SIGN-CHANGING SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH GENERALIZED CHERN-SIMONS GAUGE THEORY

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Abstract. We study the existence and asymptotic behavior of least energy sign-changing solutions for the nonlinear Schrödinger equation coupled with the Chern-Simons gauge theory

$$\begin{cases} -\Delta u + \omega u + \lambda \displaystyle\sum_{j=1}^{k} \Big(\frac{h^2(|x|)}{|x|^2} u^{2(j-1)} + \frac{1}{j} \int_{|x|}^{\infty} \frac{h(s)}{s} u^{2j}(s) ds \Big) u = f(u) \ \text{ in } \ \mathbb{R}^2, \\ u \in H_r^1(\mathbb{R}^2), \end{cases}$$

where $\omega,\ \lambda > 0$ are constants, $k \in \mathbb{N}^+$ and

$$h(s) = \int_0^s \frac{r}{2} u^2(r) dr.$$

Under some suitable assumptions on $f \in C(\mathbb{R})$, with the help of the Gagliardo-Nirenberg inequality, we apply the constraint minimization argument to obtain a least energy sign-changing solution u_λ with precisely two nodal domains. Furthermore, we prove that the energy of u_λ is strictly larger than two times of the ground state energy and analyze the asymptotic behavior of u_λ as $\lambda \searrow 0^+$. Our results cover and improve the existing ones for the gauged nonlinear Schrödinger equation when $k \equiv 1$.

1. Introduction and main results

In this paper, we consider the following nonlinear Schrödinger equation coupled with the Chern-Simons gauge theory

$$\begin{cases} -\Delta u + \omega u + \lambda \displaystyle\sum_{j=1}^{k} \Big(\frac{h^2(|x|)}{|x|^2} u^{2(j-1)} + \frac{1}{j} \int_{|x|}^{\infty} \frac{h(s)}{s} u^{2j}(s) ds \Big) u = f(u) \ \text{ in } \ \mathbb{R}^2, \\ u \in H_r^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $\omega > 0$, $\lambda > 0$ is constant representing the strength of interaction potentials, $k \in \mathbb{N}^+$ and

$$h(s) = \int_0^s \frac{r}{2} u^2(r) dr.$$

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It is generally known that (1.1) can be applied to search for standing waves for a nonlinear generalized Chern-Simons-Schrödinger system

$$\begin{cases} i\psi_t - e\phi\psi + (\nabla - ie\mathbf{A})^2\psi - e^2|\mathbf{A}|^2 \sum_{j=2}^k \frac{1}{j}|\psi|^{2(j-1)}\psi + f(\psi) = 0, \\ \kappa(\partial_2 A^1 - \partial_1 A^2) = \frac{e}{2}|\psi|^2, \\ \kappa(\partial_2\phi + \partial_t A^2) + e^2 \sum_{j=2}^k \frac{1}{j}|\psi|^{2j}A^1 = e\mathfrak{F}[\overline{\psi}(\partial_1\psi - ieA^1\psi)], \\ -\kappa(\partial_1\phi + \partial_t A^1) + e^2 \sum_{j=2}^k \frac{1}{j}|\psi|^{2j}A^2 = e\mathfrak{F}[\overline{\psi}(\partial_2\psi - ieA^2\psi)], \end{cases} \tag{1.2}$$

where $\psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ is the time-dependent wave function, $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the electric potential, $\mathbf{A} = (A^1, A^2) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the magnetic potential, e stands for the strength of the interaction with the electro-magnetic field (see [23] for example) and $\kappa \in \mathbb{R}$ is a parameter which controls the Chern-Simons term. When we consider the static case, that is, $\psi = \psi(x)$ and $A^i = A^i(x)$, and the Coulomb gauge $\mathrm{div}\mathbf{A} = 0$ whcih gives that $\mathrm{div}(\mathbf{A}u^2) = \mathbf{A} \cdot \nabla u^2$, then the solitary wave is a solution of the form $\psi = \exp(-i\omega t)u(x)$ if $u : \mathbb{R}^2 \to \mathbb{R}$ solves

$$\begin{cases} -\Delta u + \omega u + e\phi u + e^2|\mathbf{A}|^2 \sum_{j=1}^k \frac{1}{j}u^{2(j-1)}u = f(u), \\ \mathbf{A} \cdot \nabla u^2 = 0, \ \ \kappa(\partial_2 A^1 - \partial_1 A^2) = \frac{e}{2}u^2, \\ \kappa\partial_1\phi = e^2 \sum_{j=1}^k \frac{1}{j}u^{2j}A^2, \ \ -\kappa\partial_2\phi = e^2 \sum_{j=1}^k \frac{1}{j}u^{2j}A^1. \end{cases} \tag{1.3}$$

If $u : \mathbb{R}^2 \to \mathbb{R}$ is radially symmetric, by [25, Lemma 3.3], $\mathbf{A}$ can be written as

$$\mathbf{A} = \Big(\frac{e}{\kappa}\frac{h(|x|)}{|x|^2}x_2, -\frac{e}{\kappa}\frac{h(|x|)}{|x|^2}x_1\Big), \ \text{ where } \ x = (x_1, x_2) \in \mathbb{R}^2. \tag{1.4}$$

Let's insert (1.4) into the third equation in (1.3), then there holds

$$\frac{1}{s}h'(s) = \frac{1}{2}u^2(s).$$

Assuming that $h(0) = 0$, which is necessary to have $\mathbf{A}$ smooth, one has

$$h(|x|) = \int_0^{|x|} \frac{s}{2}u^2(s)ds.$$

Moreover, by the last two equations in (1.3), one has

$$\nabla\phi = \frac{e^2}{\kappa}\sum_{j=1}^k \frac{1}{j}u^{2j}(A^2, -A^1) = -\Big(\frac{e^3}{\kappa^2}\frac{h(|x|)}{|x|^2}\sum_{j=1}^k \frac{1}{j}u^{2j}\Big)x, \ \text{ with } \ x = (x_1, x_2) \in \mathbb{R}^2.$$

Hence, if we assume $\lim_{|x|\to+\infty} \phi(|x|) = 0$, there holds

$$\phi(|x|) = \frac{e^3}{\kappa^2}\sum_{j=1}^k \frac{1}{j}\int_{|x|}^{\infty} \frac{h(s)}{s}u^{2j}(s)ds.$$

Combing the above facts, to study (1.3), it is enough to consider (1.1) with $\lambda \triangleq e^4/\kappa^2$. We refer the reader to [26, 27, 23, 19] and their references therein for the details concerning the derivation and physical backgrounds of (1.1).

When $k \equiv 1$, (1.1) can be reduced to the following nonlinear Schrödinger equation of gauged type, which is also known as the Chern-Simons-Schrödinger system,

$$-\Delta u+\omega u+\lambda\Big(\frac{h^2(|x|)}{|x|^2}+\int_{|x|}^{\infty}\frac{h(s)}{s}u^2(s)ds\Big)u=f(u)\ \text{ in }\ \mathbb{R}^2. \tag{1.5}$$

From a mathematical point of view, the existence result with respect to the Cauchy problem corresponding to (1.5) has been investigated in [8], being later improved in [33, 34]. For $f(u)=|u|^{p-2}u$ in (1.5), Byeon-Huh-Seok [9] established the existence of ground state solutions for $p>4$ by a suitable constraint minimization argument, existence and nonexistence results depending on $\lambda>0$ for $p=4$, and the existence of minimizers under L^2-constraint for some $p\in(2,4)$. In [39], the authors showed that there exists a sharp constant $\omega_0>0$ such that the corresponding variational functional to (1.5) is bounded from below if $\omega\geq\omega_0$ and not bounded from below for $\omega\in(0,\omega_0)$ when $f(u)=|u|^{p-2}u$ with $p\in(2,4)$. Meanwhile, the authors in [18] obtained the multiple results when $f(u)$ is a Berestycki-Gallouët-Kavian type nonlinearity [7] and it is the planar version of the well-known Berestycki-Lions type nonlinearity [5, 6]. There are some other interesting and meaningful works on ground state, semiclasssic and normalized solutions, etc. (see [22, 30, 24, 25, 48, 40, 10, 29, 28, 47, 21, 36, 42] for example).

