A qualitative analysis of an almost-periodic Beddington-DeAngelis predator-prey system with mutual interference and time delay☆

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Abstract

Of concern is the existence and uniqueness of a predator-prey model with Beddington-DeAngelis functional response and mutual interference. By constructing a suitable Lyapunov function and using the comparison theorem of ordinary differential equation, we prove that the existence, permanence and uniqueness of a positive globally attractive almost periodic solution of the model. A more general and frequent example will be offered to describe our main theorem.

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1. Introduction

With the development of modern agriculture, ecological agriculture has become an inevitable trend of agricultural development. The construction of ecological agriculture is to combine the development of agriculture with the improvement of the environment and form a virtuous circle of mutual promotion and coordinated development. Population ecology is a science that studies the relationship between population and environment. It is also one of the basic theories of environmental ecology. The emphasis of the study are the law of spatial distribution, quantity dynamics of population and its regulatory mechanism. It is the link between different levels of modern ecology. Population is the basic unit of species existence, the basic component of biocenosis and the basis of ecosystem research. Since the 1950s, thanks to the investigation of population mathematical systems, population ecology have been further developed and formed a new branch of population mathematical ecology.

Population density is a significant factor to adjust the balance between species. Meanwhile, interspecific adjustment, the restriction process of predation, parasitism and interspecific competition for common resource factors on population density, is one of the contents of density adjustment. The competitive relationships between different populations, or interspecific interactions, are extremely complex. It can generally be divided into two categories: one is mutually beneficial, called positive interaction; the other is antagonistic, called negative interaction. Mutualism and predator-prey relationships are extreme cases of positive and negative interaction, respectively. There are also various transitional types between these two extreme types. For example, Guan and Chen [31], in 2019, have studied a two species amensalism system with weak Allee effect and Beddington-DeAngelis

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functional response

$$\begin{cases} \dot{x}(t) = x\left(a_1 - b_1 x - \dfrac{cy}{1+mx+ny}\right), \\ \dot{y}(t) = y\left(a_2 - b_2 y\right)\dfrac{y}{u+y}. \end{cases} \tag{1.1}$$

In 1924-1926, Bohr [4, 5, 6] has established the theory about almost periodic function (APF) systematically. During the immediate decade, following Bohr's research, numerous significant works were finished to APF. We refer researchers to van Kampen [7], Bochner [8, 9] and von Neumann [10]. Almost periodic differential equations (APDEs) can be founded in various fields to characterize some phenomena such as celestial mechanics, mechanical vibration, electric or ecology system, engineering technology and so on. In view of its extensive applications from science to engineering, APDEs has been developed rapidly during the past three decades. Despite a lot of works devoted to the qualitative properties of periodic solutions (see [21]), but the study of almost periodic solutions can obtain a more general and extensive application in real world because of the different time-dependent coefficients in time period. As we know, the traditional tools of solving the qualitative problems of periodic model cannot be used to solve the same problems of almost periodic issues due to the compactness of operator. Furthermore, some results are obtained in recent decades, but there have still many unresolved problems, some of them were not even mentioned in literatures. Therefore, we claim that it will be significative to begin the investigation of almost periodic differential equations. In the field of biological dynamic, several useful researches on the APDEs have been published such as hematopoiesis system [22, 23, 24, 25, 26, 27], cellular neural networks [28, 29] and so on. During the past two decades, many researchers pay more attention to the basic theory of almost periodic function. Specially, Bright [34] has developed tight estimates theory for general averaging and applied to APDEs. Furthermore, Liu and Wang [32] extended Favard separation method, a significant approach to investigate almost periodic solutions of linear differential equations, to stochastic differential equations. In addition, Campos and Tarallo [33] have studied asymptotic dichotomies of APDEs and extended to higher dimensions.

What we mainly concern in this article is the qualitative properties of a predator-prey system with Beddington-DeAngelis functional response

$$\frac{k_i(t)x(t)y(t)}{a(t)+c(t)x(t)+d(t)y(t)}. \tag{1.2}$$

The earliest introduction of this type of functional response was offered by two biologists, Beddington [11] and DeAngelis [12], respectively at the same time. As we know, there exist several famous predator-prey model such as Michaelis-Menten-type models (so called ratio-dependent models) [17, 18], Lotka-Volterra models [19, 20] and so on. Moreover, Holling-type II models

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{k_1(t)x(t)y(t)}{a(t)+c(t)x(t)}, \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{k_2(t)x(t)y(t)}{a(t)+c(t)x(t)}, \end{cases} \tag{1.3}$$

similar to Beddington-DeAngelis models, have attracted more and more discussion.

Since one of most significant subjects in ecosystem dynamics is the properties of almost periodic solutions. The existence, uniqueness and stability of almost periodic solutions have been pay more attention by mathematicians. The readers can refer to [13, 14, 15, 16] and the references cited therein. To the best of our knowledge, almost no one is concerned about the qualitative properties of almost periodic solution of a Beddington-DeAngelis predator-pry model. In this paper, stimulated by above statements, we will investigate an almost periodic predator-prey model with Beddington-DeAngelis functional response and mutual interference as follows, which is the generalization of the system (1.3)

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{k_1(t)x(t)y(t)}{a(t) + c(t)x(t) + d(t)y(t)}, \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{k_2(t)x(t)y(t)}{a(t) + c(t)x(t) + d(t)y(t)}. \end{cases} \tag{1.4}$$

Here $x(t)$ and $y(t)$, respectively, denote the population density of the prey and predator at time t. As far as we can survey, this is the pioneering work to investigate the qualitative properties of almost-periodic solution for a Beddington-DeAngelis predator-prey system with mutual interference and time delay.

Throughout the entire article, we suppose that the following conditions hold:

(H_1) The biological coefficients $a(t)$, $c(t)$, $d(t)$ $b_i(t)$, $k_i(t)$ and $r_i(t)$ $(i = 1, 2)$ are almost periodic continuous functions;

(H_2) all the parameters of the almost periodic model (1.4) satisfy the following two conditions:

$$\max_{i=1,2}\{a^u, c^u, d^u, b_i^u, k_i^u, r_i^u\} < +\infty, \tag{1.5}$$

$$\min_{i=1,2}\{a^l, c^l, d^l, b_i^l, k_i^l, r_i^l\} > 0. \tag{1.6}$$

The organization of the rest part of this article is as follows: Section 2 contains some lemmas and the existence theorem of almost periodic solutions of (1.4). We offer the proof of the uniqueness of almost periodic solutions of (1.4) in Section 3. Section 4 discuss the situation of time delay. In Section 5, we offer two examples to describe the applicability of our main results. The last section will provide some remarks and the orientations of investigation.

2. Permanence

This section is mainly focused on some preliminary results and lemmas which will be applied in what follows.

Lemma 2.1 (Lemma 2.1, [1]). *Both the positive cone $\mathbb{R}_+^2 = \{(x, y)|x > 0, y > 0\}$ and the nonnegative cone $\mathbb{R}_\star^2 = \{(x, y)|x \geq 0, y \geq 0\}$ of $\mathbb{R}^2$ are invariant with respect to (1.4).*

Lemma 2.2. *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq) x(a - bx)$, when $t \geq 0$ and $x(0) > 0$, we deduce*

$$\liminf_{t\to\infty} x(t) \geq \frac{a}{b} \left(\limsup_{t\to\infty} x(t) \leq \frac{a}{b} \right). \tag{2.1}$$

Proof. The proof of Lemma 2.2 is similar to that of the proof of Lemma 2.2 of [2], and we omit the detail here. □

Theorem 2.1. *Assume that model (1.4) fulfills the following two conditions:*

(H_1) $r_1^l > \frac{k_1^u M_2}{a^l}$,

(H_2) $r_2^u < \frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2}$.

Then model (1.4) possesses permanence which implies that any positive solution $(x(t), y(t))$ to the model (1.4) fulfills

$$\begin{aligned} 0 < m_1 \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le M_1, \\ 0 < m_2 \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le M_2. \end{aligned} \tag{2.2}$$

Proof. Based on the first equation of the model (1.4), we deduce that:

$$\dot{x}(t) \le x(t)(r_1^u - b_1^l x(t)). \tag{2.3}$$

Employing Lemma 2.2 into (2.3), we obtain

$$\limsup_{t\to\infty} x(t) \le \frac{r_1^u}{b_1^l} = M_1. \tag{2.4}$$

According to (2.4), there possesses a positive constant T_1 large enough for enough small constant $\varepsilon > 0$ such that

$$x(t) \le M_1 + \varepsilon \tag{2.5}$$

for all $t \ge T_1$. We denote $y(t) = \frac{1}{u(t)} > 0$. Applying the second equation of the model (1.4) and (2.5) leads to

$$\begin{aligned} \dot{u}(t) =& u(t)\left(r_2(t) + b_2(t)\frac{1}{u(t)}\right) - \frac{k_2(t)x(t)u^2(t)}{a(t)u(t) + c(t)x(t)u(t) + d(t)} \\ \ge& u(t)\left(b_2(t)\frac{1}{u(t)} - \frac{k_2(t)x(t)u(t)}{a(t)u(t) + c(t)x(t)u(t) + d(t)}\right) \\ \ge& u(t)\left(b_2^l\frac{1}{u(t)} - \frac{k_2^u(M_1+\varepsilon)}{a^l + c^l(M_1+\varepsilon)}\right) \end{aligned} \tag{2.6}$$

for any $t \ge T_1$. If we set $\varepsilon \to 0$ in (2.6), we have

$$\dot{u}(t) \ge u(t)\left(b_2^l\frac{1}{u(t)} - \frac{k_2^u M_1}{a^l + c^l M_1}\right). \tag{2.7}$$

Therefore, based on the transformation $y(t) = \frac{1}{u(t)}$, we get

$$\dot{y}(t) = -\frac{\dot{u}(t)}{u^2(t)} \le -\frac{1}{u(t)}\left(b_2^l\frac{1}{u(t)} - \frac{k_2^u M_1}{a^l + c^l M_1}\right) = y(t)\left(\frac{k_2^u M_1}{a^l + c^l M_1} - b_2^l y(t)\right). \tag{2.8}$$

Using Lemma 2.2 to (2.8) leads to

$$\limsup_{t\to\infty} y(t) \le \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)} = M_2. \tag{2.9}$$

Based on the last inequality, there possesses a $T_2 > T_1$ such that

$$y(t) \le M_2 + \varepsilon \tag{2.10}$$

for above small enough constant $\varepsilon > 0$. From the first equation of system (1.4), we obtain

$$\dot{x}(t) \ge x(t)(r_1^l - b_1^u x(t)) - \frac{k_1^u x(t)(M_2+\varepsilon)}{a^l} = x(t)\left(\left(r_1^l - \frac{k_1^u(M_2+\varepsilon)}{a^l}\right) - b_1^u x(t)\right). \tag{2.11}$$

If we set $\varepsilon \to 0$ in (2.11), we have

$$\dot{x}(t) \geq x(t)\left(\left(r_1^l - \frac{k_1^u M_2}{a^l}\right) - b_1^u x(t)\right). \tag{2.12}$$

Based on (H_1) and Lemma 2.2 to (2.12), we derive that:

$$\liminf_{t\to\infty} x(t) \geq \frac{r_1^l - \frac{k_1^u M_2}{a^l}}{b_1^u} = \frac{r_1^l}{b_1^u} - \frac{k_1^u}{a^l b_1^u}\cdot\frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)} = m_1. \tag{2.13}$$

