Solutions of sum-type singular fractional q-integro-differential equation with m-point boundary value using quantum calculus

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Abstract

In this study, we investigate the sum-type singular nonlinear fractional q-integro-differential m-point boundary value problem. The existence of positive solutions is obtained by the properties of the Green function, standard Caputo q-derivative, Riemann-Liouville fractional q-integral and the means of a fixed point theorem on a real Banach space $(\mathcal{X}, \|.\|)$ which has a partially order by using a cone $P \subset \mathcal{X}$. The proofs are based on solving the operator equation $\mathcal{O}_1 x + \mathcal{O}_2 x = x$ such that the operator $\mathcal{O}_1$, $\mathcal{O}_2$ are r-convex, sub-homogeneous, respectively and define on cone P. As applications, we provide an example illustrating the primary effects.

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1 Introduction

It is known that the subject of q-difference equations introduced by Jackson in 1910 [19]. After it, some researchers studied q-difference equations [1, 2, 4, 5, 9, 13, 15, 22, 35, 38, 42]. On the other hand, it published recently many modern works on integro-differential equations by using different views and fractional derivatives which young researchers could use main idea of the works for their works (see for example, [10, 3, 12, 20, 32, 36].

In 2012, Ahmad et al. studied the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD_q^\alpha u(t) = T\left(t, u(t)\right)$ with boundary conditions $\alpha_1 u(0) - \beta_1 D_q u(0) =$

$\gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, where $\alpha \in (1,2]$, $\alpha_i, \beta_i, \gamma_i$, η_i are real numbers, for $i = 1,2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$ ([5]). In 2013, Zhao el al. reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t, u(t)) = 0$ with boundary conditions $u(1) = \mu I_q^\beta u(\eta)$ and $u(0) = 0$ for almost all $t \in (0,1)$, where $q \in (0,1)$, $\alpha \in (1,2]$, $\beta \in (0,2]$, $\eta \in (0,1)$, μ is positive real number, D_q^α is the q-derivative of Riemann-Liouville and real-values continuous map u defined on $I \times [0,\infty)$ ([42]). In 2014, Ahmad et al. investigated the problem ${}^cD_q^\beta({}^cD_q^\gamma + \lambda)u(t) = pf(t,u(t)) + kI_q^\xi g(t,u(t))$ with boundary conditions $\alpha_1 u(0) - \beta_1 (t^{(1-\gamma)} D_q u(0))\big|_{t=0} = \sigma_1 u(\eta_1)$ and $\alpha_2 u(1) + \beta_2 D_q u(1) = \sigma_2 u(\eta_2)$, where $t, q \in [0,1]$, ${}^cD_q^\beta$ is the fractional Caputo q-derivative, $0 < \beta, \gamma \le 1$, $I_q^\xi(.)$ denotes the Riemann-Liouville integral with $\xi \in (0,1)$, f and g are given continuous functions, λ and p, k are real constants, $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ and $\eta_i \in (0,1)$ for $i = 1,2$ ([4]). In 2019, Samei *et al.* [37] discussed the fractional hybrid q-differential inclusions

$$ {}^cD_q^\alpha \left(\frac{x}{f\left(t, x, I_q^{\alpha_1} x, \cdots, I_q^{\alpha_n} x\right)} \right) \in F\left(t, x, I_q^{\beta_1} x, \cdots, I_q^{\beta_k} x\right), $$

with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$, where $1 < \alpha \le 2$, $q \in (0,1)$, $x_0, x_1 \in \mathbb{R}$, $\alpha_i > 0$, for $i = 1,2,\dots,n$, $\beta_j > 0$, for $j = 1,2,\dots,k$, $n,k \in \mathbb{N}$, ${}^cD_q^\alpha$ denotes Caputo type q-derivative of order α, I_q^β denotes Riemann-Liouville type q-integral of order β, $f : J \times \mathbb{R}^n \to (0,\infty)$ is continuous and $F : J \times \mathbb{R}^k \to P(\mathbb{R})$ is multifunction. Also, Ntouyas *et al.* [25] by applying definition of the fractional q-derivative of the Caputo type and the fractional q-integral of the RiemannLiouville type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional q-integro-differential equations under some boundary conditions

$$ {}^cD_q^\alpha x(t) = w\big(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^cD_q^{\beta_1} x(t), {}^cD_q^{\beta_2} x(t), \dots, {}^cD_q^{\beta_n} x(t)\big). $$

