Numerical Solution of the coupled Burgers' equation by Trigonometric B-spline Collocation Method

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Abstract

In the present study, the coupled Burgers' equation is going to be solved numerically by presenting a new technique based on collocation finite element method in which trigonometric cubic and quintic B-splines are used as approximate functions. In order to support the present study, three test problems given with appropriate initial and boundary conditions are studied. The newly obtained results are compared with some of the other published numerical solutions available in the literature. The accuracy of the proposed method is discussed by computing the error norms L_2 and L_∞. A linear stability analysis of the approximation obtained by the scheme shows that the method is unconditionally stable.

Keywords: Finite element method, collocation method, coupled Burgers' equation, trigonometric B-splines.

AMS classification: 35Q51, 74J35, 33F10.

1 Introduction

The investigation of the movement of particles inside a fluid dates back to Einstein and even before that to Brown. In his study, Esipov [1] considered the effect of gravity on the particles. He concluded that if the particles are heavier than the surrounding fluid the resulting movement is called sedimentation otherwise it is called creaming. In this phenomena, coupled Burgers Equation (cBE) plays an important role in describing the sedimentation and also the evaluation of the scaled volume concentration of two different kinds particles in fluid suspensions or colloids under the affect of gravity[2]. cBE is derived by Esipov [1] as one of the important flow equations and having rich dynamics. He stated that the velocity of sedimentation depends on the volume fraction of the constituting particles and leads to Burgers-like equations for concentration profiles. In fact, cBE is widely known as the simple form of the Navier-Stokes equation because of the fact that it involves both the nonlinear convection and viscosity terms. The cBE is given in the following form

$$U_t - U_{xx} + k_1 U U_x + k_2 (UV)_x = 0 \qquad x \in [a,b], \qquad t \in [0,T] \tag{1a}$$
$$V_t - V_{xx} + k_1 V V_x + k_3 (UV)_x = 0 \qquad x \in [a,b], \qquad t \in [0,T] \tag{1b}$$

where k_1, k_2 and k_3 are real constants and x and t are differentiations with respect to space and time, respectively. Here $U(x,t)$ and $V(x,t)$ are the unknown functions to be determined, while U_t and V_t are unsteady terms, UU_x and VV_x are nonlinear terms and finally U_{xx} and V_{xx} are diffusive terms. The equation is going to be considered with the following initial

$$U(x,0) = f(x), \qquad a \leq x \leq b$$
$$V(x,0) = g(x), \qquad a \leq x \leq b$$

and boundary conditions

$$U(a,t) = f_1(a,t), \qquad U(b,t) = f_2(b,t), ...t \in [0,T]$$
$$V(a,t) = g_1(a,t), \qquad V(b,t) = g_2(b,t), ...t \in [0,T]$$

where $f(x), g(x), f_1(a,t), f_2(b,t), g_1(a,t)$ and $g_2(b,t)$ are predefined functions [2].

The exact solutions of cBE for a wide range of initial and boundary conditions are not available. Thus, there is a need for finding numerical and approximate solutions of the equation. In the literature there are several studies about cBE in order to find out its more characteristics. Among others, Kaya [3] has considered a coupled system of viscous Burgers' equations with appropriate initial values using the decomposition method. Abdou and Soliman [4] used variational iteration method for the solutions of Burger's and coupled Burger's equations. Dehghan et al. [5] have applied a technique which is a combination of Adomian decomposition method and Pade approximation for solving coupled Burgers' equations. Khater et al [6] have obtained numerical solutions of the coupled Burgers' equation by the Chebyshev collocation methods. Rashid and Ismail [7] have used the Fourier pseudo-spectral method for finding the approximate solutions of the coupled Burgers' equation. Mittal and Arora [8] proposed a numerical method for the numerical solution of a coupled system of viscous Burgers' equation with appropriate initial and boundary conditions, by using the cubic B-spline collocation scheme on the uniform mesh points. Sadek and Kucuk [9] have described a methodology for solving optimal pointwise control of a couped system of Burgers' equations. Jia et al. [10] have discussed a H^1-Galerkin finite element method for the coupled Burgers equations and derived the optimal error estimates of the semi-discrete and fully discrete schemes of the cBE. Desai and Pradhan [11] have obtained the exact solution of Burgers' equation and coupled Burger's equation using Homotopy Perturbation Method. Kutluay and Uçar [12] have solved coupled Burgers' equation by a Galerkin quadratic B-spline FEM. Srivastava et al. [13] have proposed a fully implicit finite-difference method for the numerical solutions of one dimensional coupled Burgers' equations on the uniform mesh points. Kumar and Pandit [14] have proposed a composite numerical scheme based on finite difference and Haar wavelets to solve time dependent coupled Burgers' equation with appropriate initial and boundary conditions. Mittal and Tripathi [15] have proposed a collocation-based numerical scheme to obtain approximate solutions of coupled Burgers' equations. Siraj-ul-Islam et.al [16] have formulated a simple classical radial basis functions (RBFs) collocation (Kansa) method for the numerical solution of the Korteweg-de Vries equations, coupled Burgers' equations, and quasi non-linear hyperbolic equations. Abdullah er al [17] have developed a numerical procedure dependent on the cubic B-spline and the Hermite formula for the coupled viscous Burgers' equation. Mittal and Jiwari [18] have solved the coupled viscous Burgers' equations by using the differential quadrature method. Bhatt and Khaliq [19] have introduced two new modified fourth-order exponential time differencing Runge-Kutta (ETDRK) schemes in combination witha global fourth-order compact finite difference scheme(in space) for direct integration of nonlinear coupled viscous Burgers' equations in their original form without using any transformations or linearization techniques. Raslan et al. [20] have used cubic trigonometric B-spline (CTB) functions are used to set up the collocation method for finding solutions of a coupled system of Burgers' equation with appropriate initial and boundary conditions. Onarcan and Hepson [21] stated that trigonometric B-spline functions of higher degrees have advantages over lower ones since they can be used as approximate functions in the numerical methods if the differential equation include higher order derivatives. They have used quintic trigonometric B-splines to get numerical solutions of the coupled Burgers' equation. Zhang et al [22] have made the first attempt to extend the improved backward substitution method for solving unsteady nonlinear coupled Burgers' equations. Kapoor [23] has offered a review of the Homotopy perturbation method to fetch the analytical solution of coupled 1D non-linear Burgers' equation. Nazir et al [24] presented a new cubic B-spline (CBS) approximation technique for the numerical treatment of coupled viscous Burgers' equations arising in the study of fluid dynamics, continuous stochastic processes, acoustic transmissions and aerofoil flow theory. Başhan [25] has dealth with a numerical treatment of the coupled viscous Burgers' equation in the presence of very large Reynolds number using two effective methods. The last but not the least, Uçar et al. [26] have sought numerical solutions and stability anlaysis of modified Burgers equation via modified cubic B-spline Differential Quadtarure Methods.

In the present article, the cBE is going to be handled using finite element trigonometric B-spline cubic collocation method. During the solution process, a new type linearization technique is going to be utilized to overcome the nonlinear term appearing in the equation. Then the newly obtained results are going to be compared with some of those available in the literature.

2 Implementation of the method

cBE is generally given in the following form

$$\begin{aligned} U_t - U_{xx} + k_1 U U_x + k_2 (UV)_x = 0 \\ V_t - V_{xx} + k_1 V V_x + k_3 (UV)_x = 0 \end{aligned}$$

in which t is time, x is the space coordinate and μ is a positive parameter. For the considered problems, the appropriate boundary conditions will be chosen as

$$\begin{aligned} &U(x,0) = f(x) \\ &V(x,0) = g(x), \qquad a \leq x \leq b \end{aligned}$$

and

$$\begin{aligned} &U(a,t) = f_1(a,t), \qquad U(b,t) = f_2(b,t) \\ &V(a,t) = g_1(a,t), \qquad V(b,t) = g_2(b,t), \qquad t > 0 \end{aligned}$$

For the solution process, it is considered that the solution interval $[a,b]$ is divided into N finite elements having equal lengths using the nodal points $x_m,\ m = 0(1)N$ in such a way that $a = x_0 < x_1 \cdots < x_N = b$ and $h = (x_{m+1} - x_m)$.