Based on (2.13), for above enough small constant $\varepsilon > 0$, there possesses a $T_3 \geq T_2$ such that

$$x(t) \geq m_1 - \varepsilon \tag{2.14}$$

for all $t \geq T_3$. Hence, by setting the estimates (2.10) and (2.14) into the second equation of (1.4), we have

$$\begin{aligned}\dot{y}(t) =& y(t)\left[(-r_2(t) - b_2(t)y(t)) + \frac{k_2(t)x(t)}{a(t)+c(t)x(t)+d(t)y(t)}\right]\\ \geq& y(t)\left[\left(\frac{k_2^l(m_1-\varepsilon)}{a^u + c^u(M_1+\varepsilon)+d^u(M_2+\varepsilon)} - r_2^u\right) - b_2^u y(t)\right].\end{aligned} \tag{2.15}$$

If we take the limit by setting $\varepsilon \to 0$, we deduce that:

$$\dot{y}(t) \geq y(t)\left[\left(\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} - r_2^u\right) - b_2^u y(t)\right]. \tag{2.16}$$

Based on (H_2) and Lemma 2.2 to (2.16), we derive that:

$$\liminf_{t\to\infty} y(t) \geq \frac{\frac{k_2^l m_1}{a^u+c^u M_1+d^u M_2} - r_2^u}{b_2^u} = \frac{k_2^l m_1}{b_2^u[a^u + c^u M_1 + d^u M_2]} - \frac{r_2^u}{b_2^u} = m_2. \tag{2.17}$$

According to (2.17), there possesses a $T_4 \geq T_3$ which can guarantee that

$$y(t) \geq m_2 - \varepsilon \tag{2.18}$$

for above $\varepsilon > 0$ and for all $t \geq T_4$.

Together with Eqs. (2.4), (2.9), (2.13) and (2.17), we can draw a conclusion that system (1.4) possesses permanence under the hypothesis of the Theorem 2.1. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Assume that $\mathcal{S}$ stand for the set of all solutions $\mathbf{w}(t) = (x(t), y(t))^T$ of (1.4) on $\mathbb{R}$ fulfilling $m_1 \leq x(t) \leq M_1$, $m_2 < y(t) < M_2$ for $t \in \mathbb{R}$. Then $\mathcal{S} \neq \varnothing$.*

Proof. According to the theory of almost periodic function, there possesses a sequence $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that

$$\begin{aligned}&a(t+t_n) \to a(t),\ c(t+t_n) \to c(t),\ d(t+t_n) \to d(t),\\ &b_i(t+t_n) \to b_i(t),\ k_i(t+t_n) \to k_i(t),\ r_i(t+t_n) \to r_i(t),\ i = 1, 2,\end{aligned} \tag{2.19}$$

uniformly on $\mathbb{R}$ as $n \to \infty$. Assume that $w(t)$ is a solution of system (1.4) fulfilling $m_1 \leq x(t) \leq M_1$, $m_2 < y(t) < M_2$ for $t > \mathbb{T}$. It is obvious that the sequence $\mathbf{w}(t+t_n) = (x(t+t_n), y(t+t_n))^T$ is equicontinuous and uniformly bounded on each bounded subset of $\mathbb{R}$. Applying Ascoli's theorem leads to

$$\lim_{k\to\infty} \mathbf{w}(t+t_k) = \lim_{k\to\infty} (x(t+t_k), y(t+t_k))^T = \mathbf{z}(t) = (z_1(t), z_2(t)), \tag{2.20}$$

where $\mathbf{w}(t+t_k)$ represents a subsequence of $\mathbf{w}(t+t_n)$ uniformly on each bounded subset, $\mathbf{z}(t)$ stands for a continuous function. For any given $\mathbb{T}_1 \in \mathbb{R}$. For all positive integer n, we can suppose that $t_k + \mathbb{T}_1 \geq \mathbb{T}$. Therefore, if $t \geq 0$, we obtain

$$\begin{aligned} x(t+t_k+\mathbb{T}_1) - x(t_k+\mathbb{T}_1) = \int_{\mathbb{T}_1}^{t+\mathbb{T}_1} & x(s+t_k)\,(r_1(s+t_k) - b_1(s+t_k)x(s+t_k) \\ & - \frac{k_1(s+t_k)y(s+t_k)}{a(s+t_k)+c(s+t_k)x(s+t_k)+d(s+t_k)y(s+t_k)}\bigg), \\ y(t+t_k+\mathbb{T}_1) - y(t_k+\mathbb{T}_1) = \int_{\mathbb{T}_1}^{t+\mathbb{T}_1} & y(s+t_k)\,(-r_2(s+t_k) - b_2(s+t_k)y(s+t_k) \\ & + \frac{k_2(s+t_k)x(s+t_k)}{a(s+t_k)+c(s+t_k)x(s+t_k)+d(s+t_k)y(s+t_k)}\bigg). \end{aligned} \tag{2.21}$$

If we let $n \to \infty$ in (2.19), together with Lebesgue's dominated convergence theorem, we derive that:

$$\begin{aligned} z_1(t+\mathbb{T}_1) - z_1(\mathbb{T}_1) &= \int_{\mathbb{T}_1}^{t+\mathbb{T}_1} z_1(s)\left(r_1(s) - b_1(s)z_1(s) - \frac{k_1(s)z_2(s)}{a(s)+c(s)z_1(s)+d(s)z_2(s)}\right), \\ z_2(t+\mathbb{T}_1) - z_2(\mathbb{T}_1) &= \int_{\mathbb{T}_1}^{t+\mathbb{T}_1} z_2(s)\left(-r_2(s) - b_2(s)z_2(s) + \frac{k_2(s)z_1(s)}{a(s)+c(s)z_1(s)+d(s)z_2(s)}\right) \end{aligned} \tag{2.22}$$

for all $t \geq 0$. $\mathbb{T}_1 \in \mathbb{R}$ is selected randomly. For this reason, system (1.4) possesses a solution $\mathbf{z}(t) = (z_1(t), z_2(t))$ on $\mathbb{R}$. It is obvious that $m_1 \leq x(t) \leq M_1$, $m_2 < y(t) < M_2$ for $t \in \mathbb{R}$. Therefore, $\mathbf{z}(t) \in \mathcal{S}$. This ends the proof of Theorem 2.1. □

3. Existence and uniqueness of almost periodic solution of system (1.4)

In this section, we focus on the existence of a unique almost periodic solution of system (1.4). At first, we introduce the accurate definition of almost periodic function.

Definition 3.1 (Definition 1.1, [3]). *A function $f(t)$ is termed to be almost periodic (Bohr) if for any given $\varepsilon > 0$, the set*

$$\mathcal{T}(f,\varepsilon) = \{\tau; |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\} \tag{3.1}$$

is relatively dense, i.e., it is possible to discover an ε-translation constant $l = l(\varepsilon) > 0$ such that there possesses an ε-translation number $\tau = \tau(\varepsilon) \in \mathcal{T}(f,\varepsilon)$ in any interval with length $l(\varepsilon)$ such that

$$|f(t+\tau) - f(t)| < \varepsilon \tag{3.2}$$

is fulfilled for any $t \in \mathbb{R}$.

Theorem 3.1. *Assume that all conditions of Theorem 2.1 are hold and further that the parameters of (1.4) fulfill the under-mentioned conditions:*

(C_1) $b_1^l > \frac{k_1^u c^u M_2 + a^u k_2^u}{(a^l)^2}$;

(C_2) $b_2^l > \frac{k_1^u d^u M_2}{(a^l)^2} + \frac{k_1^u}{a^l}$.

If $\mathbf{w}_1(t) = (x_1(t), y_1(t))^T$ and $\mathbf{w}_2(t) = (x_2(t), y_2(t))^T$ stand for, respectively, two positive solutions of (1.4), we obtain

$$\lim_{t\to\infty} |\mathbf{w}_1(t) - \mathbf{w}_2(t)| = \mathbf{0}. \tag{3.3}$$

Proof. We assume that system (1.4) possesses two positive solutions $\mathbf{w}_1(t) = (x_1(t), y_1(t))^T$ and $\mathbf{w}_2(t) = (x_2(t), y_2(t))^T$. In view of (C_1), there possesses a small enough positive constant ε guarantee that

$$\begin{aligned} &m_1 - \varepsilon < x_1 < M_1 + \varepsilon,\ m_1 - \varepsilon < x_2 < M_1 + \varepsilon, \\ &m_2 - \varepsilon < y_1 < M_2 + \varepsilon,\ m_2 - \varepsilon < y_2 < M_2 + \varepsilon. \end{aligned} \tag{3.4}$$

We focus on the upper right derivatives of

$$\mathcal{V}_1(t) = \left| \ln \frac{x_1(t)}{x_2(t)} \right|. \tag{3.5}$$

It is obvious by a direct computation with (3.4) that

$$\begin{aligned}
&D^+\mathcal{V}_1(t) \\
=&-|x_1(t) - x_2(t)| b_1(t) - \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_1(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_2(t)}{a(t) + c(t)x_2(t) + d(t)y_2(t)} \right) k_1(t) \\
=&-|x_1(t) - x_2(t)| b_1(t) + \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_2(t)}{a(t) + c(t)x_2(t) + d(t)y_2(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_2(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} \right) k_1(t) \\
&+ \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_2(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t)) y_1(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} \right) k_1(t) \\
\le&-|x_1(t) - x_2(t)| b_1(t) + \frac{k_1(t)c(t)y_2(t)}{a(t)(a(t) + c(t)x_1(t) + d(t)y_1(t))} |x_1(t) - x_2(t)| \\
&+ \frac{k_1(t)d(t)y_2(t)}{a(t)(a(t) + c(t)x_1(t) + d(t)y_1(t))} |y_1(t) - y_2(t)| + \frac{k_1(t)}{a(t)} |y_2(t) - y_1(t)| \\
=&-\left(b_1(t) - \frac{k_1(t)c(t)y_2(t)}{a(t)(a(t) + c(t)x_1(t) + d(t)y_1(t))} \right) |x_1(t) - x_2(t)| \\
&+ \left(\frac{k_1(t)d(t)y_2(t)}{a(t)(a(t) + c(t)x_1(t) + d(t)y_1(t))} + \frac{k_1(t)}{a(t)} \right) |y_1(t) - y_2(t)|.
\end{aligned} \tag{3.6}$$

By the same method, we define

$$\mathcal{V}_2(t) = \left| \ln \frac{y_1(t)}{y_2(t)} \right|. \tag{3.7}$$

It is obvious by a direct computation with (3.4) that

$$\begin{aligned}
&D^+\mathcal{V}_2(t) \\
=&-|y_1(t) - y_2(t)| b_2(t) + \left(\frac{\operatorname{sgn}(y_1(t) - y_2(t)) x_1(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} - \frac{\operatorname{sgn}(y_1(t) - y_2(t)) x_2(t)}{a(t) + c(t)x_2(t) + d(t)y_2(t)} \right) k_2(t) \\
\le& k_2(t)\operatorname{sgn}(y_1(t) - y_2(t)) \frac{(a(t) + d(t)y_2(t))(x_1(t) - x_2(t)) + d(t)x_2(t)(y_2(t) - y_1(t))}{[a(t) + c(t)x_1(t) + d(t)y_1(t)][a(t) + c(t)x_2(t) + d(t)y_2(t)])} \\
&-|y_1(t) - y_2(t)| b_2(t) \\
\le&-b_2(t)|y_1(t) - y_2(t)| + \frac{k_2|x_1(t) - x_2(t)|}{a(t) + c(t)x_1(t) + d(t)y_1(t)}.
\end{aligned} \tag{3.8}$$