Very recently, both the authors in [31, 21, 51, 35, 16] are concerned with the existence of sign-changing solutions for (1.5) with the following necessary assumption

$$f(t)/t^5\ \text{ is increasing on }(0,+\infty)\text{ and decreasing on }(-\infty,0), \tag{1.6}$$

or, there exists a constant $\theta\in[0,1)$ such that for any $t>0$ and $\tau\in\mathbb{R}\backslash\{0\}$

$$\Big\{\frac{f(t\tau)}{(t\tau)^5}-\frac{f(\tau)}{\tau^5}\Big\}\text{sign}(t-1)+\frac{\theta\omega|1-t^4|}{(t\tau)^4}\geq 0. \tag{1.7}$$

As explained in [51], (1.7) is weaker than (1.6). However, (1.7) strongly depends on the fact that $\theta\in[0,1)$ in which $\theta=1$ is excluded. Hence, one of novel features of this paper is to consider the case $\theta=1$ in (1.7). We point out here that, to the best of our knowledge, there seems to be no results for (1.1) with $k\geq 2$ on existence of sign-changing solutions, even on existence of nontrivial solutions. In this paper, we try to establish the existence of least energy sign-changing solutions for (1.1) with $k\in\mathbb{N}^+$ and a general nonlinearity. To achieve this aim, without loss of generality, we suppose that $f(t)\in C(\mathbb{R})$ vanishes in $t\in(-\infty,0]$ and satisfies the following assumptions

(f_1) $|f(t)|\leq C_0(1+|t|^{p-1})$ *for some constants* $C_0>0$ *and* $2(k+2)<p<+\infty$*;*
(f_2) $f(t)=o(t)$ *as* $t\to 0^+$*;*
(f_3) $f(t)/t^{2k+3}\to+\infty$ *as* $t\to+\infty$*;*
(f_4) *for any* $t>0$ *and* $\tau\in\mathbb{R}\backslash\{0\}$ *there holds*

$$\Big\{\frac{f(t\tau)}{(t\tau)^{2k+3}}-\frac{f(\tau)}{\tau^{2k+3}}\Big\}\text{sign}(t-1)+\frac{\omega|1-t^{2(k+1)}|}{(t\tau)^{2(k+1)}}\geq 0. \tag{1.8}$$

Before stating the main results in this paper, we introduce some notations. We denote by $H_r^1(\mathbb{R}^2)\triangleq\{u\in H^1(\mathbb{R}^2):u(x)=u(|x|)\}$ to be the work space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \nabla v + \omega uv)dx \text{ and } \|u\| = \sqrt{(u, u)}.$$

Let $|\cdot|_p$ with $1 \leq p < +\infty$ be the norm of the usual Lebesgue space $L^p(\mathbb{R}^2)$. Throughout this paper, we shall denote by C and C_i $(i = 0, 1, 2, \cdots)$ for the various positive constants whose exact value may change from lines to lines but are not essential to the analysis of the problem. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function spaces, respectively. For each $\rho > 0$ and every $x \in \mathbb{R}^2$, $B_\rho(x)$ represents the ball of radius ρ centered at x, that is, $B_\rho(x) := \{y \in \mathbb{R}^2 : |y - x| < \rho\}$.

Define the variational functional $I_\lambda : H_r^1(\mathbb{R}^2) \to \mathbb{R}$ associated with (1.1) by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2)dx + \lambda \sum_{j=1}^{k} \frac{1}{2j}\int_{\mathbb{R}^2} \frac{u^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2}u^2(s)ds\Big)^2 dx - \int_{\mathbb{R}^2} F(u)dx,$$

where and in the sequel $F(t) = \int_0^t f(s)ds$. Inspired by the results in [9, 50], we will prove that $I_\lambda \in C^1(H_r^1(\mathbb{R}^2), \mathbb{R})$ whose critical points are solutions of (1.1) and its derivative can be computed as

$$\begin{aligned}\langle I'_\lambda(u), v\rangle &= \int_{\mathbb{R}^2} (\nabla u \nabla v + \omega uv)dx + \lambda \sum_{j=1}^{k}\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^{2(j-1)}uvdx \\ &\quad + \lambda \sum_{j=1}^{k} \frac{1}{j}\int_{\mathbb{R}^2} \frac{u^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2}u^2(s)ds\Big)\Big(\int_0^{|x|} su(s)v(s)ds\Big)dx - \int_{\mathbb{R}^2} f(u)vdx \\ &= \int_{\mathbb{R}^2} (\nabla u \nabla v + \omega uv)dx + \lambda \sum_{j=1}^{k}\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^{2(j-1)}uvdx \\ &\quad + \lambda \sum_{j=1}^{k} \frac{1}{j}\int_{\mathbb{R}^2} \Big(\int_{|x|}^{+\infty} \frac{h(s)}{s}u^{2j}(s)ds\Big)u(x)v(x)dx - \int_{\mathbb{R}^2} f(u)vdx,\end{aligned}$$

for any $u, v \in H_r^1(\mathbb{R}^2)$, where the Fubini's theorem is used. In particular, we have

$$\langle I'_\lambda(u), u\rangle = \|u\|^2 + \lambda \sum_{j=1}^{k} \frac{j+2}{j}\int_{\mathbb{R}^2} \frac{u^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2}u^2(s)ds\Big)^2 dx - \int_{\mathbb{R}^2} f(u)udx, \ \ \forall u \in H_r^1(\mathbb{R}^2).$$

Furthermore, we say $u \in H_r^1(\mathbb{R}^2)$ is a sign-changing solution of (1.1) if it is a solution of (1.1) and satisfies $u^\pm \neq 0$, where

$$u^+(x) \triangleq \max\{u, 0\} \text{ and } u^-(x) \triangleq \min\{u, 0\}.$$

Here, a solution $u \in H_r^1(\mathbb{R}^2)$ is called a least energy sign-changing solution of (1.1) if it possesses the least energy among all sign-changing solutions, namely,

$$I_\lambda(u) = \inf\{I_\lambda(v) : \ v \in H_r^1(\mathbb{R}^2) \text{ is a sign-changing solution of (1.1)}\}.$$

For every $u = u^+ + u^- \in H_r^1(\mathbb{R}^2)$, by some elementary computations, it is obvious to find that

$$
\begin{aligned}
I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) &+ \lambda \sum_{j=1}^{k} \frac{1}{j} \int_{\mathbb{R}^2} \frac{u^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big) \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big) dx \\
&+ \lambda \sum_{j=1}^{k} \frac{1}{2j} \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big)^2 dx \\
&+ \lambda \sum_{j=1}^{k} \frac{1}{2j} \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big)^2 dx. \qquad (1.9)
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
\langle I'_\lambda(u), u^+ \rangle = \langle I'_\lambda(u^+), u^+ \rangle &+ \lambda \sum_{j=1}^{k} \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big)^2 dx \\
&+ 2\lambda \sum_{j=1}^{k} \frac{1+j}{j} \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big) \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big) dx \\
&+ 2\lambda \sum_{j=1}^{k} \frac{1}{j} \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} u^2(s) ds \Big) \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big) dx, \qquad (1.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle I'_\lambda(u), u^- \rangle = \langle I'_\lambda(u^-), u^- \rangle &+ \lambda \sum_{j=1}^{k} \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big)^2 dx \\
&+ 2\lambda \sum_{j=1}^{k} \frac{1+j}{j} \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} |u^+(s)|^2 ds \Big) \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big) dx \\
&+ 2\lambda \sum_{j=1}^{k} \frac{1}{j} \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2} \Big(\int_0^{|x|} \frac{s}{2} u^2(s) ds \Big) \Big(\int_0^{|x|} \frac{s}{2} |u^-(s)|^2 ds \Big) dx. \qquad (1.11)
\end{aligned}
$$

Equation (1.1) is usually regarded as a nonlocal Schrödinger equation because of the appearance the Cherm-Simons term

$$
\lambda \sum_{j=1}^{k} \Big(\frac{h^2(|x|)}{|x|^2} u^{2(j-1)} + \frac{1}{j} \int_{|x|}^{\infty} \frac{h(s)}{s} u^{2j}(s) ds \Big) u
$$

which yields that (1.1) is never a pointwise identity any longer such that there are some mathematical difficulties that make the study of it more interesting. Actually, (1.1) with $\lambda \equiv 0$ belongs to the following semilinear Schrödinger equations

$$
-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \qquad (1.12)
$$

whose variational $\overline{I}_V(u): H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\overline{I}_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2]dx - \int_{\mathbb{R}^N} F(x,u)dx, \text{ where } F(x,u) = \int_0^u f(x,t)dt.$$

Over the last several decades, (1.12) has been extensively considered under the variant assumptions on $V(x)$ and $f(x,u)$ via variational methods, see [45, 46, 5, 11, 2, 3, 4, 14, 1, 15, 12] and their references therein for example. To look for sign-changing solutions of (1.12), as pointed out in [20, 43, 49], the approaches heavily rely on the following decompositions

$$\overline{I}_V(u) = \overline{I}_V(u^+) + \overline{I}_V(u^-),\ \langle \overline{I}_V(u), u^+\rangle = \langle \overline{I}_V(u^+), u^+\rangle,\ \langle \overline{I}_V(u)u^-\rangle = \langle \overline{I}_V(u^-), u^-\rangle. \quad (1.13)$$

In view of (1.9)-(1.11), the functional I_λ does not satisfy the decompositions in (1.13). Thus, there are some essential differences in investigating the existence of sign-changing solutions for (1.1) between $\lambda > 0$ and $\lambda = 0$.

In the present paper, by using a suitable constraint minimization argument, we obtain the existence and asymptotic behavior of least energy sign-changing solutions for (1.1), which can be viewed as a generalization and improvement to the results in [31, 21, 51].

The main results in this paper can be stated as follows.