2.1 Cubic Trigonometric B-spline Basis

Cubic trigonometric B-spline functions $T_m^3(x)$ form a basis over the region $a \leq x \leq b$ and vanish outside the interval $[x_{m-2}, x_{m+2}]$. These cubic trigonometric B-spline functions $T_m^3(x)$, $(m = -1(1)N+1)$, at the knots x_m are defined over the interval $[a,b]$ by [27]

$$T_m^3(x) = \frac{1}{\theta} \left\{ \begin{array}{ccc} \rho^3(x_{m-2}) & , & x_{m-2} \leq x \leq x_{m-1} \\ -\rho^2(x_{m-2})\rho(x_m) & & \\ -\rho(x_{m-2})\rho(x_{m+1})\rho(x_{m-1}) & , & x_{m-1} \leq x \leq x_m \\ -\rho(x_{m+2})\rho^2(x_{m-1}) & & \\ \rho(x_{m-2})\rho^2(x_{m+1}) & & \\ +\rho(x_{m+2})\rho(x_{m-1})\rho(x_{m+1}) & , & x_m \leq x \leq x_{m+1} \\ +\rho^2(x_{m+2})\rho(x_m) & & \\ -\rho^3(x_{m+2}) & , & x_{m+1} \leq x \leq x_{m+2} \\ 0 & , & \text{otherwise} \end{array} \right.$$

in which

$$\rho(x_m) = \sin\left(\frac{x - x_m}{2}\right),\ \theta = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right),\ m = 0(1)N.$$

The set of trigonometric cubic B-splines $\left\{T_{-1}^3(x), T_0^3(x), \ldots, T_{N+1}^3(x)\right\}$ forms a basis for the smooth functions defined over [a,b]. Therefore, an approximation solution $U_N(x,t)$ and $V_N(x,t)$ can be written in terms of the trigonometric cubic B- splines as trial functions:

$$U(x,t) \approx U_N(x,t) = \sum_{m=-1}^{N+1} T_m^3(x)\delta_m(t) \tag{2}$$

$$V(x,t) \approx V_N(x,t) = \sum_{m=-1}^{N+1} T_m^3(x)\sigma_m(t) \tag{3}$$

where $\delta_m(t)$'s are unknown, time dependent quantities to be determined from the boundary and trigonometric cubic B-spline collocation conditions. Each trigonometric cubic B-spline covers four elements so that each element $[x_m, x_{m+1}]$ is covered by four trigonometric cubic B-splines. For this problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$. Using the nodal values $U_m, U_m^{'}$ and $U_m^{''}$ and $V_m, V_m^{'}$ and $V_m^{''}$are given in terms of the parameter δ_m by:

$$\begin{aligned}
U_m &= U(x_m) = \alpha_1\delta_{m-1} + \alpha_2\delta_m + \alpha_1\delta_{m+1}\\
U_m^{'} &= U'(x_m) = \beta_1\delta_{m-1} + \beta_1\delta_{m+1}\\
U_m^{''} &= U''(x_m) = \gamma_1\delta_{m-1} + \gamma_2\delta_m + \gamma_1\delta_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
V_m &= V(x_m) = \alpha_1\sigma_{m-1} + \alpha_2\sigma_m + \alpha_1\sigma_{m+1}\\
V_m^{'} &= V'(x_m) = \beta_1\sigma_{m-1} + \beta_1\sigma_{m+1}\\
V_m^{''} &= V''(x_m) = \gamma_1\sigma_{m-1} + \gamma_2\sigma_m + \gamma_1\sigma_{m+1}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= \sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right), & \alpha_2 &= \frac{2}{(1+2\cos(h))},\\
\beta_1 &= -\frac{3\csc\left(\frac{3h}{2}\right)}{4}, & \beta_2 &= \frac{3\csc\left(\frac{3h}{2}\right)}{4}\\
\gamma_1 &= \frac{3((1+3\cos(h))\csc^2(\frac{h}{2}))}{16\left(2\cos(\frac{h}{2})+\cos\left(\frac{3h}{2}\right)\right)}, & \gamma_2 &= -\frac{3\cot^2\left(\frac{h}{2}\right)}{(2+4\cos(h))}.
\end{aligned}$$

2.2 Quintic Trigonometric B-spline Basis

Now, quintic trigonometric B-spline functions $T_m^5(x)$ form a basis over the region $a \le x \le b$ and vanish outside the interval $[x_{m-3}, x_{m+3}]$. These quintic trigonometric B-spline base functions $T_m^5(x), m = -2(1)N+2$ are defined at the nodes x_m by [27]

$$T_m^5(x) = \frac{1}{\theta}\begin{cases} \rho^5(x_{m-3}) & , \quad x_{m-3} \le x \le x_{m-2} \\ \\ -\rho^4(x_{m-3})\rho(x_{m-1}) - \rho^3(x_{m-3})\rho(x_m)\rho(x_{m-2}) & \\ -\rho^2(x_{m-3})\rho(x_{m+1})\rho^2(x_{m-2}) - \rho(x_{m-3})\rho(x_{m+2})\rho^3(x_{m-2}) & , \quad x_{m-2} \le x \le x_{m-1} \\ -\rho(x_{m-3})\rho^4(x_{m-2}) & \\ \\ \rho^3(x_{m-3})\rho^2(x_m) + \rho^2(x_{m-3})\rho(x_{m+1})\rho(x_{m-2})\rho(x_m) & \\ +\rho^2(x_{m-3})\rho^2(x_{m+1})\rho(x_{m-1}) + \rho(x_{m-3})\rho(x_{m+2})\rho^2(x_{m-2})\rho^2(x_m) + & \\ \rho(x_{m-3})\rho(x_{m+2})\rho(x_{m-2})\rho(x_{m+1})\rho(x_{m-1}) + \rho(x_{m-3})\rho^2(x_{m+2})\rho^2(x_{m-1}) & , \quad x_{m-1} \le x \le x_m \\ +\rho(x_{m+3})\rho^3(x_{m-2})\rho(x_m) + \rho(x_{m+3})\rho^2(x_{m-2})\rho(x_{m+1})\rho(x_{m-1}) & \\ +\rho(x_{m+3})\rho(x_{m-2})\rho(x_{m+2})\rho^2(x_{m-1}) + \rho^2(x_{m+3})\rho^3(x_{m-1}) & \\ \\ -\rho^2(x_{m-3})\rho^3(x_{m+1}) - \rho(x_{m-3})\rho(x_{m+2})\rho(x_{m-2})\rho^2(x_{m+1}) & \\ -\rho(x_{m-3})\rho^2(x_{m+2})\rho(x_{m-1})\rho(x_{m+1}) - \rho(x_{m-3})\rho^3(x_{m+2})\rho(x_m) & \\ -\rho(x_{m+3})\rho^2(x_{m-2})\rho^2(x_{m-+1}) - \rho(x_{m+3})\rho(x_{m-2})\rho(x_{m+2})\rho(x_{m-1})\rho(x_{m+1}) & , \quad x_m \le x \le x_{m+1} \\ -\rho(x_{m+3})\rho(x_{m-2})\rho^2(x_{m+2})\rho(x_m) - \rho^2(x_{m+3})\rho^2(x_{m-1})\rho(x_{m+1}) - & \\ \rho^2(x_{m+3})\rho(x_{m-1})\rho(x_{m+2})\rho(x_m) - \rho^3(x_{m+3})\rho^2(x_m) & \\ \\ \rho(x_{m-3})\rho^4(x_{m+2}) + \rho(x_{m+3})\rho(x_{m-2})\rho^3(x_{m+2}) + \rho^2(x_{m+3})\rho(x_{m-1})\rho^2(x_{m+2}) & , \quad x_{m+1} \le x \le x_{m+2} \\ +\rho^3(x_{m+3})\rho(x_m)\rho(x_{m+2}) + \rho^4(x_{m+3})\rho(x_{m+1}) & \\ \\ -\rho^5(x_{m+3}) & , \quad x_{m+2} \le x \le x_{m+3} \\ \\ 0 & , \quad otherwise \end{cases}$$

in which

$\rho(x_m) = \sin(\dfrac{x - x_m}{2})$, for $m = 0(1)N, \theta = \sin(\dfrac{h}{2})\sin(h)\sin(\dfrac{3h}{2})\sin(2h)\sin(\dfrac{5h}{2})$,