In 2020, Liang *et al.* [26] investigated the existence of solutions for a nonlinear problems regular and singular fractional q-differential equation

$$ {}^cD_q^\alpha f(t) = w\big(t, f(t), f'(t), {}^cD_q^\beta f(t)\big), $$

with conditions $f(0) = c_1 f(1)$, $f'(0) = c_2 {}^cD_q^\beta f(1)$ and $f^{(k)}(0) = 0$ for $2 \le k \le n-1$, here $n-1 < \alpha < n$ with $n \ge 3$, $\beta, q, c_1 \in (0,1)$, $c_2 \in (0, \Gamma_q(2-\beta))$, function w is a L^κ-Carathéodory, $w(t, x_1, x_2, x_3)$ may be singular and ${}^cD_q^\alpha$ the fractional Caputo type q-derivative. Similar results have been presented in other studies [18, 34, 38].

In this article, motivated by main idea of the works such [14, 30, 7], and among these achievements, we are going to stretch out the singular fractional q-integro-differential equation

$$ \begin{aligned} &D_q^\alpha x(t) + w_1\bigg(t, x(t), (\Omega x)(t), D_q^{\gamma_i} x(t), \dots, D_q^{\gamma_{k_2}} x(t)\bigg) \\ &\qquad\qquad + w_2\bigg(t, x(t), (\Omega x)(t), D_q^{\gamma_1} x(t), \dots, D_q^{\gamma_{k_2}} x(t)\bigg) = 0, \qquad (1.1) \\ &(\Omega x)(t) = \int_0^t \mu(t, qs) f\big(t, s, x(s), D_q^{\beta_1} x(s), \dots, D_q^{\beta_{k_1}} x(s)\big)\, d_q s \end{aligned} $$

with m-point boundary conditions $D_q^{\beta_i}x(0) = D_q^{\gamma_j}x(0) = 0$ for i, j in N_{k_1}, N_{k_2}, , respectively, here $N_k = \{1, 2, 3, \dots, k\}$, $D_q^{\gamma_{k_2}+1}x(0) = D_q^{\gamma_{k_2}+2}x(0) = D_q^{\gamma_{k_2}+3}x(0) = 0$ and

$$D_q^{\gamma_{k_2}+3}x(1) = \sum_{k=1}^{m-2} \eta_k D_q^{\gamma_{k_2}+3}x(\tau_k),$$

where $t \in J = (0, 1)$, $\alpha \in (n-1, n]$ with n more than or equal to five, $0 < \beta_1 < \beta_2 < \cdots < \beta_{k_1} \leq \gamma_{k_2}$,

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{k_2},$$

$\alpha - \gamma_{k_2} \in (4, 5]$, $\eta_k \in (0, \infty)$, $\tau_k \in (0, 1)$ such that $\tau_k < \tau_k + 1$, here $k \in N_{m-2}$, also

$$d = \sum_{k=1}^{m-2} \eta_k \tau_k^{\alpha-\gamma_{k_2}-4} < 1,$$

$\mu : \overline{J}^2 \to [0, \infty)$, $f : \overline{J}^2 \times \mathbb{R}^{k_1+1} \to [0, \infty)$, positive real-valued functions w_i define on $(0, 1] \times \mathbb{R}^{k_2+2}$ are continuous such that $\lim_{t\to 0^+} w_i(t, ., ., \dots, .) = +\infty$, that is, w_i are singular at $t = 0$ and D_q is the Riemann-Liouville fractional q-derivative.

2 Preliminaries

First, we point out some of the materials on the fractional q-calculus and fundamental results of it which needed in the next sections (for more information, consider [28, 6, 19]). Then, some well-known theorems of fixed point theorem and definition are expressed.