Now we introduce the following function

$$\mathcal{V}(t)=\mathcal{V}_1(t)+\mathcal{V}_2(t)=\left|\ln\frac{x_1(t)}{x_2(t)}\right|+\left|\ln\frac{y_1(t)}{y_2(t)}\right|. \tag{3.9}$$

Thus, it is known via a simple calculation that

$$\begin{aligned}D^+\mathcal{V}(t)\leq&-\left(b_1(t)-\frac{k_1(t)c(t)y_2(t)}{a(t)(a(t)+c(t)x_1(t)+d(t)y_1(t))}-\frac{k_2}{a(t)+c(t)x_1(t)+d(t)y_1(t)}\right)|x_1(t)-x_2(t)|\\&-\left(b_2(t)-\frac{k_1(t)d(t)y_2(t)}{a(t)(a(t)+c(t)x_1(t)+d(t)y_1(t))}-\frac{k_1(t)}{a(t)}\right)|y_1(t)-y_2(t)|\\\leq&-\left(b_1^l-\frac{k_1^uc^u(M_2+\varepsilon)+k_2^ua^u}{(a^l)^2}\right)|x_1(t)-x_2(t)|-\left(b_2^l-\frac{k_1^ud^u(M_2+\varepsilon)}{(a^l)^2}-\frac{k_1^u}{a^l}\right)|y_1(t)-y_2(t)|\end{aligned} \tag{3.10}$$

In view of conditions (C_1), we can effortlessly deduce that there possesses an small enough $\varepsilon>0$ guarantee

$$\begin{aligned}\rho_1(\varepsilon)&=b_1^l-\frac{k_1^uc^u(M_2+\varepsilon)+k_2^ua^u}{(a^l)^2}>\varepsilon,\\\rho_2(\varepsilon)&=b_2^l-\frac{k_1^ud^u(M_2+\varepsilon)}{(a^l)^2}-\frac{k_1^u}{a^l}>\varepsilon.\end{aligned} \tag{3.11}$$

Therefore, we discover that there exists two constants $\varepsilon>0$ and $\alpha(\varepsilon)>0$ such that

$$D^+\mathcal{V}(t)+\varepsilon|x_1(t)-x_2(t)|+\alpha(\varepsilon)|y_1(t)-y_2(t)|\leq 0 \tag{3.12}$$

Integrating the last inequality from $\mathbb{T}$ to t, one gets

$$\mathcal{V}(t)+\varepsilon|x_1(t)-x_2(t)|+\alpha(\varepsilon)|y_1(t)-y_2(t)|<\mathcal{V}(\mathbb{T})<+\infty. \tag{3.13}$$

Thus,

$$\begin{aligned}\limsup_{t\to\infty}\int_{\mathbb{T}}^t|x_1(s)-x_2(s)|ds&<\frac{\mathcal{V}(\mathbb{T})}{\varepsilon}<+\infty,\\\limsup_{t\to\infty}\int_{\mathbb{T}}^t|y_1(s)-y_2(s)|ds&<\frac{\mathcal{V}(\mathbb{T})}{\alpha(\varepsilon)}<+\infty.\end{aligned} \tag{3.14}$$

That means

$$\lim_{t\to\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to\infty}|y_1(t)-y_2(t)|=0. \tag{3.15}$$

This ends the proof of Theorem 3.1. □

Theorem 3.2. *Assume that all conditions proposed by Theorem 3.1 hold. Then there possesses a unique almost periodic solution (APS) of (1.4).*

Proof. In view of Theorem 2.2, there possesses a positive bounded solution $\mathbf{v}(t)=(v_1(t),v_2(t))^T$ for $t\geq 0$. Let us denote that $\mathbf{v}(t)$ stand for a solution of system (1.4). Thus, we can construct a sequence $\{t_{k_s}\}$, $\lim\limits_{k_s\to\infty}t_{k_s}=\infty$, to guarantee that $\mathbf{v}(t+t_{k_s})=(v_1(t+t_{k_s}),v_2(t+t_{k_s}))^T$ represents a solution of the undermentioned model:

$$\begin{cases}\dot{x}(t)=x(t)(r_1(t+t_{k_s})-b_1(t+t_{k_s})x(t))-\dfrac{k_1(t+t_{k_s})x(t)y(t)}{a(t+t_{k_s})+c(t+t_{k_s})x(t)+d(t+t_{k_s})y(t)},\\\dot{y}(t)=y(t)(-r_2(t+t_{k_s})-b_2(t+t_{k_s})y(t))+\dfrac{k_2(t+t_{k_s})x(t)y(t)}{a(t)+c(t+t_{k_s})x(t)+d(t+t_{k_s})y(t)},\end{cases} \tag{3.16}$$

Together with Theorem 2.1, we can discover that $\dot{v}_1(t+t_{k_s})$ and $\dot{v}_2(t+t_{k_s})$ are uniformly bounded too. Hence, $v_1(t+t_{k_s})$ and $v_2(t+t_{k_s})$ are equicontinuous. Then there, together with Ascoli's theorem, possess two subsequence $\{v_1(t+t'_{k_s})\} \subseteq \{v_1(t+t_{k_s})\}$ and $\{v_2(t+t'_{k_s})\} \subseteq \{v_2(t+t_{k_s})\}$ which can guarantee that there have two positive constant M, N and $K(\epsilon)$ for any enough small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with the property that when k_s, M, $N > K(\epsilon)$, one gets

$$|v_1(t+t_M) - v_1(t+t_{k_s})| < \varepsilon_1, \ |v_2(t+t_N) - v_2(t+t_{k_s})| < \varepsilon_2. \tag{3.17}$$

By setting $\epsilon = \max\{\varepsilon_1, \varepsilon_2\}$, we obtain that $v_1(t)$ and $v_2(t)$ represent asymptotically APFs. Moreover, $v_1(t)$ (or $v_2(t)$), defined on $\mathbb{R}$, are the sun function of $A_1(t+t_k)$ (or $A_2(t+t_k)$) and $B_1(t+t_k)$ (or $B_2(t+t_k)$), where $A_1(t+t_k)$ (or $A_2(t+t_k)$) is a continuous function and $B_1(t+t_k)$ (or $B_2(t+t_k)$) is an APF, such that for all $t \in \mathbb{R}$,

$$v_1(t+t_k) = A_1(t+t_k) + B_1(t+t_k), \ v_2(t+t_k) = A_2(t+t_k) + B_2(t+t_k), \tag{3.18}$$

where

$$\lim_{k\to\infty} A_1(t+t_k) = 0 = \lim_{k\to\infty} A_2(t+t_k), \ \lim_{k\to\infty} B_1(t+t_k) = B_1(t), \ \lim_{k\to\infty} B_2(t+t_k) = B_2(t). \tag{3.19}$$

Hence, $B_i(t)$ $(i = 1, 2)$ stand for two APFs. It shows that for $i = 1, 2$, $\lim\limits_{k\to\infty} \dot{v}_i(t+t_k) = B_i(t)$. Furthermore,

$$\begin{aligned}
\lim_{k\to\infty} \dot{v}_i(t+t_k) &= \lim_{k\to\infty}\lim_{\sigma\to 0} \frac{v_i(t+t_k+\sigma) - v_i(t+t_k)}{\sigma} = \lim_{\sigma\to 0}\lim_{k\to\infty} \frac{v_i(t+t_k+\sigma) - v_i(t+t_k)}{\sigma} \\
&= \lim_{\sigma\to 0} \frac{v_i(t+\sigma) - v_i(t)}{\sigma} = \lim_{\sigma\to 0} \frac{B_i(t+\sigma) - B_i(t)}{\sigma}
\end{aligned} \tag{3.20}$$

What we can deduce is that for $i = 1, 2$ the limit $\dot{B}_i(t)$ exist. In the rest part of this proof, we will deduce that $(B_1(t), B_2(t))^T$ represents an APS of (1.4). Based on (2.19), together with the statement proposed by Theorem 2.2, it is effortless to check that $\lim\limits_{n\to\infty} v_i(t+t_n) = B_i(t)$ for $i = 1, 2$. Hence,

$$\begin{aligned}
\dot{B}_1(t) &= \lim_{n\to\infty} \dot{v}_1(t+t_n) \\
&= \lim_{n\to\infty} v_1(t+t_n) \Big[(r_1(t+t_n) - b_1(t+t_n)v_1(t+t_n)) \\
&\quad - \frac{k_1(t+t_n)v_2(t+t_n)}{a(t+t_n) + c(t+t_n)v_1(t+t_n) + d(t+t_n)v_2(t+t_n)}\Big] \\
&= B_1(t)\left[(r_1(t) - b_1(t)B_1(t)) - \frac{k_1(t)B_2(t)}{a(t) + c(t)B_1(t) + d(t)B_2(t)}\right].
\end{aligned}$$

By the same method, we have

$$\dot{B}_2(t) = B_2(t)(-r_2(t) - b_2(t)B_2(t)) + \frac{k_2(t)B_1(t)B_2(t)}{a(t) + c(t)B_1(t) + d(t)B_2(t)}.$$

It means that $(B_1(t), B_2(t))^T$ fulfilled (1.4) and is a positive APS. Then there possesses a unique APS of (1.4) together with Theorem 3.1. This ends the proof of Theorem 3.2. □

Remark 3.1. *Suppose that $d(t) \equiv 0$, then system (1.4), equivalent to (1.3), represents an almost periodic Holling-type II predator-prey reaction-diffusion model. We can discover that all results established by this article can be also applied to Holling-type II system.*

4. Time delay system

Recently, Zhang et al. [35] investigated an interesting system with impulsive effects and time delays

$$\begin{cases} \dfrac{dx_i(t)}{dt} = x_i(t)\left[a_i - b_i(t)x_i(t-\tau_i(t)) + \displaystyle\sum_{j=1, j\neq i}^{n} c_{ij}(t)\frac{x_j(t)}{1+x_j(t)}\right],\ t \neq t_k, \\ x_i(t_k^+) = (1+h_{ik})x_i(t_k),\ k \in \mathbb{Z}^+,\ i = 1, 2, \cdots, n, \end{cases}$$

under initial conditions

$$x_i(\eta) = \psi_i(\eta),\ \eta \in [-\tau, 0],\ \psi(\eta) \in C([-\tau, 0], \mathbb{R}^+),\ i = 1, 2, \cdots, n. \tag{4.1}$$

Thus, we claim that it will be interesting to consider the qualitative properties of an almost-periodic Beddington-DeAngelis predator-prey system with mutual interference and time delay since it can extend the predecessors results.

In this section, we propose and study the following time-delay model

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1(t) - b_1(t)x(t-\tau_1(t)) - \dfrac{k_1(t)y(t)}{a(t)+c(t)x(t)+d(t)y(t)}\right), \\ \dot{y}(t) = y(t)\left(-r_2(t) - b_2(t)y(t-\tau_2(t)) + \dfrac{k_2(t)x(t)}{a(t)+c(t)x(t)+d(t)y(t)}\right), \end{cases} \tag{4.2}$$

As far as we can survey, this is the first article to study the existence and global stability of almost periodic positive solution of Beddington-DeAngelis predator-prey model with time-delay and mutual interference.