Let $U_N(x,t)$ and $V_N(x,t)$ be approximate solution to $U(x,t)$ and $V(x,t)$ defined as

$$U(x,t) \approx U_N(x,t) = \sum_{m=-2}^{N+2} T_m^5(x)\delta_m(t), \tag{4}$$

$$V(x,t) \approx V_N(x,t) \sum_{m=-2}^{N+2} T_m^5(x)\sigma_m(t) \tag{5}$$

and

$$\begin{aligned}
U_m &= U(x_m) = a_1\delta_{m-2} + a_2\delta_{m-1} + a_3\delta_m + a_2\delta_{m+1} + a_1\delta_{m+2} \\
U'_m &= U'(x_m) = b_1\delta_{m-2} + b_2\delta_{m-1} - b_2\delta_{m+1} - b_1\delta_{m+2} \\
U''_m &= U''(x_m) = c_1\delta_{m-2} + c_2\delta_{m-1} + c_3\delta_m + c_2\delta_{m+1} + c_1\delta_{m+2} \\
U'''_m &= U'''(x_m) = d_1\delta_{m-2} + d_2\delta_{m-1} - d_2\delta_{m+1} - d_1\delta_{m+2} \\
U_m^{(4)} &= U^{(4)}(x_m) = e_1\delta_{m-2} + e_2\delta_{m-1} + e_3\delta_m + e_2\delta_{m+1} + e_1\delta_{m+2}
\end{aligned}$$

where

$$a_1 = \sin^5(\frac{h}{2})/\theta$$

$$a_2 = 2\sin^5(\frac{h}{2})\cos(\frac{h}{2})(16\cos^2(\frac{h}{2}) - 3)/\theta$$

$$a_3 = 2(1 + 48\cos^4(\frac{h}{2}) - 16\cos^2(\frac{h}{2}))\sin^5(\frac{h}{2})/\theta$$

$$b_1 = (-5/2)\sin^4(\frac{h}{2})\cos(\frac{h}{2})/\theta$$

$$b_2 = -5\sin^4(\frac{h}{2})\cos^2(\frac{h}{2})(8\cos^2(\frac{h}{2}) - 3)/\theta$$

$$c_1 = (5/4)\sin^3(\frac{h}{2})(5\cos^2(\frac{h}{2}) - 1)/\theta$$

$$c_2 = (5/2)\sin^3(\frac{h}{2})\cos(\frac{h}{2})(-15\cos^2(\frac{h}{2}) + 3 + 16\cos^4(\frac{h}{2}))/\theta$$

$$c_3 = (-5/2)\sin^3(\frac{h}{2})(16\cos^6(\frac{h}{2}) - 5\cos^2(\frac{h}{2}) + 1)/\theta$$

$$d_1 = (-5/8)\sin^2(\frac{h}{2})\cos(\frac{h}{2})(25\cos^2(\frac{h}{2}) - 13)/\theta$$

$$d_2 = (-5/4)\sin^2(\frac{h}{2})\cos^2(\frac{h}{2})(8\cos^4(\frac{h}{2}) - 35\cos^2(\frac{h}{2}) + 15)/\theta$$

$$e_1 = (5/16)(125\cos^4(\frac{h}{2}) - 114\cos^2(\frac{h}{2}) + 13)\sin(\frac{h}{2})/\theta$$

$$e_2 = (-5/8)\sin(\frac{h}{2})\cos(\frac{h}{2})(176\cos^6(\frac{h}{2}) - 137\cos^4(\frac{h}{2}) - 6\cos^2(\frac{h}{2}) + 15)/\theta$$

$e_3 = (5/8)(92\cos^6(\dfrac{h}{2}) - 117\cos^4(\dfrac{h}{2}) + 62\cos^2(\dfrac{h}{2}) - 13)(-1 + 4\cos^2(\dfrac{h}{2}))\sin(\dfrac{h}{2})/\theta$

The same approximations can be obtained for V and its derivatives by replacing δ_m's by σ_m's.

Now, we are going to discretize the cBE given as

$$\begin{aligned} U_t - U_{xx} + k_1 U U_x + k_2 (UV)_x &= 0 \\ V_t - V_{xx} + k_1 V V_x + k_3 (UV)_x &= 0 \end{aligned}$$

For this purpose, we have implemented the Crank-Nicolson type scheme for space discretization and forward finite difference scheme for the time discretization. Firstly the equation is discretized as,

$$\begin{aligned} &\frac{U^{n+1} - U^n}{\Delta t} - \frac{(U_{xx})^{n+1} + (U_{xx})^n}{2} + k_1 \frac{(UU_x)^{n+1} + (UU_x)^n}{2} \\ &\qquad + k_2 \frac{((UV)_x)^{n+1} + ((UV)_x)^n}{2} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{V^{n+1} - V^n}{\Delta t} - \frac{(V_{xx})^{n+1} + (V_{xx})^n}{2} + k_1 \frac{(VV_x)^{n+1} + (VV_x)^n}{2} \\ &\qquad + k_2 \frac{((UV)_x)^{n+1} + ((UV)_x)^n}{2} = 0 \end{aligned}$$

where a linearization technique is used at the left hand side of the Eq. (8) to linearize the nonlinear terms as $U = Z_i$ and $V = G_i$ they would be like that. After these linearizations, Eq. (8) becomes

$$\begin{aligned} &\frac{U^{n+1} - U^n}{\Delta t} - \frac{(U_{xx})^{n+1} + (U_{xx})^n}{2} + k_1 Z_i \frac{(U_x^{n+1} + U_x^n)}{2} \\ &\qquad + k_2 G_i \frac{(U_x^{n+1} + U_x^n)}{2} + k_2 Z_i \frac{(V_x^{n+1} + V_x^n)}{2} = 0 \end{aligned}$$

$$\begin{aligned} &\frac{V^{n+1} - V^n}{\Delta t} - \frac{(V_{xx})^{n+1} + (V_{xx})^n}{2} + k_1 G_i \frac{(V_x^{n+1} + V_x^n)}{2} \\ &\qquad + k_3 G_i \frac{(U_x^{n+1} + U_x^n)}{2} + k_3 Z_i \frac{(V_x^{n+1} + V_x^n)}{2} = 0. \end{aligned}$$

When they are rearranged, they become as follows

$$\begin{aligned} &\frac{2U^{n+1}}{\Delta t} - U_{xx}^{n+1} + k_1 Z_i U_x^{n+1} + k_2 G_i U_x^{n+1} + k_2 Z_i V_x^{n+1} \\ &\qquad = \frac{2U^n}{\Delta t} + U_{xx}^n - k_1 Z_i U_x^n - k_2 G_i U_x^n - k_2 Z_i V_x^n \end{aligned} \tag{7}$$

$$\begin{aligned} &2\frac{V^{n+1}}{\Delta t} - V_{xx}^{n+1} + k_1 G_i V_x^{n+1} + k_3 G_i U_x^{n+1} + k_3 Z_i V_x^{n+1} \\ &\qquad = 2\frac{V^n}{\Delta t} + V_{xx}^n - k_1 G_i V_x^n - k_3 G_i U_x^n - k_3 Z_i V_x^n. \end{aligned}$$

2.3 Cubic Trigonometric B-spline Collocation Method (CTBCM) and Quintic Trigonometric B-spline Collocation Method (QTBCM)

By using cubic approximations given by (2)−(3) and their derivatives in (10), we obtain the following iterative scheme:

$$\begin{aligned}&(g1)\ \delta_{i-1}^{n+1}+(g2)\ \sigma_{i-1}^{n+1}+(g3)\ \delta_{i}^{n+1}+(g4)\ \sigma_{i}^{n+1}+(g5)\ \delta_{i+1}^{n+1}+(g6)\ \sigma_{i+1}^{n+1}\\ &=(h1)\ \delta_{i-1}^{n}+(h2)\ \sigma_{i-1}^{n}+(h3)\ \delta_{i}^{n}+(h4)\ \sigma_{i}^{n}+(h5)\ \delta_{i+1}^{n}+(h6)\ \sigma_{i+1}^{n}\end{aligned}$$

$$\begin{aligned}&(g7)\ \delta_{i-1}^{n+1}+(g8)\ \sigma_{i-1}^{n+1}+(g9)\ \delta_{i}^{n+1}+(g10)\ \sigma_{i}^{n+1}+(g11)\ \delta_{i+1}^{n+1}+(g12)\ \sigma_{i+1}^{n+1}\\ &=(h7)\ \delta_{i-1}^{n}+(h8)\ \sigma_{i-1}^{n}+(h9)\ \delta_{i}^{n}+(h10)\ \sigma_{i}^{n}+(h11)\ \delta_{i+1}^{n}+(h12)\ \sigma_{i+1}^{n}\end{aligned}$$

where

$g1 = 2\alpha_1/\Delta t - \gamma_1 + k_1 Z_i\beta_1 + k_2 G_i\beta_1$ $h1 = 2\alpha_1/\Delta t + \gamma_1 - k_1 Z_i\beta_1 - k_2 G_i\beta_1$
$g2 = Z_i k_2\beta_1$ $h2 = -Z_i k_2\beta_1$
$g3 = 2\alpha_2/\Delta t - \gamma_2$ $h3 = 2\alpha_2/\Delta t + \gamma_2$
$g4 = 0$ $h4 = 0$
$g5 = 2\alpha_1/\Delta t - \gamma_1 + k_1 Z_i\beta_2 + k_2 G_i\beta_2$ $h5 = 2\alpha_1/\Delta t + \gamma_1 - k_1 Z_i\beta_2 - k_2 G_i\beta_2$
$g6 = k_2 Z_i\beta_2$ $h6 = -k_2 Z_i\beta_2$
$g7 = k_3 G_i\beta_1$ $h7 = -k_3 G_i\beta_1$
$g8 = 2\alpha_1/\Delta t - \gamma_1 + k_1 G_i\beta_1 + k_3 Z_i\beta_1$ $h8 = 2\alpha_1/\Delta t + \gamma_1 - k_1 G_i\beta_1 - k_3 Z_i\beta_1$
$g9 = 0$ $h9 = 0$
$g10 = 2\alpha_2/\Delta t - \gamma_2$ $h10 = 2\alpha_2/\Delta t + \gamma_2$
$g11 = k_3 G_i\beta_2$ $h11 = -k_3 G_i\beta_2$
$g12 = 2\alpha_1/\Delta t - \gamma_1 + k_1 G_i\beta_2 + k_3 Z_i\beta_2$ $h12 = 2\alpha_1/\Delta t + \gamma_1 - k_1 G_i\beta_2 - k_3 Z_i\beta_2$

This iterative scheme results in a system of equations involving $(2N+6)$ unknowns and $(2N+2)$ equations. Using the boundary conditions of the problem, the unknowns δ_{-1}, σ_{-1} from the left boundary and the unknowns δ_{N+1}, σ_{N+1} from the right boundary are eliminated and a solvable system of equations is obtained.