Let $q \in (0, 1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [19]. The power function $(x - y)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(x - y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ for $n \geq 1$ and $(x - y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [1]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(x - y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty}(x - yq^k)/(x - yq^{\alpha+k}).$$

If $y = 0$, then it is clear that $x^{(\alpha)} = x^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(z) = (1 - q)^{(z-1)}/(1 - q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0, -1, -2, \cdots\}$ [19]. Note that, $\Gamma_q(z + 1) = [z]_q\Gamma_q(z)$. The value of q-Gamma function, $\Gamma_q(z)$, for input values q and z with counting the number of sentences n in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order n which show in Algorithm 2. The q-derivative of function f, is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3 ([1]). Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ ([1]). The q-integral of a function f defined on $[0, b]$ is defined by

$$I_q f(x) = \int_0^x f(s) d_q s = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k)$$

for $0 \le x \le b$, provided the series is absolutely converges [1]. The q-derivative of function f is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3 [1]. If $a \in [0,b]$, then $\int_a^b f(u) d_q u = (1-q)\sum_{k=0}^{\infty} q^k \left[bf(bq^k) - af(aq^k)\right]$ whenever the series exists [1]. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1}h))(x)$ for $n \ge 1$ and $g \in C([0,b])$ [1]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ ([1]. The fractional Riemann-Liouville type q-integral of the function f on J for $\alpha \ge 0$ is defined by $(I_q^0 f)(t) = f(t)$ and $(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s$ for $t \in J$ and $\alpha > 0$ [15]. Also, the Caputo fractional q-derivative of a function f is defined by

$$\begin{aligned}\left({}^c D_q^\alpha f\right)(t) &= \left(I_q^{[\alpha]-\alpha}(D_q^{[\alpha]} f)\right)(t)\\ &= \frac{1}{\Gamma_q\left([\alpha]-\alpha\right)}\int_0^t (t-qs)^{([\alpha]-\alpha-1)}\left(D_q^{[\alpha]} f\right)(s) d_q s,\end{aligned} \tag{2.1}$$

where $t \in J$ and $\alpha > 0$ [15]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $\left(D_q^\alpha(I_q^\alpha f)\right)(x) = f(x)$, where $\alpha, \beta \ge 0$ [15]. By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 4.

Throughout this article, we denote $L^1(0,1)$, $L^1[0,1]$, $C(0,1)$, $C[0,1]$, $C^1(0,1)$, $C^1[0,1]$ by $\mathcal{L}$, $\overline{\mathcal{L}}$, $\mathcal{A}$, $\overline{\mathcal{A}}$, $\mathcal{B}$, $\overline{\mathcal{B}}$, respectively. You can find the following lemmas in the [27, 23, 21].

Lemma 2.1. *If $x \in \mathcal{A} \cap \mathcal{L}$ with $D_q^\alpha x \in \mathcal{A} \cap \mathcal{L}$, then $I_q^\alpha D_q^\alpha x(t) = x(t) + \sum_{i=1}^n c_i t^{\alpha-i}$, where n is the smallest integer greater than or equal to α and c_i is some real number.*

Lemma 2.2. *If $x \in \overline{\mathcal{L}}$ and $\alpha > \beta > 0$, then $I_q^\alpha I_q^\beta x(t) = I_q^{\alpha+\beta} x(t)$, $D_q^\beta I_q^\alpha x(t) = I_q^{\alpha-\beta} x(t)$ and $D_q^\beta I_q^\alpha x(t) = x(t)$. If $\alpha > 0$ and $\gamma > -1$, then $D_q^\alpha t^\gamma = \frac{\Gamma_q(\gamma+1)}{\Gamma_q(\gamma-\alpha+1)} t^{\gamma-\alpha}$. Also, $I^\alpha x \in \overline{\mathcal{A}}$ for all $\alpha > 0$ and $x \in \overline{\mathcal{A}}$.*

Lemma 2.2 implies next result.

