Nonexistence of sign-changing solutions for some elliptic and parabolic inequalities

(Running title: Nonexistence of sign-changing solutions)

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Abstract. We establish nonexistence of nontrivial solutions (including sign-changing ones) for some partial differential inequalities of elliptic and parabolic type containing nonlinear terms that depend on the positive and negative part of the sought function in different ways. Systems of elliptic inequalities with similar structure are also considered. The proofs are based on the test function method.

Keywords: nonexistence, sign-changing solutions, positive part, negative part.

1 Introduction

Starting from the 1960s, many mathematicians have considered sufficient conditions for nonexistence of nontrivial (distinct from zero or some other constant a. e.) solutions to nonlinear partial differential equations and inequalities in respective functional classes. A method for studying this problem based on use of special test functions was suggested by S. Pohozaev [1] and developed in his joint works with E. Mitidieri, V. Galaktionov, and other authors (see, in particular, monographs [2], [3]), as well as in some works of the authors of the present paper (see [4], [5], and references there). These papers dealt mostly with inequalities where the nonlinear terms depended on the absolute value of the sought function. Here we modify the test function method in order to obtain sufficient conditions for nonexistence of nontrivial solutions for some quasilinear elliptic inequalities containing terms that depend on the positive and negative parts of the sought function in a different way.

The rest of the paper consists of three sections. In §2, we prove nonexistence of nontrivial solutions for some scalar quasilinear elliptic inequalities, in §3, for respective systems, and in §4, for some parabolic problems.

2 Scalar inequalities

We will use the notation $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$. We consider the quasilinear elliptic inequality

$$-\Delta_p u \geq a(x)u_+^{q_1} + b(x)u_-^{q_2} \quad (x \in \mathbb{R}^n), \tag{2.1}$$

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where $a(x) \geq c_1|x|^{\beta_1}$, $b(x) \geq c_2|x|^{\beta_2}$ with some constants $c_1, c_2 > 0$, $\beta_1, \beta_2 \in \mathbb{R}$ for all $x \in \mathbb{R}^n$.

Definition 2.1. We will say that a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, for which there exists a $\lambda_0 > 0$ such that $u_+ \in L^{q_1-\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, $u_- \in L^{q_2+\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, and $|Du|^p u^{\pm\lambda-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\lambda \in [0, \lambda_0]$, satisfies inequality (2.1) in the weak (distributional) sense, if for any nonnegative test function $\varphi \in C^1_0(\mathbb{R}^n)$ there holds the following inequality:

$$\int\limits_{\mathbb{R}^n} |Du|^{p-2}(Du, D\varphi)\, dx \geq \int\limits_{\mathbb{R}^n} (a(x)u_+^{q_1} + b(x)u_-^{q_2})\varphi\, dx. \tag{2.2}$$

Theorem 2.1. *Let* $\min(q_1, q_2) > p - 1$ *and*

$$n - \frac{\beta_i(p-1) - pq_i}{q_i - p + 1} \leq 0, \ \ i = 1, 2. \tag{2.3}$$

Then inequality (2.1) *has no weak solutions distinct from the identical zero a. e.*

Proof. Introduce a family of test functions $\varphi_\eta \in C^1_0(\mathbb{R}^n; [0,1])$ of the form

$$\varphi_R(x) = \psi_R^\varkappa(x) \tag{2.4}$$

with $\varkappa > \frac{pq}{q-p}$, where $q = \min(q_1, q_2)$, and $\psi_R \in C^1_0(\mathbb{R}^n; [0,1])$ such that

$$\psi_R(x) = \begin{cases} 1 & (|x| \leq R), \\ 0 & (|x| \geq 2R), \end{cases} \tag{2.5}$$

and there exists a constant $c > 0$ such that

$$|D\psi_R(x)| \leq cR^{-1} \qquad (x \in \mathbb{R}^n). \tag{2.6}$$

Suppose that there exists a solution with $u_+ \not\equiv 0$. Substituting $\varphi(x) = u_{\varepsilon,+}^{-\lambda}(x)\varphi_R(x)$ with $\lambda \in (0, \lambda_0]$ into (2.2), we get

$$\int\limits_{\mathbb{R}^n} u_+^{q_1} u_{\varepsilon,+}^{-\lambda}(1+|x|)^{\beta_1}\varphi_R\, dx \leq \int\limits_{\mathbb{R}^n} (|Du|^{p-2}Du, D(u_{\varepsilon,+}^{-\lambda}\varphi_R))\, dx =$$

$$= -\lambda \int\limits_{\mathbb{R}^n} u_{\varepsilon,+} \, {}^{+\lambda-1}|Du_+|^p\varphi_R\, dx + \int\limits_{\mathbb{R}^n} |Du_+|^{p-2}(Du_+, D\varphi_R)\, dx \leq$$

$$\leq -\lambda \int\limits_{\mathbb{R}^n} u_{\varepsilon,+}^{-\lambda-1}|Du_+|^p\varphi_R\, dx + \int\limits_{\mathbb{R}^n} |Du_+|^{p-1}|D\varphi_R|\, dx.$$

We represent the integrand in the right-hand side of the obtained relation in the form

$$|Du_+|^{p-1}|D\varphi_R| = \left(\frac{\lambda p}{p-1}u_+^{-\lambda-1}\varphi_R\right)^{\frac{p-1}{p}} |Du_+|^{p-1} \cdot |D\varphi_\eta| \cdot \left(\frac{\lambda p}{p-1}u_+^{-\lambda-1}\varphi_R\right)^{\frac{1-p}{p}}.$$

