

ZERO-HOPF BIFURCATIONS AND CHAOS OF QUADRATIC JERK SYSTEMS

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ABSTRACT. The purpose of this paper is to propose some coefficient conditions, characterizing the stability of periodic solutions bifurcated from zero-Hopf bifurcations of the general quadratic jerk system, and apply these theoretical results to a special jerk system in order to predict chaos. First, we characterize the zero-Hopf bifurcations of the general quadratic jerk system in \mathbb{R}^3 . The coefficient conditions on stability of periodic solutions are obtained via the averaging theory of first order. Next, we apply the theoretical results to a two-parameter jerk system. Finally special attention is paid to a jerk system with one non-negative parameter ϵ and one non-linearity. By studying the continuation of periodic solution initiating at the zero-Hopf bifurcation, we numerically find a sequence of period doubling bifurcations which leads to the creation of chaotic attractor.

1. INTRODUCTION

It is well known that equilibria play an important role in studying the dynamics of nonlinear differential systems. For example, one method for detecting chaos is the Shilnikov's method [1], which is based on the homoclinic orbit of a saddle-focus, ensuring the existence of a horseshoe in the neighbourhood of this orbit and consequently the birth of chaotic motions.

Quadratic systems are some of the simplest after linear ones. There are many examples exhibiting rich dynamical behaviors, see for instance the Lorenz system [2], the Chen system [3], the Liu system [4], the Rössler system [5], the Rikitake system [6], the Lü system [7], the Maxwell-Bloch system [8], among many others.

Let's consider the general quadratic jerk system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5yx + a_6zx + a_7y^2 + a_8zy + a_9z^2, \end{cases} \quad (1)$$

where $a_i \in \mathbb{R}$, $0 \leq i \leq 9$. It contains some particular cases, such as the normal form of the triple-zero bifurcation [9], the Genesio system [10, 11], the Michelson system [12, 13] and its generalization [14], among several others. For the special case $a_4 = a_5 = a_6 = 0$, it is proved in [15] that the solution and its derivatives are bounded. Many cases of system (1) can exhibit self-excited attractors or hidden attractors under parameter variation.

Let $\Delta = a_1^2 - 4a_0a_4$ and $a_4 \neq 0$. The system has no real equilibria if $\Delta < 0$; the situation with a hidden attractor [16] is a special case. It has a single non-hyperbolic equilibrium if $\Delta = 0$, for some chaotic examples see [17, 18]. By the way, if $a_4 = 0$ the system also has a single equilibrium; see [19] for some chaotic examples with a stable equilibrium; see [20] for the simplest chaotic case with an unstable equilibrium. In this paper we focus our attention on the case: $a_4 \neq 0$. Without loss of generality, from now on we assume $a_4 > 0$ in system (1).

In general, a zero-Hopf equilibrium is a singular point of the 3D system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues; the zero-Hopf bifurcation is a two-parameter unfolding of the system with a zero-Hopf equilibrium if the two parameters cross through the critical values. Some complicated invariant sets of the unfolding can be bifurcated from the isolated zero-Hopf equilibrium under some conditions, which may implies a local birth of chaos, see [21, 22] and the reference therein.

The study of zero-Hopf bifurcations in low dimensional systems has attracted intense interest in recent years, see for instance [23, 24]. One approach is to pass to normal form, but it requires some efforts, for more details see [26]. Here we will study the zero-Hopf bifurcations of jerk system via the averaging theory of the first order. For the averaging theory of first order and second order, see [27, 28]. For the averaging theory of other orders, see for instance [29–31].

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As far as we know, the study of zero-Hopf bifurcations of system (1) has not been completely made in the current literature, i.e. only some special cases are considered, see for instance [14, 17, 24]. In [17] the authors proved the existence of zero-Hopf bifurcation at the origin of the following system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2, \end{cases} \quad (2)$$

where $a_3^2 + 4a_2 < 0, a_4 \neq 0$, for which the resulting periodic solution is unstable. In [24] the authors studied the zero-Hopf bifurcation at the origin of the following system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = (a_3 + a_6 x) z + (a_2 + a_5 x) y - x^2, \end{cases} \quad (3)$$

provided an estimate of the periodic solutions and their kind of stability.

We now recall the classical averaging theory. Consider the following initial value problems

$$\frac{dx}{dt} = \epsilon F_1(t, x) + \epsilon^2 F_2(t, x, \epsilon), \quad x(0) = x_0, \quad (4)$$

and

$$\frac{dy}{dt} = \epsilon g(y), \quad y(0) = x_0, \quad (5)$$

with $x, y \in D \subseteq \mathbf{R}^n, t \geq 0$ and $\epsilon \in (0, \epsilon_0]$. Moreover we assume that F_1, F_2 are T -periodic in t , and

$$g(y) = \frac{1}{T} \int_0^T F_1(t, y) dt. \quad (6)$$

Theorem 1. [28] *Consider systems (4), (5) and suppose that:*

- (1) $F_1, D_x F_1, D_{xx} F_1$ and $D_x F_2$ are continuous and bounded by a constant (independent of ϵ) in $[0, \infty) \times D$.
- (2) F_1 and F_2 are T -periodic in t , with T independent of ϵ .
- (3) $y(t)$ is contained in an (ϵ -independent) interior subset of D on the time-scale $1/\epsilon$.