Theorem 1.1. *Assume that* $(f_1)-(f_4)$ *hold, then* (1.1) *possesses at least one least energy sign-changing solution* $u_\lambda \in H_r^1(\mathbb{R}^2)$*, which has precisely two nodal domains.*

Theorem 1.2. *Assume that* $(f_1)-(f_4)$ *hold and let* $u_\lambda \in H_r^1(\mathbb{R}^2)$ *be a least energy sign-changing solution of* (1.1) *obtained in Theorem 1.1, then* $c_\lambda > 0$ *is achieved and* $I_\lambda(u_\lambda) > 2c_\lambda$*, where*

$$c_\lambda \triangleq \inf\{I_\lambda(u) : u \in \mathcal{N}_\lambda\} \text{ and } \mathcal{N}_\lambda \triangleq \{u \in H_r^1(\mathbb{R}^2)\backslash\{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

Theorem 1.3. *Assume that* $(f_1)-(f_4)$ *hold, then for each sequence* $\{\lambda_n\} \subset (0,+\infty)$ *with* $\lambda_n \searrow 0$ *as* $n \to \infty$*, there exist a subsequence, still denoted by itself, and* $u_0 \in H_r^1(\mathbb{R}^2)$ *such that* u_{λ_n} *obtained in Theorem 1.1 converges strongly in* $H_r^1(\mathbb{R}^2)$ *to* u_0*. Moreover,* u_0 *is a least energy sign-changing solution of the following problem*

$$\begin{cases} -\Delta u + \omega u = f(u) \text{ in } \mathbb{R}^2, \\ u \in H_r^1(\mathbb{R}^2), \end{cases} \quad (1.14)$$

which changes sign only once.

Remark 1.4. *Similar to the discussions in* [51]*,* (f_4) *with* $k = 1$ *exactly implies that*

$$f(\tau)\tau - 6F(\tau) + 2\omega\tau^2 \geq 0, \quad \forall \tau \in \mathbb{R},$$

which indicates that for every $u \in H_r^1(\mathbb{R}^2)$ *there holds*

$$6I_\lambda(u) - \langle I'_\lambda(u), u\rangle \geq 2|\nabla u|_2^2 + \int_{\mathbb{R}^2} [f(u)u - 6F(u) + 2\omega u^2]dx \geq 2|\nabla u|_2^2.$$

Obviously, we cannot use the method in [51] *to show that* I_λ *is coercive on the set*

$$\mathcal{M}_\lambda \triangleq \{u \in H_r^1(\mathbb{R}^2) : u^\pm \neq 0 \text{ and } \langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u), u^-\rangle = 0\}. \quad (1.15)$$

Remark 1.5. *As far as we are concerned, there seems to be no results concerning sign-changing solutions for* (1.1) *until now. We would like to highlight that Theorems 1.1- 1.3 can be suitable for the model nonlinearity*

$$f(t)=\begin{cases} 0, & \text{if } t\le 0,\\ t^{p-1}+\alpha t^{q-1}, & \text{if } t>0,\end{cases}$$

where $2<q<2(k+2)<p<+\infty$ *and* $\alpha\in\mathbb{R}$ *satisfy*

$$0<\alpha\le \frac{(2k+4-q)(q-2)}{(p-2k-4)(p-2)}\Big[\frac{2(k+1)(q-2)}{(p-q)(p-2k-4)}\Big]^{\frac{p-q}{p-2}}.$$

One could easily verify that each $f(t)$ *does not satisfy the Nehari type monotone condition corresponding to* (1.1)

$$f(t)/t^{2k+3}\ \text{ is increasing on } (0,+\infty) \text{ and decreasing on } (-\infty,0),\quad \forall k\in\mathbb{N}^+.$$

On the other hand, $f(t)$ *(even the case* $k\equiv 1$*) cannot be applied to* (1.7) *proposed in* [51]. *Therefore, we have to introduce some new analytic techniques to deal with the challenges appearing in this paper.*

To complete this section, we sketch the proofs of our main results as follows. In the proof of Theorem 1.1, inspired by [17, 4], we apply the constrain minimization problem

$$m_\lambda \triangleq \inf\{I_\lambda(u): u\in\mathcal{M}_\lambda\}, \tag{1.16}$$

to establish the existence of least energy solutions of (1.1), where $\mathcal{M}_\lambda$ is given by (1.15). Firstly, we follow [51, Lemma 2.4] to show that $\mathcal{M}_\lambda\neq\emptyset$, where we have to take some delicate analysis for $k\ge 2$. However, since (f_4) just implies that (see Section 3 below)

$$f(\tau)\tau-2(k+2)F(\tau)+(k+1)\omega\tau^2\ge 0,\quad \forall\tau\in\mathbb{R}, \tag{1.17}$$

it's difficult to prove that $m_\lambda>0$ in a standard way. To overcome it, we make full use of the Gagliardo-Nirenberg inequality (see for instance [38, Theorem in Lecture II]), there exists a best constant $K_{\text{opt}}>0$, and for any $s\in(2,+\infty)$, such that

$$|u|_s^s\le K_{\text{opt}}|\nabla u|_2^{s-2}|u|_2^2,\quad \forall u\in H^1(\mathbb{R}^2). \tag{1.18}$$

Secondly, because of (1.17), we take advantage of the cerebrated Vanishing lemma due to P. L. Lions (see e.g. [32, 50]) to prove that I_λ is coercive on $\mathcal{M}_\lambda$ which indicates that every minimizing sequence of m_λ is bounded in $H_r^1(\mathbb{R}^2)$. Then, by virtue of the Vanishing lemma again, we show that m_λ can be achieved by a nontrivial $u_\lambda\in H_r^1(\mathbb{R}^2)$. Finally, we modify the proof of [19, Propositin 4.9], in which a nonnegative solution was obtained, to prove that u_λ is indeed a solution of (1.1). So, we complete the proof of Theorem 1.1.

As to the proofs of Theorems 1.2 and 1.3, there are no essential differences from the counterparts in [31, 51] except the boundedness of $\{u_{\lambda_n}\}$. In view of the analytic skills in the proof of Theorem 1.1, we can derive the boundedness of $\{u_{\lambda_n}\}$ in a very similar way. Up to now, we can prove the main results successfully in this paper.

The paper is organized as follows. In Section 2, we provide some basic properties for the Chern-Simons term. Section 3 is devoted to the proofs of Theorems 1.1-1.3.

2. Variational settings and Preliminaries

In this section, we formulate (1.1) as a variational problem and prepare some preliminary results. First, for every $u \in H_r^1(\mathbb{R}^2)$ and $k \in \mathbb{N}^+$, we define, for short, the following quantities,

$$E_j(u) \triangleq \int_{\mathbb{R}^2} \frac{u^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}u^2 dr\Big)^2 dx, \quad \forall j \in \{1,2,\cdots,k\}.$$

In view of [9, Proposition 4.1], one has

$$\int_0^{|x|} \frac{r}{2}u^2(r)dr \le \frac{|x|}{2\sqrt{2}}\Big(\int_0^{|x|} ru^4(r)dr\Big)^{1/2} \le \frac{|x|}{4\pi\sqrt{2}}\Big(\int_{\mathbb{R}^2}|u|^4 dx\Big)^{1/2},$$

which implies that

$$E_j(u) \le \frac{1}{32\pi^2}\int_{\mathbb{R}^2} u^{2j}dx \int_{\mathbb{R}^2}|u|^4 dx, \quad \forall j \in \{1,2,\cdots,k\}. \tag{2.1}$$

Therefore, arguing as [9, Appendix A], we can show that $E_j(u) \in C^1(H_r^1(\mathbb{R}^2),\mathbb{R})$ for any $j \in \{1,2,\cdots,k\}$ and then every critical point is a weak solution of (1.1). Moreover, we can obtain the following lemma.

Lemma 2.1. *([9, Lemma 3.2]) Suppose that there is a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ converging weakly to a function $u \in H_r^1(\mathbb{R}^2)$ as $n \to \infty$. Then, for every $\varphi \in H_r^1(\mathbb{R}^2)$, $\{E_j(u_n)\}$, $\{E_j'(u_n)\varphi\}$ and $\{E_j'(u_n)u_n\}$ converge up to a subsequence to $E_j(u)$, $E'(u)\varphi$ and $E_j'(u)u$ for any $j \in \{1,2,\cdots,k\}$, respectively, as $n \to \infty$.*

3. Proofs of Theorems 1.1-1.3

In this section, we give the detail proofs of Theorems 1.1-1.3 and always assume (f_1)–(f_4) are satisfied for simplicity. Firstly, we give some elementary, but very important for the proofs of our main results, facts:

$$\begin{aligned} &\zeta(\theta) \triangleq \theta^{2(k+2)} - (k+2)\theta^2 + k + 1 > \zeta(1) = 0, \forall\theta \in (0,1)\cup(1,+\infty),\\ &\xi_j(\theta) \triangleq (j+2)\theta^{2(k+2)} - (k+2)\theta^{2(j+2)} + k - j > \xi_j(1) = 0,\ \forall\theta \in (0,1)\cup(1,+\infty), \end{aligned} \tag{3.1}$$

where $j \in \{1,2,\cdots,k\}$. For every $j \in \{1,2,\cdots,k\}$, we define

$$h_{j+1,1}(s,t) \triangleq (j+1)s^{2(k+2)} + t^{2(k+2)} - (k+2)s^{2j+2}t^2 + k - j,\ \forall(s,t) \in (0,+\infty)\times(0,+\infty),$$
$$h_{1,j+1}(s,t) \triangleq s^{2(k+2)} + (j+1)t^{2(k+2)} - (k+2)s^{2j+2}t^2 + k - j,\ \forall(s,t) \in (0,+\infty)\times(0,+\infty),$$
$$h_{j,2}(s,t) \triangleq js^{2(k+2)} + 2t^{2(k+2)} - (k+2)s^{2j}t^4 + k - j,\ \forall(s,t) \in (0,+\infty)\times(0,+\infty),$$
$$h_{2,j}(s,t) \triangleq 2s^{2(k+2)} + jt^{2(k+2)} - (k+2)s^{2j}t^4 + k - j,\ \forall(s,t) \in (0,+\infty)\times(0,+\infty).$$