2

Using the Young inequality

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'}, \ \ a, b > 0, \ \ s > 1,$$

where

$$a = \left(\frac{\lambda p}{2(p-1)} u_{\varepsilon,+}^{-\lambda-1} \varphi_R \right)^{\frac{p-1}{p}} |Du_+|^{p-1},$$

$$b = |D\varphi_R| \cdot \left(\frac{\lambda p}{2(p-1)} u_{\varepsilon,+}^{-\lambda-1} \varphi_R \right)^{\frac{1-p}{p}}, \ \ s = \frac{p-1}{p}, \ \ s' = p,$$

we get

$$\int\limits_{\mathbb{R}^n} u_+^{q_1} u_{\varepsilon,+}^{-\lambda} (1+|x|)^{\beta_1} \varphi_R \, dx + \lambda \int\limits_{\mathbb{R}^n} u_+^{-\lambda-1} |Du_+|^p \varphi_R \, dx \leq$$

$$\leq \frac{\lambda}{2} \int\limits_{\mathbb{R}^n} u_+^{-\lambda-1} |Du_+|^p \varphi_R \, dx + c(\lambda) \int\limits_{\mathbb{R}^n} u_{\varepsilon,+}^{-\lambda+p-1} |D\varphi_R|^p \varphi_R^{1-p} \, dx,$$

i. e.,

$$\int\limits_{\mathbb{R}^n} u_+^{q_1} u_{\varepsilon,+}^{-\lambda} (1+|x|)^{\beta_1} \varphi_R \, dx + \frac{\lambda}{2} \int\limits_{\mathbb{R}^n} u_+^{-\lambda-1} |Du_+|^p \varphi_R \, dx \leq$$

$$\leq c(\lambda) \int\limits_{\mathbb{R}^n} u_{\varepsilon,+}^{-\lambda+p-1} |D\varphi_R|^p \varphi_R^{1-p} \, dx$$

and after passing to the limit as $\varepsilon \to +0$ (which is admissible by the Lebesgue dominated convergence theorem)

$$\int\limits_{\mathbb{R}^n} u_+^{q_1-\lambda} (1+|x|)^{\beta_1} \varphi_R \, dx + \frac{\lambda}{2} \int\limits_{\mathbb{R}^n} u_+^{-\lambda-1} |Du_+|^p \varphi_R \, dx \leq$$

$$\leq c(\lambda) \int\limits_{\mathbb{R}^n} u_+^{-\lambda+p-1} |D\varphi_R|^p \varphi_R^{1-p} \, dx, \tag{2.7}$$

whence

$$\int\limits_{\mathbb{R}^n} u_+^{q_1-\lambda} (1+|x|)^{\beta_1} \varphi_R \, dx \leq c(\lambda) \int\limits_{\mathbb{R}^n} u_+^{-\lambda+p-1} |D\varphi_R|^p \varphi_R^{1-p} dx. \tag{2.8}$$

Now represent the integrand in the right-hand side of (2.8) as

$$u_+^{-\lambda+p-1} |D\varphi_\eta|^p \varphi_\eta^{1-p} = (u_+ \rho^{-\frac{\beta_1}{q_1-\lambda}} \varphi_\eta^{\frac{1}{q_1-\lambda}})^{-\lambda+p-1} \cdot |D\varphi_\eta|^p \rho^{\frac{\beta_1(-\lambda+p-1)}{q_1-\lambda}} \varphi_\eta^{\frac{(1-p)(q_1-\lambda)+\lambda-p+1}{q_1-\lambda}}.$$

Applying again the Young inequality with parameters

$$a = (u_+ (1+|x|)^{-\frac{\beta_1}{q_1-\lambda}} \varphi_R^{\frac{1}{q_1-\lambda}})^{-\lambda+p-1},$$

$$b = |D\varphi_R|^p (1 + |x|)^{\frac{\beta_1(-\lambda+p-1)}{q_1-\lambda}} \varphi_R^{\frac{(1-p)(q_1-\lambda)+\lambda-p+1}{q_1-\lambda}},$$

$$s = \frac{q_1 - \lambda}{-\lambda + p - 1}, \quad s' = \frac{q_1 - \lambda}{q_1 - p + 1},$$

we arrive at

$$\frac{1}{2} \int\limits_{\mathbb{R}^n} u_+^{q_1-\lambda}(1 + |x|)^{\beta_1}\varphi_R \, dx \le c(\lambda) \int\limits_{\mathbb{R}^n} |D\varphi_R|^{\frac{p(q_1-\lambda)}{q_1-p+1}}(1 + |x|)^{-\frac{\beta_1(-\lambda+p-1)}{q_1-p+1}}\varphi_R^{1-\frac{p(q_1-\lambda)}{q_1-p+1}} \, dx.$$

Using (2.4)–(2.5), we replace the left-hand side of this inequality by a smaller quantity

$$\frac{1}{2} \int\limits_{B_R(0)} u_+^{q_1-\lambda}(1 + |x|)^{\beta_1}\varphi_R \, dx = \frac{1}{2} \int\limits_{B_R(0)} u_+^{q_1-\lambda}(1 + |x|)^{\beta_1} \, dx,$$

and rewrite the right-hand side in the form

$$\int\limits_{B_{2R}(0)} |D\varphi_R|^{\frac{p(q_1-\lambda)}{q_1-p+1}}(1 + |x|)^{-\frac{\beta_1(p-1-\lambda)}{q_1-p+1}}\varphi_R^{1-\frac{p(q_1-\lambda)}{q_1-p+1}} \, dx.$$