In a similar way, but now using quintic approximations given by (4)−(5) and their derivatives in (10), we obtain the following iterative scheme:

$$\begin{aligned}&(m1)\delta_{i-2}^{n+1}+(m2)\sigma_{i-2}^{n+1}+(m3)\delta_{i-1}^{n+1}+(m4)\sigma_{i-1}^{n+1}+(m5)\delta_{i}^{n+1}+(m6)\sigma_{i}^{n+1}\\ &\qquad+(m7)\delta_{i+1}^{n+1}+(m8)\sigma_{i+1}^{n+1}+(m9)\delta_{i+2}^{n+1}+(m10)\sigma_{i+2}^{n+1}\\ &=(f1)\delta_{i-2}^{n}+(f2)\sigma_{i-2}^{n}+(f3)\delta_{i-1}^{n}+(f4)\sigma_{i-1}^{n}+(f5)\delta_{i}^{n}+(f6)\sigma_{i}^{n}+(f7)\delta_{i+1}^{n}\\ &\qquad+(f8)\sigma_{i+1}^{n}+(f9)\delta_{i+2}^{n}+(f10)\sigma_{i+2}^{n}\end{aligned}$$

$$\begin{aligned}&(m11)\delta_{i-2}^{n+1}+(m12)\sigma_{i-2}^{n+1}+(m13)\delta_{i-1}^{n+1}+(m14)\sigma_{i-1}^{n+1}+(m15)\delta_{i}^{n+1}\\ &\qquad+(m16)\sigma_{i}^{n+1}+(m17)\delta_{i+1}^{n+1}+(m18)\sigma_{i+1}^{n+1}+(m19)\delta_{i+2}^{n+1}+(m20)\sigma_{i+2}^{n+1}\\ &=(f11)\delta_{i-2}^{n}+(f12)\sigma_{i-2}^{n}+(f13)\delta_{i-1}^{n}+(f14)\sigma_{i-1}^{n}+(f15)\delta_{i}^{n}+(f16)\sigma_{i}^{n}\\ &\qquad+(f17)\delta_{i+1}^{n}+(f18)\sigma_{i+1}^{n}+(f19)\delta_{i+2}^{n}+(f20)\sigma_{i+2}^{n}\end{aligned}$$

where

$$m1 = 2a_1/\Delta t - c_1 + k_1 Z_i b_1 + k_2 G_i b_1$$
$$m2 = k_2 Z_i b_1$$
$$m3 = 2a_2/\Delta t - c_2 + k_1 Z_i b_2 + k_2 G_i b_2$$
$$m4 = k_2 Z_i b_2$$
$$m5 = 2a_3/\Delta t - c_3$$
$$m6 = 0$$
$$m7 = 2a_2/\Delta t - c_2 - k_1 Z_i b_2 - k_2 G_i b_2$$
$$m8 = -k_2 Z_i b_2$$
$$m9 = 2a_1/\Delta t - c_1 - k_1 Z_i b_1 - k_2 G_i b_1$$
$$m10 = -k_2 Z_i b_1$$
$$m11 = k_3 G_i b_1$$
$$m12 = 2a_1/\Delta t - c_1 + k_1 G_i b_1 + k_3 Z_i b_1$$
$$m13 = k_3 G_i b_2$$
$$m14 = 2a_2/\Delta t - c_2 + k_1 G_i b_2 + k_3 Z_i * b_2$$
$$m15 = 0$$
$$m16 = 2a_3/\Delta t - c_3$$
$$m17 = -k_3 G_i b_2$$
$$m18 = 2a_2/\Delta t - c_2 - k_1 G_i b_2 - k_3 Z_i b_2$$
$$m19 = -k_3 G_i b_1$$
$$m20 = 2a_1/\Delta t - c_1 - k_1 G_i b_1 - k_3 Z_i b_1$$

$$f1 = 2a_1/\Delta t + c_1 - k_1 Z_i * b_1 - k_2 G_i b_1$$
$$f2 = -k_2 Z_i b_1$$
$$f3 = 2a_2/\Delta t + c_2 - k_1 Z_i b_2 - k_2 G_i b_2$$
$$f4 = -k_2 Z_i b_2$$
$$f5 = 2a_3/\Delta t + c_3$$
$$f6 = 0$$
$$f7 = 2a_2/\Delta t + c_2 + k_1 Z_i b_2 + k_2 G_i b_2$$
$$f8 = k_2 Z_i b_2$$
$$f9 = 2a_1/\Delta t + c_1 + k_1 Z_i b_1 + k_2 G_i b_1$$
$$f10 = k_2 Z_i b_1$$
$$f11 = -k_3 G_i b_1$$
$$f12 = 2a_1/\Delta t + c_1 - k_3 Z_i b_1 - k_1 G_i b_1$$
$$f13 = -k_3 G_i b_2$$
$$f14 = 2a_2/\Delta t + c_2 - k_3 Z_i b_2 - k_1 G_i b_2$$
$$f15 = 0$$
$$f16 = 2a_3/\Delta t + c_3$$
$$f17 = k_3 G_i b_2$$
$$f18 = 2a_2/\Delta t + c_2 + k_3 Z_i b_2 + k_1 G_i b_2$$
$$f19 = k_3 G_i b_1$$
$$f20 = 2a_1/\Delta t + c_1 + k_3 Z_i b_1 + k_1 G_i * b_1$$

This iterative scheme results in a system of equations involving $(2N + 10)$ unknowns and $(2N + 2)$ equations. Using the boundary conditions of the problem, the unknowns δ_{-2}, σ_{-2}, δ_{-1}, σ_{-1} from the left boundary and the unknowns δ_{N+2}, σ_{N+2}, δ_{N+1}, σ_{N+1} from the right boundary are eliminated and solvable system is obatined.

Now utilizing this scheme, we are going to carry out our calculations until the desired time level. But for this, first of all we need the initial values of the unknowns at time $t = 0$. The following section will deal with this step of the solution process.

2.4 Initial state

The initial vector d^0 is determined from the initial and boundary conditions. For CTBCM, the approximation (2) must be rewritten as

$$U_N(x,0)= \sum_{m=-1}^{N+1} \delta_m^0(t) T_m^3(x), \quad V_N(x,0)= \sum_{m=-1}^{N+1} \sigma_m^0(t) T_m^3(x)$$

and for QTBCM, the approximation (4) must be as

$$U_N(x,0)= \sum_{m=-2}^{N+2} \delta_m^0(t) T_m^5(x), \quad V_N(x,0)= \sum_{m=-2}^{N+2} \sigma_m^0(t) T_m^5(x)$$

where the δ_m^0's and σ_m^0's are unknown initial parameters.

We require the initial numerical approximation $U_N(x,0)$ and $V_N(x,0)$ satisfy the following initial conditions:

$$U_N(x_i,0) = f(x_i), \qquad i = 0,1,...,N$$

$$V_N(x_i,0) = g(x_i), \qquad i = 0,1,...,N$$

respectively. While this requirement results in $(2N+2)$ equations and $(2N+6)$ unknowns for CTBCM, it results in $(2N + 2)$ equations and $(2N + 10)$ unknowns for QTBCM. Using $U'(x_i,0) = f'(x_i)$ and $V'(x_i,0) = g'(x_i)$ for CTBCM; and using $U'(x_i,0) = f'(x_i)$, $V'(x_i,0) = g'(x_i)$, $U''(x_i,0) = f''(x_i)$, $V''(x_i,0) = g''(x_i)$ for QTBCM, because the first and the second derivatives of the approximate initial conditions shall agree with those of the exact initial conditions, the initial vector d^0 is obtained by means of the matrix equation in the following form

$$Wd^0 = b.$$

Here the coefficent matrix W and the right hand side column vector b are obtained for CTBCM and QTBCM accordingly. The solution of system of equations results in the initial values of δ_m^0 and σ_m^0. Thus one can start the iterative procedure to find out the next time values of δ_m and σ_m.