We claim that for any $(s,t) \in (0,+\infty)\times(0,+\infty)$ and $j \in \{1,2,\cdots,k\}$, there holds

$$h_{j+1,1}(s,t) \ge 0,\ h_{1,j+1}(s,t) \ge 0,\ h_{j,2}(s,t) \ge 0 \text{ and } h_{2,j}(s,t) \ge 0. \tag{3.2}$$

We just give the explanation of $h_{j+1,1}(s,t) \ge 0$ for every $(s,t) \in (0,+\infty)\times(0,+\infty)$ and $j \in \{1,2,\cdots,k\}$. In fact, we split it into two cases.

Case I: $j = k$. Then it suffices to show that for $j \in \{1, 2, \cdots, k\}$,

$$\overline{h}_{j+1}(s,t) = (j+1)s^{2(j+2)} + t^{2(j+2)} - (j+2)s^{2j+2}t^2 \geq 0, \ \ \forall (s,t) \in (0,+\infty) \times (0,+\infty),$$

which is equivalent to

$$(j+1)(s/t)^{2(j+2)} - (j+2)(s/t)^{2j+2} + 1 \geq 0, \ \ \forall (s,t) \in (0,+\infty) \times (0,+\infty).$$

Similar to (3.1), we know that $\overline{h}_{j+1}(s,t) > 0$ for any $(s,t) \in (0,+\infty) \times (0,+\infty)$ and $s \neq t$. Obviously, $\overline{h}_{j+1}(s,t) = 0$ when $s = t$. So, the claim is true.

Case II: $j \neq k$. It follows from some simple computations that

$$\begin{cases} \frac{\partial}{\partial s} h_{j+1,1}(s_0,t_0) = 2(k+2)(j+1)s_0^{2k+3} - 2(k+2)(j+1)s_0^{2j+1}t_0^2 = 0, \\ \frac{\partial}{\partial t} h_{j+1,1}(s_0,t_0) = 2(k+2)t_0^{2k+3} - 2(k+2)s_0^{2j+2}t_0 = 0, \end{cases}$$

which yields that $(s_0,t_0) = (1,1)$ since $s_0 > 0$ and $t_0 > 0$. One the other hand,

$$\det \begin{pmatrix} \frac{\partial^2}{\partial s^2} h_{j+1,1} & \frac{\partial^2}{\partial s \partial t} h_{j+1,1} \\ \frac{\partial^2}{\partial t \partial s} h_{j+1,1} & \frac{\partial^2}{\partial t^2} h_{j+1,1} \end{pmatrix}_{(s_0,t_0)} = 16(k+2)^2(j+1)[(k+1)^2 - j(k+1) - (j+1)].$$

Obviously, $\frac{\partial^2}{\partial s^2} h_{j+1,1}(s_0,t_0) > 0$ and $\det \mathbf{H}_{h_{j+1,1}}(s_0,t_0) > 0$ for every $j < k$, where $\mathbf{H}_{h_{j+1,1}}$ is the Hesse matrix of $h_{j+1,1}$. Then we have $h_{j+1,1}(s,t) \geq h_{j+1,1}(s_0,t_0) = 0$.

As a consequence of (3.1), we have

$$G(\tau,u) \triangleq \int_{\mathbb{R}^2} g(\tau,u)dx \geq 0, \ \forall \tau \in (0,+\infty) \text{ and } u \in H_r^1(\mathbb{R}^2), \tag{3.3}$$

where

$$g(\tau,u) = \frac{1-\tau^{2(k+2)}}{2(k+2)} f(u)u + F(\tau u) - F(u) + \frac{\omega\zeta(\tau)}{2(k+2)} u^2. \tag{3.4}$$

To get (3.3), we prove that $g(\tau,u) \geq 0$ for any $\tau \in (0,+\infty)$ and $u \in H_r^1(\mathbb{R}^2)$. Indeed,

$$\begin{aligned} \frac{\partial}{\partial \tau} g(\tau,u) &= f(\tau u)u - \tau^{2k+3} f(u)u + \omega\tau(\tau^{2(k+1)} - 1)u^2 \\ &= \tau^{2k+3} u^{2(k+2)} \left\{ \left[\frac{f(\tau u)}{(\tau u)^{2k+3}} - \frac{f(u)}{u^{2k+3}} \right] + \frac{\omega[\tau^{2(k+1)} - 1]}{(\tau u)^{2k+2}} \right\}. \end{aligned}$$

In view of (1.8), one can find that $g_\tau(\tau,u) < 0$ for any $\tau \in (0,1)$, and $g_\tau(\tau,u) > 0$ for any $\tau \in (1,+\infty)$. So, $g(\tau,u) > g(1,u) = 0$ for every $\tau \in (0,1) \cup (1,+\infty)$. By virtue of (f_2) and letting $\tau \to 0$ in (3.4), there holds

$$f(u)u - 2(k+2)F(u) + (k+1)\omega u^2 \geq 0, \ \ \forall u \in H_r^1(\mathbb{R}^2). \tag{3.5}$$

Combing (3.1)-(3.3), we obtain the following lemma.

Lemma 3.1. *For every $u = u^+ + u^- \in H_r^1(\mathbb{R}^2)$ and $(s,t) \in (0,+\infty) \times (0,+\infty)$, we have*

$$\begin{aligned} &I_\lambda(u) - I_\lambda(su^+ + tu^-) - \frac{1-s^{2(k+2)}}{2(k+2)} \langle I'_\lambda(u), u^+ \rangle - \frac{1-t^{2(k+2)}}{2(k+2)} \langle I'_\lambda(u), u^- \rangle \\ &\geq \frac{\zeta(s)}{2(k+2)} |\nabla u^+|_2^2 + \frac{\zeta(t)}{2(k+2)} |\nabla u^-|_2^2. \end{aligned} \tag{3.6}$$

Proof. For every $j \in \{1, 2, \cdots, k\}$, we define

$$E_j^{+,+,+} \triangleq \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^+(r)|^2 dr\Big)^2 dx,\ E_j^{-,-.-} \triangleq \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^-(r)|^2 dr\Big)^2 dx,$$

$$E_j^{+,+,-} \triangleq \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^+(r)|^2 dr\Big)\Big(\int_0^{|x|} \frac{r}{2}|u^-(r)|^2 dr\Big)dx,$$

$$E_j^{+,-,-} \triangleq \int_{\mathbb{R}^2} \frac{|u^+|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^-(r)|^2 dr\Big)^2 dx,\ E_j^{-,+,+} \triangleq \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^+(r)|^2 dr\Big)^2 dx,$$

$$E_j^{-,+,-} \triangleq \int_{\mathbb{R}^2} \frac{|u^-|^{2j}}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^+(r)|^2 dr\Big)\Big(\int_0^{|x|} \frac{r}{2}|u^-(r)|^2 dr\Big)dx.$$

Then, some direct computations give us that

$$\begin{aligned}
&I_\lambda(u) - I_\lambda(su^+ + tu^-) - \frac{1-s^{2(k+2)}}{2(k+2)}\langle I'_\lambda(u), u^+\rangle - \frac{1-t^{2(k+2)}}{2(k+2)}\langle I'_\lambda(u), u^-\rangle \\
&= \frac{\zeta(s)}{2(k+2)}|\nabla u^+|_2^2 + \frac{\zeta(t)}{2(k+2)}|\nabla u^-|_2^2 + \frac{\lambda}{2(k+2)}\sum_{j=1}^k \frac{1}{j}[\xi_j(s)E_j^{+,+,+} + \xi_j(t)E_j^{-,-,-}] \\
&\quad + \frac{\lambda}{k+2}\sum_{j=1}^k \frac{1}{j}\Big(h_{j+1,1}(s,t)E_j^{+,+,-} + h_{1,j+1}(s,t)E_j^{-,+,-}\Big) \\
&\quad + \frac{\lambda}{2(k+2)}\sum_{j=1}^k \frac{1}{j}\Big(h_{j,2}(s,t)E_j^{+,-,-} + h_{2,j}(s,t)E_j^{-,+,+}\Big) + G(s,u^+) + G(t,u^-),
\end{aligned}$$

which together with (3.1), (3.2) and (3.3) yields the desired result. □

Lemma 3.2. *Given a function $u \in H_r^1(\mathbb{R}^2)$ with $u^\pm \neq 0$, then there exists a unique pair $(s_u, t_u) \in (0, +\infty) \times (0, +\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.*