Taking into account condition (2.6), we get

$$\int\limits_{B_R(0)} u_+^{q_1-\lambda}(1 + |x|)^{\beta_1} \, dx \le cR^{n-\frac{p(q_1-\lambda)+\beta_1(p-1-\lambda)}{q_1-p+1}},$$

which yields a contradiction as $R \to \infty$, if in (2.3) a strong inequality holds for $i = 1$ and $\lambda > 0$ is sufficiently small. If in (2.3) we have an equality for $i = 1$, repeating the same arguments for inequality (2.7), we obtain

$$\int\limits_{B_R(0)} u_+^{q_1-\lambda}(1 + |x|)^{\beta_1} \, dx + \frac{\lambda}{2} \int\limits_{B_R(0)} u_+^{-\lambda-1}|Du_+|^p\varphi_R \, dx \le cR^{n-\frac{p(q_1-\lambda)+\beta_1(p-1-\lambda)}{q_1-p+1}},$$

whence

$$\int\limits_{B_R(0)} u_+^{-\lambda-1}|Du_+|^p\varphi_R \, dx \le cR^{n-\frac{p(q_1-\lambda)+\beta_1(p-1-\lambda)}{q_1-p+1}}. \tag{2.9}$$

Then, substituting $\varphi(x) = \varphi_R(x)$ into (2.2), we have

$$\int\limits_{\mathbb{R}^n} u_+^{q_1}(1 + |x|)^{\beta_1}\varphi_R \, dx \le \int\limits_{\mathbb{R}^n} (|Du|^{p-2}Du, D\varphi_R) \, dx \le \int\limits_{\mathbb{R}^n} |Du|^{p-1} \cdot |D\varphi_R| \, dx$$

and by the Hölder inequality

$$\int\limits_{\mathbb{R}^n} u_+^{q_1}(1+|x|)^{\beta_1}\varphi_R\, dx \le \left(\int\limits_{\mathbb{R}^n} u_+^{-\lambda-1}|Du_+|^p\varphi_R\, dx\right)^{\frac{p-1}{p}} \times$$

$$\times \left(\int\limits_{\mathbb{R}^n} u_+^{(\lambda+1)(p-1)}|D\varphi_R|^p\varphi_R^{1-p}\, dx\right)^{\frac{1}{p}}.$$

Using (2.9) and applying once more the Hölder inequality, and then condition (2.6), we get

$$\int\limits_{\mathbb{R}^n} u_+^{q_1}(1+|x|)^{\beta_1}\varphi_R\, dx \le cR^{\frac{(p-1)(n(q_1-p+1)-p(q_1-\lambda)-\beta_1(p-1-\lambda))}{p(q_1-p+1)}} \left(\int\limits_{\mathrm{supp}|D\varphi_R|} u_+^{q_1}(1+|x|)^{\beta_1}\varphi_R dx\right)^{\frac{p-1}{pq_1}} \times$$

$$\times \left(\int\limits_{\mathrm{supp}|D\varphi_R|} |D\varphi_R|^{\frac{pq_1}{q_1-(p-1)(\lambda+1)}}\left((1+|x|)^{\beta_1(\lambda+1)}\varphi_R^{q_1+1}\right)^{\frac{1-p}{q_1-(p-1)(\lambda+1)}}\, dx\right)^{\frac{q_1-(p-1)(\lambda+1)}{pq_1}} \le$$

$$\le cR^{\frac{(p-1)(n(q_1-p+1)-p(q_1-\lambda)-\beta_1(p-1-\lambda))+n(q_1-(p-1)(\lambda+1))-pq_1+\beta_1(1-p)(\lambda+1)}{pq_1}} \left(\int\limits_{\mathrm{supp}|D\varphi_R|} u_+^{q_1}(1+|x|)^{\beta_1}\varphi_R dx\right)^{\frac{p-1}{pq_1}} =$$

$$= c\left(\int\limits_{\mathrm{supp}|D\varphi_R|} u_+^{q_1}(1+|x|)^{\beta_1}\varphi_R dx\right)^{\frac{p-1}{pq_1}},$$

where the right-hand side tends to 0 as $R \to \infty$ similarly to (2.9), which again yields a contradiction completing the proof of the fact that $u_+ \equiv 0$ a. e. Similarly, using test functions $\varphi(x) = u_-^\lambda(x)\varphi_R(x)$, we prove that $u_- \equiv 0$ a. e. This completes the proof of the theorem.

3 Systems of elliptic inequalities

Consider a system of quasilinear elliptic inequalities

$$\begin{cases} -\Delta_p u \ge a(x)v_+^{q_1} + b(x)v_-^{q_2} & (x \in \mathbb{R}^n), \\ -\Delta_q v \ge c(x)u_+^{p_1} + d(x)u_-^{p_2} & (x \in \mathbb{R}^n), \end{cases} \tag{3.1}$$

where $p, q, p_1, q_1, p_2, q_2 > 1, p - 1 < p_1,\ q - 1 < q_1$, a, b, $c\, d$ are nonnegative functions such that $a(x) \ge c_1(1+|x|)^\alpha$, $b(x) \ge c_2(1+|x|)^\beta$, $c(x) \ge c_3(1+|x|)^\gamma$, $d(x) \ge c_4(1+|x|)^\delta$ for $x \in \mathbb{R}^n$, $c_1, \ldots, c_4 > 0$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Definition 3.1. A pair of functions $(u, v)\colon u, v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \times W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$, for which there exists a $\lambda_0 > 0$ such that $u_+ \in L^{q_1-\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, $u_- \in L^{q_2+\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, $v_+ \in L^{p_1-\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, $v_- \in L^{p_2+\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, $|Du|^p v^{\pm\lambda-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $|Du|^q u^{\pm\lambda-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\lambda \in [0, \lambda_0]$, is called a weak solution of system (3.1) if they satisfy the integral inequalities

$$-\int_{\mathbb{R}^n} |Du|^{p-2}(Du, D\varphi_1)\,dx \geq \int_{\mathbb{R}^n} (a(x)v_+^{q_1} + b(x)v_-^{q_2})\varphi_1\,dx,$$
$$\int_{\mathbb{R}^n} |Dv|^{q-2}(Dv, D\varphi_2)\,dx \geq \int_{\mathbb{R}^n} (c(x)u_+^{p_1} + d(x)u_-^{p_2})\varphi_2\,dx \tag{3.2}$$

for all nonnegative test functions $\varphi_1, \varphi_2 \in C^1_0(\mathbb{R}^n)$.