Then we have the following statements:

- (a) $x(t) - y(t) = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$ on the time-scale $1/\epsilon$.
- (b) If p is an equilibrium of the averaged system (5) whereas

$$\text{Det } D_y g(p) \neq 0,$$

then there exists a T -periodic solution $\varphi(t, \epsilon)$ of system (4), which is close to p such that

$$\lim_{\epsilon \rightarrow 0} \varphi(t, \epsilon) = p.$$

- (c) *The stability or instability of the periodic solution $\varphi(t, \epsilon)$ is the same as that of the equilibrium p . In fact, the equilibrium p has the stability behaviour of the Poincaré map associated to the periodic solution $\varphi(t, \epsilon)$.*

The paper is organized as follows. In the next section, zero-Hopf bifurcation of a 3D system is studied. In Section 3, the results derived in the previous section are applied to system (1). Finally, as an application, we consider the zero-Hopf bifurcations and chaos of some special systems.

2. ZERO-HOPF BIFURCATION OF A 3D SYSTEM

Let's consider the following system

$$\begin{cases} \frac{du}{dt} = -\delta v, \\ \frac{dv}{dt} = \delta u + \alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4 \epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7 \epsilon v + \alpha_8 w^2 + \alpha_9 \epsilon w, \\ \frac{dw}{dt} = \beta (\alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4 \epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7 \epsilon v + \alpha_8 w^2 + \alpha_9 \epsilon w), \end{cases} \quad (7)$$

where $\delta, \beta > 0$, $\alpha_i \in \mathbb{R}$, $1 \leq i \leq 9$, $\alpha_6, \alpha_9 \neq 0$, $\alpha_8 > 0$, and $\epsilon \in [0, \epsilon_0)$ with ϵ_0 small enough. For this system, we consider linear stability and zero-Hopf bifurcation of the equilibrium $O(0, 0, 0)$.

Proposition 1. *If $\alpha_7 < 0$, $\alpha_9 < 0$, and $\epsilon > 0$ is sufficiently small, the origin of system (7) is asymptotically stable.*

Proof. The characteristic polynomial of system (7) at the origin is

$$P(\lambda) = \lambda^3 - \epsilon (\beta \alpha_9 + \alpha_7) \lambda^2 + (\alpha_4 \epsilon + \delta) \delta \lambda - \epsilon \alpha_9 \beta \delta^2. \quad (8)$$

According to the Routh-Hurwitz criterion and the assumptions $\epsilon, \delta, \beta > 0$ for system (7), the origin is asymptotically stable if

$$\alpha_7 < -\frac{\alpha_4 \alpha_9 \beta \epsilon}{\alpha_4 \epsilon + \delta}, \quad \alpha_9 < 0, \quad \alpha_4 > -\frac{\delta}{\epsilon}. \quad (9)$$

Since $\epsilon > 0$ and sufficiently small, $\alpha_7 < 0$, $\alpha_9 < 0$ implies (9), so that the origin is asymptotically stable. \square

System (7) has an isolated zero-Hopf equilibrium at the origin for $\epsilon = 0$. In the next section, this system will be shown to be a prototype for studying the zero-Hopf bifurcations of the system (1).

Under the cylindrical coordinates,

$$u = r \cos(\theta), v = r \sin(\theta), w = w, \quad (10)$$

where $r > 0$, system (7) becomes

$$\begin{cases} \frac{dr}{dt} = \sin \theta (p_1(\theta) r^2 + p_2(\theta) wr + p_3(\theta) \epsilon r + \alpha_8 w^2 + \alpha_9 \epsilon w), \\ \frac{d\theta}{dt} = \delta + q_1(\theta) r + q_2(\theta) \frac{w^2}{r} + q_3(\theta) \frac{\epsilon w}{r} + q_4(\theta) w + q_5(\theta) \epsilon, \\ \frac{dw}{dt} = \beta (p_1(\theta) r^2 + p_2(\theta) wr + p_3(\theta) \epsilon r + \alpha_8 w^2 + \alpha_9 \epsilon w), \end{cases} \quad (11)$$

where

$$p_1(\theta) = (\alpha_5 - \alpha_1) \sin^2 \theta + \alpha_2 \sin \theta \cos \theta + \alpha_1, \quad (12)$$

$$p_2(\theta) = \alpha_3 \cos \theta + \alpha_6 \sin \theta, \quad (13)$$

$$p_3(\theta) = \alpha_4 \cos \theta + \alpha_7 \sin \theta, \quad (14)$$

and

$$q_1(\theta) = -\alpha_2 \sin^3 \theta + (\alpha_5 - \alpha_1) \cos \theta \sin^2 \theta + \alpha_2 \sin \theta + \alpha_1 \cos \theta, \quad (15)$$

$$q_2(\theta) = \alpha_8 \cos \theta, \quad (16)$$

$$q_3(\theta) = \alpha_9 \cos \theta, \quad (17)$$

$$q_4(\theta) = -\alpha_3 \sin^2 \theta + \alpha_6 \cos \theta \sin \theta + \alpha_3, \quad (18)$$

$$q_5(\theta) = -\alpha_4 \sin^2 \theta + \alpha_7 \cos \theta \sin \theta + \alpha_4. \quad (19)$$