3 Stability Analysis

In this section, CTBCM has been applied to the first equation (1a) in Eq.(1) then $Z_i = \widehat{U}$ and $G_i = \widehat{V}$ are taken, and von Neumann method has been applied to the the following scheme

$$\begin{aligned}&(g1)\delta_{i-1}^{n+1} + (g2)\sigma_{i-1}^{n+1} + (g3)\delta_i^{n+1} + (g4)\sigma_i^{n+1} + (g5)\delta_{i+1}^{n+1} + (g6)\sigma_{i+1}^{n+1}\\ &= (h1)\delta_{i-1}^{n} + \sigma_{i-1}^{n}(h2) + (h3)\delta_i^{n} + (h4)\sigma_i^{n} + (h5)\delta_{i+1}^{n} + (h6)\sigma_{i+1}^{n}\end{aligned}$$

where

$$\begin{aligned}
g1 &= \frac{2\alpha_1}{\Delta t} - \gamma_1 + k_1\widehat{U}\beta_1 + k_2\widehat{V}\beta_1\\
g2 &= k_2 Z_{\dot{I}}\beta_1\\
g3 &= \frac{2}{\Delta t}\alpha_2 - \gamma_2\\
g4 &= 0\\
g5 &= \frac{2}{\Delta t}\alpha_1 - \gamma_1 + k_1\widehat{U}\beta_2 + k_2\widehat{V}\beta_2\\
g6 &= k_2\widehat{U}\beta_2
\end{aligned}$$

and

$$\begin{aligned}
h1 &= \frac{2\alpha_1}{\Delta t} + \gamma_1 - k_1\widehat{U}\beta_1 - k_2\widehat{V}\beta_1\\
h2 &= -k_2\widehat{U}\beta_1\\
h3 &= \frac{2}{\Delta t}\alpha_2 + \gamma_2\\
h4 &= 0\\
h5 &= \frac{2}{\Delta t}\alpha_1 + \gamma_1 - k_1\widehat{U}\beta_2 - k_2\widehat{V}\beta_2\\
h6 &= -k_2\widehat{U}\beta_2
\end{aligned}$$

In this scheme, A and B are harmonic amplitudes, $\phi = kh$, k is mode number, $i = \sqrt{-1}$ and g is the amplification factor, when in place of δ_i^n and σ_i^n the following notations are used

$$\begin{aligned}
\delta_i^n &= A\zeta\exp(ij\phi)\\
\sigma_i^n &= B\zeta\exp(ij\phi)\\
g &= \frac{\zeta^{n+1}}{\zeta^n}
\end{aligned}$$

and the required changes are made, the following equality

$$g = \frac{X_2 + iY}{X_1 - iY} \tag{8}$$

is obtained. Here

$$X_1 = A\left[2\left(\alpha_1 - \frac{\gamma_1\Delta t}{2}\right)\cos\phi + \left(\alpha_2 - \frac{\gamma_2\Delta t}{2}\right)\right],$$
$$X_2 = A\left[2\left(\alpha_1 + \frac{\gamma_1\Delta t}{2}\right)\cos\phi + \left(\alpha_2 + \frac{\gamma_2\Delta t}{2}\right)\right]$$

and

$$Y = \left[\sin\varphi\left(A\left(\frac{\beta_2 k_2\Delta t}{2}\widehat{V} + \frac{\beta_2 k_1\Delta t}{2}\widehat{U} - \frac{\beta_1 k_2\Delta t}{2}\widehat{V} + \frac{\beta_1 k_1\Delta t}{2}\widehat{U}\right) + B\left(-\frac{\beta_1 k_2\Delta t}{2}\widehat{U} + \frac{\beta_2 k_2\Delta t}{2}\widehat{U}\right)\right)\right]$$

Since the numerical scheme obtained as a result of the linearization technique applied in the study given with Ref.[20] for the coupled Burgers equation is similar to the numerical scheme given in this study (16), similarly, $|g| \leq 1$ is found as shown in Ref. [20]. From here the scheme is unconditionally stable. Since U and V are symmetrical, similar results are obtained in the second equation (1b) in the coupled Burgers system given by (1).

4 Numerical examples and results

In this section, three common test problems about the cBE are going to be solved and the results will be compared with some of those available in the literature. If the exact solution of the test problem is available, then the accuracy of the numerical method is going to be controlled by using the error norms L_2 and L_∞ given as follows, respectively:

$$L_2 = \sqrt{\frac{\sum_{i=0}^{N}\left|U_i - (U_N)_i\right|^2}{\sum_{i=0}^{N}\left|U_i\right|^2}}, \qquad L_\infty = \max_{1\leq i\leq N}\left|U_i - (U_N)_i\right|$$

4.1 Test Problem 1

First of all, the cBE is considered for $k_1 = -2$ and $k_2 = k_3 = 1$ with the following initial and boundary conditions

$$U(x,0) = \sin(x), \quad V(x,0) = \sin(x)$$

and

$$U(-\pi,t) = U(\pi,t) = V(-\pi,t) = V(\pi,t) = 0$$

For this problem the analytical solution is

$$U(x,t) = V(x,t) = e^{-t}\sin(x)$$

Table 1: Comparison of the calculated error norms L_2 and $L_\infty$of Problem 1 with $\Delta t = 0.01$ for $N = 50$, 100, and 200 at different times on $[-\pi, \pi]$.

	CTBCM					
	$N = 50$		$N = 100$		$N = 200$	
t	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	$1.22694e-04$	$1.10799e-04$	$3.00195e-05$	$2.71628e-05$	$6.87807e-06$	$6.22353e-06$
0.5	$6.13619e-04$	$3.71445e-04$	$1.50107e-04$	$9.10443e-05$	$3.43908e-05$	$2.08591e-05$
1.0	$1.22762e-03$	$4.50723e-04$	$3.00236e-04$	$1.10451e-04$	$6.87828e-05$	$2.53038e-05$
1.5	$1.84199e-03$	$4.10192e-04$	$4.50388e-04$	$1.00495e-04$	$1.03176e-04$	$2.30217e-05$
2.0	$2.45674e-03$	$3.31827e-04$	$6.00562e-04$	$8.12772e-05$	$1.37570e-04$	$1.86181e-05$
2.5	$3.07186e-03$	$2.51656e-04$	$7.50759e-04$	$6.16260e-05$	$1.71966e-04$	$1.41158e-05$
3.0	$3.68737e-03$	$1.83221e-04$	$9.00978e-04$	$4.48570e-05$	$2.06363e-04$	$1.02742e-05$
	QTBCM					
0.1	$1.00394e-06$	$9.07529e-07$	$8.43609e-07$	$7.63698e-07$	$8.33815e-07$	$7.54645e-07$
0.5	$5.00073e-06$	$3.03764e-06$	$4.21016e-06$	$2.55768e-06$	$4.16519e-06$	$2.52833e-06$
1.0	$9.95481e-06$	$3.67096e-06$	$8.40074e-06$	$3.09647e-06$	$8.32072e-06$	$3.06399e-06$
1.5	$1.48628e-05$	$3.32544e-06$	$1.25719e-05$	$2.81095e-06$	$1.24666e-05$	$2.78454e-06$
2.0	$1.97251e-05$	$2.67730e-06$	$1.67237e-05$	$2.26810e-06$	$1.66029e-05$	$2.24934e-06$
2.5	$2.45423e-05$	$2.02065e-06$	$2.08562e-05$	$1.71566e-06$	$2.07296e-05$	$1.70342e-06$
3.0	$2.93146e-05$	$1.46401e-06$	$2.49695e-05$	$1.24586e-06$	$2.48467e-05$	$1.23839e-06$

To start the initialization process, the needed initial and boundary conditions are obtained from the analytical solution. Table 1 shows a comparison of the calculated error norms L_2 and L_∞ of Problem 1 with $\Delta t = 0.01$ for $N =$ 50, 100, and 200 at different times on $[-\pi, \pi]$. From the table it is seen that using quintic B-spline base functions instead of cubic ones produces much better results. Table 2 shows a comparison of the calculated error norms L_2 and L_∞ of Problem 1 with $N = 100$ at different times. In Table 3, a comparison of the calculated error norms L_∞ of Problem 1 for $U(x,t) = V(x,t)$ with $\Delta t = 0.01$ for $N = 50$ at different times is presented. From the table, it is clear that the present results obtained by CTBCM are in good agreement with those of compared ones, and the present results obtained by QTCBCM are much better than all of the compared ones. Table 4 presents a comparison of the calculated error norms L_2 and L_∞ of Problem 1 with results from [8] and [12] with $\Delta t = 0.001$ for $N = 200$ and 400 at different times. Again from the table it is obvious that while the results obtained by CTBCM are in good agreement with those of compared ones, the results obtained by QTBCM are much more better than all of the compared ones. Figure 1 shows numerical simulations of Problem 1 for values of $N = 100$, $\Delta t = 0.001$, $k_1 = -2$, $k_2 = k_3 = 1$ at times $t = 1, 2$ and 3. In fact, both the exact and numerical solutions of the problem are drawn on this diagram, but the curves are indistinguishable since they are very close to each other.