Theorem 3.1. *Let* $\min\limits_{i,j=1,2} p_i q_j > (p-1)(q-1)$ *and* $\min\limits_{i=1,\ldots,4} a_i > n$, $\min\limits_{i=1,\ldots,4} b_i > n$, *where*

$$a_1 = \frac{pp_1 + (\alpha(q-1) + q + \gamma)(p-1)}{p_1 q_1 - (p-1)(q-1)}, \quad a_2 = \frac{pp_1 + (\beta(p-1) + q + \delta)(p-1)}{p_2 q_1 - (p-1)(q-1)},$$
$$a_3 = \frac{pp_1 + (\alpha(q-1) + q + \gamma)(p-1)}{p_1 q_2 - (p-1)(q-1)}, \quad a_4 = \frac{pp_2 + (\beta(p-1) + q + \delta)(p-1)}{p_2 q_2 - (p-1)(q-1)},$$
$$b_1 = \frac{qq_1 + (\alpha(q-1) + q + \gamma)(p-1)}{p_1 q_1 - (p-1)(q-1)}, \quad b_2 = \frac{qq_1 + (\beta(p-1) + q + \delta)(p-1)}{p_2 q_1 - (p-1)(q-1)},$$
$$b_3 = \frac{qq_1 + (\alpha(q-1) + q + \gamma)(p-1)}{p_1 q_2 - (p-1)(q-1)}, \quad b_4 = \frac{qq_2 + (\beta(p-1) + q + \delta)(p-1)}{p_2 q_2 - (p-1)(q-1)}.$$

Then system (3.1) *has no nontrivial weak solutions.*

Proof. Let $\varphi_R \in C^\infty_0(\overline{\mathbb{R}}^n; \mathbb{R}_+)$ be the family of test functions from the previous section with $\varkappa > 0$ sufficiently large.

Substituting $\varphi_1(x) = u^\lambda_{\varepsilon,+}(x)\varphi_R(x)$ into the first inequality (3.2) and $\varphi_2(x) = v^{-\lambda}_{\varepsilon,+}(x)\varphi_R(x)$ into the second one so that $\varepsilon > 0$ and $\max\{1-p, 1-q\} < -\lambda < 0$, we get

$$\int (a(x)v_+^{q_1} + b(x)v_-^{q_2})u^\lambda_{\varepsilon,+}\varphi_R\,dx \leq \lambda \int |Du_+|^p u^{-\lambda-1}_{\varepsilon,+}\varphi_R\,dx + \int (|Du_+|^{p-1}|D\varphi_R|u^{-\lambda}_{\varepsilon,+}\,dx, \tag{3.3}$$

$$\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})v^\lambda_{\varepsilon,+}\varphi_R\,dx \leq \lambda \int |Dv_+|^q v^{\lambda-1}_{\varepsilon,+}\varphi_R\,dx + \int (|Dv_+|^{q-1}|D\varphi_R|v^{-\lambda}_{\varepsilon,+}\,dx. \tag{3.4}$$

Applying the Young inequality to the first terms in the right-hand sides of the obtained relations yields

$$\int (a(x)v_+^{q_1} + b(x)v_-^{q_2})u^\lambda_{\varepsilon,+}\varphi_R\,dx + \frac{\lambda}{2}\int |Du_+|^p u^{-\lambda-1}_{\varepsilon,+}\varphi_R\,dx \leq c_\lambda \int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{-\lambda+p-1}_{\varepsilon,+}\,dx,$$

$$\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})v^{-\lambda}_{\varepsilon,+}\varphi_R\,dx + \frac{|\lambda|}{2}\int |Dv_+|^q v^{-\lambda-1}_{\varepsilon,+}\varphi_R\,dx \leq d_\lambda \int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v^{-\lambda+q-1}_{\varepsilon,+}\,dx,$$

and after passing to the limit as $\varepsilon \to 0_+$ to

$$\int (a(x)v_+^{q_1} + b(x)v_-^{q_2})u_+^\lambda \varphi_R \, dx + \frac{\lambda}{2} \int |Du_+|^p u_+^{-\lambda-1} \varphi_R \, dx \le c_\lambda \int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{-\lambda+p-1} \, dx, \quad (3.5)$$

$$\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})v_+^{-\lambda} \varphi_R \, dx + \frac{\lambda}{2} \int |Dv_+|^q v_+^{-\lambda-1} \varphi_R \, dx \le d_\lambda \int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v_+^{-\lambda+q-1} \, dx, \quad (3.6)$$

where the constants c_λ and d_λ are positive and depend only on p, q, and λ.

Similarly, substituting $\varphi_1(x) = u_{\varepsilon,-}^\lambda(x)\varphi_R(x)$ into the first inequality (3.2) and $\varphi_2(x) = v_{\varepsilon,-}^{-\lambda}(x)\varphi_R(x)$ into the second one so that $\varepsilon > 0$ and $\max\{1-p, 1-q\} < -\lambda < 0$, we get

$$\int (a(x)v_+^{q_1} + b(x)v_-^{q_2})u_-^\lambda \varphi_R \, dx + \frac{\lambda}{2} \int |Du_-|^p u_-^{\lambda-1} \varphi_R \, dx \le e_\lambda \int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_-^{\lambda+p-1} \, dx, \quad (3.7)$$

$$\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})v_-^\lambda \varphi_R \, dx + \frac{|\lambda|}{2} \int |Dv_+|^q v_-^{\lambda-1} \varphi_R \, dx \le f_\lambda \int \frac{|D\varphi_R|^q}{\varphi_R^{q-1}} v_-^{\lambda+q-1} \, dx, \quad (3.8)$$

where the constants e_λ and f_λ are positive and depend only on p, q, and λ.