Using the rescaling $r = \epsilon R$, $w = \epsilon W$, system (11) becomes

$$\begin{cases} \frac{dR}{dt} = \epsilon \sin \theta (p_1(\theta) R^2 + p_2(\theta) WR + p_3(\theta) R + \alpha_8 W^2 + \alpha_9 W), \\ \frac{d\theta}{dt} = \delta + \epsilon \left(q_1(\theta) R + q_2(\theta) \frac{W^2}{R} + q_3(\theta) \frac{W}{R} + q_4(\theta) W + q_5(\theta) \right), \\ \frac{dW}{dt} = \epsilon \beta (p_1(\theta) R^2 + p_2(\theta) WR + p_3(\theta) R + \alpha_8 W^2 + \alpha_9 W). \end{cases} \quad (20)$$

Therefore the solutions of system (20) in the region $\frac{d\theta}{dt} > 0$ can be studied through the solutions of the following system

$$\begin{cases} \frac{dR}{d\theta} = F_1(\theta, R, W) \epsilon + O(\epsilon^2), \\ \frac{dW}{d\theta} = F_2(\theta, R, W) \epsilon + O(\epsilon^2), \end{cases} \quad (21)$$

where

$$F_1(\theta, R, W) = \frac{\sin \theta (p_1(\theta) R^2 + p_2(\theta) W R + p_3(\theta) R + \alpha_8 W^2 + \alpha_9 W)}{\delta}, \quad (22)$$

$$F_2(\theta, R, W) = \frac{\beta (p_1(\theta) R^2 + p_2(\theta) W R + p_3(\theta) R + \alpha_8 W^2 + \alpha_9 W)}{\delta}. \quad (23)$$

Using the same notations as in the Theorem 1, we have $t = \theta$, $T = 2\pi$, $x = (R, W)^T$, and

$$F(\theta, R, W) = \begin{pmatrix} F_1(\theta, R, W) \\ F_2(\theta, R, W) \end{pmatrix}, \quad f(R, W) = \begin{pmatrix} f_1(R, W) \\ f_2(R, W) \end{pmatrix}, \quad (24)$$

where

$$f_i(R, W) = \frac{1}{2\pi} \int_0^{2\pi} F_i(\theta, R, W) d\theta, \quad i = 1, 2. \quad (25)$$

Thus for system (21), we have

$$f_1(R, W) = \frac{R(\alpha_6 W + \alpha_7)}{2\delta}, \quad (26)$$

$$f_2(R, W) = \frac{\beta [(\alpha_1 + \alpha_5) R^2 + 2\alpha_8 W^2 + 2\alpha_9 W]}{2\delta}. \quad (27)$$

Hence the averaged system reads

$$\begin{cases} \frac{dR}{d\theta} = f_1(R, W) \epsilon, \\ \frac{dW}{d\theta} = f_2(R, W) \epsilon, \end{cases} \quad (28)$$

and therefore the unique equilibrium of (28) with $R > 0$ is

$$(R^*, W^*) = \left(\frac{1}{|\alpha_6|} \sqrt{\frac{2\alpha_7(\alpha_6\alpha_9 - \alpha_7\alpha_8)}{\alpha_1 + \alpha_5}}, -\frac{\alpha_7}{\alpha_6} \right), \quad (29)$$

where

$$\alpha_7(\alpha_6\alpha_9 - \alpha_7\alpha_8)(\alpha_1 + \alpha_5) > 0. \quad (30)$$

Computing the Jacobian of f_1, f_2 at this point, we get

$$\det \frac{\partial (f_1, f_2)}{\partial (R, W)} \Big|_{(R, W) = (R^*, W^*)} = -\frac{\alpha_7(\alpha_6\alpha_9 - \alpha_7\alpha_8)\beta}{\alpha_6\delta^2}, \quad (31)$$

which is non-zero due to $\beta > 0$ and (30). We consider the stability of the equilibrium (R^*, W^*) , for which the characteristic polynomial is

$$P(\lambda) = \lambda^2 + b_1 \lambda + b_2, \quad (32)$$

where

$$\begin{aligned} b_1 &= -\frac{\beta(\alpha_6\alpha_9 - 2\alpha_7\alpha_8)}{\alpha_6\delta}, \\ b_2 &= -\frac{\beta(\alpha_6\alpha_9 - \alpha_7\alpha_8)\alpha_7}{\delta^2\alpha_6}. \end{aligned}$$

It is obvious that $b_2 \neq 0$ due to $\beta > 0$ and (30). Thus (R^*, W^*) is non-degenerate.

According to the Routh-Hurwitz criterion, the polynomial (32) has all its roots with negative real parts if and only if

$$b_1 > 0, \quad b_2 > 0, \quad (33)$$

which implies that the equilibrium (R^*, W^*) is an attractor. Using the Trace-Determinant criteria, the equilibrium is a repeller if

$$b_1 < 0, \quad b_2 > 0; \quad (34)$$

and it is a saddle point if

$$b_2 < 0. \quad (35)$$

Let's consider the case $\alpha_9 > 0$. In view of condition (30) and the basic assumptions $\beta > 0, \delta > 0, \alpha_8 > 0$ for system (7). By solving (33), (34) and (35), respectively, we get the following conclusions:

