1-\sigma)-1} \int_0^\infty \frac{|\ln t|^{\lambda_1}(\min\{1,t\})^{\lambda_2} t^{\sigma_1-1}}{(1+t)^{\lambda_3}\big(\max\{1,t\}\big)^{\lambda_4}} dt\\ &= x^{p(1-\sigma)-1} K(\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)x^{p(1-\sigma)-1}.\end{aligned}$$

Similarly, we easily get $\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,y) = K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)y^{q(1-\sigma_1)-1}$.

Lemma 2.3 If there exists a constant $M>0$, such that the inequality

$$\begin{aligned}I :&= \int_0^\infty \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)g(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} dxdy\\ &\le M\Big\{\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\Big\}^{\frac{1}{p}} \Big\{\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\Big\}^{\frac{1}{q}}\end{aligned} \tag{2.7}$$

holds true, then we have

$$\sigma+\sigma_1 = \lambda_3+\lambda_4-\lambda_2. \tag{2.8}$$

Proof. Letting $\gamma = \sigma+\sigma_1+\lambda_2-\lambda_3-\lambda_4$, we discussed the situation as follows:

1° If $\gamma<0$, for $0<\varepsilon<-\gamma$, setting the following two functions

$$f_\varepsilon(x) = \begin{cases} x^{\sigma-1+\frac{\varepsilon}{p}}, & x\in(0,1]\\ 0, & x\in(1,\infty)\end{cases}, g_\varepsilon(y) = \begin{cases} y^{\sigma_1-1+\frac{\varepsilon}{q}}, & y\in(0,1]\\ 0, & y\in(1,\infty)\end{cases}.$$

Then, we find

$$J_1 := \Big[\int_0^\infty x^{p(1-\sigma)-1} f_\varepsilon^p(x)dx\Big]^{\frac{1}{p}} \Big[\int_0^\infty y^{q(1-\sigma_1)-1} g_\varepsilon^q(y)dy\Big]^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{2.9}$$

Besides, note the facts that $\sigma_1+\lambda_2>0$, setting $\frac{y}{x}=t$, and Fubini's theorem (cf. [11]), we find

$$\begin{aligned}
I_1 :&= \int_0^\infty \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f_\varepsilon(x)g_\varepsilon(y)dxdy\\
&= \int_0^1 x^{\sigma-1+\frac{\varepsilon}{p}}dx \int_0^1 \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} y^{\sigma_1-1+\frac{\varepsilon}{q}}dy\\
&= \int_0^1 x^{\gamma-1+\varepsilon}dx \int_0^{x^{-1}} \frac{|\ln t|^{\lambda_1}(\min\{1,t\})^{\lambda_2} t^{\sigma_1-1+\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}(\max\{1,t\})^{\lambda_4}}dt\\
&\geq \int_0^1 x^{\gamma-1+\varepsilon}dx \int_0^1 \frac{(\ln\frac{1}{t})^{\lambda_1} t^{\sigma_1+\lambda_2-1+\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt\\
&= \Gamma(\lambda_1+1)\sum_{n=0}^\infty \frac{\binom{-\lambda_3}{n}}{(n+\sigma_1+\lambda_2+\frac{\varepsilon}{q})^{\lambda_1+1}} \int_0^1 x^{\gamma-1+\varepsilon}dx.
\end{aligned} \tag{2.10}$$

By (2.10), (2.7) and (2.9), we deduce that

$$\begin{aligned}
&\Gamma(\lambda_1+1)\sum_{n=0}^\infty \frac{\binom{-\lambda_3}{n}}{(n+\sigma_1+\lambda_2+\frac{\varepsilon}{q})^{\lambda_1+1}} \int_0^1 x^{\gamma-1+\varepsilon}dx \leq I_1\\
&= \int_0^\infty \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dxdy\\
&\leq M\Big[\int_0^\infty x^{p(1-\sigma)-1} f_\varepsilon^p(x)dx\Big]^{\frac{1}{p}} \Big[\int_0^\infty y^{q(1-\sigma_1)-1} g_\varepsilon^q(y)dy\Big]^{\frac{1}{q}} = M\frac{1}{\varepsilon} < \infty.
\end{aligned}$$

In view of $\gamma+\varepsilon<0$, we get $\int_0^1 x^{\gamma-1+\varepsilon}dx=\infty$. Then, we get a contradictory expression as $\infty \leq I_1 < \infty$.