Further, using the test function $\varphi = \varphi_R$ in (3.2), similarly to the previous argument, we obtain the relations

$$\int (a(x)v_+^{q_1} + b(x)v_-^{q_2})\varphi_R \, dx \le \int |Du|^{p-1}|D\varphi_R| \, dx = \\ \int |Du_+|^{p-1}|D\varphi_R| \, dx + \int |Du_-|^{p-1}|D\varphi_R| \, dx, \quad (3.9)$$

$$\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})\varphi_R \, dx \le \int |Dv|^{q-1}|D\varphi_R| \, dx = \\ \int |Dv_+|^{q-1}|D\varphi_R| \, dx + \int |Dv_-|^{q-1}|D\varphi_R| \, dx. \quad (3.10)$$

Apply the Hölder inequality to each integral in the right-hand side of the obtained relations. Taking into account (3.5)–(3.8), we get

$$\int |Du_+|^{p-1}|D\varphi_R| \, dx \le \\ \le \left(\int |Du_+|^p u_+^{\lambda-1} \varphi_R \, dx \right)^{\frac{p-1}{p}} \times \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{(1-\lambda)(p-1)} \, dx \right)^{\frac{1}{p}} \le \\ \le g_\lambda \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{-\lambda+p-1} \, dx \right)^{\frac{p-1}{p}} \times \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{(1-\lambda)(p-1)} \, dx \right)^{\frac{1}{p}} \quad (3.11)$$

and similar relations for the integrals

$$\int |Du_-|^{p-1}|D\varphi_R| \, dx, \quad \int |Dv_+|^{q-1}|D\varphi_R| \, dx, \quad \int |Dv_-|^{q-1}|D\varphi_R| \, dx.$$

Applying the Hölder inequality with the exponent $r > 1$ to the first integral in the right-hand side of (3.11), we obtain

$$\left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{-\lambda+p-1} \, dx \right)^{\frac{p-1}{p}} \leq \left(\int c(x) u_+^{(\lambda+p-1)r} \varphi_R \, dx \right)^{\frac{p-1}{pr}} \left(\int c^{-\frac{r'}{r}}(x) \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}} \, dx \right)^{\frac{p-1}{pr'}},$$

(3.12)

where $\dfrac{1}{r} + \dfrac{1}{r'} = 1$.

Choosing the exponent r so that

$$(\lambda + p - 1)r = p_1,$$

(3.13)

from (3.11) and (3.12) we have

$$\int |Du_+|^{p-1} |D\varphi_R| \, dx \leq g_\lambda \left(\int c(x) u_+^{p_1} \varphi_R \, dx \right)^{\frac{p-1}{pr}} \times$$

$$\times \left(\int c^{-\frac{r'}{r}}(x) \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}} \, dx \right)^{\frac{p-1}{pr'}} \left(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{(1-\lambda)(p-1)} \, dx \right)^{\frac{1}{p}}.$$

(3.14)

Applying the Hölder inequality with the exponent $y > 1$ to the last integral in the right-hand side of (3.14), we have

$$\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u_+^{(1-\lambda)(p-1)} \, dx \leq \left(\int c(x) u_+^{(1-\lambda)(p-1)y} \varphi_R \, dx \right)^{\frac{1}{y}} \left(\int c^{-\frac{y'}{y}}(x) \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}} \, dx \right)^{\frac{1}{y'}},$$

(3.15)

where $\dfrac{1}{y} + \dfrac{1}{y'} = 1$.

Choosing y in (3.15) according to (3.19) so that

$$(1 - \lambda)(p - 1)y = p_1,$$

(3.16)

and taking into account (3.14), we arrive at the estimate

$$\int |Du_+|^{p-1} |D\varphi_R| \, dx \leq g_\lambda \left(\int c(x) u_+^{p_1} \varphi_R \, dx \right)^{\frac{p-1}{pr}} \left(\int c^{-\frac{r'}{r}}(x) \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}} \, dx \right)^{\frac{p-1}{pr'}} \times$$

$$\times \left(\int c(x) u_+^{p_1} \varphi_R \, dx \right)^{\frac{1}{py}} \left(\int c^{-\frac{y'}{y}}(x) \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}} \, dx \right)^{\frac{1}{py'}},$$

which implies

$$\int |Du_+|^{p-1} |D\varphi_R| \, dx \leq g_\lambda \left(\int (c(x) u_+^{p_1} + d(x) u_-^{p_2}) \varphi_R \, dx \right)^{\frac{p-1}{pr} + \frac{1}{py}} \times$$

(3.17)

$$\times \left(\int c^{-\frac{r'}{r}}(x) \frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}} \, dx \right)^{\frac{p-1}{pr'}} \left(\int c^{-\frac{y'}{y}}(x) \frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}} \, dx \right)^{\frac{1}{py'}},$$

where the exponents r and y are chosen according to (3.13), (3.16). Substituting these expressions for the exponents, due to the choice of φ_R and conditions on $c(x)$ and $d(x)$ we get

$$\int |Du_+|^{p-1}|D\varphi_R|\,dx \le$$
$$\le g_\lambda \left(\int (c(x)u_+^{p_1} + d(x)u_-^{p_2})\varphi_R\,dx \right)^{\frac{p-1}{q_1}} \times R^{\frac{\gamma(1-p)-p+n(q_1-p+1)}{q_1}}. \quad (3.18)$$

We obtain similar estimates for

$$\int |Du_-|^{p-1}|D\varphi_R|\,dx, \ \int |Dv_+|^{q-1}|D\varphi_R|\,dx, \ \int |Dv_-|^{q-1}|D\varphi_R|\,dx.$$