- (a) The equilibrium is an attractor if one of the following conditions hold:
 - (a₁) $\alpha_6 < 0, \quad \alpha_7 < 0, \quad \frac{\alpha_9 \alpha_6}{2 \alpha_7} < \alpha_8 < \frac{\alpha_9 \alpha_6}{\alpha_7};$
 - (a₂) $\alpha_6 > 0, \quad \alpha_7 > 0, \quad \alpha_8 > \frac{\alpha_9 \alpha_6}{\alpha_7}.$
- (b) The equilibrium is a repeller if one of the following conditions hold:
 - (b₁) $\alpha_6 < 0, \quad \alpha_7 < 0, \quad \alpha_8 < \frac{\alpha_6 \alpha_9}{2 \alpha_7};$
 - (b₂) $\alpha_6 > 0, \quad \alpha_7 < 0.$
- (c) The equilibrium is a saddle point if one of the following conditions holds:
 - (c₁) $\alpha_6, \alpha_7 < 0, \quad \alpha_8 > \frac{\alpha_6 \alpha_9}{\alpha_7};$
 - (c₂) $\alpha_6 < 0, \quad \alpha_7 > 0;$
 - (c₃) $\alpha_6, \alpha_7 > 0, \quad \alpha_8 < \frac{\alpha_9 \alpha_6}{\alpha_7}.$

Theorem 2. Assume that $\alpha_9 > 0$ and (30). Then for system (7), a periodic solution borns at the origin when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:

- (1) In case (a), the periodic solution is asymptotically stable;
- (2) In case (b), the periodic solution is unstable;
- (3) In case (c), the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

Proof. According to Theorem 1, for $\epsilon > 0$ and sufficiently small, system (21) has a periodic solution $(R(\theta, \epsilon), W(\theta, \epsilon))$, such that

$$(R(0, \epsilon), W(0, \epsilon)) = (R^*, W^*) + \mathcal{O}(\epsilon).$$

Using the coordinates (R, θ, W) , the periodic solution $(R(t, \epsilon), \theta(t, \epsilon), W(t, \epsilon))$ of system (20) satisfies

$$(R(0, \epsilon), \theta(0, \epsilon), W(0, \epsilon)) = (R^*, 0, W^*) + \mathcal{O}(\epsilon).$$

Now the periodic solution $(r(t, \epsilon), \theta(t, \epsilon), w(t, \epsilon))$ of system (11) satisfies

$$(r(0, \epsilon), \theta(0, \epsilon), w(0, \epsilon)) = (\epsilon R^*, 0, \epsilon W^*) + \mathcal{O}(\epsilon^2).$$

Finally, in system (7) the periodic solution $(u(t, \epsilon), v(t, \epsilon), w(t, \epsilon))$ satisfies

$$(u(0, \epsilon), v(0, \epsilon), w(0, \epsilon)) = (\epsilon R^*, 0, \epsilon W^*) + \mathcal{O}(\epsilon^2).$$

The stability of this solution follows from that of the equilibrium (R^*, W^*) in the averaged system (28). Thus we complete the proof. \square

Similarly, the conclusions for the case $\alpha_9 < 0$ are listed below:

- (d) The equilibrium is an attractor if one of the following conditions holds:
 - (d₁) $\alpha_6 < 0, \quad \alpha_7 > 0, \quad \alpha_8 < \frac{\alpha_9 \alpha_6}{2 \alpha_7};$
 - (d₂) $\alpha_6 > 0, \quad \alpha_7 > 0.$
- (e) The equilibrium is a repeller if one of the following conditions holds:
 - (e₁) $\alpha_6 < 0, \quad \alpha_7 > 0, \quad \frac{\alpha_9 \alpha_6}{2 \alpha_7} < \alpha_8 < \frac{\alpha_9 \alpha_6}{\alpha_7};$
 - (e₂) $\alpha_6 > 0, \quad \alpha_7 < 0, \quad \alpha_8 > \frac{\alpha_9 \alpha_6}{\alpha_7}.$
- (f) The equilibrium is a saddle point if one of the following conditions holds:
 - (f₁) $\alpha_6 < 0, \quad \alpha_7 < 0;$
 - (f₂) $\alpha_6 > 0, \quad \alpha_7 < 0, \quad \alpha_8 < \frac{\alpha_9 \alpha_6}{\alpha_7};$
 - (f₃) $\alpha_6 < 0, \quad \alpha_7 > 0, \quad \alpha_8 > \frac{\alpha_9 \alpha_6}{\alpha_7}.$

Using the same arguments, we can concluded that:

Theorem 3. Assume that $\alpha_9 < 0$ and (30). Then for system (7), a periodic solution borns at the origin when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:

- (1) In case (d), the periodic solution is asymptotically stable.
- (2) In case (e), the periodic solution is unstable.
- (3) In case (f), the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

Remark 1. According to Proposition 1, there are three sub-cases for system (7): (e_2) , (f_1) and (f_2) , in which the origin is asymptotically stable when $\epsilon > 0$ is sufficiently small.

Remark 2. Recall the expressions for $f_1(R, W)$, $f_2(R, W)$ in (26) and (27), and the assumptions for (7) at the beginning of this section: $\alpha_6, \alpha_9 \neq 0$ and $\alpha_8 > 0$. So $f_1(R, W)$, $f_2(R, W)$ can't be identically zero. Thus there is no need for considering the averaging theory of second order.

3. ZERO-HOPF BIFURCATIONS IN THE SYSTEM (1)

Recalling that we have assumed that $a_4 > 0$ in the introduction section. Thus system (1) has two equilibria, $E_1 = (x_1, 0, 0)$ and $E_2 = (x_2, 0, 0)$, where

$$x_1 = -\frac{a_1}{2a_4} + \frac{\sqrt{\Delta}}{2a_4}, \quad x_2 = -\frac{a_1}{2a_4} - \frac{\sqrt{\Delta}}{2a_4}, \quad (36)$$

where $\Delta = a_1^2 - 4a_0a_4$.