4.1. Permanence

Theorem 4.1. *Assume that system (4.2) satisfies the following conditions:*

(H_1) $r_1^l > \frac{k_1^u}{d^l}$,

(H_2) $\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} > r_2^u$,

Then model (4.2) possesses permanence which means that any positive solution $(x(t), y(t))$ to the model (4.2) fulfills

$$0 < m_1 \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M_1,$$
$$0 < m_2 \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq M_2,$$

where

$$M_1 = \frac{r_1^u}{b_1^l}\exp\left(r_1^u\tau\right),$$
$$m_1 = \frac{r_1^l - \frac{k_1^u}{d^l}}{b_1^u}\exp\left\{\left[\left(r_1^l - \frac{k_1^u}{d^l}\right) - b_1^u M_1\right]\tau\right\},$$
$$M_2 = \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)}\exp\left(\frac{k_2^u M_1}{a^l + c^l M_1}\tau\right),$$
$$m_2 = \left(\frac{k_2^l m_1}{b_2^u\left(a^u + c^u M_1 + d^u M_2\right)} - \frac{r_2^u}{b_2^u}\right)\exp\left\{\left[\left(\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} - r_2^u\right) - b_2^u M_2\right]\tau\right\}.$$

Proof. Based on the first equation of the system (4.2), we deduce that:

$$\dot{x}(t) \leq x(t) \left[r_1^u - b_1^l x(t - \tau_1(t)) \right], \ t > \tau. \tag{4.3}$$

Here and subsequently, $x(\bar{t})$ denotes any local maximal value of $x(t)$. Thanks to (4.3), we obtain that

$$0 = \dot{x}(\bar{t}) \leq x(\bar{t}) \left[r_1^u - b_1^l x(\bar{t} - \tau_1(\bar{t})) \right]. \tag{4.4}$$

Based on (4.4), we obatin

$$x(\bar{t} - \tau_1(\bar{t})) \leq \frac{r_1^u}{b_1^l}, \ t > \tau. \tag{4.5}$$

Integrating both sides of (4.3) on interval $[\bar{t} - \tau_1(\bar{t}), \bar{t}]$, we get that

$$\ln \frac{x(\bar{t})}{x(\bar{t} - \tau_1(\bar{t}))} \leq \int_{\bar{t}-\tau_1(\bar{t})}^{\bar{t}} \left[r_1^u - b_1^l x(t - \tau_1(t)) \right] dt \leq r_1^u \tau. \tag{4.6}$$

Together with (4.5) and (4.6), we deduce that

$$x(\bar{t}) \leq \frac{r_1^u}{b_1^l} \exp\left(r_1^u \tau \right) \equiv M_1. \tag{4.7}$$

We need to notice that $x(\bar{t})$ stands for any local maximal value of $x(t)$, hence there possesses a $T_1 > \tau$, for $t > T_1$, one has

$$x(t) \leq M_1. \tag{4.8}$$

From the first equation of system (4.2), we have

$$\dot{x}(t) \geq x(t) \left[\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u x(t - \tau_1(t)) \right], \ t > \tau. \tag{4.9}$$

Here and subsequently, $x(\tilde{t})$ denotes any local minimal value of $x(t)$. Thanks to (4.9), we obtain that

$$0 = \dot{x}(\tilde{t}) \geq x(\tilde{t}) \left[\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u x(\tilde{t} - \tau_1(\tilde{t})) \right]. \tag{4.10}$$

Based on (4.10), we obatin

$$x(\tilde{t} - \tau_1(\tilde{t})) \geq \frac{r_1^l - \frac{k_1^u}{d^l}}{b_1^u}. \tag{4.11}$$

Integrating both sides of (4.9) on interval $[\tilde{t} - \tau_1(\tilde{t}), \tilde{t}]$, noticing that $\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u x(\tilde{t} - \tau_1(\tilde{t})) \leq 0$, we obtain

$$\ln \frac{x(\tilde{t})}{x(\tilde{t} - \tau_1(\tilde{t}))} \geq \int_{\tilde{t}-\tau_1(\tilde{t})}^{\tilde{t}} \left[\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u x(t - \tau_1(t)) \right] dt \geq \left[\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u M_1 \right] \tau. \tag{4.12}$$

Together with (4.11) and (4.12), we deduce that

$$x(\tilde{t}) \geq \frac{r_1^l - \frac{k_1^u}{d^l}}{b_1^u} \exp\left\{ \left[\left(r_1^l - \frac{k_1^u}{d^l} \right) - b_1^u M_1 \right] \tau \right\} \equiv m_1 \tag{4.13}$$

Hence there possesses a $T_2 > \tau$, for $t > T_2$, one has

$$x(t) \geq m_1.$$

We denote $y(t) = \frac{1}{u(t)} > 0$ and $y(t-\tau_2(t)) = \frac{1}{u(t-\tau_2(t))} > 0$. Applying the second equation of the system (4.2) leads to

$$\begin{aligned}\dot{u}(t) =& u(t)\left(r_2(t) + \frac{b_2(t)}{u(t-\tau_2(t))} - \frac{k_2(t)x(t)u(t)}{a(t)u(t)+c(t)x(t)u(t)+d(t)}\right)\\ \geq& u(t)\left(\frac{b_2(t)}{u(t-\tau_2(t))} - \frac{k_2(t)x(t)u(t)}{a(t)u(t)+c(t)x(t)u(t)+d(t)}\right)\\ \geq& u(t)\left(\frac{b_2^l}{u(t-\tau_2(t))} - \frac{k_2^u M_1}{a^l + c^l M_1}\right).\end{aligned} \tag{4.14}$$

Therefore, based on the transformation $y(t) = \frac{1}{u(t)}$, we get

$$\dot{y}(t) = -\frac{\dot{u}(t)}{u^2(t)} \leq -\frac{1}{u(t)}\left(\frac{b_2^l}{u(t-\tau_2(t))} - \frac{k_2^u M_1}{a^l + c^l M_1}\right) = y(t)\left[\frac{k_2^u M_1}{a^l + c^l M_1} - b_2^l y(t-\tau_2(t))\right]. \tag{4.15}$$

Here and subsequently, $y(\bar{t})$ denotes any local maximal value of $y(t)$. Thanks to (4.15), we obtain that

$$0 = \dot{y}(\bar{t}) \leq y(\bar{t})\left[\frac{k_2^u M_1}{a^l + c^l M_1} - b_2^l y(\bar{t}-\tau_2(\bar{t}))\right]. \tag{4.16}$$

Based on (4.16), we obatin

$$y(\bar{t}-\tau_2(\bar{t})) \leq \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)}. \tag{4.17}$$

Integrating the last inequality (4.17) on interval $[\bar{t}-\tau_2(\bar{t}), \bar{t}]$, we have

$$\ln\frac{y(\bar{t})}{y(\bar{t}-\tau(\bar{t}))} \leq \int_{\bar{t}-\tau(\bar{t})}^{\bar{t}}\left[\frac{k_2^u M_1}{a^l + c^l M_1} - b_2^l y(t-\tau_2(t))\right]dt \leq \frac{k_2^u M_1}{a^l + c^l M_1}\tau. \tag{4.18}$$

Together with (4.17) and (4.18), we deduce that

$$y(\bar{t}) \leq \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)}\exp\left(\frac{k_2^u M_1}{a^l + c^l M_1}\tau\right) \equiv M_2. \tag{4.19}$$

We need to notice that $y(\bar{t})$ stands for any local maximal value of $y(t)$, hence there possesses a $T_3 > \tau$, for $t > T_3$, one has

$$y(t) \leq M_2. \tag{4.20}$$

From the second equation of system (4.2), we have

$$\dot{y}(t) \geq y(t)\left[\left(\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} - r_2^u\right) - b_2^u y(t-\tau_2(t))\right], \ t > \tau. \tag{4.21}$$

Here and subsequently, $y(\tilde{t})$ denotes any local minimal value of $y(t)$. Thanks to (4.21), we obtain that

$$0 = \dot{y}(\tilde{t}) \geq \dot{y}(\tilde{t})\left[\left(\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} - r_2^u\right) - b_2^u y(\tilde{t}-\tau_2(\tilde{t}))\right]. \tag{4.22}$$

Based on (4.22), we obatin

$$y(\tilde{t}-\tau_2(\tilde{t})) \geq \frac{k_2^l m_1}{b_2^u\left(a^u+c^u M_1+d^u M_2\right)}-\frac{r_2^u}{b_2^u}. \tag{4.23}$$

Integrating both sides of (4.21) on interval $[\tilde{t}-\tau_2(\tilde{t}),\tilde{t}]$, noticing that

$$\left(\frac{k_2^l m_1}{a^u+c^u M_1+d^u M_2}-r_2^u\right)-b_2^u y(\tilde{t}-\tau_2(\tilde{t})) \leq 0,$$

we obtain

$$\begin{aligned}\ln \frac{y(\tilde{t})}{y(\tilde{t}-\tau_2(\tilde{t}))} &\geq \int_{\tilde{t}-\tau_2(\tilde{t})}^{\tilde{t}}\left[\left(\frac{k_2^l m_1}{a^u+c^u M_1+d^u M_2}-r_2^u\right)-b_2^u y(t-\tau_2(t))\right] dt \\ &\geq\left[\left(\frac{k_2^l m_1}{a^u+c^u M_1+d^u M_2}-r_2^u\right)-b_2^u M_2\right] \tau .\end{aligned} \tag{4.24}$$

Together with (4.23) and (4.24), we deduce that

$$y(\tilde{t}) \geq\left(\frac{k_2^l m_1}{b_2^u\left(a^u+c^u M_1+d^u M_2\right)}-\frac{r_2^u}{b_2^u}\right) \exp \left\{\left[\left(\frac{k_2^l m_1}{a^u+c^u M_1+d^u M_2}-r_2^u\right)-b_2^u M_2\right] \tau\right\} \equiv m_2. \tag{4.25}$$

Hence there possesses a $T_4>\tau$, for $t>T_4$, one has

$$y(t) \geq m_2.$$

This ends the proof of Theorem 4.1. □

Theorem 4.2. *Assume that $\mathcal{S}$ stand for the set of all solutions $\mathbf{w}(t)=(x(t), y(t))^T$ of (4.2) on $\mathbb{R}$ fulfilling $m_1 \leq x(t) \leq M_1$, $m_2<y(t)<M_2$ for $t \in \mathbb{R}$. Then $\mathcal{S} \neq \varnothing$.*

The proof of Theorem 4.2 is similar to the proof of Theorem 2.2. Hence, it is omitted.