Proof. For the fixed $u \in H_r^1(\mathbb{R}^2)$ with $u^\pm \neq 0$, we consider the following vector function

$$V(s,t) \triangleq (\langle I'_\lambda(su^+ + tu^-), su^+\rangle, \langle I'_\lambda(su^+ + tu^-), tu^-\rangle),\ \forall (s,t) \in (0,+\infty)\times(0,+\infty),$$

where

$$\begin{aligned}
\langle I'_\lambda(su^+ + tu^-), su^+\rangle &= s^2\|u^+\|^2 + \lambda\sum_{j=1}^k \frac{j+2}{j}s^{2(j+2)}E_j^{+,+,+} - \int_{\mathbb{R}^2} f(su^+)su^+ dx \\
&+ \lambda\sum_{j=1}^k \frac{1}{j}\Big[2s^4t^{2j}E_j^{-,+,+} + 2(j+1)s^{2(j+1)}t^2E_j^{+,+,-} + js^{2j}t^4E_j^{+,-,-} + 2s^2t^{2(j+1)}E_j^{-,+,-}\Big],
\end{aligned}$$

$$\begin{aligned}
\langle I'_\lambda(su^+ + tu^-), tu^-\rangle &= t^2\|u^-\|^2 + \lambda\sum_{j=1}^k \frac{j+2}{j}t^{2(j+2)}E_j^{-,-,-} - \int_{\mathbb{R}^2} f(tu^-)tu^- dx \\
&+ \lambda\sum_{j=1}^k \frac{1}{j}\Big[js^4t^{2j}E_j^{-,+,+} + 2s^{2(j+1)}t^2E_j^{+,+,-} + 2s^{2j}t^4E_j^{+,-,-} + 2(j+1)s^2t^{2(j+1)}E_j^{-,+,-}\Big].
\end{aligned}$$

Recalling that (2.1), then we can apply $(f_1)-(f_3)$ to show that there exist two constants $0<R_1<R_2<+\infty$ such that

$$\begin{cases} \langle I'_\lambda(R_1u^+ + R_1u^-), R_1u^+\rangle > 0, \ \langle I'_\lambda(R_1u^+ + R_1u^-), R_1u^-\rangle > 0, \\ \langle I'_\lambda(R_2u^+ + R_2u^-), R_2u^+\rangle < 0, \ \langle I'_\lambda(R_2u^+ + R_2u^-), R_2u^-\rangle < 0. \end{cases}$$

Hence, by the monotonicity of $s>0$ (resp. $t>0$) if $t>0$ (resp. $s>0$) is fixed, one has

$$\begin{cases} \langle I'_\lambda(R_1u^+ + tu^-), R_1u^+\rangle > 0 \text{ and } \langle I'_\lambda(su^+ + R_1u^-), R_1u^-\rangle > 0, \quad \forall s,t\in[R_1,R_2], \\ \langle I'_\lambda(R_2u^+ + tu^-), R_2u^+\rangle < 0 \text{ and } \langle I'_\lambda(su^+ + R_2u^-), R_2u^-\rangle < 0, \quad \forall s,t\in[R_1,R_2]. \end{cases}$$

It follows from the Miranda's theorem [37] (or, [31, Lemma 2.4]) that there exists a pair $(s_u,t_u)\in(R_1,R_2)\times(R_1,R_2)$ such that $V(s_u,t_u)=(0,0)$ which gives $s_uu^+ + t_uu^-\in\mathcal{M}_\lambda$.

We next prove that the pair $(s_u,t_u)\in(0,+\infty)\times(0,+\infty)$ is unique. Arguing it indirectly, there exist two pairs, (s_i,t_i) with $i=1,2$, such that $v_i\triangleq s_iu^+ + t_iu^-\in\mathcal{M}_\lambda$. Then

$$\begin{aligned} &I_\lambda(v_1)-I_\lambda(v_2)-\frac{s_1^{2(k+2)}-s_2^{2(k+2)}}{2(k+2)s_1^{2(k+2)}}\langle I'_\lambda(v_1), s_1u^+\rangle-\frac{t_1^{2(k+2)}-t_2^{2(k+2)}}{2(k+2)t_1^{2(k+2)}}\langle I'_\lambda(v_1), t_1u^-\rangle \\ &\quad\geq \frac{s_1^2\zeta(s_2/s_1)}{2(k+2)}|\nabla u^+|_2^2+\frac{t_1^2\zeta(t_2/t_1)}{2(k+2)}|\nabla u^-|_2^2+G(s_2/s_1, s_1u^+)+G(t_2/t_1, t_1u^-), \end{aligned}$$

and

$$\begin{aligned} &I_\lambda(v_2)-I_\lambda(v_1)-\frac{s_2^{2(k+2)}-s_1^{2(k+2)}}{2(k+2)s_2^{2(k+2)}}\langle I'_\lambda(v_2), s_2u^+\rangle-\frac{t_2^{2(k+2)}-t_1^{2(k+2)}}{2(k+2)t_2^{2(k+2)}}\langle I'_\lambda(v_2), t_2u^-\rangle \\ &\quad\geq \frac{s_2^2\zeta(s_1/s_2)}{2(k+2)}|\nabla u^+|_2^2+\frac{t_2^2\zeta(t_1/t_2)}{2(k+2)}|\nabla u^-|_2^2+G(s_1/s_2, s_2u^+)+G(t_1/t_2, t_2u^-). \end{aligned}$$

Since $\langle I'_\lambda(v_1), s_1u^+\rangle=\langle I'_\lambda(v_2), s_2u^+\rangle=0$, and in view of (3.1) and (3.3), we can derive a contradiction by adding the above two inequalities. The proof is complete. □

Lemma 3.3. *There are two constants $\rho^\pm\in(0,+\infty)$ such that $\|u^\pm\|\geq\rho^\pm$ for any $u\in\mathcal{M}_\lambda$. Moreover, $m_\lambda=\inf\{I_\lambda(u): u\in\mathcal{M}_\lambda\}>0$. Similarly, $c_\lambda=\inf\{I_\lambda(u): u\in\mathcal{N}_\lambda\}>0$.*

Proof. By $(f_1)-(f_2)$, for any $\epsilon>0$, there exists a constant $C_\epsilon>0$ such that

$$\max\{f(t)t, F(t)\}\leq\epsilon t^2+C_\epsilon|t|^p \text{ for any } t\in\mathbb{R} \text{ and } p>2(k+2). \tag{3.7}$$

For any $u\in\mathcal{M}_\lambda$ which gives $\langle I'_\lambda(u), u^\pm\rangle=0$, combing (1.10)-(1.11) and (3.7), we obtain

$$\|u^\pm\|^2\leq\int_{\mathbb{R}^2}f(u^\pm)u^\pm dx\leq\frac{1}{2}\|u^\pm\|^2+C_2\|u^\pm\|^p,$$

which gives the desired result. Let $\{u_n\}\subset\mathcal{M}_\lambda$ be a minimizing sequence of m_λ, that is, $\{u_n\}\subset\mathcal{M}_\lambda$ satisfies $I_\lambda(u_n)\to m_\lambda$ as $n\to\infty$. To proceed the proof, we split it into two cases.

Case 1: Either $\overline{\rho}^+\triangleq\inf_{n\in\mathbb{N}}|\nabla u_n^+|_2^2>0$ or $\overline{\rho}^-\triangleq\inf_{n\in\mathbb{N}}|\nabla u_n^-|_2^2>0$. It follows from (3.5) that

$$m_\lambda=\lim_{n\to\infty}I_\lambda(u_n)=\lim_{n\to\infty}\left[I_\lambda(u_n)-\frac{1}{2(k+2)}\langle I'_\lambda(u_n), u_n\rangle\right]\geq\frac{k+1}{2(k+2)}(\overline{\rho}^+ + \overline{\rho}^-)>0.$$

Case 2: $\overline{\rho}^{\pm} \triangleq \inf_{n\in\mathbb{N}} |\nabla u_n^{\pm}|_2^2 = 0$. By the first part of this lemma, up to a subsequence if necessary, we can conclude that

$$|\nabla u_n^{\pm}|_2^2 \to 0 \text{ and } |u_n^{\pm}|_2^2 \geq \frac{1}{2}(\rho^{\pm})^2 > 0. \tag{3.8}$$

Let's define

$$s_n \triangleq \sqrt{2}|u_n^+|_2^{-1} \text{ and } t_n \triangleq \sqrt{2}|u_n^-|_2^{-1}.$$

By (3.8), we obtain $\{s_n\}$, $\{t_n\}$ are bounded and then

$$\lim_{n\to\infty} s_n^2|\nabla u_n^+|_2^2 = \lim_{n\to\infty} t_n^2|\nabla u_n^-|_2^2 = 0. \tag{3.9}$$