Proposition 2. There is a eight-parameter family of the system (1), for which the equilibria $E_{1,2}$ are both zero-Hopf equilibria. Namely

$$a_0 = \frac{a_1^2}{4a_4}, \quad a_3 = \frac{a_6a_1}{2a_4} \quad (37)$$

with

$$a_2 < \frac{a_1a_5}{2a_4}. \quad (38)$$

The two equilibria merge into a single equilibrium $(-\frac{a_1}{2a_4}, 0, 0)$ at the bifurcation, with the spectrum $\lambda_{1,2} = \pm i\sqrt{\frac{a_1a_5}{2a_4} - a_2}$, $\lambda_3 = 0$.

Proof. The Jacobian matrix at E_1 is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2a_4x_1 + a_1 & a_5x_1 + a_2 & a_6x_1 + a_3 \end{bmatrix},$$

whose characteristic polynomial is

$$P(\lambda) = \lambda^3 + g_2\lambda^2 + g_1\lambda + g_0,$$

where

$$g_0 = -2a_4x_1 - a_1, \quad g_1 = -a_5x_1 - a_2, \quad g_2 = -a_6x_1 - a_3.$$

By imposing the conditions for the zero-Hopf bifurcation, i.e., $g_0 = g_2 = 0$ and $g_1 > 0$, and after some tedious algebra, we obtain the conditions (37) and (38). The analysis for E_2 is similar, and we omit it. The rest part of this proposition can be verified by direct computation. Thus we complete the proof. \square

Set the parameters in system (1) as follows:

$$\begin{cases} a_0 = s_0\epsilon^2 + \frac{a_{1,0}^2}{4a_{4,0}}, \\ a_1 = s_1\epsilon^2 + a_{1,0}, \\ a_2 = s_2\epsilon + a_{2,0}, \\ a_3 = s_3\epsilon + \frac{a_{1,0}a_{6,0}}{2a_{4,0}}, \\ a_4 = s_4\epsilon^2 + a_{4,0}, \\ a_k = s_k\epsilon + a_{k,0}, \quad 5 \leq k \leq 9, \end{cases} \quad (39)$$

and

$$\begin{cases} s_0 < -\frac{a_{1,0}(a_{1,0}s_4 - 2a_{4,0}s_1)}{4a_{4,0}^2}, \\ a_{4,0} > 0, \\ a_{2,0} < \frac{a_{5,0}a_{1,0}}{2a_{4,0}}. \end{cases} \quad (40)$$

where $\epsilon > 0$ sufficiently small, $a_{i,0}, s_i, 0 \leq i \leq 9$ are all real parameters.

With the foregoing conditions, x_1 and x_2 become

$$x_1 = -\frac{a_{1,0}}{2a_{4,0}} + \frac{\eta\epsilon}{2a_{4,0}} + \mathcal{O}(\epsilon^2), \quad x_2 = -\frac{a_{1,0}}{2a_{4,0}} - \frac{\eta\epsilon}{2a_{4,0}} + \mathcal{O}(\epsilon^2), \quad (41)$$

where

$$\eta = \sqrt{2a_{1,0}s_1 - 4a_{4,0}s_0 - \frac{a_{1,0}^2s_4}{a_{4,0}}}. \quad (42)$$

It should be noted that the developing of $x_{1,2}$ up to ϵ^1 is sufficient for applying the averaging theory of first order. Let

$$\xi = \sqrt{2a_{1,0}a_{4,0}a_{5,0} - 4a_{2,0}a_{4,0}^2}. \quad (43)$$

Both ξ and η are positive due to the conditions (40).

3.1. The periodic solution arising from E_1 . Under conditions (39) and (40), we can use the transformation

$$\begin{cases} x = -\frac{4a_{4,0}^2}{\xi^2}v + w + x_1, \\ y = -\frac{2a_{4,0}}{\xi}u, \\ z = v, \end{cases} \quad (44)$$

which transforms system (1) to

$$\begin{cases} \frac{du}{dt} = -\delta v, \\ \frac{dv}{dt} = \delta u + \alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(\eta)\epsilon w + \dots, \\ \frac{dw}{dt} = \beta [\alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(\eta)\epsilon w] + \dots, \end{cases} \quad (45)$$

where

$$\delta = \sqrt{\frac{a_{5,0}a_{1,0}}{2a_{4,0}} - a_{2,0}}, \quad \beta = \frac{4a_{4,0}^2}{\xi^2}, \quad (46)$$

$$\alpha_1 = \frac{4a_{7,0}a_{4,0}^2}{\xi^2}, \quad (47)$$

$$\alpha_2 = -\frac{2a_{4,0}(\xi^2 a_{8,0} - 4a_{4,0}^2 a_{5,0})}{\xi^3}, \quad (48)$$

$$\alpha_3 = -\frac{2a_{5,0}a_{4,0}}{\xi}, \quad (49)$$

$$\alpha_4(\eta) = -\frac{a_{5,0}\eta - s_5 a_{1,0} + 2s_2 a_{4,0}}{\xi}, \quad (50)$$

$$\alpha_5 = \frac{a_{9,0}\xi^4 - 4a_{6,0}a_{4,0}^2\xi^2 + 16a_{4,0}^5}{\xi^4}, \quad (51)$$

$$\alpha_6 = \frac{a_{6,0}\xi^2 - 8a_{4,0}^3}{\xi^2}, \quad (52)$$

$$\alpha_7(\eta) = \frac{a_{6,0}\eta\xi^2 - 8\eta a_{4,0}^3 - s_6 a_{1,0}\xi^2 + 2s_3\xi^2 a_{4,0}}{2\xi^2 a_{4,0}}, \quad (53)$$

$$\alpha_8 = a_{4,0}, \quad (54)$$

$$\alpha_9(\eta) = \eta, \quad (55)$$

and we have dropped terms that are not used in the following analysis.