4.2. Global asymptotical stability

Theorem 4.3. *Assume that all conditions of Theorem 4.1 are hold and further that the parameters of (4.2) fulfill the under-mentioned conditions:*

(H_3) $\liminf\limits_{t \rightarrow+\infty} L_i(t)>0,\ i=1,2,$

where

$$\begin{aligned}L_1(t)= & \left(b_1^l-\frac{k_1^u c^u M_2+k_2^u a^l}{a^l\left[a^l+c^l m_1+d^l m_2\right]}\right)-\left(r_1(t)+M_1 b_1(t)+\frac{k_1(t) M_2}{a^l+c^l m_1+d^l m_2}\right) \int_t^{\phi_1^{-1}(t)} b_1(u) d u \\ & -\frac{k_1(t) c^u M_1 M_2}{\left(a^l+c^l m_1+d^l m_2\right)^2} \int_t^{\phi_1^{-1}(t)} b_1(u) d u-\frac{M_1 b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))} \int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} b_1(u) d u \\ & -\frac{k_2(t)[a^u+d^u M_2] M_2}{\left(a^l+c^l m_1+d^l m_2\right)^2} \int_t^{\phi_2^{-1}(t)} b_2(u) d u\end{aligned}$$

and

$$L_2(t) = b_2^l - \frac{k_1^u}{a^l}\left(1 + \frac{d^u M_2}{a^l + c^l m_1 + d^l m_2}\right) - \left(r_2(t) + b_2(t)M_2 + \frac{k_2(t)M_1}{a^l + c^l m_1 + d^l m_2}\right)\int_t^{\phi_2^{-1}(t)} b_2(u)du$$
$$- \frac{k_2(t)d^u M_1 M_2}{(a^l + c^l m_1 + d^l m_2)^2}\int_t^{\phi_2^{-1}(t)} b_2(u)du - \frac{M_2 b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} b_2(u)du$$
$$- \frac{k_1(t)(a^u + c^u)M_1^2}{(a^l + c^l m_1 + d^l m_2)^2}\int_t^{\phi_1^{-1}(t)} b_1(u)du,$$

in which ϕ_i^{-1} stands for the inverse function of $\phi_i(t) = t - \tau_i(t)(i = 1, 2)$, respectively.

If $\mathbf{w}_1(t) = (x_1(t), y_1(t))^T$ and $\mathbf{w}_2(t) = (x_2(t), y_2(t))^T$ stand for, respectively, two positive solutions of (4.2), we obtain

$$\lim_{t\to\infty} |\mathbf{w}_1(t) - \mathbf{w}_2(t)| = \mathbf{0}. \tag{4.26}$$

Proof. We assume that system (4.2) possesses two positive solutions $\mathbf{w}_1(t) = (x_1(t), y_1(t))^T$ and $\mathbf{w}_2(t) = (x_2(t), y_2(t))^T$. Thanks to Theorem 4.1, there has a positive constant T, such that

$$m_1 \le x_i \le M_1,\ m_2 \le y_i \le M_2,\ i = 1, 2. \tag{4.27}$$

Throughout the proof,

$$V_{11}(t) = |\ln x_1(t) - \ln x_2(t)|.$$

Thus, we obtain the upper right derivative of V_{11} along system (4.2)

$$\begin{aligned} D^+V_{11}(t) =& -\operatorname{sgn}(x_1(t) - x_2(t))\left[x_1(t - \tau_1(t)) - x_2(t - \tau_1(t))\right] b_1(t) \\ & - \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t))y_1(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t))y_2(t)}{a(t) + c(t)x_2(t) + d(t)y_2(t)}\right) k_1(t) \\ =& -\operatorname{sgn}(x_1(t) - x_2(t))\left[x_1(t - \tau_1(t)) - x_2(t - \tau_1(t))\right] b_1(t) \\ & + \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t))y_2(t)}{a(t) + c(t)x_2(t) + d(t)y_2(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t))y_2(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)}\right) k_1(t) \\ & + \left(\frac{\operatorname{sgn}(x_1(t) - x_2(t))y_2(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)} - \frac{\operatorname{sgn}(x_1(t) - x_2(t))y_1(t)}{a(t) + c(t)x_1(t) + d(t)y_1(t)}\right) k_1(t). \end{aligned}$$

By applying the following inequality $-\operatorname{sgn}(a)\cdot b \le -|a| + |a - b|(a, b \in \mathbb{R})$ we get that

$$\begin{aligned} D^+V_{11}(t) \le& - b_1(t)\,|x_1(t) - x_2(t)| + b_1(t)\left|\int_{t-\tau_1(t)}^t (\dot{x}_1(s) - \dot{x}_2(s))ds\right| + \frac{k_1(t)}{a(t)}|y_2(t) - y_1(t)| \\ & + \frac{k_1(t)c(t)y_2(t)|x_1(t) - x_2(t)|}{a(t)[a(t) + c(t)x_1(t) + d(t)y_1(t)]} + \frac{k_1(t)d(t)y_2(t)|y_1(t) - y_2(t)|}{a(t)[a(t) + c(t)x_1(t) + d(t)y_1(t)]} \\ \le& - b_1^l\,|x_1(t) - x_2(t)| + b_1(t)\left|\int_{t-\tau_1(t)}^t (\dot{x}_1(s) - \dot{x}_2(s))ds\right| + \frac{k_1^u}{a^l}|y_2(t) - y_1(t)| \\ & + \frac{k_1^u c^u M_2}{a^l[a^l + c^l m_1 + d^l m_2]}|x_1(t) - x_2(t)| + \frac{k_1^u d^u M_2}{a^l[a^l + c^l m_1 + d^l m_2]}|y_1(t) - y_2(t)|. \end{aligned} \tag{4.28}$$

By substituting the first equation of (4.2) into (4.28), we deduce that

$$
\begin{aligned}
&D^{+}V_{11}(t)\\
\le&-b_1^l\,|x_1(t)-x_2(t)|+\frac{k_1^uc^uM_2|x_1(t)-x_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^ud^uM_2|y_1(t)-y_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|\\
&+b_1(t)\left|\int_{t-\tau_1(t)}^{t}\left\{x_1(s)\left[r_1(s)-b_1(s)x_1(s-\tau_1(s))-\frac{k_1(s)y_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right]\right.\right.\\
&\left.\left.-x_2(s)\left[r_1(s)-b_1(s)x_2(s-\tau_1(s))-\frac{k_1(s)y_2(s)}{a(s)+c(s)x_2(s)+d(s)y_2(s)}\right]\right\}ds\right|\\
=&-b_1^l\,|x_1(t)-x_2(t)|+\frac{k_1^uc^uM_2|x_1(t)-x_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^ud^uM_2|y_1(t)-y_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|\\
&+b_1(t)\left|\int_{t-\tau_1(t)}^{t}\left\{\left[r_1(s)-b_1(s)x_1(s-\tau_1(s))-\frac{k_1(s)y_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right](x_1(s)-x_2(s))\right.\right.\\
&-b_1(s)x_2(s)\left[x_1(s-\tau_1(s))-x_2(s-\tau_1(s))\right]\\
&\left.\left.-x_2(s)\left[\frac{k_1(s)y_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}-\frac{k_1(s)y_2(s)}{a(s)+c(s)x_2(s)+d(s)y_2(s)}\right]\right\}ds\right|.
\end{aligned}
\tag{4.29}
$$

It follows from (4.29) that for $t \ge T+\tau$

$$
\begin{aligned}
&D^{+}V_{11}(t)\\
\le&-b_1^l\,|x_1(t)-x_2(t)|+\frac{k_1^uc^uM_2|x_1(t)-x_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^ud^uM_2|y_1(t)-y_2(t)|}{a^l[a^l+c^lm_1+d^lm_2]}+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|\\
&+b_1(t)\int_{t-\tau_1(t)}^{t}\left\{\left[r_1(s)+b_1(s)x_1(s-\tau_1(s))+\frac{k_1(s)y_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right]|x_1(s)-x_2(s)|\right.\\
&+b_1(s)x_2(s)\left|x_1(s-\tau_1(s))-x_2(s-\tau_1(s))\right|+\frac{k_1(s)a(s)M_1^2}{(a^l+c^lm_1+d^lm_2)^2}\left|y_2(s)-y_1(s)\right|\\
&\left.+\frac{k_1(s)c(s)M_1^2\left|y_2(s)-y_1(s)\right|}{(a^l+c^lm_1+d^lm_2)^2}+\frac{k_1(s)c(s)M_1M_2\left|x_2(s)-x_1(s)\right|}{(a^l+c^lm_1+d^lm_2)^2}\right\}ds\\
\le&-b_1^l\,|x_1(t)-x_2(t)|+\frac{k_1^uc^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|x_1(t)-x_2(t)|+\frac{k_1^ud^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|y_1(t)-y_2(t)|\\
&+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|+b_1(t)\int_{t-\tau_1(t)}^{t}F_1(s)ds\\
=&-b_1^l\,|x_1(t)-x_2(t)|+\frac{k_1^uc^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|x_1(t)-x_2(t)|+\frac{k_1^ud^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|y_1(t)-y_2(t)|\\
&+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|+b_1(t)\left[P_1(t)-P_1(\phi_1(t))\right],
\end{aligned}
\tag{4.30}
$$

where

$$
\begin{aligned}
F_1(s)=&\left[r_1(s)+M_1b_1(s)+\frac{k_1(s)M_2}{a^l+c^lm_1+d^lm_2}\right]|x_1(s)-x_2(s)|+M_1b_1(s)\left|x_1(s-\tau_1(s))-x_2(s-\tau_1(s))\right|\\
&+\frac{k_1(s)a^uM_1^2\left|y_2(s)-y_1(s)\right|}{(a^l+c^lm_1+d^lm_2)^2}+\frac{k_1(s)c^uM_1^2\left|y_2(s)-y_1(s)\right|}{(a^l+c^lm_1+d^lm_2)^2}+\frac{k_1(s)c^uM_1M_2\left|x_2(s)-x_1(s)\right|}{(a^l+c^lm_1+d^lm_2)^2}
\end{aligned}
$$

and $P_1(s)$ denotes a primitive function of $F_1(s)$.

In what follows, we denote

$$V_{12}(t)=\int_t^{\phi_1^{-1}(t)}\int_{\phi_1(u)}^t b_1(u)F_1(s)dsdu. \tag{4.31}$$

It is obvious by a direct calculation that

$$V_{12}(t)=\int_t^{\phi_1^{-1}(t)} b_1(u)\left[P_1(t)-P_1(\phi_1(u))\right]du=P_1(t)\int_t^{\phi_1^{-1}(t)} b_1(u)du-\int_t^{\phi_1^{-1}(t)} b_1(u)P_1(\phi_1(u))du.$$

Therefore, we get that for $t\geq T+\tau$

$$\begin{aligned}D^+V_{12}(t)=&F_1(t)\int_t^{\phi_1^{-1}(t)} b_1(u)du+P_1(t)\left[\frac{b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(t)}-b_1(t)\right]-\left[\frac{b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(t)}P_1(t)-b_1(t)P_1(\phi_1(t))\right]\\=&F_1(t)\int_t^{\phi_1^{-1}(t)} b_1(u)du-b_1(t)\left[P_1(t)-P_1(\phi_1(t))\right].\end{aligned} \tag{4.32}$$

From now on, we define

$$V_{13}(t)=M_1\int_{t-\tau_1(t)}^t\int_{\phi_1^{-1}(u)}^{\phi_1^{-1}(\phi_1^{-1}(u))}\frac{b_1(s)b_1(\phi_1^{-1}(u))}{\dot{\phi}_1(\phi_1^{-1}(u))}\left|x_1(u)-x_2(u)\right|dsdu. \tag{4.33}$$