Since $\{u_n\} \subset \mathcal{M}_\lambda$, by (3.1) and (3.3), we know that

$$I_\lambda(u_n) = \max_{s,t>0} I_\lambda(su_n^+ + tu_n^-). \tag{3.10}$$

Choosing $\epsilon = 1/4$ in (3.7), combing (3.9) and (3.10), we derive

$$\begin{aligned} I_\lambda(u_n) &\geq I_\lambda(s_nu_n^+ + t_nu_n^-) \geq \frac{s_n^2}{4}|u_n^+|_2^2 + \frac{t_n^2}{4}|u_n^-|_2^2 - C_4 s_n^p|u_n^+|_p^p - C_4 t_n^p|u_n^-|_p^p \\ &\geq 1 - C_5(s_n|\nabla u_n^+|_2)^{p-2}s_n^2|u_n^+|_2^2 - C_5(t_n|\nabla u_n^-|_2)^{p-2}t_n^2|u_n^-|_2^2 = 1 + o(1), \end{aligned}$$

which gives the desired result, where we have used (1.18). The proof is complete. □

Lemma 3.4. *The variational functional I_λ is coercive on $\mathcal{M}_\lambda$.*

Proof. Arguing it indirectly, we may assume that there exists a sequence $\{u_n\} \subset \mathcal{M}_\lambda$ such that $I_\lambda(u_n) \to m_\lambda$ and $\|u_n\| \to +\infty$ as $n \to \infty$. Set $v_n = u_n/\|u_n\|$, then $\|v_n\| \equiv 1$. Passing to a subsequence, there exists a function $v \in H_r^1(\mathbb{R}^2)$ such that $v_n \rightharpoonup v$ in $H_r^1(\mathbb{R}^2)$, $v_n \to v$ in $L^s(\mathbb{R}^2)$ with $s \in (2, +\infty)$, and $v_n \to v$ a.e. in $\mathbb{R}^2$. We claim that there exist two constants $R > 0$ and $\delta > 0$ such that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx \geq \delta. \tag{3.11}$$

If not, by using Lion's vanishing lemma (see e.g. [32, 50]), one can conclude that $v_n \to 0$ in $L^s(\mathbb{R}^2)$ for any $s \in (2, +\infty)$. For a fixed constant $L > \sqrt{2(1 + m_\lambda)}$, by (3.7), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F(Lv_n)dx = 0,$$

which implies that

$$\lim_{n\to\infty} I_\lambda(Lv_n) \geq \frac{L^2}{2} - \lim_{n\to\infty} \int_{\mathbb{R}^2} F(Lv_n)dx = \frac{L^2}{2}. \tag{3.12}$$

Since $\{u_n\} \subset \mathcal{M}_\lambda$, by (3.10) we derive

$$I_\lambda(u_n) = \max_{s,t>0} I_\lambda(su_n^+ + tu_n^-) \geq I_\lambda(L\|u_n\|^{-1}u_n^+ + L\|u_n\|^{-1}u_n^-) = I_\lambda(Lv_n).$$

By (3.12), we can get a contradiction because of the choice of L. So, (3.11) holds true and there exists $y_n \in \mathbb{R}^2$ such that $\int_{B_R(0)} \overline{v}_n^2 dx \geq \delta/2 > 0$, where $\overline{v}_n = v_n(\cdot - y_n)$. Clearly, $\|\overline{v}_n\| = \|v_n\| = 1$, then there is a function $\overline{v} \neq 0$ such that $\overline{v}_n \rightharpoonup \overline{v}$ in $H_r^1(\mathbb{R}^2)$, $\overline{v}_n \to \overline{v}$ in

$L^s(\mathbb{R}^2)$ with $s \in (2, +\infty)$, and $\overline{v}_n \to \overline{v}$ a.e. in $\mathbb{R}^2$ in the sense of a subsequence. For every $x \in \{x \in \mathbb{R}^2 : \overline{v}(x) \neq 0\}$, we derive $|u_n(\cdot - y_n)| = \|u_n\| \cdot |\overline{v}_n(x)| \to \infty$ as $n \to \infty$. It follows from (f_3) that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^{2(k+2)}} dx = \liminf_{n\to\infty} \int_{\mathbb{R}^2} \frac{F(u_n(\cdot - y_n))}{|u_n(\cdot - y_n)|^{2(k+2)}} |\overline{v}_n|^{2(k+2)} dx = +\infty. \tag{3.13}$$

Since $I_\lambda(u_n) \to m_\lambda > 0$ as $n \to \infty$, combing (2.1) and (3.13) we have

$$0 = \limsup_{n\to\infty} \frac{I_\lambda(u_n)}{\|u_n\|^{2(k+2)}} \leq \frac{C}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^{2(k+2)}} dx = -\infty,$$

a contradiction. The proof is complete. □

Lemma 3.5. *Let $u \in H^1_r(\mathbb{R}^2)$ be a minimizer of $I_\lambda(u)$ under the constraint $\mathcal{M}_\lambda$, then u is a least energy sign-changing solution of* (1.1).

Proof. We follow the idea used in [41, Theorem 2.2] and [44, Theorem 1.1]. Let $u \in \mathcal{M}_\lambda$ be a minimizer of the functional $I_\lambda|_{\mathcal{M}_\lambda}$. Then from Lemmas 3.1 and 3.2, one has

$$I_\lambda(u) = \inf_{v \in H^1_r(\mathbb{R}^2) \text{ with } v^\pm \neq 0} \max_{s,t>0} I_\lambda(sv^+ + tv^-) = \inf_{v \in \mathcal{M}_\lambda} I_\lambda(v). \tag{3.14}$$

Suppose by contradiction that u is not a weak solution of (1.1), then there exists a function $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that

$$\langle I'_\lambda(u), \varphi \rangle < -1.$$

Choosing $\varepsilon > 0$ sufficiently small so that

$$\langle I'_\lambda(su^+ + tu^- + \sigma\varphi), \varphi \rangle \leq -\frac{1}{2}, \text{ for all } |s-1|,\ |t-1|,\ |\sigma| \leq \varepsilon. \tag{3.15}$$

Let $\eta(s, t)$ be a cutoff function such that $0 \leq \eta(s, t) \leq 1$ for any $(s, t) \in (0, +\infty) \times (0, +\infty)$ and

$$\eta(s,t) = \begin{cases} 1, & \text{if } |s-1| \leq \frac{1}{2}\varepsilon \text{ and } |t-1| \leq \frac{1}{2}\varepsilon, \\ 0, & \text{if } |s-1| \geq \varepsilon \text{ or } |t-1| \leq \varepsilon. \end{cases}$$

For every $(s, t) \in (0, +\infty) \times (0, +\infty)$, we define a path $v : \mathbb{R}^+ \times \mathbb{R}^+ \to H^1_r(\mathbb{R}^2)$ by

$$v(s,t) = \begin{cases} su^+ + tu^-, & \text{if } |s-1| \geq \varepsilon \text{ or } |t-1| \leq \varepsilon \\ su^+ + tu^- + \varepsilon\eta(s,t)\varphi, & \text{if } |s-1| < \varepsilon \text{ and } |t-1| < \varepsilon. \end{cases}$$

We then claim that

$$\max_{s,t>0} I_\lambda(v(s,t)) < m_\lambda. \tag{3.16}$$

If $|s-1| \geq \varepsilon$ or $|t-1| \leq \varepsilon$, then $v(s, t) = su^+ + tu^-$, and Lemma 3.2 implies that

$$I_\lambda(v(s,t)) = I_\lambda(su^+ + tu^-) < I_\lambda(u) = m_\lambda.$$

If $|s-1| < \varepsilon$ and $|t-1| < \varepsilon$, by (3.15)

$$\begin{aligned} I_\lambda(v(s,t)) &= I_\lambda(su^+ + tu^-) + \int_0^\varepsilon \langle I'_\lambda(su^+ + tu^- + \sigma\eta(s,t)\varphi), \eta(s,t)\varphi \rangle d\sigma \\ &\leq I_\lambda(su^+ + tu^-) - \frac{1}{2}\varepsilon\eta(s,t) < m_\lambda, \end{aligned}$$

yielding that (3.16) holds. Hence, there exists a constant $\epsilon \in (0, 1-\varepsilon)$ such that

$$\max_{\epsilon \le s,t \le 2-\epsilon} I_\lambda(v(s,t)) = \overline{m}_\lambda < m_\lambda. \tag{3.17}$$

Similar to Lemma 3.2, set

$$\overline{V}(s,t) \triangleq (\langle I'_\lambda(v(s,t)), v^+(s,t)\rangle, \langle I'_\lambda(v(s,t)), v^-(s,t)\rangle).$$

It follows from some standard computations that

$$\begin{cases} \langle I'_\lambda(v(\epsilon,t)), v^+(\epsilon,t)\rangle > 0 \text{ and } \langle I'_\lambda(v(2-\epsilon,t)), v^+(2-\epsilon,t)\rangle < 0, & \forall t \in [\epsilon, 2-\epsilon], \\ \langle I'_\lambda(v(s,\epsilon)), v^-(s,\epsilon)\rangle > 0 \text{ and } \langle I'_\lambda(v(s,2-\epsilon)), v^-(s,2-\epsilon)\rangle < 0, & \forall s \in [\epsilon, 2-\epsilon]. \end{cases}$$

which together with the Miranda's theorem [37] (or, [31, Lemma 2.4]) that yields a pair $(s_0, t_0) \in (\epsilon, 2-\epsilon)\times(\epsilon, 2-\epsilon)$ such that $\overline{V}(s_0, t_0) = (0, 0)$. So, $s_0u^+ + t_0u^- + \varepsilon\eta(s_0, t_0)\varphi \in \mathcal{M}_\lambda$, which is a contradiction to (3.17). The proof is complete. □