Now the system (45) is in the form of (7) with $\alpha_9(\eta) = \eta > 0$. Applying Theorem 2 to it, we get the following conclusions.

Theorem 4. *Let $\alpha_i = \alpha_i(\eta)$, $i = 4, 7, 9$ and assume that the conditions (30), (39), and (40) hold. Then for system (1), a periodic solution borns at E_1 when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:*

In case (a), the periodic solution is asymptotically stable;

In case (b), the periodic solution is unstable;

In case (c), the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

3.2. The periodic solution arising from E_2 . Under conditions (39) and (40), we can use the transformation

$$\begin{cases} x = -\frac{4a_{4,0}^2}{\xi^2}v + w + x_2, \\ y = -\frac{2a_{4,0}}{\xi}u, \\ z = v, \end{cases} \quad (56)$$

which transforms system (1) to

$$\begin{cases} \frac{du}{dt} = -\delta v, \\ \frac{dv}{dt} = \delta u + \alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(-\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(-\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(-\eta)\epsilon w + \dots, \\ \frac{dw}{dt} = \beta [\alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(-\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(-\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(-\eta)\epsilon w] + \dots \end{cases} \quad (57)$$

System (57) is in the form of (7) with $\alpha_9(-\eta) = -\eta < 0$. Applying Theorem 3 to it, we get the following conclusions.

Theorem 5. *Let $\alpha_i = \alpha_i(-\eta)$, $i = 4, 7, 9$ and assume that the conditions (30), (39), and (40) hold. Then for system (1), a periodic solution borns at E_2 when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:*

In case (d), the periodic solution is asymptotically stable;

In case (e), the periodic solution is unstable;

In case (f), the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

4. ZERO-HOPF BIFURCATION AND CHAOS FOR A CLASS OF SYSTEMS

From (37), we know that a_0 and a_3 are the parameters of the zero-Hopf bifurcations in system (1). Therefore we may choose $a_1 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$, $a_2 = -7$ and $a_4 = 1$. This yields the following system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = a_0 - 7y + a_3z + x^2, \end{cases} \quad (58)$$

where we may assume that $a_0 \leq 0$. It belongs to the class JD_2 , see [32, 33].

According to Proposition 2, zero-Hopf bifurcations occur simultaneously at

$$a_0 = 0, \quad a_3 = 0 \quad (59)$$

for the equilibria $E_1 = (\sqrt{-a_0}, 0, 0)$, $E_2 = (-\sqrt{-a_0}, 0, 0)$ of system (58). At this bifurcation point, system (58) is reduced to

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = -7y + x^2. \end{cases} \quad (60)$$

and the two equilibria merge into a single equilibrium $O(0, 0, 0)$. As in Section 3, we consider the following perturbed system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = a_0 + a_2 y + a_3 z + a_4 x^2, \end{cases} \quad (61)$$

where

$$\begin{cases} a_0 = s_0 \epsilon^2, \\ a_2 = s_2 \epsilon - 7, \\ a_3 = s_3 \epsilon, \\ a_4 = s_4 \epsilon^2 + 1, \\ s_0 < 0. \end{cases} \quad (62)$$

This is a special case of system (1), and the other parameters $a_1, a_5, a_6, a_7, a_8, a_9$ in the form of (39) are identically zero. As in Subsection 3.1, the involved quantities are

$$\begin{aligned} \eta &= 2\sqrt{-s_0}, \quad \xi = 2\sqrt{7}, \quad \delta = \sqrt{7}, \quad \beta = \frac{4}{\xi^2}, \\ \alpha_1 &= \alpha_2 = \alpha_3 = 0, \quad \alpha_4(\eta) = \alpha_4(-\eta) = -\frac{2s_2}{\xi}, \quad \alpha_5 = \frac{16}{\xi^4}, \quad \alpha_6 = -\frac{8}{\xi^2}, \\ \alpha_7(\eta) &= -\frac{4\eta}{\xi^2} + s_3, \quad \alpha_7(-\eta) = \frac{4\eta}{\xi^2} + s_3, \quad \alpha_8 = 1, \quad \alpha_9(\eta) = \eta, \quad \alpha_9(-\eta) = -\eta. \end{aligned}$$

With the conditions (62), the equilibria of system (61) become $E_1 = (x_1, 0, 0)$ and $E_2 = (x_2, 0, 0)$, where

$$x_1 = \frac{\eta \epsilon}{2} + O(\epsilon^2), \quad x_2 = -\frac{\eta \epsilon}{2} + O(\epsilon^2), \quad (63)$$

are changing with respect to ϵ . Applying the Theorems 4 and 5 to system (61), we get the following conclusions, respectively.