4.2 Test Problem 2

In the second test problem, the numerical solutions of cBE are obtained for $k_1 = 2$ with different values of k_2 and k_3 at various time levels. For the second test problem the exact solutions are

$$U(x,t) = a_0 - 2A\left(\frac{2k_2 - 1}{4k_2k_3 - 1}\right)\tan h\left(A\left(x - 2At\right)\right)$$

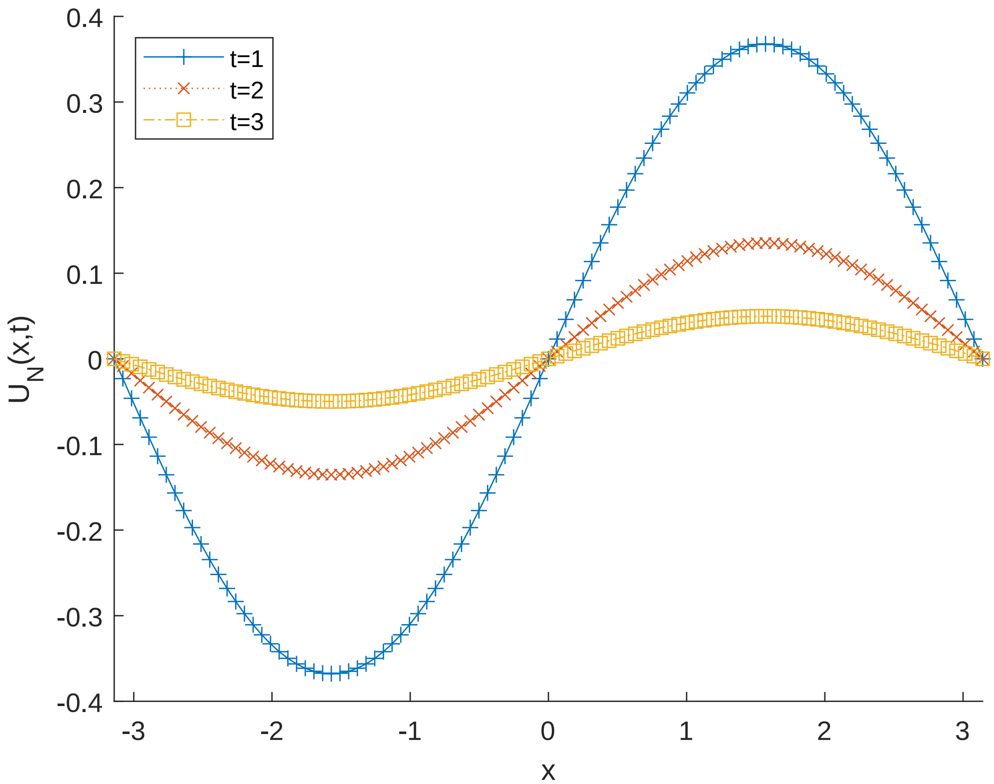


Figure 1: Numerical simulations of Problem I for values of $N = 100$, $\Delta t = 0.001$, $k_1 = -2$, $k_2 = k_3 = 1$ at times $t = 1, 2$ and 3.

Table 2: Comparison of the calculated error norms L_2 and L_∞ of Problem 1 with $N = 100$ at different times.

	CTBCM					
	$\Delta t = 0.01$		$\Delta t = 0.005$		$\Delta t = 0.001$	
t	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	$3.00195e-05$	$2.71628e-05$	$3.06440e-05$	$2.77278e-05$	$3.08438e-05$	$2.79086e-05$
0.5	$1.50107e-04$	$9.10443e-05$	$1.53229e-04$	$9.29383e-05$	$1.54229e-04$	$9.35443e-05$
1.0	$3.00236e-04$	$1.10451e-04$	$3.06482e-04$	$1.12748e-04$	$3.08481e-04$	$1.13484e-04$
1.5	$4.50388e-04$	$1.00495e-04$	$4.59758e-04$	$1.02586e-04$	$4.62757e-04$	$1.03255e-04$
2.0	$6.00562e-04$	$8.12772e-05$	$6.13058e-04$	$8.29684e-05$	$6.17057e-04$	$8.35096e-05$
2.5	$7.50759e-04$	$6.16260e-05$	$7.66381e-04$	$6.29084e-05$	$7.71380e-04$	$6.33188e-05$
3.0	$9.00978e-04$	$4.48570e-05$	$9.19728e-04$	$4.57906e-05$	$9.25728e-04$	$4.60893e-05$
	QTBCM					
0.1	$8.43609e-07$	$7.63698e-07$	$2.18900e-07$	$1.98165e-07$	$1.89960e-08$	$1.71967e-08$
0.5	$4.21016e-06$	$2.55768e-06$	$1.09245e-06$	$6.63669e-07$	$9.48025e-08$	$5.75928e-08$
1.0	$8.40074e-06$	$3.09647e-06$	$2.17983e-06$	$8.03476e-07$	$1.89165e-07$	$6.97252e-08$
1.5	$1.25719e-05$	$2.81095e-06$	$3.26217e-06$	$7.29389e-07$	$2.83089e-07$	$6.32960e-08$
2.0	$1.67237e-05$	$2.26810e-06$	$4.33949e-06$	$5.88530e-07$	$3.76579e-07$	$5.10724e-08$
2.5	$2.08562e-05$	$1.71566e-06$	$5.41181e-06$	$4.45185e-07$	$4.69634e-07$	$3.86329e-08$
3.0	$2.49695e-05$	$1.24586e-06$	$6.47916e-06$	$3.23280e-07$	$5.62259e-07$	$2.80541e-08$

Table 3: Comparison of the calculated error norms L_∞ of Problem 1 for $U(x,t) = V(x,t)$ with $\Delta t = 0.01$ for $N = 50$ at different times.

	Present		[12]	[15]	[18]	[21]
t	CTBCM	QTBCM				
0.5	3.71445×10^{-4}	3.03764×10^{-6}	2.2662×10^{-5}	1.1030×10^{-4}	1.51688×10^{-4}	7.9881×10^{-4}
1.0	4.50723×10^{-4}	3.67096×10^{-6}	1.4617×10^{-5}	1.3368×10^{-4}	1.83970×10^{-4}	9.6837×10^{-4}
2.0	3.31827×10^{-4}	2.67730×10^{-6}	7.3805×10^{-6}	9.8182×10^{-5}	1.35250×10^{-4}	7.1154×10^{-4}
3.0	1.83221×10^{-4}	1.46401×10^{-6}	4.0272×10^{-6}	1.0298×10^{-5}	7.46014×10^{-5}	3.9213×10^{-4}

$$V(x,t) = a_0\left(\frac{2k_3-1}{2k_2-1}\right) - 2A\left(\frac{2k_2-1}{4k_2k_3-1}\right)\tan h\left(A\left(x-2At\right)\right)$$

For this problem the initial and boundary conditions are taken from the exact solutions as follows

$$U(x,0) = a_0 - 2A\left(\frac{2k_2-1}{4k_2k_3-1}\right)\tan h\left(Ax\right)$$
$$V(x,0) = a_0\left(\frac{2k_3-1}{2k_2-1}\right) - 2A\left(\frac{2k_2-1}{4k_2k_3-1}\right)\tan h\left(Ax\right)$$

here $a_0 = 0.05$ and $A = \frac{1}{2}\left(\frac{a_0(4k_2k_3-1)}{2k_2-1}\right)$. The numerical simulations are run on the domain $[-10, 10]$ at time intervals $\Delta t = 0.01$.

While Table 5 shows a comparison of the computed error norms L_2 and $L_\infty$of Problem 2 using cubic trigonometric B-splines with $\Delta t = 0.01$ for $k_2 = 0.1$ and $k_3 = 0.3$ at various times, Table 6 a comparison of the computed error norms L_2 and L_∞ of Problem 2 using quintic trigonometric B-splines with $\Delta t = 0.01$ for $k_2 = 0.1$ and $k_3 = 0.3$ at various times. From the tables, it is clearly seen that there is a noticeable decrease in both of the error norms L_2 and L_∞.when mesh sizes decrease. Table 7 a comparison of the calculated error norms L_2 and L_∞ with those from other authors for $U_N(x,t)$ and $V_N(x,t)$ of Problem 2 for $\Delta t = 0.01$ and $N = 100$. From the table, it is clearly seen that the present results are in good agreement with those of compared ones.

Table 4: Comparison of the calculated error norms L_2 and L_∞ of Problem 1 with results from [8] and [12] with $\Delta t = 0.001$ for $N = 200$ and 400 at different times.