2° If $\gamma>0$, for $0<\varepsilon<\gamma$, setting the following two functions

$$\tilde{f}_\varepsilon(x) = \begin{cases} x^{\sigma-1-\frac{\varepsilon}{p}}, & x\in[1,\infty)\\ 0, & x\in(0,1)\end{cases}, \quad \tilde{g}_\varepsilon(y) = \begin{cases} y^{\sigma_1-1-\frac{\varepsilon}{q}}, & y\in[1,\infty)\\ 0, & y\in(0,1)\end{cases}.$$

Then, we find

$$\tilde{J}_1 := \Big[\int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_\varepsilon^p(x)dx\Big]^{\frac{1}{p}} \Big[\int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_\varepsilon^q(y)dy\Big]^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{2.11}$$

Setting $\frac{y}{x}=t$, we find

$$\begin{aligned}
\tilde{I}_1 :&= \int_0^\infty \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \tilde{f}_\varepsilon(x)\tilde{g}_\varepsilon(y)dxdy\\
&= \int_1^\infty x^{\sigma-1-\frac{\varepsilon}{p}}dx \int_1^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} y^{\sigma_1-1-\frac{\varepsilon}{q}}dy\\
&= \int_1^\infty x^{\gamma-1-\varepsilon}dx \int_{x^{-1}}^\infty \frac{|\ln t|^{\lambda_1}(\min\{1,t\})^{\lambda_2} t^{\sigma_1-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}(\max\{1,t\})^{\lambda_4}}dt\\
&\geq \int_1^\infty x^{\gamma-1-\varepsilon}dx \int_1^\infty \frac{(\ln t)^{\lambda_1} t^{\sigma_1-\lambda_4-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt\\
&= \int_1^\infty x^{\gamma-1-\varepsilon}dx \int_0^1 \frac{(\ln\frac{1}{u})^{\lambda_1} u^{\lambda_3+\lambda_4-\sigma_1-1+\frac{\varepsilon}{q}}}{(1+u)^{\lambda_3}}du\\
&= \Gamma(\lambda_1+1)\sum_{n=0}^\infty \frac{\binom{-\lambda_3}{n}}{(n+\lambda_3+\lambda_4-\sigma_1+\frac{\varepsilon}{q})^{\lambda_1+1}} \int_1^\infty x^{\gamma-1-\varepsilon}dx.
\end{aligned} \tag{2.12}$$

By (2.12), (2.7) and (2.11), we deduce that

$$\Gamma(\lambda_1+1)\sum_{n=0}^{\infty}\frac{\binom{-\lambda_3}{n}}{(n+\lambda_3+\lambda_4-\sigma_1+\frac{\varepsilon}{q})^{\lambda_1+1}}\int_1^{\infty}x^{\gamma-1-\varepsilon}dx\le\tilde{I}_1$$
$$=\int_0^{\infty}\int_0^{\infty}\frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}\tilde{f}_\varepsilon(x)\tilde{g}_\varepsilon(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dxdy$$
$$\le M\Big[\int_0^{\infty}x^{p(1-\sigma)-1}\tilde{f}_\varepsilon^p(x)dx\Big]^{\frac{1}{p}}\Big[\int_0^{\infty}y^{q(1-\sigma_1)-1}\tilde{g}_\varepsilon^q(y)dy\Big]^{\frac{1}{q}}=M\frac{1}{\varepsilon}<\infty.$$

In view of $\gamma-\varepsilon>0$, we get $\int_1^{\infty}x^{\gamma-1-\varepsilon}dx=\infty$. Then, we get a contradictory expression as $\infty\le\tilde{I}_1<\infty$. To sum up the above discussions, only $\gamma=0$, that is $\sigma+\sigma_1=\lambda_3+\lambda_4-\lambda_2$.

Lemma 2.4 If $\sigma+\sigma_1=\lambda_3+\lambda_4-\lambda_2$, then we have $M\ge K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.