It is obvious by a direct computation that

$$\begin{aligned}D^+V_{13}(t)=&\frac{M_1b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} b_1(u)du\left|x_1(t)-x_2(t)\right|\\&-M_1b_1(t)\left|x_1(t-\tau_1(t))-x_2(t-\tau_1(t))\right|\int_t^{\phi_1^{-1}(t)} b_1(u)du\end{aligned} \tag{4.34}$$

for $t\geq T+\tau$. By abuse of notation, we continue to write V_1 for the sum of V_{11}, V_{12} and V_{13}. Thus, it follows from (4.30), (4.32) and (4.34) that

$$\begin{aligned}&D^+V_1(t)\\\leq&-b_1^l\left|x_1(t)-x_2(t)\right|+\frac{k_1^uc^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|x_1(t)-x_2(t)|+\frac{k_1^ud^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|y_1(t)-y_2(t)|\\&+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|+F_1(t)\int_t^{\phi_1^{-1}(t)} b_1(u)du+\frac{M_1b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} b_1(u)du\left|x_1(t)-x_2(t)\right|\\&-M_1b_1(t)\left|x_1(t-\tau_1(t))-x_2(t-\tau_1(t))\right|\int_t^{\phi_1^{-1}(t)} b_1(u)du\\=&-b_1^l\left|x_1(t)-x_2(t)\right|+\frac{k_1^uc^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|x_1(t)-x_2(t)|+\frac{k_1^ud^uM_2}{a^l[a^l+c^lm_1+d^lm_2]}|y_1(t)-y_2(t)|\\&+\frac{k_1^u}{a^l}|y_2(t)-y_1(t)|+\left\{\left[r_1(t)+M_1b_1(t)+\frac{k_1(t)M_2}{a^l+c^lm_1+d^lm_2}\right]\left|x_1(t)-x_2(t)\right|\right.\\&\left.+\frac{k_1(t)a^uM_1^2\left|y_2(t)-y_1(t)\right|}{\left(a^l+c^lm_1+d^lm_2\right)^2}+\frac{k_1(t)c^uM_1^2\left|y_2(t)-y_1(t)\right|}{\left(a^l+c^lm_1+d^lm_2\right)^2}+\frac{k_1(t)c^uM_1M_2\left|x_2(t)-x_1(t)\right|}{\left(a^l+c^lm_1+d^lm_2\right)^2}\right\}\int_t^{\phi_1^{-1}(t)} b_1(u)du\\&+\frac{M_1b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} b_1(u)du\left|x_1(t)-x_2(t)\right|\end{aligned} \tag{4.35}$$

for $t \geq T + \tau$.

By the same method, we define

$$V_{21}(t) = |\ln y_1(t) - \ln y_2(t)| .$$

It is obvious by a direct computation that

$$\begin{aligned}
&D^+V_{21}(t) \\
=&-\operatorname{sgn}(y_1(t) - y_2(t))\,[y_1(t-\tau_1(t)) - y_2(t-\tau_1(t))]\,b_2(t) \\
&+\left(\frac{\operatorname{sgn}(y_1(t)-y_2(t))x_1(t)}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - \frac{\operatorname{sgn}(y_1(t)-y_2(t))x_2(t)}{a(t)+c(t)x_2(t)+d(t)y_2(t)}\right)k_2(t) \\
\leq& k_2(t)\operatorname{sgn}(y_1(t)-y_2(t))\frac{(a(t)+d(t)y_2(t))(x_1(t)-x_2(t)) + d(t)x_2(t)(y_2(t)-y_1(t))}{[a(t)+c(t)x_1(t)+d(t)y_1(t)][a(t)+c(t)x_2(t)+d(t)y_2(t)])} \\
&- b_2(t)\,|y_1(t)-y_2(t)| + b_2(t)\left|\int_{t-\tau_2(t)}^{t}(\dot{y}_1(s)-\dot{y}_2(s))ds\right| \\
\leq& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| + b_2(t)\left|\int_{t-\tau_2(t)}^{t}(\dot{y}_1(s)-\dot{y}_2(s))ds\right| \\
=& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| \\
&+ b_2(t)\left|\int_{t-\tau_2(t)}^{t}\left\{y_1(s)\left[-r_2(s) - b_2(s)y_1(s-\tau_2(s)) + \frac{k_2(s)x_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right]\right.\right. \\
&\left.\left.-y_2(s)\left[-r_2(s) - b_2(s)y_2(s-\tau_1(s)) + \frac{k_2(s)x_2(s)}{a(s)+c(s)x_2(s)+d(s)y_2(s)}\right]\right\}ds\right| \\
=& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| \\
&+ b_2(t)\left|\int_{t-\tau_2(t)}^{t}\left\{\left[-r_2(s) - b_2(s)y_1(s-\tau_2(s)) + \frac{k_2(s)x_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right](y_1(s)-y_2(s))\right.\right. \\
&-b_2(s)y_2(s)\,[y_1(s-\tau_2(s)) - y_2(s-\tau_2(s))] \\
&\left.\left.-y_2(s)\left[\frac{k_2(s)x_2(s)}{a(s)+c(s)x_2(s)+d(s)y_2(s)} - \frac{k_2(s)x_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right]\right\}ds\right| \\
\leq& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| \\
&+ b_2(t)\left|\int_{t-\tau_2(t)}^{t}\left\{\left[r_2(s) + b_2(s)y_1(s-\tau_2(s)) + \frac{k_2(s)x_1(s)}{a(s)+c(s)x_1(s)+d(s)y_1(s)}\right]|y_1(s)-y_2(s)|\right.\right. \\
&+b_2(s)y_2(s)\,|y_1(s-\tau_2(s)) - y_2(s-\tau_2(s))| + y_2(s)\left[\frac{k_2(s)a(s)\,(x_2(s)-x_1(s))}{(a^l+c^lm_1+d^lm_2)^2}\right. \\
&\left.\left.+\frac{k_2(s)d(s)x_2(s)\,(y_1(s)-y_2(s))}{(a^l+c^lm_1+d^lm_2)^2} + \frac{k_2(s)d(s)y_2(s)\,(x_2(s)-x_1(s))}{(a^l+c^lm_1+d^lm_2)^2}\right]\right\}ds \\
\leq& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| + b_2(t)\int_{t-\tau_2(t)}^{t}F_2(s)ds \\
=& \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\,|y_1(t)-y_2(t)| + b_2(t)\,[P_2(t) - P_2(\phi_2(t))],
\end{aligned} \tag{4.36}$$

where

$$F_2(s) = \left[r_2(s) + b_2(s)M_2 + \frac{k_2(s)M_1}{a^l + c^l m_1 + d^l m_2}\right] |y_1(s) - y_2(s)| + b_2(s)M_2 |y_1(s - \tau_2(s)) - y_2(s - \tau_2(s))|$$
$$+ \left[\frac{k_2(s)a^u M_2 |x_2(s) - x_1(s)|}{(a^l + c^l m_1 + d^l m_2)^2} + \frac{k_2(s)d^u M_1 M_2 |y_1(s) - y_2(s)|}{(a^l + c^l m_1 + d^l m_2)^2} + \frac{k_2(s)d^u M_2^2 |x_2(s) - x_1(s)|}{(a^l + c^l m_1 + d^l m_2)^2}\right]$$

and $P_2(s)$ denotes a primitive function of $F_2(s)$.

In what follows, we denote

$$V_{22}(t) = \int_t^{\phi_2^{-1}(t)} \int_{\phi_2(u)}^t b_2(u)F_2(s)dsdu. \tag{4.37}$$

It is obvious by a direct calculation that

$$\begin{aligned} V_{22}(t) &= \int_t^{\phi_2^{-1}(t)} b_2(u)\left[P_2(t) - P_2(\phi_2(u))\right] du \\ &= P_2(t) \int_t^{\phi_2^{-1}(t)} b_2(u)du - \int_t^{\phi_2^{-1}(t)} b_2(u)P_2(\phi_2(u))du. \end{aligned}$$

Therefore, we get that for $t \geq T + \tau$

$$\begin{aligned} D^+V_{22}(t) &= F_2(t) \int_t^{\phi_2^{-1}(t)} b_2(u)du + P_2(t)\left[\frac{b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(t)} - b_2(t)\right] - \left[\frac{b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(t)} P_2(t) - b_2(t)P_2(\phi_2(t))\right] \\ &= F_2(t) \int_t^{\phi_2^{-1}(t)} b_2(u)du - b_2(t)\left[P_2(t) - P_2(\phi_2(t))\right]. \end{aligned} \tag{4.38}$$

From now on, we define

$$V_{23}(t) = M_2 \int_{t-\tau_2(t)}^t \int_{\phi_2^{-1}(u)}^{\phi_2^{-1}(\phi_2^{-1}(u))} \frac{b_2(s)b_2(\phi_2^{-1}(u))}{\dot{\phi}_2(\phi_2^{-1}(u))} |y_1(u) - y_2(u)| dsdu. \tag{4.39}$$

It is obvious by a direct computation that

$$\begin{aligned} D^+V_{23}(t) &= \frac{M_2 b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))} \int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} b_2(u)du\, |y_1(t) - y_2(t)| \\ &\quad - M_2 b_2(t) |y_1(t - \tau_2(t)) - y_2(t - \tau_2(t))| \int_t^{\phi_2^{-1}(t)} b_2(u)du \end{aligned} \tag{4.40}$$

for $t \geq T + \tau$. By abuse of notation, we continue to write V_2 for the sum of V_{21}, V_{22} and V_{23}. Thus, it follows from (4.36), (4.38) and (4.40) that

$$
\begin{aligned}
& D^+V_2(t) \\
\le & \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\left|y_1(t)-y_2(t)\right| + F_2(t)\int_t^{\phi_2^{-1}(t)} b_2(u)du \\
& + \frac{M_2 b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} b_2(u)du\left|y_1(t)-y_2(t)\right| \\
& - M_2 b_2(t)\left|y_1(t-\tau_2(t))-y_2(t-\tau_2(t))\right|\int_t^{\phi_2^{-1}(t)} b_2(u)du \\
= & \frac{k_2(t)|x_1(t)-x_2(t)|}{a(t)+c(t)x_1(t)+d(t)y_1(t)} - b_2(t)\left|y_1(t)-y_2(t)\right| + \frac{M_2 b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} b_2(u)du\left|y_1(t)-y_2(t)\right| \\
& + \left\{\left[r_2(t)+b_2(t)M_2+\frac{k_2(t)M_1}{a^l+c^l m_1+d^l m_2}\right]\left|y_1(t)-y_2(t)\right| + \left[\frac{k_2(t)a^u M_2\left|x_2(t)-x_1(t)\right|}{\left(a^l+c^l m_1+d^l m_2\right)^2}\right.\right. \\
& \left.\left. + \frac{k_2(t)d^u M_1 M_2\left|y_1(t)-y_2(t)\right|}{\left(a^l+c^l m_1+d^l m_2\right)^2} + \frac{k_2(t)d^u M_2^2\left|x_2(t)-x_1(t)\right|}{\left(a^l+c^l m_1+d^l m_2\right)^2}\right]\right\}\int_t^{\phi_2^{-1}(t)} b_2(u)du
\end{aligned}
\tag{4.41}
$$