Combining these estimates with (3.9) and (3.10) and introducing the notation

$$A = A(R) = \int (a(x)v_+^{q_1} + b(x)v_-^{q_2})\varphi_R\,dx, \quad B = B(R) = \int (c(x)u_+^{p_1} + d(x)u_-^{p_2})\varphi_R\,dx,$$

we obtain the estimates

$$\begin{cases} A \le C_\lambda \left(B^{\frac{p-1}{q_1}} R^{\frac{\gamma(1-p)-p+n(q_1-p+1)}{q_1}} + B^{\frac{p-1}{q_2}} R^{\frac{\delta(1-p)-p+n(q_1-p+1)}{q_1}} \right), \\ B \le D_\lambda \left(A^{\frac{q-1}{p_1}} R^{\frac{\alpha(1-q)-q+n(p_1-q+1)}{p_1}} + A^{\frac{q-1}{p_2}} R^{\frac{\beta(1-q)-q+n(p_2-q+1)}{p_2}} \right), \end{cases} \quad (3.19)$$

where the constants C_λ and D_λ are positive and depend only on the parameters of the inequalities under consideration and on λ. Therefore in each inequality (3.19) at least one summand in the right-hand side is greater or equal to a half of the left-hand side. Hence we get

$$A(R) \le E_\lambda R^{n - \min\limits_{i=1,\dots,4} a_i}, \quad B(R) \le F_\lambda R^{n - \min\limits_{i=1,\dots,4} b_i},$$

where the constants E_λ and F_λ are positive and depend only on the parameters of the inequalities under consideration and on λ.

Taking $R \to +\infty$, we come to a contradiction, which proves the theorem.

4 Parabolic problems

Let $u_0 \in C(\mathbb{R}^n)$. We will use the notation $u_{0,+} = \max\{u_0, 0\}$, $u_{0,-} = -\min\{u_0, 0\}$, $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$. Consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|Du|^{p-2}Du) \ge a(x,t)u_+^{q_1} + b(x,t)u_-^{q_2} & ((x,t) \in \mathbb{R}^n \times \mathbb{R}_+), \\ u(x,0) = u_0(x) & (x \in \mathbb{R}^n), \end{cases} \quad (4.1)$$

where $a(x,t) \ge c_1(1 + |x|)^{-\beta_1}$, $b(x,t) \ge c_2(1 + |x|)^{-\beta_2}$ with some constants $c_1, c_2 > 0$, $\beta_1, \beta_2 \in \mathbb{R}$ for all $x \in \mathbb{R}^n$.

We define its weak solutions as follows.

Definition 4.1. A nonnegative function $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ is called a weak (local) solution of problem (4.1) if it satisfies the integral inequality

$$\int\limits_{t_0}^{t_1} \int\limits_{\mathbb{R}^n} (a(x)u_+^{q_1} + b(x)u_-^{q_2})\psi \, dx \, dt - \int\limits_{\mathbb{R}^n} (u(x,t_1)\psi(x,t_1) - u(x,t_0)\psi(x,t_0)) \, dx \leq$$
$$\leq \int\limits_{t_0}^{t_1} \int\limits_{\mathbb{R}^n} \left(-u\frac{\partial \psi}{\partial t} + |Du|^{p-2}(Du, D\psi) \right) dx \, dt \tag{4.2}$$

for some $t^* > 0$, for all t_0, t_1 such that $0 \leq t_0 < t_1 \leq t^*$, and for any nonnegative function $\psi \in C^1((\mathbb{R}^n) \times [t_0, t_1])$ such that for all $t \in [t_0, t_1]$ there holds $\psi(\cdot, t) \in C_0^1(\mathbb{R}^n)$, and there exists a $\lambda_0 > 0$ such that $u_+ \in L_{\text{loc}}^{q_1-\lambda}(\mathbb{R}^n \times \mathbb{R}_+)$, $u_- \in L_{\text{loc}}^{q_2+\lambda}(\mathbb{R}^n \times \mathbb{R}_+)$ $|Du|^p u^{\pm\lambda-1} \in L_{\text{loc}}^1(\mathbb{R}^n \times \mathbb{R}_+)$ for $\lambda \in [0, \lambda_0]$, and $\lim\limits_{t \to 0_+} u(x,t) = u_0(x)$ for all $x \in \mathbb{R}^n$. The supremum of all possible values $\tau = t_1 - t_0$ is alled the *life time* of solution u. If it is infinite, the solution is called *global*.

Remark 4.1. If u is sufficiently regular, inequality (4.2) can be obtained from (4.1) by integration by parts.

Sufficient conditions for the nonexistence of nontrivial weak solutions of the Cauchy problem (4.1) in this sense can be formulated as follows.

Theorem 4.1. *Let $p > 1$, $\min(q_1, q_2) > p - 1$,*

$$2(p - \beta_i) < \theta(q_i - p + 1) \ (i = 1, 2) \tag{4.3}$$

and

$$n + \frac{p - \beta_i}{q_i - p + 1} \leq 0 \ (i = 1, 2), \tag{4.4}$$

and the initial function u_0 is nonnegative.

Then problem (4.1) has no global weak solutions u in $\mathbb{R}^n \times \mathbb{R}_+$ apart from the identical zero a.e.