Proof. Taking the functions $\tilde{f}_\varepsilon(x)$ and $\tilde{g}_\varepsilon(y)$ of Lemma 2.3, and setting $\frac{y}{x}=t$, we find

$$\tilde{I}\varepsilon=\varepsilon\int_0^{\infty}\int_0^{\infty}\frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}\tilde{f}_\varepsilon(x)\tilde{g}_\varepsilon(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dxdy$$
$$=\varepsilon\int_1^{\infty}x^{\sigma-1-\frac{\varepsilon}{p}}dx\Big[\int_1^{\infty}\frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}y^{\sigma_1-1-\frac{\varepsilon}{q}}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dy\Big]$$
$$=\varepsilon\int_1^{\infty}x^{-1-\varepsilon}dx\Big[\int_{x^{-1}}^{\infty}\frac{|\ln t|^{\lambda_1}(\min\{1,t\})^{\lambda_2}t^{\sigma_1-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}(\max\{1,t\})^{\lambda_4}}dt\Big]$$
$$=\varepsilon\int_1^{\infty}x^{-1-\varepsilon}dx\Big[\int_0^1\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt+\int_1^{\infty}\frac{(\ln t)^{\lambda_1}t^{\sigma_1-\lambda_4-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt\Big]-$$
$$\varepsilon\int_1^{\infty}x^{-1-\varepsilon}dx\int_0^{x^{-1}}\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt=H_1-H_2 \tag{2.13}$$

By the proof of lemma 2.1, we have

$$H_1=\varepsilon\int_1^{\infty}x^{-1-\varepsilon}dx\Big[\int_0^1\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt+\int_1^{\infty}\frac{(\ln t)^{\lambda_1}t^{\sigma_1-\lambda_4-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt\Big]$$
$$=\Gamma(\lambda_1+1)\sum_{n=0}^{\infty}\left[\frac{\binom{-\lambda_3}{n}}{\big(n+\sigma_1+\lambda_2-\frac{\varepsilon}{q}\big)^{\lambda_1+1}}+\frac{\binom{-\lambda_3}{n}}{\big(n+\lambda_3+\lambda_4-\sigma_1+\frac{\varepsilon}{q}\big)^{\lambda_1+1}}\right].$$

Noticing the fact $0<t<\frac{1}{x}<1\Rightarrow t^{\frac{\sqrt{\varepsilon}}{q}}<(\frac{1}{x})^{\frac{\sqrt{\varepsilon}}{q}}=x^{-\frac{\sqrt{\varepsilon}}{q}}$, we have

$$H_2=\varepsilon\int_1^{\infty}x^{-1-\varepsilon}dx\int_0^{x^{-1}}\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}}}{(1+t)^{\lambda_3}}dt$$
$$<\varepsilon\int_1^{\infty}x^{-1}dx\int_0^{x^{-1}}\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}-\frac{\sqrt{\varepsilon}}{q}+\frac{\sqrt{\varepsilon}}{q}}}{(1+t)^{\lambda_3}}dt$$
$$<\varepsilon\int_1^{\infty}x^{-1-\frac{\sqrt{\varepsilon}}{q}}dx\int_0^{x^{-1}}\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}-\frac{\sqrt{\varepsilon}}{q}}}{(1+t)^{\lambda_3}}dt$$
$$\le\varepsilon\int_1^{\infty}x^{-1-\frac{\sqrt{\varepsilon}}{q}}dx\int_0^{1}\frac{\big(\ln\frac{1}{t}\big)^{\lambda_1}t^{\sigma_1+\lambda_2-1-\frac{\varepsilon}{q}-\frac{\sqrt{\varepsilon}}{q}}}{(1+t)^{\lambda_3}}dt$$
$$=q\sqrt{\varepsilon}\Gamma(1+\lambda_1)\sum_{n=0}^{\infty}\frac{\binom{-\lambda_3}{n}}{\big(n+\sigma_1+\lambda_2-\frac{\varepsilon+\sqrt{\varepsilon}}{q}\big)^{\lambda_1+1}}.$$

Returning H_1 and H_2 to (2.13), and letting $\varepsilon\to0^+$. By (2.4) we obtain

$$\tilde{I}_1\varepsilon>K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)(1-o(1))\,(\varepsilon\to0^+).$$

Moreover, we have

$$M = M\tilde{J}_1\varepsilon > \tilde{I}_1\varepsilon > K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4)(1 - o(1)).$$