for $t \ge T+\tau$. We construct the following Lyapunov function

$$V(t) = V_1(t) + V_2(t).$$

It follows from (4.35) and (4.41) that

$$
\begin{aligned}
& D^+V(t) = D^+V_1(t) + D^+V_2(t) \\
\le & -\left[\left(b_1^l - \frac{k_1^u c^u M_2 + k_2^u a^l}{a^l\left[a^l+c^l m_1+d^l m_2\right]}\right) - \left(r_1(t)+M_1 b_1(t)+\frac{k_1(t)M_2}{a^l+c^l m_1+d^l m_2}\right)\int_t^{\phi_1^{-1}(t)} b_1(u)du\right. \\
& - \frac{k_1(t)c^u M_1 M_2}{\left(a^l+c^l m_1+d^l m_2\right)^2}\int_t^{\phi_1^{-1}(t)} b_1(u)du - \frac{M_1 b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} b_1(u)du \\
& \left. - \frac{k_2(t)[a^u+d^u M_2]M_2}{\left(a^l+c^l m_1+d^l m_2\right)^2}\int_t^{\phi_2^{-1}(t)} b_2(u)du\right]\left|x_1(t)-x_2(t)\right| \\
& - \left[b_2^l - \frac{k_1^u}{a^l}\left(1+\frac{d^u M_2}{a^l+c^l m_1+d^l m_2}\right) - \left(r_2(t)+b_2(t)M_2+\frac{k_2(t)M_1}{a^l+c^l m_1+d^l m_2}\right)\int_t^{\phi_2^{-1}(t)} b_2(u)du\right. \\
& - \frac{k_2(t)d^u M_1 M_2}{\left(a^l+c^l m_1+d^l m_2\right)^2}\int_t^{\phi_2^{-1}(t)} b_2(u)du - \frac{M_2 b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} b_2(u)du \\
& \left. - \frac{k_1(t)(a^u+c^u)M_1^2}{\left(a^l+c^l m_1+d^l m_2\right)^2}\int_t^{\phi_1^{-1}(t)} b_1(u)du\right]\left|y_1(t)-y_2(t)\right| \\
= & -\left(L_1(t)\left|x_1(t)-x_2(t)\right| + L_2(t)\left|y_1(t)-y_2(t)\right|\right).
\end{aligned}
\tag{4.42}
$$

Based on condition (H_3), there have positive constants α_1, α_2 and $T_0 \ge T+\tau$ such that

$$L_1 \ge \alpha_1 > 0,\ L_2 \ge \alpha_2 > 0. \tag{4.43}$$

If $\alpha^* = \min\{\alpha_1, \alpha_2\}$, we get from (4.42) and (4.43)

$$D^+V(t) \le -\alpha^* \left(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\right). \tag{4.44}$$

Integrating the last inequality from T_0 to t, we have

$$V(t) + \alpha^* \int_{T_0}^{t} \left(|x_1(u) - x_2(u)| + |y_1(u) - y_2(u)|\right) du \le V(T_0)$$

for $t \ge T_0$. Hence, $V(t)$ represents a bounded function on interval $[T_0, +\infty)$. In addition,

$$\int_{T_0}^{+\infty} \left(|x_1(u) - x_2(u)| + |y_1(u) - y_2(u)|\right) du \le +\infty.$$

Together with system (4.2) and Theorem 4.1, we can deduce that $x_1(t) - x_2(t)$, $y_1(t) - y_2(t)$ and their derivatives are bounded on interval $[T_0, +\infty)$. In other word, $(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|)$ is uniformly continuous. Thanks to Barbalar Lemma [30], we deduce that

$$\lim_{t\to+\infty} \left(|x_1(u) - x_2(u)| + |y_1(u) - y_2(u)|\right) = 0.$$

That is,

$$\lim_{t\to+\infty} |x_1(u) - x_2(u)| = \lim_{t\to+\infty} |y_1(u) - y_2(u)| = 0.$$

Thus, the positive almost periodic solution of (4.2) remains globally asymptotically stable. This finishes the proof of Theorem 4.3. □

4.3. Existence and uniqueness of almost periodic solution

Theorem 4.4. *Assume that all conditions proposed by Theorem 4.1 and Theorem 4.3 hold. Then there possesses a unique almost periodic solution (APS) of (4.2).*

The proof of Theorem 4.4 is similar to the proof of Theorem 3.2. Hence, it is omitted.

5. Examples

In this section, we illustrate two examples to verify the feasibility of our theorem.

Example 5.1. *In this example we consider the following system*

$$\begin{cases} \dot{x}(t) = x(t)\left[(10 + \cos\sqrt{3}t) - (1.9 + 0.1\cos t)x(t)\right] \\ \qquad - \dfrac{(1 + 0.5\cos t)x(t)y(t)}{(3.99 + 0.01\cos\sqrt{6}t) + (0.8 + 0.2\cos t)x(t) + (0.03 + 0.01\cos\sqrt{7}t)y(t)}, \\ \dot{y}(t) = y(t)\left[-(0.02 + 0.01\cos\sqrt{5}t) - (2.5 + 0.5\cos t)y(t)\right] \\ \qquad + \dfrac{(3 + \cos\sqrt{2}t)x(t)y(t)}{(3.99 + 0.01\cos\sqrt{6}t) + (0.8 + 0.2\cos t)x(t) + (0.03 + 0.01\cos\sqrt{7}t)y(t)}, \end{cases} \tag{5.1}$$

In this example, we obtain $a^u = 4$, $a^l = 3.98$, $b_1^u = 2$, $b_1^l = 1.8$, $b_2^u = 3$, $b_2^l = 2$, $c^u = 1$, $c^l = 0.6$, $d^u = 0.04$, $d^l = 0.02$, $k_1^u = 1.5$, $k_1^l = 0.5$, $k_2^u = 4$, $k_2^l = 2$, $r_1^u = 11$, $r_1^l = 9$, $r_2^u = 0.03$, $r_2^l = 0.01$.

Moreover,

$$
\begin{aligned}
r_1^l &= 9 > \frac{k_1^u M_2}{a^l} \approx 0.6024, \\
r_2^u &= 0.03 < \frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} \approx 0.8253, \\
M_1 &= \frac{r_1^u}{b_1^l} \approx 6.1111, \\
M_2 &= \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)} = \frac{4 \cdot 6.1111}{2 \cdot (3.98 + 0.6 \cdot 6.1111)} \approx 1.5984, \\
m_1 &= \frac{r_1^l}{b_1^u} - \frac{k_1^u k_2^u M_1}{a^l b_1^u b_2^l (a^l + c^l M_1)} = \frac{9}{2} - \frac{1.5 \cdot 4 \cdot 6.1111}{3.98 \cdot 2 \cdot 2 \cdot (3.98 + 0.6 \cdot 6.1111)} \approx 4.1988, \\
m_2 &= \frac{k_2^l m_1}{b_2^u(a^u + c^u M_1 + d^u M_2)} - \frac{r_2^u}{b_2^u} \approx \frac{2 \cdot 4.1988}{3 \cdot (4 + 1 \cdot 6.1111 + 0.04 \cdot 1.5984)} - 0.01 \approx 0.2651.
\end{aligned}
\tag{5.2}
$$

Due to (5.2), one gets

$$
\begin{aligned}
1.8 = b_1^l &> \frac{k_1^u c^u M_2 + a^u k_2^u}{(a^l)^2} \approx 1.1614, \\
2 = b_2^l &> \frac{k_1^u d^u M_2}{(a^l)^2} + \frac{k_1^u}{a^l} \approx 0.3830.
\end{aligned}
$$

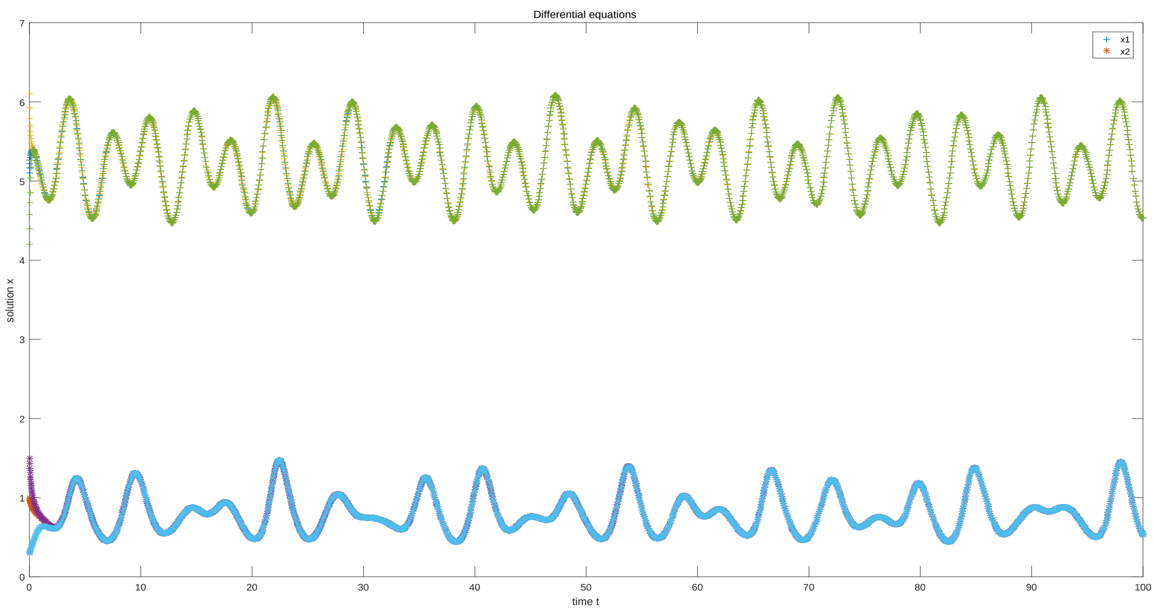

Figure 5.1: Numeric simulation of the prey $x(t)$ and the predator $y(t)$ of (4.2) with the initial conditions $(x(0), y(0))^T = (5.1, 1.0)^T$, $(x(0), y(0))^T = (6.1, 1.5)^T$ and $(x(0), y(0))^T = (4.2, 0.3)^T$.