Proof of Theorem 1.1. Let $\{u_n\} \subset \mathcal{M}_\lambda$ be a minimizing sequence of I_λ under the constraint $\mathcal{M}_\lambda$, namely $\{u_n\} \subset \mathcal{M}_\lambda$ and $I_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. By Lemma 3.4, the sequence $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^2)$ and then there exists a function $u_\lambda \in H^1_r(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_\lambda$ in $H^1_r(\mathbb{R}^2)$, $u_n \to u_\lambda$ in $L^s(\mathbb{R}^2)$ with $s \in (2, +\infty)$, and $u_n \to u_\lambda$ a.e. in $\mathbb{R}^2$. In view of (1.10) and (1.11), by using $\{u_n\} \subset \mathcal{M}_\lambda$ and Lemma 3.3, one has

$$0 < (\rho^\pm)^2 \le \liminf_{n\to\infty} \|u_n^\pm\|^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^2} f(u_n^\pm)u_n^\pm dx = \int_{\mathbb{R}^2} f(u_\lambda^\pm)u_\lambda^\pm dx,$$

which shows that $u_\lambda^\pm \neq 0$. Moreover, by Lemma 2.1, one can easily observe that

$$\langle I'_\lambda(u_\lambda), u_\lambda^\pm\rangle \le \liminf_{n\to\infty} \langle I'_\lambda(u_n), u_n^\pm\rangle = 0. \tag{3.18}$$

Then combing the Fatou's lemma, Lemma 2.1, (3.3), (3.6) and (3.18), for every $(s,t) \in (0,+\infty)\times(0,+\infty)$, there holds

$$\begin{aligned} m_\lambda &= \liminf_{n\to\infty} I_\lambda(u_n) = \liminf_{n\to\infty} [I_\lambda(u_n) - \frac{1}{2(k+2)}\langle I'_\lambda(u_n), u_n\rangle] \\ &= \liminf_{n\to\infty} \left[\frac{1}{2(k+2)}|\nabla u_n|_2^2 + \frac{\lambda}{2(k+2)}\sum_{j=1}^{k}\frac{k-j}{k}E_j(u_n) + G(0,u_n)\right] \\ &\ge I_\lambda(u_\lambda) - \frac{1}{2(k+2)}\langle I'_\lambda(u_\lambda), u_\lambda^+\rangle - \frac{1}{2(k+2)}\langle I'_\lambda(u_\lambda), u_\lambda^-\rangle \\ &\ge I_\lambda(su_\lambda^+ + tu_\lambda^-) - \frac{s^{2(k+2)}}{2(k+2)}\langle I'_\lambda(u_\lambda), u_\lambda^+\rangle - \frac{t^{2(k+2)}}{2(k+2)}\langle I'_\lambda(u_\lambda), u_\lambda^-\rangle \\ &\ge I_\lambda(su_\lambda^+ + tu_\lambda^-) \ge m_\lambda, \text{ by the arbitrariness of } (s,t) \in (0,+\infty)\times(0,+\infty), \end{aligned}$$

which yields that $\langle I'_\lambda(u_\lambda), u_\lambda^+\rangle = \langle I'_\lambda(u_\lambda), u_\lambda^-\rangle = 0$ and $I_\lambda(u_\lambda) = m_\lambda$. Recalling that Lemma 3.5, we know that u_λ is indeed a least energy sign-changing solution solution of (1.1). We apply the arguments in [13] to determine the number of nodal domains of u_λ. If u_λ has more than two nodal domains, say, D_1, D_2 are positive domains, and D_3 is a negative domain. Then $u_\lambda|_{D_1\cup D_3} \in \mathcal{M}_\lambda$ and $u_\lambda|_{D_2} \in \mathcal{N}_\lambda$, thus $I_\lambda(u_\lambda) \ge m_\lambda + c_\lambda$, a contradiction. □

Proof of Theorem 1.2. Proceeding as the proof of Theorem 1.1, there exists a function $v_\lambda \in H_r^1(\mathbb{R}^2)$ such that $I'_\lambda(v_\lambda) = 0$ and $I_\lambda(v_\lambda) = c_\lambda > 0$ (see Lemma 3.3). From Theorem 1.1, we know that (1.1) has a least energy sign-changing solution $u_\lambda \in H_r^1(\mathbb{R}^2)$ which changes sign only once. It is similar to Lemma 3.2 that there exist two constants $s_\lambda, t_\lambda > 0$ such that $s_\lambda u_\lambda^+ \in N_\lambda$ and $t_\lambda u_\lambda^- \in N_\lambda$. By using (1.9) and (3.14), one has

$$m_\lambda = I_\lambda(u_\lambda) \geq I_\lambda(s_\lambda u_\lambda^+ + t_\lambda u_\lambda^-) > I_\lambda(s_\lambda u_\lambda^+) + I_\lambda(t_\lambda u_\lambda^-) \geq 2c_\lambda.$$

The proof is complete. □

Proof of Theorem 1.3. For each $\lambda > 0$, let $u_\lambda = u_\lambda^+ + u_\lambda^-$ with $u_\lambda^\pm \neq 0$ be a least energy sign-changing solution of (1.1) obtained in Theorem 1.1, which changes sign only once. Choosing a sequence $\{\lambda_n\} \subset (0,1)$ to satisfy $\lambda_n \searrow 0$ as $n \to \infty$. We denote $\{u_n\}$ and $\{m_n\}$ by $\{u_{\lambda_n}\}$ and $\{m_{\lambda_n}\}$ for simplicity, respectively.

Step 1. The sequences $\{m_n\}$ and $\{u_n\}$ are bounded in $\mathbb{R}^+$ and $H_r^1(\mathbb{R}^2)$, respectively. Firstly, we prove that $\{m_{\lambda_n}\}$ is bounded. Note that the idea comes from [51, Theorem 1.3], but we have to take some delicate analysis when $k \in \mathbb{N}^+$ with $k \geq 2$. Let $w_0 \in C_0^\infty(\mathbb{R}^2)$ with $w_0^\pm \neq 0$. From $(f_1) - (f_3)$, there exist $k+1$ constants $\beta_0 > 0$ and $\beta_j \geq j^{-1}E_j(w_0)$ for any $j \in \{1, 2, \cdots, k\}$ such that

$$\begin{cases} \int_{\mathbb{R}^2} F(sw_0^+)dx \geq \sum_{j=1}^k \beta_j s^{2(j+2)} - \beta_0, & \forall s \in (0, +\infty), \\ \int_{\mathbb{R}^2} F(tw_0^-)dx \geq \sum_{j=1}^k \beta_j t^{2(j+2)} - \beta_0, & \forall t \in (0, +\infty). \end{cases} \tag{3.19}$$

In view of the notations used in the proof of Lemma 3.1, we have

$$\begin{aligned} E_j(w_0) = {} & E_j^{+,+,+}(w_0) + 2E_j^{+,+,-}(w_0) + E_j^{+,-,-}(w_0) \\ & + E_j^{-,+,+}(w_0) + 2E_j^{-,+,-}(w_0) + E_j^{-,-,-}(w_0). \end{aligned} \tag{3.20}$$

Obviously, there exist two constants $M_1, M_2 \in (0, +\infty)$ such that

$$\begin{cases} \max_{s>0}\left(\frac{s^2}{2}\|w_0^+\|^2 - \sum_{j=1}^k \frac{1}{2j}E_j^{+,+,+}(w_0)s^{2(j+2)}\right) \triangleq M_1, \\ \max_{t>0}\left(\frac{t^2}{2}\|w_0^-\|^2 - \sum_{j=1}^k \frac{1}{2j}E_j^{-,-,-}(w_0)t^{2(j+2)}\right) \triangleq M_2. \end{cases} \tag{3.21}$$

Similar to the Case I of (3.2), for any $(s,t) \in (0,+\infty) \times (0,+\infty)$ there holds

$$H_{i,j}(s,t) \geq 0, \quad i \in \{1,2,3,4\} \text{ and } j \in \{1, 2, \cdots, k\}, \tag{3.22}$$

where

$$\begin{aligned} &H_{1,j}(s,t) \triangleq s^{2(j+2)} - s^{2j+2}t^2 + t^{2(j+2)}, \; H_{2,j}(s,t) \triangleq 2s^{2(j+2)} - s^{2j}t^4 + 2t^{2(j+2)}, \\ &H_{3,j}(s,t) \triangleq 2s^{2(j+2)} - s^4t^{2j} + 2t^{2(j+2)}, \; H_{4,j}(s,t) \triangleq s^{2(j+2)} - s^2t^{2j+2} + t^{2(j+2)}. \end{aligned}$$