Letting $\varepsilon \to 0^+$, it follows that $M \geq K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

3 Main results

Theorem 3.1 Assuming $M > 0$, the following conditions(i)-(iv) are equivalent:
(i) The following inequality holds

$$J := \left[\int_0^\infty y^{p\sigma_1 - 1}\Big(\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dx\Big)^p dy\right]^{\frac{1}{p}} < M\Big[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\Big]^{\frac{1}{p}}. \tag{3.1}$$

(ii) The following inequality holds

$$I := \int_0^\infty \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)g(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dxdy < M\Big\{\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\Big\}^{\frac{1}{q}}. \tag{3.2}$$

(iii) The constants σ, σ_1 meet the following condition:

$$\sigma + \sigma_1 = \lambda_3 + \lambda_4 - \lambda_2. \tag{3.3}$$

(iv) When $\sigma + \sigma_1 = \lambda_3 + \lambda_4 - \lambda_2$, we have

$$K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = K(\sigma_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

and

$$M \geq K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4). \tag{3.4}$$

Proof. (i) $\Rightarrow$ (ii). By the weighted Hölder's inequality (cf. [12]), we have

$$I = \int_0^\infty \Big(y^{\sigma_1 - \frac{1}{p}}\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dx\Big)\Big(y^{\frac{1}{p}-\sigma_1}g(y)\Big)dy < J\Big[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\Big]^{\frac{1}{q}}.$$

Then by (3.1), We derive (3.2).

(ii) $\Rightarrow$ (iii). By Lemma2.3, we have equation (3.3).

(iii) $\Rightarrow$ (iv). We have proved that $K(\sigma, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = K(\sigma_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ in Lemma 2.1. By Lemma 2.4, we have (3.4).

(iv) $\Rightarrow$ (i). By Hölder inequality, and Lemma 2.2, we have

$$\begin{aligned}&\Big(\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dx\Big)^p \\ &= \left\{\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\Big[\frac{y^{\frac{\sigma_1-1}{p}}}{x^{\frac{\sigma-1}{q}}}f(x)\Big]\Big[\frac{x^{\frac{\sigma-1}{q}}}{y^{\frac{\sigma_1-1}{p}}}\Big]dx\right\}^p \\ &\leq \int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f^p(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\frac{y^{\sigma_1-1}}{x^{\frac{p(\sigma-1)}{q}}}dx \times\end{aligned}$$

$$
\begin{aligned}
&\Big[\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\frac{x^{\sigma-1}}{y^{\frac{q(\sigma_1-1)}{p}}}dx\Big]^{\frac{p}{q}}\\
&=\Big\{\omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3\lambda_4,y)\Big\}^{\frac{p}{q}}\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f^p(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\frac{y^{\sigma_1-1}}{x^{\frac{p(\sigma-1)}{q}}}dx\\
&=K^{\frac{p}{q}}(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)y^{1-p\sigma_1}\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f^p(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\frac{y^{\sigma_1-1}}{x^{\frac{p(\sigma-1)}{q}}}dx. \qquad (3.5)
\end{aligned}
$$

We first certify that (3.5) takes the form of a strict inequality. Suppose that (3.5) takes the form of equality for a $y\in(0,\infty)$, then by the condition that Hölder inequality holds the equation , there exist constants A and B, which are not all zero, such that

$$
A\frac{y^{\sigma_1-1}}{x^{\frac{p(\sigma-1)}{q}}}f^p(x)=B\frac{x^{\sigma-1}}{y^{\frac{q(\sigma_1-1)}{p}}}g^q(y)\quad a.e.in(0,\infty)\times(0,\infty).
$$

We might as well assume $A\neq 0$, it follows that

$$
x^{p(1-\sigma)-1}f^p(x)=\big[y^{\frac{q(\lambda_3+\lambda_4-\lambda_2)}{2}}g^q(y)\big]\frac{B}{Ax}\quad a.e.in(0,\infty).
$$

For the generalized integral $\int_0^\infty[y^{\frac{q(\lambda_3+\lambda_4-\lambda_2)}{2}}g^q(y)]\frac{B}{Ax}dx$ is divergent, which contradicts the fact that $0<\int_0^\infty \varphi(x)f^p(x)dx<\infty$.