It follows that all conditions, together with the last two inequalities and system (5.1) proposed by Theorem 3.1. Therefore, system (5.1) has a positive, unique, globally attractive, APS. Under the initial conditions $(x(0), y(0))^T = (5.1, 1.0)^T$, $(x(0), y(0))^T = (6.1, 1.5)^T$ *and* $(x(0), y(0))^T = (4.2, 0.3)^T$, *the population dynamic of* $(x(t), y(t))^T$ *is offered by Figure 5.1. By the graph, we can effortlessly discover that* $(x(t), y(t))^T$ *is asymptotic to the unique, APS of the model (5.1).*

Example 5.2. *In this example we consider the following time-delay system*

$$\begin{cases} \dot{x}(t) = x(t)\left[(0.3-0.05\sin\sqrt{2}t)-(0.25-0.04\cos t)x(t-0.01)-\dfrac{0.13y(t)}{1+x(t)+y(t)}\right], \\ \dot{y}(t) = y(t)\left[-(0.05+0.01\cos\sqrt{0.2}t)-(2.5-1.3\sin\sqrt{0.2}t)y(t-0.02)+\dfrac{(9+\cos\sqrt{2}t)x(t)}{1+x(t)+y(t)}\right], \end{cases} \tag{5.3}$$

In this example, we obtain $a^u = a^l = 1$, $b_1^u = 0.29$, $b_1^l = 0.21$, $b_2^u = 3.8$, $b_2^l = 1.2$, $c^u = c^l = 1$, $d^u = d^l = 1$, $k_1^u = k_1^l = 0.13$, $k_2^u = 10$, $k_2^l = 8$, $r_1^u = 0.35$, $r_1^l = 0.25$, $r_2^u = 0.06$, $r_2^l = 0.04$, $\tau = 0.02$. *Moreover,*

$$\begin{aligned} & r_1^l = 0.25 > \frac{k_1^u}{d^l} = 0.13, \\ & \frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} \approx 0.0940 > r_2^u = 0.06, \\ & M_1 = \frac{r_1^u}{b_1^l}\exp\left(r_1^u\tau\right) \approx 1.6784, \\ & m_1 = \frac{r_1^l - \frac{k_1^u}{d^l}}{b_1^u}\exp\left\{\left[\left(r_1^l - \frac{k_1^u}{d^l}\right) - b_1^u M_1\right]\tau\right\} \approx 0.4108, \\ & M_2 = \frac{k_2^u M_1}{b_2^l(a^l + c^l M_1)}\exp\left(\frac{k_2^u M_1}{a^l + c^l M_1}\tau\right) \approx 5.9191, \\ & m_2 = \left(\frac{k_2^l m_1}{b_2^u\left(a^u + c^u M_1 + d^u M_2\right)} - \frac{r_2^u}{b_2^u}\right)\exp\left\{\left[\left(\frac{k_2^l m_1}{a^u + c^u M_1 + d^u M_2} - r_2^u\right) - b_2^u M_2\right]\tau\right\} \approx 0.0057. \end{aligned} \tag{5.4}$$

Due to (5.4), one gets

$$\liminf_{t\to+\infty} L_1(t) > 0.2 > 0, \quad \liminf_{t\to+\infty} L_2(t) > 0.5 > 0.$$

It follows that all conditions, together with the last two inequalities and system (5.3) proposed by Theorem 4.3. Therefore, system (5.3) has a positive, unique, globally attractive, APS. Under the initial conditions $(x(0), y(0))^T = (0.1, 0.2)^T$, $(x(0), y(0))^T = (1, 2)^T$ *and* $(x(0), y(0))^T = (3, 4)^T$, *the population dynamic of* $(x(t), y(t))^T$ *is offered by Figure 5.2. By the graph, we can effortlessly discover that* $(x(t), y(t))^T$ *is asymptotic to the unique, APS of the model (5.3).*

6. Conclusion

In this research, we investigate the existence and uniqueness of almost periodic solution of a famous predator-prey reaction-diffusion system, the denominated predator-prey model with Beddington-DeAngelis functional response, assuming continuous and almost periodic parameters. In the future, effective methods to study the other properties of almost periodic solution will be the goal of us.

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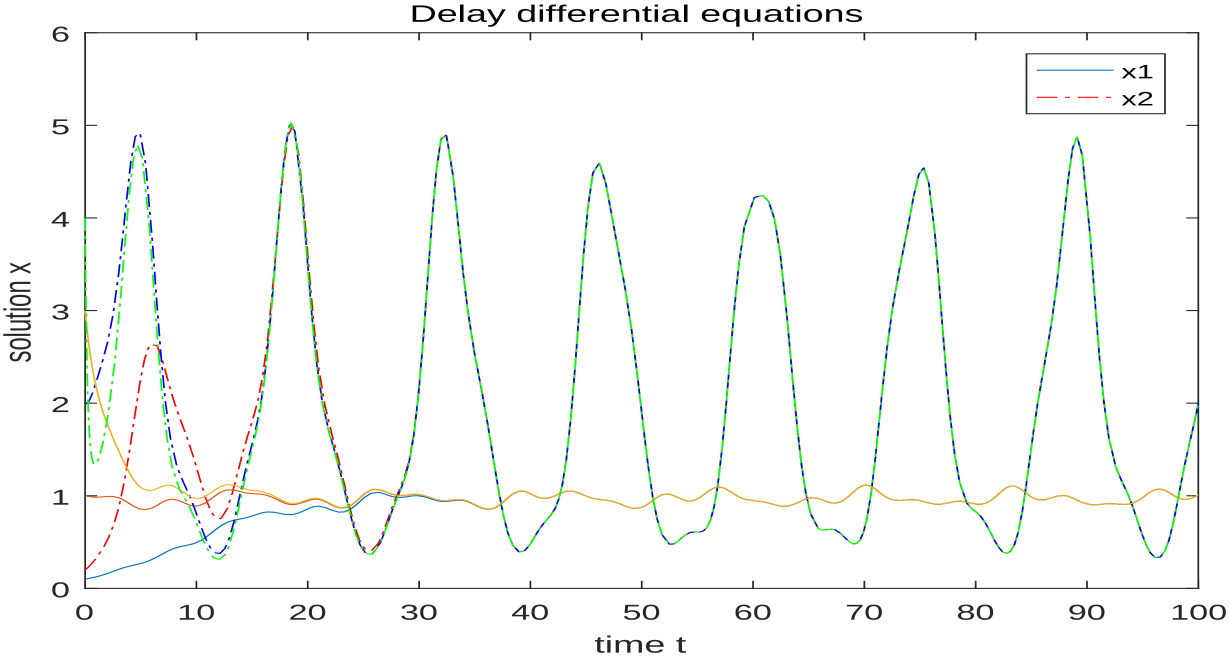

Figure 5.2: Numeric simulation of the prey $x(t)$ and the predator $y(t)$ of (4.2) with the initial conditions $(x(0), y(0))^T = (0.1, 0.2)^T$, $(x(0), y(0))^T = (1, 2)^T$ and $(x(0), y(0))^T = (3, 4)^T$.

References

[1] C. X. Shen, *Permanence and global attractivity of the food-chain system with Holling IV type functional response*, Appl. Math. Comput. **194** (1) (2007), 179–185.

[2] F. D. Chen, *On a nonlinear non-autonomous predator-prey model with diffusion and distributed delay*, J. Comput. Appl. Math. **180** (1) (2005) 33–49.

[3] C. Y. He, *Almost Periodic Differential Equations*, Higher Education Press, 1992, in Chinese.

[4] H. Bohr, *Zur theorie der fastperiodischen funktionen. I*, Acta Math. **45** (1924) 29–127, in German.

[5] H. Bohr, *Zur theorie der fastperiodischen funktionen. II*, Acta Math. **46** (1925) 101–214, in German.

[6] H. Bohr, *Zur theorie der fastperiodischen funktionen. III*, Acta Math. **47** (1926) 237–281, in German.

[7] E.R. van Kampen, *Almost periodic functions and compact groups*, Ann. of Math. (2) **37** (1936) 78–91.

[8] S. Bochner, *Beiträge zur theorie der fastperiodischen funktionen, I. Funktionen einer Variablen*, Math. Ann. **96** (1927) 119–147, in German.

[9] S. Bochner, *Abstrakte fastperiodische funktionen*, Acta Math. **61** (1933) 149–184, in German.

[10] J. von Neumann, *Almost periodic functions in a group. I*, Trans. Amer. Math. Soc. **36** (1934) 445–492.

[11] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, J. Animal Ecol. **44** (1975) 331-340.

[12] D. L. DeAngelis, R. A. Golstein, R. V. O'Neil, *A model for trophic interaction*, Ecology **56** (1975) 881-892.

[13] Z. Li, M. A. Han, F. D. Chen, *Almost periodic solutions of a discrete almost periodic logistic equation with delay*, Appl. Math. Comput. **232** (2014), 743–751.

[14] T. Diagana, H. Zhou, *Existence of positive almost periodic solutions to the hematopoiesis model*, Appl. Math. Comput. **274** (2016), 644–648.

[15] Y. J. Wang, *Periodic and almost periodic solutions of a nonlinear single species discrete model with feedback control*, Appl. Math. Comput. **219** (2013), 5480–5486.

[16] J. Rui, B. C. Liu, *Almost-periodic solutions of an almost-periodically forced wave equation*, J. Math. Anal. Appl. **451** (2017), 629–658.

[17] R. Arditi, L. R. Ginzburg, *Coupling in predator-prey dynamics: ratio-dependence*, J. Theor. Biol. **139** (1989), 311–326.

[18] H. I. Freedman, R. M. Mathsen, *Persistence in predator-prey systems with ratio-dependent predator influence*, Bull. Math. Biol. **55** (4) (1993), 817–827.

[19] B. Liu, Y. Zhang, L. Chen, *Dynamic complexities in a Lotka-Volterra predator-prey model concerning impulsive control strategy*, Int. J. Bifurcat. Chaos Appl. Sci. Eng. **15** (2) (2005), 517–531.

[20] B. Liu, Y. Zhang, L. Chen, *The dynamical behavior of a Lotka-Volterra predator-prey model concerning integrate pest management*, Nonlinear Anal. Real World Appl. **6** (2005), 227–243.

[21] D. M. Luo, *The study of global stability of a periodic Beddington-DeAngelis and Tanner predator-prey model*, Results in Mathematics **74** (101) (2019), 1–18.

[22] T. Diagana, H. Zhou, *Existence of positive almost periodic solutions to the hematopoiesis model*, Appl. Math. Comput. **274** (2016) 644–648.

[23] H. S. Ding, H. M. N'Guérékata, J. J. Nieto, *Weighted pseudo almost periodic solutions to a class of discrete hematopoiesis model*, Rev. Mat. Complut. **26** (2013) 427–443.

[24] H. S. Ding, Q. L. Liu, J. J. Nieto, *Existence of positive almost periodic solutions to a class of hematopoiesis model*, Appl. Math. Model. **40** (2016) 3289–3297.

[25] B. W. Liu, *New results on the positive almost periodic solutions for a model of hematopoiesis*, Nonlinear Anal. Real World Appl. **17** (2014) 252–264.

[26] H. Zhang, M. Q. Yang, L. J. Wang, *Existence and exponential convergence of the positive almost periodic solution for a model of hematopoiesis*, Appl. Math. Lett. **26** (2013) 38–42.

[27] H. Zhou, W. Wang, Z. F. Zhou, *Positive almost periodic solution for a model of hematopoiesis with infinite time delays and a nonlinear harvesting term*, Abstr. Appl. Anal. (2013) 146729.

[28] H. Zhang, J. Y. Shao, *Almost periodic solutions for cellular neural networks with time-varying delays in leakage terms*, Appl. Math. Comput. **219** (2013) 11471–11482.

[29] J. Gao, Q. R. Wang, L. W. Zhang, *Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales*, Appl. Math. Comput. **237** (2014) 639–649.

[30] I. Barbalat, *System dequations differentielles doscillations nonlinears*, Rev. Roumaine Math. Pures Appl. **4** (1959) 267–270.

[31] X. Guan, F. Chen, *Dynamical analysis of a two species amensalism model with Beddington-DeAngelis functional response and Allee effect on the second species*, Nonlinear Anal. Real World Appl. **48** (2019), 71–93.

[32] Z. Liu, W. Wang, *Favard separation method for almost periodic stochastic differential equations*, J. Differ. Equa. **260** (2016), 8109–8136.

[33] J. Campos, M. Tarallo, *Asymptotically dichotomic almost periodic differential equations*, J. Differ. Equa. **263** (2017), 1323–1386.

[34] I. Bright, *Tight estimates for general averaging applied to almost-periodic differential equations*, J. Differ. Equa. **246** (2009), 2922–2937.

[35] H. Zhang, Y. Li, B. Jing, W. Zhao, *Global stability of almost periodic solution of multispecies mutualism system with time delays and impulsive effects*, Appl. Math. Comput. **232** (2014) 1138–1150.