Lemma 2.3. *Assume that $x(t) = I_q^{\gamma_{k_2}} y(t)$ where $y(t) \in \overline{\mathcal{A}}$. Then the problem (1.1) reduces to the problem*

$$\begin{aligned} D_q^{\alpha-\gamma_{k_2}} y(t) &+ w_1\bigg(t, I_q^{\gamma_{k_2}} y(t), \int_0^t \mu(t,s) f\big(t,s, I_q^{\gamma_{k_2}} y(s), I_q^{\gamma_{k_2}-\beta_1} y(s), \dots, I_q^{\gamma_{k_2}-\beta_{k_1}} y(s)\big) ds,\\ &\qquad I_q^{\gamma_{k_2}-\gamma_1} y(t), I_q^{\gamma_{k_2}-\gamma_2} y(t), \dots, I_q^{\gamma_{k_2}-\gamma_{k_2-1}} y(t), y(t)\bigg)\\ &+ w_2\bigg(t, I_q^{\gamma_{k_2}} y(t), \int_0^t \mu(t,s) f\big(t,s, I_q^{\gamma_{k_2}} y(s), I_q^{\gamma_{k_2}-\beta_1} y(s), \dots, I_q^{\gamma_{k_2}-\beta_{k_1}} y(s)\big) ds,\\ &\qquad I_q^{\gamma_{k_2}-\gamma_1} y(t), I_q^{\gamma_{k_2}-\gamma_2} y(t), \dots, I_q^{\gamma_{k_2}-\gamma_{k_2-1}} y(t), y(t)\bigg) = 0,\end{aligned} \tag{2.2}$$

where $t \in J$, with boundary conditions $y(0) = y'(0) = y''(0) = y'''(0) = 0$ and

$$y'''(1) = \sum_{k=1}^{m-2} \eta_k y'''(\tau_k).$$

Moreover, if $y \in \overline{\mathcal{A}}$ is a positive solution of the problem (2.2)*, then this implies that $x(t) = I_q^{\gamma_{k_2}} y(t)$ is a positive solution for the problem* (1.1)*.*

Theorem 2.4. *([17]) Suppose that $(\mathcal{X}, \rho)$ be a complete metric space which has an order $\leq$. Also, let self-map F define on $\mathcal{X}$ an increasing and $x_n \leq x$ for each natural number n whenever $\{x_n\}$ is an increasing sequence belongs to $\mathcal{X}$ with $x_n \to x$. Then F has a fixed point whenever there exists $x_0 \in \mathcal{X}$ such that $x_0 \leq Fx_0$ and there exists a continuous and increasing function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$ and $\rho(F(x), F(y)) \leq \rho(x, y) - \phi(\rho(x, y))$ for each $x \geq y$. In addition to, F has a unique fixed point whenever there exists $z \in \mathcal{X}$ which is comparable to x and y, for each $x, y \in \mathcal{X}$.*

An operator $\mathcal{O} : P \to P$ is said to be r-concave whenever $\mathcal{O}(tx) \geq t^r \mathcal{O}x$ for any $x \in P$ and $t \in J$, here $P \subset \mathcal{X}$ is a cone and $r \in [0, 1)$ [41]. Also, $\mathcal{O} : P \to P$ is called homogeneous, sub-homogeneous whenever $\mathcal{O}(rx) = r\mathcal{O}x$ for each $r \in (0, \infty)$, whenever $\mathcal{O}(tx) \geq t\mathcal{O}x$ for all $t \in J$, respectively, for $x \in P$ [41].

Theorem 2.5. *([41]) Let $P \subseteq \mathcal{X}$ be a normal cone, $\mathcal{O}_1, \mathcal{O}_2 : P \to P$ are increasing r-concave map, increasing sub-homogeneous operator, respectively. Assume that there is $z > \vartheta$ such that $\mathcal{O}_1 z$ and $\mathcal{O}_2 z$ belong to P_z. Then the operator equation $\mathcal{O}_1 x + \mathcal{O}_2 x = x$ has a unique solution x^* in P_k whenever there exists $\delta_0 > 0$ such that $\mathcal{O}_1 x \geq \delta_0 \mathcal{O}_2 x$ for each $x \in P$. Moreover, the sequence $y_n = \mathcal{O}_1 y_{n-1} + \mathcal{O}_2 y_{n-1}$ here $n \geq 1$, with initial value $y_0 \in P_k$ converges to x^*.*