Further, by (3.5), Lemma 2.2 and Fubini's theorem, we have

$$
\begin{aligned}
J&=\Big[\int_0^\infty y^{p\sigma_1-1}\Big(\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dx\Big)^p dy\Big]^{\frac{1}{p}}\\
&<K^{\frac{1}{q}}(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big[\int_0^\infty\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f^p(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\frac{y^{\sigma_1-1}}{x^{\frac{p(\sigma-1)}{q}}}dxdy\Big]^{\frac{1}{p}}\\
&=K^{\frac{1}{q}}(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big[\int_0^\infty \omega(\sigma,\sigma_1,\lambda_1,\lambda_2,\lambda_3\lambda_4,x)f^p(x)dx\Big]^{\frac{1}{p}}\\
&=K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\Big]^{\frac{1}{p}}.
\end{aligned}
$$

For $0<K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\leq M$, so (3.1) is established. □

By Theorem 3.1 and (3.5), we easily get the following conclusion:

Corollary 1 If $\sigma+\sigma_1=\lambda_3+\lambda_4-\lambda_2$, then we have the following equivalent inequalities:

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f(x)g(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dxdy\\
&<K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big\{\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\Big\}^{\frac{1}{q}}, \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty y^{p\sigma_1-1}\Big\{\int_0^\infty \frac{|\ln\frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}dx\Big\}^p dy\\
&<K^p(\sigma_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx, \qquad (3.7)
\end{aligned}
$$

where the constant factors $K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and $K^p(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ appearing on the right-hand side of (3.6) and (3.7) are the best possible.

Taking $\sigma=\sigma_1$ in Theorem 3.1, $\sigma=\sigma_1=\frac{\lambda_3+\lambda_4-\lambda_2}{2}$ are derived from $\sigma+\sigma_1=\lambda_3+\lambda_4-\lambda_2$ and $\lambda_2+\lambda_3+\lambda_4>0$ from $-\lambda_2<\sigma,\sigma_1<\lambda_3+\lambda_4$, $K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)=K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Then, we obtain the following simple and more applicable conclusions which excluding parameter variables σ,σ_1.

Corollary 2 If $\lambda_1 > -1, \lambda_2+\lambda_3+\lambda_4 > 0$, $f(x), g(y) \geq 0$, satisfying $0 < \int_0^\infty x^{p(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1} f^p(x)dx < \infty, 0 < \int_0^\infty y^{q(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1} g^q(y)dy < \infty$. Then, we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)g(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} dxdy$$
$$< K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big\{\int_0^\infty x^{p(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1} f^p(x)dx\Big\}^{\frac{1}{p}} \Big\{\int_0^\infty y^{q(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1} g^q(y)dy\Big\}^{\frac{1}{q}}, \tag{3.8}$$

and

$$\int_0^\infty y^{\frac{p(\lambda_3+\lambda_4-\lambda_2)}{2}-1}\Big\{\int_0^\infty \frac{|\ln \frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} dx\Big\}^p dy$$
$$< K^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\int_0^\infty x^{p(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1} f^p(x)dx, \tag{3.9}$$

where the constant factors $K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and $K^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ appearing on the right-hand side of (3.8) and (3.9) are the best possible.

4 Operator expressions and some particular inequalities

Setting functions as $\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma_1)-1}$ $(x, y > 0)$, apparently, $\psi^{1-p}(y) := y^{p\sigma_1-1}$. We also set the following linear spaces with norm:

$$L_{p,\varphi}(\mathbb{R}_+) := \Big\{f = f(x), x \in \mathbb{R}_+; \|f\|_{p,\varphi} = \big[\int_0^\infty \varphi(x)|f(x)|^p dx\big]^{\frac{1}{p}} < \infty\Big\},$$

$$L_{q,\psi}(\mathbb{R}_+) := \Big\{g = g(y), y \in \mathbb{R}_+; \|g\|_{q,\psi} = \big[\int_0^\infty \psi(y)|g(y)|^q dy\big]^{\frac{1}{q}} < \infty\Big\},$$

$$L_{p,\psi^{1-p}}(\mathbb{R}_+) := \Big\{h = h(y), y \in \mathbb{R}_+; \|h\|_{p,\varphi^{1-p}} = \big[\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy\big]^{\frac{1}{p}} < \infty\Big\}.$$

Specially, when $\varphi(x) = 1$, we write that $\|f\|_{p,1} = \|f\|_p$.