Denoting $w_{s,t} = sw_0^+ + tw_0^-$, we claim that

$$\max_{s,t>0} I_1(w_{s,t}) \leq M_1 + M_2 + 2\beta_0 \triangleq M_0 \in (0, +\infty), \tag{3.23}$$

where M_0 is a constant independent of $n \in \mathbb{N}$. Indeed, combing (3.19)-(3.22), we obtain the following estimate, for any $(s,t) \in (0,+\infty) \times (0,+\infty)$,

$$
\begin{aligned}
I_1(w_{s,t}) \leq{}& \frac{s^2}{2}\|w_0^+\|^2 + \frac{t^2}{2}\|w_0^-\|^2 + 2\beta_0 - \sum_{j=1}^{k} \frac{1}{j} E_j(w_0)[s^{2(j+2)} + t^{2(j+2)}] \\
&+ \sum_{j=1}^{k} \Big[\frac{1}{2j} s^{2j+4} E_j^{+,+,+}(w_0) + \frac{1}{j} s^{2j+2} t^2 E_j^{+,+,-}(w_0) + \frac{1}{2j} s^{2j} t^4 E_j^{+,-,-}(w_0) \Big] \\
&+ \sum_{j=1}^{k} \Big[\frac{1}{2j} s^4 t^{2j} E_j^{-,+,+}(w_0) + \frac{1}{j} s^2 t^{2j+2} E_j^{-,+,-}(w_0) + \frac{1}{2j} t^{2j+4} E_j^{-,-,-}(w_0) \Big] \\
={}& \Big[\frac{s^2}{2}\|w_0^+\|^2 - \sum_{j=1}^{k} \frac{1}{2j} E_j^{+,+,+}(w_0) s^{2(j+2)} \Big] + \Big[\frac{t^2}{2}\|w_0^-\|^2 - \sum_{j=1}^{k} \frac{1}{2j} E_j^{-,-,-}(w_0) t^{2(j+2)} \Big] \\
&+ 2\beta_0 - \sum_{j=1}^{k} \frac{1}{j} \Big[H_{1,j}(s,t) E_j^{+,+,-}(w_0) + H_{3,j}(s,t) E_j^{-,+,+}(w_0) \Big] \\
&- \sum_{j=1}^{k} \frac{1}{2j} \Big[H_{2,j}(s,t) E_j^{+,-,-}(w_0) + H_{4,j}(s,t) E_j^{-,+,-}(w_0) \Big] \\
\leq{}& M_1 + M_2 + 2\beta_0 = M_0 \in (0,+\infty),
\end{aligned}
$$

which implies that (3.23) holds true. Since $0 < \lambda_n < 1$, by (3.14) and (3.23) we have

$$
m_n \leq \max_{s,t>0} I_{\lambda_n}(sw_0^+ + tw_0^-) \leq \max_{s,t>0} I_1(w_{s,t}) \leq M_0.
$$

One can obtain the boundness of $\{u_n\}$ in a very similar way in the proof of Lemma 3.4, we omit it here. The proof of Step 1 is complete.

Step 2. $I_0'(u_0) = 0$ and $I_0(u_0) = m_0$.
Recalling that the Step 1, there exist a subsequence, still denoted by itself, and a function $u_0 \in H_r^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^2)$, $u_n \to u_0$ in $L^s(\mathbb{R}^2)$ with each $s \in (2,+\infty)$ and $u_n \to u_0$ a.e. $\mathbb{R}^2$. Similar to the proof in Theorem 1.1, one has $u_0^\pm \neq 0$. By $(f_1)-(f_2)$,

$$
\langle I_0'(u_0), \varphi \rangle = \lim_{n\to\infty} \langle I_{\lambda_n}'(u_n), \varphi \rangle = 0, \ \ \forall \varphi \in C_{0,r}^\infty(\mathbb{R}^2) = \{u \in C_{0,r}^\infty(\mathbb{R}^2) : u(x) = u(|x|)\},
$$

which yields that $I_0'(u_0) = 0$ in $H_r^1(\mathbb{R}^2)$. As a consequence, u_0 is a sign-changing solution of (1.14), which changes sign only once. By virtue of the Fatou's lemma, one has

$$
m_0 \leq I_0(u_0) \leq \liminf_{n\to\infty} I_{\lambda_n}(u_n) = \liminf_{n\to\infty} m_n \leq \limsup_{n\to\infty} m_n. \tag{3.24}
$$

On the other hand, let $w_0 \in H^1(\mathbb{R}^2)$ be a least energy sign-changing solution of (1.14), that is, $I_0'(w_0) = 0$ and $I_0(w_0) = m_0$. We claim that

$$
\lim_{n\to\infty} \langle I_{\lambda_n}'(w_0), w_0^+ \rangle = \lim_{n\to\infty} \langle I_{\lambda_n}'(w_0), w_0^- \rangle = 0. \tag{3.25}
$$

In fact, since $\langle I_0'(w_0), w_0^+ \rangle = \|w_0^+\|^2 - \int_{\mathbb{R}^2} f(w_0^+) w_0^+ dx = 0$, we can conclude the following fact that

$$\langle I'_{\lambda_n}(w_0), w_0^+\rangle = \lambda_n \sum_{j=1}^{k} \frac{1}{j}\Big[(j+2)E_j^{+,+,+}(w_0) + 2(j+1)E_j^{+,+,-}(w_0) + E_j^{+,-,-}(w_0) + 2E_j^{-,+,+}(w_0) + 2E_j^{-,+,-}(w_0)\Big],$$

which yielding the first part of (3.25) since $\lambda_n \to 0$ as $n \to \infty$. We can get the other part of (3.25) in the same way. Since $w_0^{\pm} \neq 0$, by Lemma 3.2, then there exists a unique pair $(s_n, t_n) \in (0, +\infty) \times (0, +\infty)$ such that $s_n w_0^+ + t_n w_0^- \in \mathcal{M}_{\lambda_n}$. We claim that

$$\text{both } \{s_n\} \text{ and } \{t_n\} \text{ are bounded in } \mathbb{R}^+. \tag{3.26}$$

We argue it indirectly and distinguish the following three cases:

Case 1. $s_n \to \infty$ and $t_n \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that $s_n \leq t_n$ for any $n \in \mathbb{N}$. By $\langle I'_{\lambda_n}(s_n w_0^+ + t_n w_0^-), t_n w_0^-\rangle = 0$, we have

$$t_n^2\|w_0^-\|^2 + \lambda_n \sum_{j=1}^{k} \frac{1}{j}\Big[(j+2)t_n^{2j+4}E_j^{-,-,-}(w_0) + 2(j+1)s_n^2 t_n^{2j+2}E_j^{-,+,-}(w_0) +2s_n^{2j}s_n^4 E_j^{+,-,-}(w_0) + s_n^4 t_n^{2j}E_j^{-,+,+}(w_0) + 2s_n^{2j+2}t_n^2 E_j^{+,+,-}(w_0)\Big] = \int_{\mathbb{R}^2} f(s_n w_0^+) s_n w_0^+ dx.$$

Dividing $t_n^{2(k+2)}$ on both sides of the above formula, we can get a contradiction by (f_3).

Case 2. $s_n \to \infty$ as $n \to \infty$ and $\{t_n\}$ is bounded. By $\langle I'_{\lambda_n}(s_n w_0^+ + t_n w_0^-), s_n w_0^+\rangle = 0$, we have

$$s_n^2\|w_0^+\|^2 + \lambda_n \sum_{j=1}^{k} \frac{1}{j}\Big[(j+2)s_n^{2j+4}E_j^{+,+,+}(w_0) + 2(j+1)s_n^{2j+2}t_n^2 E_j^{+,+,-}(w_0) +s_n^{2j}t_n^4 E_j^{+,-,-}(w_0) + 2s_n^4 t_n^{2j}E_j^{-,+,+}(w_0) + 2s_n^2 t_n^{2j+2}E_j^{-,+,-}(w_0)\Big] = \int_{\mathbb{R}^2} f(s_n w_0^+) s_n w_0^+ dx.$$

Dividing $s_n^{2(k+2)}$ on both sides of the above formula, we can get a contradiction by (f_3).

Case 3. $t_n \to \infty$ as $n \to \infty$ and $\{s_n\}$ is bounded. We take a contradiction as Case 2.

Combing (3.1), (3.6) and (3.25)-(3.26), there holds

$$\begin{aligned} m_0 &= I_0(w_0) = \limsup_{n\to\infty} I_{\lambda_n}(w_0) \\ &\geq \limsup_{n\to\infty}\Big[I_{\lambda_n}(s_n w_0^+ + t_n w_0^-) + \frac{1 - s_n^{2(k+2)}}{2(k+2)}\langle I'_{\lambda_n}(w_0), w_0^+\rangle + \frac{1 - t_n^{2(k+2)}}{2(k+2)}\langle I'_{\lambda_n}(w_0), w_0^-\rangle\Big] \\ &= \limsup_{n\to\infty} I_{\lambda_n}(s_n w_0^+ + t_n w_0^-) \geq \limsup_{n\to\infty} m_n, \end{aligned}$$

which together with (3.24) gives that $I_0(u_0) = m_0$. The proof is complete. □

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