Note that last result holds whenever $\mathcal{O}_2$ is a null operator. In this paper, we use the Banach space $\mathcal{X} = C(\overline{J})$ with the partial order $f_1 \leq f_2$ if and only if $f_1(t) \leq f_2(t)$ for all $t \in \overline{J}$ and $f_1, f_2 \in \mathcal{X}$. It has been proved $(\mathcal{X}, \leq)$ has this property that $f_n \leq f$ for all n whenever $\{f_n\}$ is an increasing sequence in $\mathcal{X}$ with $f_n \to f$ [24]. Moreover, $\max\{f_1, f_2\} \in \mathcal{X}$ for each $f_1, f_2 \in \mathcal{X}$, that is, for almost all $f_1, f_2 \in \mathcal{X}$ there exists $g \in \mathcal{X}$ which is comparable to f_1 and f_2. We consider the normal cone P which is the set of all $f \in \mathcal{X}$ such that $f(t) \geq 0$ for all t belongs to J with normal constant 1.

3 Main results

First, we state and prove the following key results.

Lemma 3.1. *The problem $D_q^{\alpha - \gamma_{k_2}} x(t) + v(t) = 0$ with the boundary conditions $x(0) = x'(0) = x''(0) = x'''(0) = 0$ and $x'''(1) = \sum_{k=1}^{m-2} \eta_k x'''(\tau_k)$ has the unique solution*

$$x(t) = \int_0^1 G_1(t, qs) v(s)\, ds + \frac{d_0 t^{\alpha - \gamma_{k_2} - 1}}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)(1 - d)} \int_0^1 G_2(\tau_k, qs) v(s)\, ds,$$

whenever there exists $v \in C(0, 1]$, and $\alpha - \gamma_{k_2} \in (4, 5]$, $d \neq 1$, here $d_0 = \sum_{k=1}^{m-2} \eta_k$, $d = \sum_{k=1}^{m-2} \eta_k \tau_k^{\alpha - \gamma_{k_2} - 4}$,

$$G_1(t, s) = \begin{cases} \frac{1}{\Gamma_q(\alpha - \gamma_{k_2})} t^{\alpha - \gamma_{k_2} - 1} (1 - qs)^{\alpha - \gamma_{k_2} - 4} - (t - qs)^{\alpha - \gamma_{k_2} - 1}, & s \leq t, \\ \frac{1}{\Gamma_q(\alpha - \gamma_{k_2})} t^{\alpha - \gamma_{k_2} - 1} (1 - qs)^{\alpha - \gamma_{k_2} - 4}, & t \leq s, \end{cases} \tag{3.1}$$

and $G_2(t, qs) = \frac{\partial^3 G_1(t,qs)}{\partial t^3}$

$$= \begin{cases} \frac{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)}{\Gamma_q(\alpha-\gamma_{k_2})}(t^{\alpha-\gamma_{k_2}-4}(1-qs)^{\alpha-\gamma_{k_2}-4} - (t-qs)^{\alpha-\gamma_{k_2}-4}), & s \le t, \\ \frac{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)}{\Gamma_q(\alpha-\gamma_{k_2})} t^{\alpha-\gamma_{k_2}-4}(1-qs)^{\alpha-\gamma_{k_2}-4}, & t \le s. \end{cases} \tag{3.2}$$

Proof. First of all, it can be seen that Lemma 2.1 implies that the solution of the problem (2.2) is

$$x(t) = -I_q^{\alpha-\gamma_{k_2}} v(t) + \sum_{i=1}^{5} c_i t^{\alpha-\gamma_{k_2}-i},$$

where c_i belongs to $\mathbb{R}$. By employing the conditions, we obtain $c_2 = c_3 = c_4 = c_5 = 0$ and

$$c_1 = \frac{1}{\Gamma_q(\alpha-\gamma_{k_2})(1-d)}\Bigg[\int_0^1 (1-qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s - \sum_{k=1}^{m-2} \eta_k \int_0^{\tau_k} (\tau_k - qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s\Bigg].$$