If $f \in L_{p,\varphi}(\mathbb{R}_+)$, a singular Hilbert-type integral operator is defined as $T : L_{p,\varphi}(\mathbb{R}_+) \to L_{p,\psi^{1-p}}(\mathbb{R}_+)$,

$$T(f)(y) := \int_0^\infty \frac{|\ln \frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f(x)dx, y \in \mathbb{R}_+.$$

For $f \in L_{p,\varphi}(\mathbb{R}_+), g \in L_{q,\psi}(\mathbb{R}_+)$, the formal inner product of Tf and g and the norm of operator T are defined as:

$$(Tf, g) := \int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f(x)g(y)dxdy,$$

$$\|T\| := \sup_{f(\neq 0)\in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}}.$$

When $\sigma + \sigma_1 = \lambda_3 + \lambda_4 - \lambda_2$, the constant factor $K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ of (3.6) is the best possible, so we obtain $\|T\| = K(\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. By Corollary 1, we have

Corollary 3 If $\sigma + \sigma_1 = \lambda_3 + \lambda_4 - \lambda_2, \varphi(x) = x^{p(1-\sigma)-1}, \psi(y) = y^{q(1-\sigma_1)-1}$, $f(>0) \in L_{p,\varphi}(\mathbb{R}_+), g(>0) \in L_{q,\psi}(\mathbb{R}_+)$, we have the following equivalent operator inequalities with norm:

$$(Tf, g) < \|T\|\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{4.1}$$

$$\|T(f)\|^p_{p,\psi^{1-p}} < \|T\|^p\|f\|^p_{p,\varphi}. \tag{4.2}$$

Specially, when $\sigma = \sigma_1$, we get $\varphi(x) = x^{p(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1}, \psi(y) = y^{q(1-\frac{\lambda_3+\lambda_4-\lambda_2}{2})-1}$, $\|T\| = K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.

Theoretically, many equivalence inequalities can be obtained by using (3.6) and (3.7), however, the values of σ, σ_1 in (3.6) and (3.7) are difficult to control. In practical application, it is simple and easy to obtain inequalities by selecting the appropriate parameter values in (3.8) and (3.9).

Example 1 1° Letting $\lambda_1 = \lambda_2 = \lambda_4 = 0, \lambda_3 = \lambda(>0), p = q = 2$, we get $\varphi(x) = x^{1-\lambda}, \psi(y) = y^{1-\lambda}$, $K(0,0,\lambda,0) = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ by calculating (2.5). Then by (3.8) and (3.9), we have (1.5) and its equivalent form. Continue to letting $\lambda = 1$, we have (1.1) and its equivalent form.

2° Letting $\lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda_4 = \lambda(>0), p = q = 2$, we get $\varphi(x) = x^{1-\lambda}, \psi(y) = y^{1-\lambda}$, $K(0,0,0,\lambda) = \frac{4}{\lambda}$ by calculating (2.5). Then by (3.8) and (3.9), we have (1.6) and its equivalent form. Continue to letting $\lambda = 1$, we have (1.2) and its equivalent form.

3° Letting $\lambda_1 = \lambda_3 = \lambda_4 = 0, \lambda_2 = \lambda(>0), p = q = 2$, we get $\varphi(x) = x^{1+\lambda}, \psi(y) = y^{1+\lambda}$, $K(0,\lambda,0,0) = \frac{4}{\lambda}$ by calculating (2.5). Then by (3.8) and (3.9), we have (1.7) and its equivalent form.

4° Letting $\lambda_1 = \lambda_3 = 1, \lambda_2 = \lambda_4 = 0, p = q = 2$, we get $\varphi(x) = \psi(y) = 1$, $K(1,0,1,0) = c_0 = 7.327724754^+$ by calculating (2.5). Then by (3.8) and (3.9), we have (1.3) and its equivalent form. Letting $\lambda_1 = \lambda_4 = 1, \lambda_2 = \lambda_3 = 0, p = q = 2$, we get $\varphi(x) = \psi(y) = 1$, $K(1,0,0,1) = 8$ by calculating (2.5). Then by (3.8) and (3.9), we have (1.4) and its equivalent form.