		Present				[8]		[12]	
		CTBCM		QTBCM					
N	t	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
200	0.1	7.70290×10^{-6}	6.96987×10^{-6}	8.99738×10^{-9}	8.14309×10^{-9}	8.21×10^{-6}	7.45×10^{-6}	0.17×10^{-6}	0.52×10^{-6}
	0.5	3.85151×10^{-5}	2.33606×10^{-5}	4.49451×10^{-8}	2.72823×10^{-8}	2.49×10^{-5}	4.10×10^{-5}	0.27×10^{-6}	0.36×10^{-6}
	1.0	7.70316×10^{-5}	2.83384×10^{-5}	8.97861×10^{-8}	3.30625×10^{-8}	3.00×10^{-5}	8.21×10^{-5}	0.36×10^{-6}	0.22×10^{-6}
400	0.1	1.91936×10^{-6}	1.73671×10^{-6}	8.37395×10^{-9}	7.57795×10^{-9}	2.05×10^{-6}	1.86×10^{-6}	0.07×10^{-6}	0.14×10^{-6}
	0.5	9.59685×10^{-6}	5.82078×10^{-6}	4.18504×10^{-8}	2.53936×10^{-8}	1.02×10^{-5}	6.22×10^{-6}	0.16×10^{-6}	0.14×10^{-6}
	1.0	1.91938×10^{-5}	7.06100×10^{-6}	8.36523×10^{-8}	3.07889×10^{-8}	2.04×10^{-5}	7.56×10^{-6}	0.15×10^{-6}	0.10×10^{-6}

Table 5: Comparison of the computed error norms L_2 and L_∞ of Problem 2 using cubic trigonometric B-splines with $\Delta t = 0.01$ for $k_2 = 0.1$ and $k_3 = 0.3$ at various times.

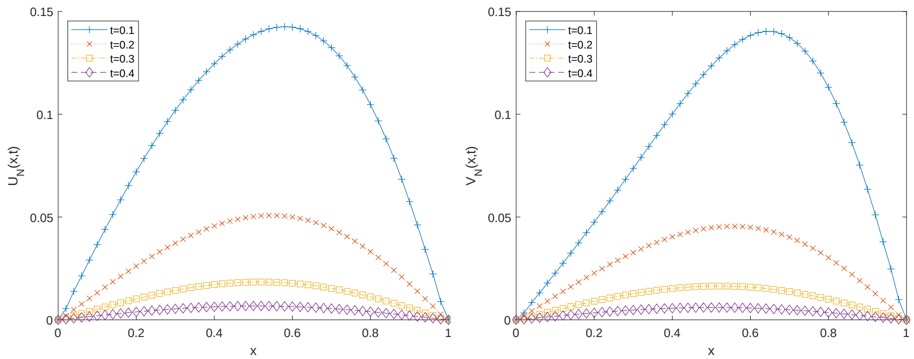
		CTBCM			
		U_N		V_N	
N	t	L_2	L_∞	L_2	L_∞
50	0.1	$5.97533e-04$	$3.86270e-05$	$6.33197e-04$	$2.35912e-05$
	0.5	$2.91102e-03$	$1.85436e-04$	$3.07924e-03$	$1.11898e-04$
	1.0	$5.70536e-03$	$3.62583e-04$	$6.02668e-03$	$2.17005e-04$
	1.5	$8.42042e-03$	$5.34584e-04$	$8.88491e-03$	$3.18514e-04$
	2.0	$1.10693e-02$	$7.03179e-04$	$1.16689e-02$	$4.16985e-04$
	2.5	$1.36595e-02$	$8.68399e-04$	$1.43872e-02$	$5.12937e-04$
	3.0	$1.61963e-02$	$1.03062e-03$	$1.70459e-02$	$6.06530e-04$
100	0.1	$4.56627e-05$	$3.12521e-06$	$8.75203e-05$	$2.51865e-06$
	0.5	$2.21646e-04$	$1.47074e-05$	$4.26445e-04$	$1.24776e-05$
	1.0	$4.33522e-04$	$2.83167e-05$	$8.35758e-04$	$2.47485e-05$
	1.5	$6.39151e-04$	$4.13487e-05$	$1.23347e-03$	$3.68667e-05$
	2.0	$8.39763e-04$	$5.39726e-05$	$1.62153e-03$	$4.88525e-05$
	2.5	$1.03606e-03$	$6.62820e-05$	$2.00108e-03$	$6.07323e-05$
	3.0	$1.22850e-03$	$7.83245e-05$	$2.37286e-03$	$7.25176e-05$
200	0.1	$9.23657e-05$	$5.60410e-06$	$6.14491e-05$	$2.86617e-06$
	0.5	$4.52154e-04$	$2.78132e-05$	$2.95049e-04$	$1.35627e-05$
	1.0	$8.89323e-04$	$5.51797e-05$	$5.71858e-04$	$2.60320e-05$
	1.5	$1.31629e-03$	$8.21730e-05$	$8.36882e-04$	$3.78299e-05$
	2.0	$1.73475e-03$	$1.08833e-04$	$1.09248e-03$	$4.91045e-05$
	2.5	$2.14567e-03$	$1.35194e-04$	$1.34005e-03$	$5.99411e-05$
	3.0	$2.54973e-03$	$1.61277e-04$	$1.58055e-03$	$7.04021e-05$

Table 6: Comparison of the computed error norms L_2 and L_∞ of Problem 2 using quintic trigonometric B-splines with $\Delta t = 0.01$ for $k_2 = 0.1$ and $k_3 = 0.3$ at various times.

		QTBCM			
		U_N		V_N	
N	t	L_2	L_∞	L_2	L_∞
50	0.1	$1.48899e-04$	$9.34184e-06$	$1.14452e-04$	$5.14174e-06$
	0.5	$7.28664e-04$	$4.57126e-05$	$5.54679e-04$	$2.40139e-05$
	1.0	$1.43245e-03$	$9.03097e-05$	$1.08243e-03$	$4.62982e-05$
	1.5	$2.11939e-03$	$1.34155e-04$	$1.59262e-03$	$6.75867e-05$
	2.0	$2.79237e-03$	$1.77405e-04$	$2.08869e-03$	$8.80182e-05$
	2.5	$3.45307e-03$	$2.20099e-04$	$2.57265e-03$	$1.07862e-04$
	3.0	$4.10259e-03$	$2.62290e-04$	$3.04587e-03$	$1.27149e-04$
100	0.1	$1.38311e-04$	$8.53910e-06$	$1.04253e-04$	$4.60873e-06$
	0.5	$6.76629e-04$	$4.20943e-05$	$5.04449e-04$	$2.19331e-05$
	1.0	$1.33020e-03$	$8.32189e-05$	$9.83626e-04$	$4.22821e-05$
	1.5	$1.96822e-03$	$1.23675e-04$	$1.44643e-03$	$6.16680e-05$
	2.0	$2.59334e-03$	$1.63564e-04$	$1.89609e-03$	$8.03133e-05$
	2.5	$3.20710e-03$	$2.02964e-04$	$2.33449e-03$	$9.83320e-05$
	3.0	$3.81053e-03$	$2.41934e-04$	$2.76291e-03$	$1.15801e-04$
200	0.1	$1.38134e-04$	$8.49587e-06$	$1.04076e-04$	$4.58839e-06$
	0.5	$6.75728e-04$	$4.18888e-05$	$5.03499e-04$	$2.18263e-05$
	1.0	$1.32847e-03$	$8.28176e-05$	$9.81729e-04$	$4.20787e-05$
	1.5	$1.96571e-03$	$1.23076e-04$	$1.44361e-03$	$6.13752e-05$
	2.0	$2.59011e-03$	$1.62783e-04$	$1.89236e-03$	$7.99252e-05$
	2.5	$3.20319e-03$	$2.02003e-04$	$2.32986e-03$	$9.78687e-05$
	3.0	$3.80599e-03$	$2.40782e-04$	$2.75740e-03$	$1.15278e-04$

Table 7: Comparison of the calculated error norms L_2 and L_∞ with those from other authors for $U_N(x,t)$ and $V_N(x,t)$ of Problem 2 for $\Delta t = 0.01$ and $N = 100$.