Theorem 6. *Let $\alpha_i = \alpha_i(\eta)$, $i = 4, 7, 9$ and assume that the conditions (30) and (62) hold. Then for system (61), a periodic solution borns at E_1 when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:*

In case (a_1) , the periodic solution is asymptotically stable;

In case (b_1) , the periodic solution is unstable;

In case (c_1) or (c_2) , the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

Theorem 7. *Let $\alpha_i = \alpha_i(-\eta)$, $i = 4, 7, 9$ and assume that the conditions (30) and (62) hold. Then for system (61), a periodic solution borns at E_2 when $\epsilon = 0$, and it exists for $\epsilon > 0$ sufficiently small. Moreover, the following statements hold:*

In case (d_1) , the periodic solution is asymptotically stable;

In case (e_1) , the periodic solution is unstable;

In case (f_1) or (f_3) , the periodic solution has a stable and an unstable manifolds, which locally are formed by two cylinders.

Consider the following system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = -\epsilon^2 - 7y - 0.129\epsilon z + x^2, \end{cases} \quad (64)$$

where $\epsilon \geq 0$ is a parameter. It contains only one control parameter and only one non-linearity.

The system has two equilibria $E_{1,2} = (\pm\epsilon, 0, 0)$. For them, zero-Hopf bifurcations occur at $\epsilon = 0$. Furthermore, Hopf bifurcation can not occur. By direct computation, we can check that the condition (a_1) in Theorem 6 is satisfied. Thus according to the theoretical expectations, there exists a stable periodic solution near the origin for ϵ small. This is shown in Fig. 1 for $\epsilon = 0.02$.

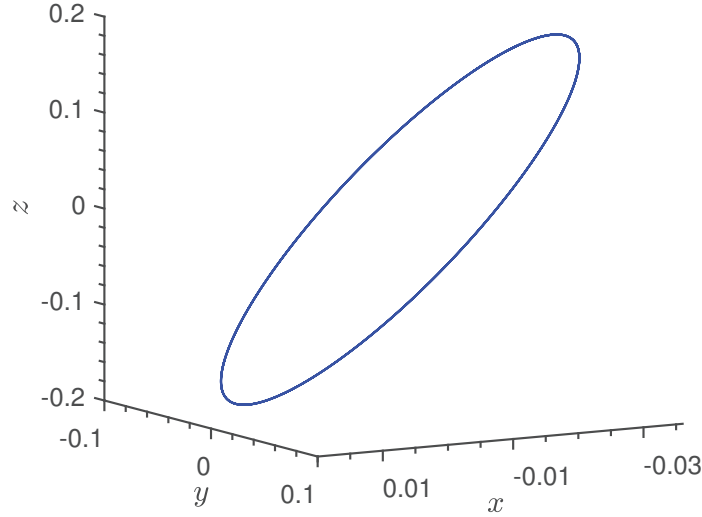


FIGURE 1. Periodic solution of system (64) when $\epsilon = 0.02$. Initial conditions are given by $x(0) = y(0) = z(0) = 0$

When $\epsilon = 7.75$, the system becomes chaotic. The Lyapunov exponents of the system are $L_1 = 0.27047, L_2 = 0, L_3 = -1.2703$, so that the Kaplan-Yorke dimension is $D_{KY} = 2.213$. It is a novel chaotic system, which is not reported in [32–34]. The largest Lyapunov exponent of it is relatively bigger than the known chaotic systems in the class JD_2 . See Fig. 2 for the chaotic attractor and Fig. 3(b) for the evolution of Lyapunov exponents versus ϵ .

In Fig. 3, the bifurcation diagram for x versus ϵ is displayed for the initial conditions $x(0) = y(0) = z(0) = 0$ on the interval $[0, 7.8]$. Starting from a zero-Hopf equilibrium, the system undergoes a zero-Hopf bifurcation at $\epsilon = 0$, period-doubling bifurcations at $\epsilon = 5.348, 6.014, 6.176, \dots$, and then becomes chaotic at about $\epsilon = 6.240$. Beyond this value there is a wide range of ϵ for which the eventual behaviour is chaotic. Several narrow periodic windows can be observed which are surrounded by regions of chaos. By the way, we find a narrow range $[6.02, 6.20]$ of the parameter ϵ in which there are coexisting attractors, see Fig. 4. In a word, despite the simplicity of system (64), it has rich dynamical behaviours: zero-Hopf bifurcations, periodic attractors, coexistence of attractors, and chaotic attractors depending on the parameter values. The pattern for transition to chaos is zero-Hopf bifurcations \mapsto periodic attractors \mapsto period doubling cascades \mapsto chaos.

5. CONCLUSIONS

In this work, we study zero-Hopf bifurcations of the general quadratic jerk system. The system has ten parameters, in which one parameter is assumed to be positive. In order to apply the averaging theory of first order, we consider perturbations of the values of the parameters for which the jerk system has a zero-Hopf equilibrium. Thus we obtain the stability conditions for the periodic solutions bifurcated from zero-Hopf bifurcations. These conditions can help to understand the formulating mechanism of chaotic attractors in the jerk system. As an application, we study zero-Hopf bifurcations of some quadratic jerk systems. The obtained