This example illustrates that (3.8) and (3.9) integrate the results of some references. In addition, we can obtain some new Hilbert-type integral inequalities with simple form by choosing suitable parameter values in (3.8) and (3.9).

Example 2 Letting $\lambda_1 = \lambda_3 = \lambda_4 = 1, \lambda_2 = 0, p = q = 2$, we get $\varphi(x) = x^{-1}$, $K(1,0,1,1) = \frac{\pi^2}{6}$ by calculating (2.5). If $f, g \geq 0, \|f\|_{2,\varphi}, \|q\|_{2,\varphi} < \infty$, Then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}| f(x) g(y)}{(x+y)\max\{x,y\}} dxdy < \frac{\pi^2}{6} \|f\|_{2,\varphi} \|q\|_{2,\varphi}, \tag{4.3}$$

$$\int_0^\infty y \Big\{ \int_0^\infty \frac{|\ln \frac{y}{x}| f(x)}{(x+y)\max\{x,y\}} dx \Big\}^2 dy < \frac{\pi^4}{36} \|f\|_{2,\varphi}^2, \tag{4.4}$$

where the constant factors $\frac{\pi^2}{6}$ and $\frac{\pi^4}{36}$ are the best possible.

Example 3 Letting $\lambda_1 = \lambda_2 \lambda_3 = 1, \lambda_4 = 0, p = q = 2$, we get $\varphi(x) = x$, $K(1,1,1,0) = \frac{\pi^2}{6}$ by calculating (2.5). If $f, g \geq 0, \|f\|_{2,\varphi}, \|q\|_{2,\varphi} < \infty$, Then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}| \min\{x,y\} f(x) g(y)}{x+y} dxdy < \frac{\pi^2}{6} \|f\|_{2,\varphi} \|q\|_{2,\varphi}, \tag{4.5}$$

$$\int_0^\infty \frac{1}{y} \Big\{ \int_0^\infty \frac{|\ln \frac{y}{x}| \min\{x,y\} f(x)}{x+y} dx \Big\}^2 dy < \frac{\pi^4}{36} \|f\|_{2,\varphi}^2, \tag{4.6}$$

where the constant factors $\frac{\pi^2}{6}$ and $\frac{\pi^4}{36}$ are the best possible.

Example 4 Letting $\lambda_1 = 0, \lambda_2 = \lambda_3 = \lambda_4 = 1, p = q = 2$, we get $\varphi(x) = 1$, $K(0,1,1,1) = 4 - \pi$ by calculating (2.5). If $f, g \geq 0, \|f\|_2, \|q\|_2 < \infty$, Then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{\min\{x,y\} f(x) g(y)}{(x+y)\max\{x,y\}} dxdy < (4-\pi) \|f\|_2 \|q\|_2, \tag{4.7}$$

$$\int_0^\infty \Big[\int_0^\infty \frac{\min\{x,y\} f(x)}{(x+y)\max\{x,y\}} dx \Big]^2 dy < (4-\pi)^2 \|f\|_2^2, \tag{4.8}$$

where the constant factors $4 - \pi$ and $(4-\pi)^2$ are the best possible.

Example 5 Letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1, p = q = 2$, we get $\varphi(x) = 1$, $K(1,1,1,1) = 8(1 - catalan) = 0.672275246^+$ by calculating (2.5). If $f, g \geq 0, \|f\|_2, \|q\|_2 < \infty$, Then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{|\ln \frac{y}{x}| \min\{x,y\} f(x) g(y)}{(x+y)\max\{x,y\}} dxdy < 8(1 - catalan) \|f\|_2 \|q\|_2, \tag{4.9}$$

$$\int_0^\infty \Big[\int_0^\infty \frac{|\ln \frac{y}{x}| \min\{x,y\} f(x)}{(x+y)\max\{x,y\}} dx\Big]^2 dy < 64(1-catalan)^2 \|f\|_2^2, \tag{4.10}$$

where the constant factors $8(1-catalan)$ and $64(1-catalan)^2$ are the best possible.

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College of science, Shao yang university,Shao yang, China 422000
Email address: liuqiongxx13@163.com