				Present		[6]	[7]	[8]	[12]
				CTBCM	QTBCM				
	t	k_2	k_3			L_2			
$U_N(x,t)$	0.5	0.1	0.30	$2.21646e-04$	$6.76629e-04$	$1.44e-03$	$3.245e-05$	$6.736e-04$	$6.783e-04$
		0.3	0.03	$2.27815e-04$	$7.45551e-04$	$6.68e-04$	$2.733e-05$	$7.326e-04$	$7.609e-04$
	1	0.1	0.30	$4.33522e-04$	$1.33020e-03$	$1.27e-03$	$2.405e-05$	$1.325e-03$	$1.334e-03$
		0.3	0.03	$4.33368e-04$	$1.46927e-03$	$1.30e-03$	$2.832e-05$	$1.452e-03$	$1.500e-03$
$V_N(x,t)$	0.5	0.1	0.30	$4.26445e-04$	$5.04449e-04$	$5.42e-04$	$2.746e-05$	$9.057e-04$	$5.101e-04$
		0.3	0.03	$4.70374e-04$	$1.32205e-03$	$1.20e-03$	$2.454e-04$	$1.591e-03$	$1.327e-03$
	1	0.1	0.30	$8.35758e-04$	$9.83626e-04$	$1.29e-03$	$3.745e-05$	$1.251e-03$	$0.995e-03$
		0.3	0.03	$9.26878e-04$	$2.60682e-03$	$2.35e-03$	$4.525e-04$	$2.250e-03$	$2.617e-03$
						L_∞			
$U_N(x,t)$	0.5	0.1	0.30	$1.47074e-05$	$4.20943e-05$	$4.38e-05$	$9.619e-04$	$4.167e-05$	$4.208e-05$
		0.3	0.03	$2.70943e-05$	$4.61427e-05$	$4.58e-05$	$4.310e-04$	$4.590e-05$	$4.703e-05$
	1	0.1	0.30	$2.83167e-05$	$8.32189e-05$	$8.66e-05$	$1.153e-03$	$8.258e-05$	$8.320e-05$
		0.3	0.03	$4.98805e-05$	$9.22995e-05$	$9.16e-05$	$1.268e-03$	$9.182e-05$	$9.409e-05$
$V_N(x,t)$	0.5	0.1	0.30	$1.24776e-05$	$2.19331e-05$	$4.99e-05$	$3.332e-04$	$1.480e-04$	$0.221e-04$
		0.3	0.03	$7.64233e-05$	$1.81391e-04$	$1.81e-04$	$1.148e-03$	$5.729e-04$	$1.818e-04$
	1	0.1	0.30	$2.47485e-05$	$4.22821e-04$	$9.92e-05$	$1.162e-03$	$4.770e-05$	$4.255e-05$
		0.3	0.03	$1.52359e-04$	$3.62676e-04$	$3.62e-04$	$1.638e-03$	$3.617e-04$	$3.636e-04$



(a) (b)

Figure 2: The behaviour of numerical solutions of Problem 3 for values of $N = 50$, $\Delta t = 0.001$, $k_1 = 2$, $k_2 = k_3 = 10$ at times $t = 0.1$, 0.2, 0.3 and 0.4.

4.3 Test Problem 3

Lastly, the cBE is considered with the following initial

$$U(x,0) = \begin{cases} \sin(2\pi x), & x \in [0, 0.5] \\ 0, & x \in (0.5, 1 \end{cases}$$

and

$$V(x,0) = \begin{cases} 0, & x \in [0, 0.5] \\ -\sin(2\pi x), & x \in (0.5, 1 \end{cases}$$

and zero boundary conditions.

Table 8 shows a comparison of the calculated U_N and V_N of Problem 3 with $N = 50$ for $k_2 = k_3 = 10$ and $k_2 = k_3 = 100$ at different times. From Table 9 a comparison of the maximum values of U_N and V_N found out by the present approach with those of [12] is presented. From the table it is obviously seen that the maximum values of U_N and V_N and the x positions at which those maximum values are attained are in good agreement with those of [12]. Those results are also presented graphically in Figures 2-3.

4.4 Conclusion

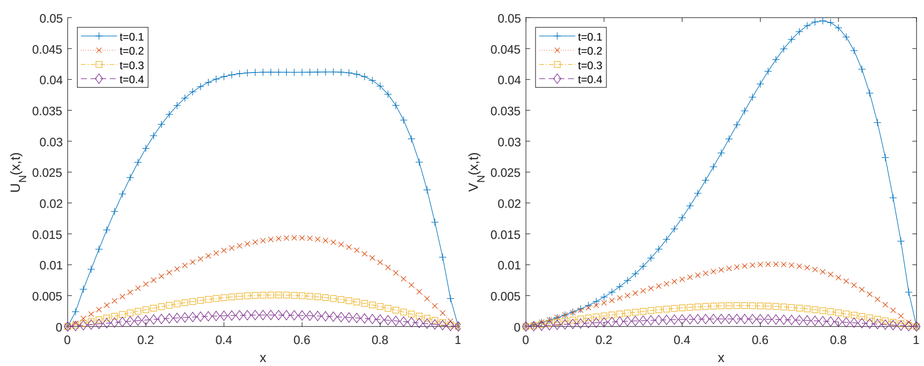
In this paper, numerical solutions of the cBE based on the trigonometric cubic B-spline finite element have been presented. Three test problems are worked out to examine the performance of the algorithms. The performance and accuracy of the method is shown by calculating the error norms L_2 and L_∞. For each linearization technique, the error norms are sufficiently small and the invariants are satisfactorily constant in all computer runs. The computed results show that the present method is a remarkably successful numerical technique for

Table 8: Comparison of the calculated U_N and V_N of Problem 3 with $N = 50$ for $k_2 = k_3 = 10$ and $k_2 = k_3 = 100$ at different times.

		$k_2 = k_3 = 10$				$k_2 = k_3 = 100$			
		CTBCM		QTBCM		CTBCM		QTBCM	
x	t	U_N	V_N	U_N	V_N	U_N	V_N	U_N	V_N
0.2	0.1	0.074267	0.048375	0.071964	0.047489	0.030210	0.004724	0.028837	0.004768
	0.2	0.027245	0.023688	0.026117	0.022749	0.007151	0.003946	0.006865	0.003832
	0.3	0.010838	0.009617	0.010195	0.009051	0.002879	0.001804	0.002722	0.001713
	0.4	0.004158	0.003703	0.003836	0.003418	0.001128	0.000729	0.001045	0.000678
0.4	0.1	0.126764	0.101431	0.124532	0.100069	0.041425	0.017768	0.040453	0.017614
	0.2	0.047173	0.041684	0.045649	0.040369	0.012684	0.007829	0.012312	0.007638
	0.3	0.018050	0.016070	0.017147	0.015271	0.004897	0.003162	0.004672	0.003025
	0.4	0.006803	0.006065	0.006343	0.005656	0.001862	0.001216	0.001743	0.001142
0.6	0.1	0.144605	0.140976	0.142334	0.138363	0.041598	0.040123	0.041159	0.039276
	0.2	0.051780	0.046621	0.049985	0.044991	0.014810	0.010363	0.014347	0.010031
	0.3	0.018732	0.016746	0.017764	0.015882	0.005240	0.003510	0.004986	0.003343
	0.4	0.006898	0.006159	0.006426	0.005738	0.001910	0.001265	0.001786	0.001184
0.8	0.1	0.107673	0.117144	0.104658	0.113107	0.039665	0.050149	0.038916	0.048358
	0.2	0.034822	0.031800	0.033143	0.030232	0.010889	0.008343	0.010378	0.007916
	0.3	0.011943	0.010713	0.011181	0.010028	0.003441	0.002374	0.003227	0.002226
	0.4	0.004312	0.003854	0.003969	0.003548	0.001206	0.000807	0.001114	0.000746

Table 9: Comparison of the maximum values of U_N and V_N of Problem 3 with those found by [12] at different times.

		Present $\Delta t = 0.001$				[12] $\Delta t = 0.001$	
		CTBCM		QTBCM			
	t	$U^N_{\max}$	x	$U^N_{\max}$	x	$U^N_{\max}$	x
$k_2 = k_3 = 10$	0.1	0.144798	0.58	0.142561	0.58	0.14348	0.58
	0.2	0.052477	0.54	0.050739	0.54	0.05252	0.54
	0.3	0.019365	0.52	0.018394	0.52	0.01945	0.52
	0.4	0.007203	0.50	0.006719	0.50	0.00724	0.50
$k_2 = k_3 = 100$	0.1	0.041859	0.48	0.041217	0.48	0.04108	0.44
	0.2	0.014820	0.58	0.014366	0.58	0.01475	0.58
	0.3	0.005353	0.54	0.005102	0.54	0.00536	0.54
	0.4	0.001984	0.52	0.001858	0.52	0.00199	0.52
		$V^N_{\max}$		$V^N_{\max}$		$V^N_{\max}$	
$k_2 = k_3 = 10$	0.1	0.143228	0.66	0.140294	0.66	0.14238	0.66
	0.2	0.047065	0.56	0.045481	0.56	0.04723	0.56
	0.3	0.017284	0.52	0.016420	0.52	0.01741	0.52
	0.4	0.006426	0.50	0.005996	0.50	0.00648	0.50
$k_2 = k_3 = 100$	0.1	0.051073	0.76	0.049495	0.76	0.04994	0.76
	0.2	0.010448	0.64	0.010087	0.64	0.01049	0.64
	0.3	0.003549	0.56	0.003386	0.56	0.00360	0.56
	0.4	0.001306	0.52	0.001226	0.52	0.00133	0.52



(a) (b)

Figure 3: The behaviour of numerical solutions of Problem 3 for values of $N = 50$, $\Delta t = 0.001$, $k_1 = 2$, $k_2 = k_3 = 100$ at times $t = 0.1$, 0.2, 0.3 and 0.4.

solving the cBE and advisable for getting numerical solutions of other types of non-linear equations.

Acknowledgments: This work has been supported by İnönü University Scientific Research Projects Unit under Grant No: FYL-2020-2341.

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