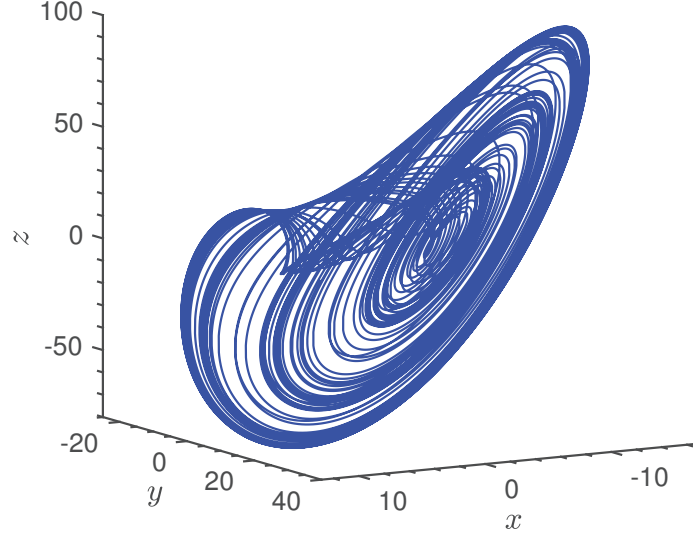


FIGURE 2. Chaotic attractor of system (64) when $\epsilon = 7.75$. Initial conditions are given by $x(0) = y(0) = z(0) = 0$

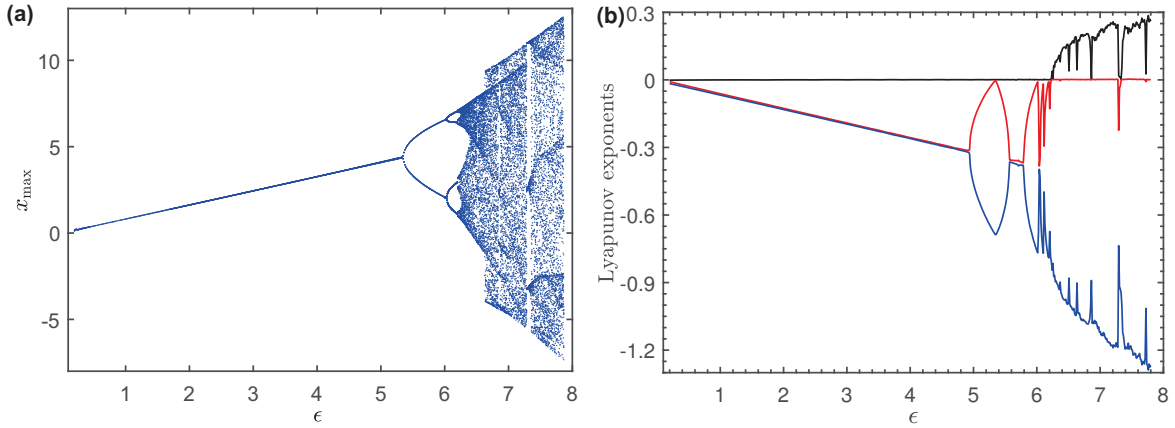


FIGURE 3. (a) Bifurcation diagram of system (64): maximum values of x versus ϵ on the interval $[0, 7.8]$. The initial conditions are $x(0) = y(0) = z(0) = 0$. (b) The corresponding Lyapunov exponents depicted as functions of ϵ on the same interval

results are applied to a one-parameter jerk system, for which period doubling cascade, coexisting attractors and chaos with narrow periodic windows are found on varying the parameter.

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COMPLIANCE WITH ETHICAL STANDARDS

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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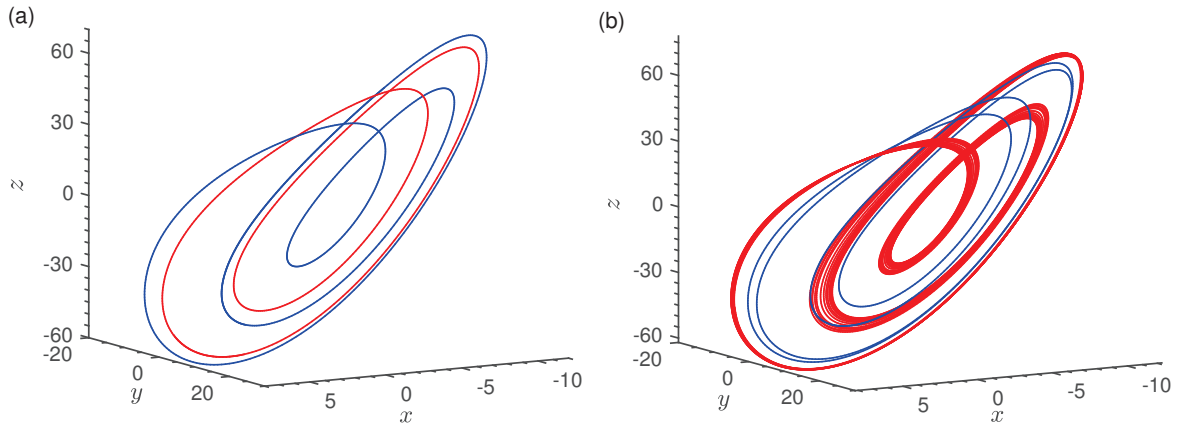


FIGURE 4. The blue attractor is generated from the initial conditions $(0, 0, 0)$, and the red one from the initial conditions $(-0.53, -0.64, 0.22)$. (a) Coexistence of two periodic attractors with $\epsilon = 6.02$. (b) Coexistence of a periodic attractor and a chaotic attractor with $\epsilon = 6.180$. The Lyapunov exponents of the chaotic attractor are $L_1 = 0.054637, L_2 = 0, L_3 = -0.85183$, so that the Kaplan-Yorke dimension is $D_{KY} = 2.0641$

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