Proof. Suppose hat a solution u of problem (4.1) does exist and consider its weak formulation (4.2) with test functions $u_+^{\gamma}(x,t)\varphi_R(x)T_\tau(t)$, where $\varphi_R = \psi_R^{\varkappa}$, $\varkappa > \max\limits_{i=1,2} \frac{pq_i}{q_i-p+1}$. In order to obtain a priori estimates of solutions to (4.1), we use a family of functions of spatial variables $\psi_R \in C_0^1(\mathbb{R}^n; [0, 1])$, with a parameter $R > 0$ such that

$$\psi_R(x) = \begin{cases} 1 & (|x| \leq R), \\ 0 & (|x| \geq 2R) \end{cases} \tag{4.5}$$

and there exists a constant $c > 0$ such that

$$|D\psi_R(x)| \leq cR^{-1} \qquad (x \in \mathbb{R}^n), \tag{4.6}$$

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and a family of time-dependent functions $T_\tau \in C^1([0,\tau];[0,1])$ with a parameter $\tau > 0$ such that

$$T_\tau(t) = \begin{cases} 1 & (0 \le t \le \tau/2), \\ 0 & (3\tau/4 \le t \le \tau) \end{cases} \tag{4.7}$$

and, moreover,

$$\int_{\tau/2}^{3\tau/4} \frac{|T_\tau'|^{r'}}{|T_\tau|^{r'-1}}\, dt \le c\tau^{1-r'} \tag{4.8}$$

with some constant $c > 0$ and $r = \dfrac{q+\gamma}{q-1}$, where $\gamma < 0$ and $|\gamma|$ is sufficiently small.

In this case inequality (4.2) and our assumption on $a(x,t)$ imply

$$c_1 \int_0^\tau \int_{\operatorname{supp}\varphi_R} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau\, dx\, dt + \int_{\operatorname{supp}\varphi_R} u_{0,+}^{\gamma+1}\varphi_R\, dx \le$$

$$\le \gamma \int_0^\tau \int_{\operatorname{supp}\varphi_R} |Du|^p u_+^{\gamma-1}\varphi_R T_\tau\, dx\, dt + \int_0^\tau \int_{\operatorname{supp}|D\varphi_R|} u_+^\gamma |Du|^{p-2}(Du, D\varphi_R)T_\tau\, dx\, dt- \tag{4.9}$$

$$- \int_0^\tau \int_{\operatorname{supp}\varphi_R} u_+(u_+^\gamma \varphi_R T_\tau')_t\, dx\, dt.$$

Integrating by parts the last term in this inequality twice, for a sufficiently regular u we get

$$\int_0^\tau \int_{\operatorname{supp}\varphi_R} u(u_+^\gamma \varphi_R T_\tau)_t dx\, dt =$$

$$= \int_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R T_\tau dx\Big|_0^\tau - \int_0^\tau \int_{\operatorname{supp}\varphi_R} \frac{\partial u}{\partial t} u_+^\gamma \varphi_R T_\tau dx\, dt =$$

$$= \int_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R T_\tau dx\Big|_0^\tau - \frac{1}{1+\gamma}\int_0^\tau \int_{\operatorname{supp}\varphi_R} (u_+^{\gamma+1})_t \varphi_R T_\tau dx\, dt =$$

$$= \left(1 - \frac{1}{1+\gamma}\right) \int_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R T_\tau dx\Big|_0^\tau + \frac{1}{1+\gamma}\int_0^\tau \int_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R(T_\tau)_t dx\, dt =$$

$$= \left(\frac{1}{1+\gamma} - 1\right) \int_{\operatorname{supp}\varphi_R} u_{0,+}^{\gamma+1}\varphi_R dx + \frac{1}{1+\gamma}\int_0^\tau \int_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R(T_\tau)_t dx\, dt.$$

For any weak solution u of problem (4.1) this inequality can be obtained by approximation

with regular functions and a standard passage to the limit. Hence, (4.9) can be rewritten as

$$
c_1 \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} u_+^{q_i+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau \, dx \, dt + \frac{1}{\gamma+1} \int\limits_{\operatorname{supp}\varphi_R} u_{0,+}^{\gamma+1}\varphi_R \, dx \le
$$

$$
\le \gamma \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} |Du|^p u_+^{\gamma-1}\varphi_R T_\tau \, dx \, dt + \int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} u_+^\gamma |Du|^{p-2}(Du, D\varphi_R) T_\tau \, dx \, dt - \quad (4.10)
$$

$$
- \frac{1}{\gamma+1} \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R T'_\tau \, dx \, dt.
$$

Further we apply the Young inequality with appropriate parameters to the second and third terms in the right-hand side:

$$
\int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} u_+^\gamma |Du|^{p-2}(Du, D\varphi_R) T_\tau \, dx \, dt \le \int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} u_+^\gamma |Du|^{p-1}|D\varphi_R| T_\tau \, dx \, dt \le
$$
$$
\le \frac{|\gamma|}{4} \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} |Du|^p u_+^{\gamma-1}\varphi_R T_\tau \, dx \, dt + c \int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} u_+^{p+\gamma-1}|D\varphi_R|^p \varphi_R^{1-p} T_\tau \, dx \, dt, \quad (4.11)
$$

$$
- \frac{1}{\gamma+1} \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R T'_\tau \, dx \, dt \le \frac{1}{|\gamma+1|} \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} u_+^{\gamma+1}\varphi_R |T'_\tau| \, dx \, dt \le
$$
$$
\le \frac{c_1}{4} \int\limits_0^\tau \int\limits_{\operatorname{supp}\varphi_R} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau \, dx \, dt + c \int\limits_0^\tau |T'_\tau|^{\frac{q_1+\gamma}{q_1-1}} T_\tau^{-\frac{\gamma+1}{q_1-1}} \, dt \int\limits_{\operatorname{supp}\varphi_R} (1+|x|)^{\frac{\beta_1(\gamma+1)}{q_1-1}}\varphi_R \, dx.
$$
$$
(4.12)
$$

Similarly the second term in the right-hand side of (4.11) can be estimated as

$$
c \int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} u_+^{p+\gamma-1}|D\varphi_R|^p \varphi_R^{1-p} T_\tau \, dx \, dt \le
$$
$$
\le \int\limits_0^\tau \int\limits_{\operatorname{supp}|D\varphi_R|} \left(\frac{c_1}{4} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R + c|D\varphi_R|^{\frac{p(q_1+\gamma)}{q_1-p+1}}(1+|x|)^{\frac{\beta_1(p+\gamma-1)}{q_1-p+1}} \varphi_R^{\frac{(1-p)(p+\gamma-1)-(q_1+\gamma)}{q_1-p+1}} \right) T_\tau \, dx \, dt,
$$
$$
(4.13)
$$

where due to (4.5)–(4.6) for $\varkappa > \frac{pq_1}{q_1-p+1}$ one has

$$
|D\varphi_R(x)|^{\frac{p(q_1+\gamma)}{q_1-p+1}} \varphi_R^{\frac{(1-p)(q_1+\gamma+1)-\gamma}{q_1-p+1}}(x) = \varkappa^p |D(\xi_{\frac{1}{R}} \cdot \psi_R)(x)|^{\frac{p(q_1+\gamma)}{q_1-p+1}}(\xi_{\frac{1}{R}} \cdot \psi_R)^{\varkappa - \frac{p(q_1+\gamma)}{q_1-p+1}}(x) \le
$$
$$
\le c(1+|x|)^{-\frac{p(q_1+\gamma)}{q_1-p+1}}(x) \quad (x \in \operatorname{supp}\varphi_R).
$$
$$
(4.14)
$$

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Combining inequalities (4.10)–(4.14), we get

$$\frac{c_1}{2} \int_0^\tau \int_{B_{2R}(0)} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau \, dx \, dt + \frac{1}{\gamma+1} \int_{\operatorname{supp}\varphi_R} u_{0,+}^{\gamma+1}\varphi_R \, dx \le$$

$$\le c \left(\int_0^\tau |T_\tau'|^{\frac{q_1+\gamma}{q_1-1}} T_\tau^{-\frac{\gamma+1}{q_1-1}} \, dt \int_{\operatorname{supp}\varphi_R} (1+|x|)^{\frac{\beta_1(\gamma+1)}{q_1-1}}\varphi_R \, dx + \int_0^\tau dt \int_{\operatorname{supp}\varphi_R} (1+|x|)^{\frac{-p(q_1+\gamma)+\beta_1(\gamma+p-1)}{q_1-p+1}} \, dx \right).$$
(4.15)

Here we take into account that $\gamma < 0$. Since u_0 is nonnegative, we have

$$\frac{c_1}{4} \int_0^\tau \int_{B_{2R}(0)} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau \, dx \, dt \le$$

$$\le c \left(\int_{\tau/2}^{3\tau/4} |T_\tau'|^{\frac{q_1+\gamma}{q_1-1}} T_\tau^{-\frac{\gamma+1}{q_1-1}} \, dt \int_{B_{2R}(0)} (1+|x|)^{\frac{\beta_1(\gamma+1)}{q_1-1}}\varphi_R \, dx + \right.$$

$$\left. + \int_0^\tau dt \int_{B_{2R}(0)} (1+|x|)^{\frac{-p(q_1+\gamma)+\beta_1(\gamma+p-1)}{q_1-p+1}} \, dx \right).$$

This implies

$$\int_0^\tau \int_{B_{2R}(0)} u_+^{q_1+\gamma}(1+|x|)^{-\beta_1}\varphi_R T_\tau \, dx \, dt \le c_3 R^n \left(\tau R^{-l_\gamma} + \tau_+^{-\frac{\gamma+1}{q_1-1}} R^{-\frac{\beta_1(\gamma+1)}{q_1-1}} \right),$$
(4.16)

where $c_3 > 0$ and

$$l_\gamma = \frac{\beta_1(\gamma+p-1) - p(q_1+\gamma)}{q_1-p+1}.$$
(4.17)

It is easy to see that the right-hand side of (4.16) attains its minimum at

$$\tau_* = \left(\frac{\gamma+1}{q_1-1} R^{\frac{l_\gamma - \beta_1(\gamma+1)}{q_1-1}} \right)^{\frac{q_1-1}{q_1+\gamma}} = c_4 R^{\frac{(\beta_1(p-2)-p(q_1-1))(\gamma+1)}{q_1-p+1}},$$
(4.18)

where $c_4 = c_4(q_1, p, \gamma) \to c(q_1, p, 0) > 0$ as $\gamma \to 0_-$. Substituting (4.18) into (4.16) and taking $R \to \infty$, by condition (4.21), if $|\gamma|$ is sufficiently small, we get $u_+ \equiv 0$ a.e. Similarly, using test functions of the form $u_-^{-\gamma}(x,t)\varphi_R(x)T_\tau(t)$, we get $u_- \equiv 0$ a.e. This completes the proof.

Under additional assumptions on the behavior of the initial function one can obtain sufficient conditions for nonexistence not only of global solutions of problem (4.1) bu for local ones as well. Namely, there holds

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Theorem 4.2. *Let $p > 1$, $\min(q_1, q_2) > p - 1$, and*

$$\beta_i(p_i - 2) - p(q_i - 1) > 0 \; (i = 1, 2). \tag{4.19}$$

Suppose that the initial function $u_0 \in C(\mathbb{R}^n)$ satisfies the inequality

$$|u_0(x)| \geq c_0(1 + |x|)^\mu \quad (x \in \mathbb{R}^n) \tag{4.20}$$

with some constants $c_0 > 0$ and $\mu \in \mathbb{R}$, so that

$$\beta_i > \mu(q_i - p + 1) + p \; (i = 1, 2). \tag{4.21}$$

Then problem (4.1) has no positive functions u in $(\mathbb{R}^n) \times [0, T]$ for any arbitrary small $T > 0$.

Proof. We combine (4.15) with (4.20) and choose $\tau = \tau^*$ from (4.18). Due to (4.19) and (4.21), this leads to a contradiction as $R \to \infty$, which proves the theorem.

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