Therefore, we can conclude that the unique solution of the problem is

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-qs)^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})} v(s)\, d_q s \\ &\quad + \frac{t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})(1-d)}\Bigg[\int_0^1 (1-qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \\ &\quad - \sum_{k=1}^{m-2} \eta_k \int_0^{\tau_k} (\tau_k - qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s\Bigg] \\ &= -\int_0^t \frac{(t-qs)^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})} v(s)\, d_q s + \Bigg[\frac{t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})} \\ &\quad + \frac{d t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})(1-d)}\Bigg] \int_0^1 (1-qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \\ &\quad - \frac{d_0 t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})(1-d)} \int_0^{\tau_k} (\tau_k - qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \\ &= \frac{1}{\Gamma_q(\alpha-\gamma_{k_2})} \\ &\quad \times \int_0^t \Big(t^{\alpha-\gamma_{k_2}-1}(1-qs)^{\alpha-\gamma_{k_2}-4} - (t-qs)^{\alpha-\gamma_{k_2}-1}\Big) v(s)\, d_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha-\gamma_{k_2})} \int_t^1 t^{\alpha-\gamma_{k_2}-1}(1-qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \\ &\quad + \frac{d_0 t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})(1-d)}\Bigg[\int_0^1 \tau_k^{\alpha-\gamma_{k_2}-4}(1-qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \end{aligned}$$

$$
\begin{aligned}
&- \int_0^{\tau_k} (\tau_k - qs)^{\alpha-\gamma_{k_2}-4} v(s)\, d_q s \Big] \\
&= \int_0^1 G_1(t,qs) v(s)\, d_q s \\
&+ \frac{d_0 t^{\alpha-\gamma_{k_2}-1}}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} \\
&\times \int_0^1 G_2(\tau_k,qs) v(s)\, d_q s.
\end{aligned}
$$

This finishes the validation. □

By simple review we can see the function G_1 in (3.1) is a continuous on $\overline{J}^2$, $G_1(t,qs) \geq 0$, $G_1(t,qs) > 0$ for each t,s belong to $\overline{J}$, J, respectively, and $G_2(t,qs) \geq 0$ for almost all t,s beloing to $\overline{J}$. Also,

$$
\frac{s(s^2-3s+3)(1-qs)^{\alpha-\gamma_{k_2}-4} t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})} \leq G_1(t,qs) \leq \frac{(1-qs)^{\alpha-\gamma_{k_2}-4} t^{\alpha-\gamma_{k_2}-1}}{\Gamma_q(\alpha-\gamma_{k_2})}
$$

for all $t,s \in \overline{J}$. In addition to,

$$
\sup_{t\in\overline{J}} \int_0^1 G_1(t,qs) s^{-\sigma}\, ds = \frac{1}{\Gamma_q(\alpha-\gamma_{k_2})} \big[B_q(1-\sigma,\alpha-\gamma_{k_2}-3) - B_q(1-\sigma,\alpha-\gamma_{k_2}) \big],
$$

and

$$
\begin{aligned}
\int_0^1 G_2(\delta,qs) s^{-\sigma}\, ds = &\frac{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)}{\Gamma_q(\alpha-\gamma_{k_2})} \\
&\times \Big[\delta^{\alpha-\gamma_{k_2}-4} - \delta^{\alpha-\gamma_{h_2}-\sigma-3} \Big] B_q\left(1-\sigma,\alpha-\gamma_{k_2}-3\right)
\end{aligned}
$$

for each $\sigma, \delta \in J$. This conclude that

$$
\begin{aligned}
\Lambda := &\sup_{t\in\overline{J}} \int_0^1 \Big[G_1(t,qs) \\
&+ \frac{d_0 t^{\alpha-\gamma_{k_2}-1}}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k,qs) \Big] s^{-\sigma} d_q s \\
= &\frac{1}{\Gamma_q(\alpha-\gamma_{k_2})} \left[\left(1 + \frac{\sum_{k=1}^{m-2} \eta_k \big(\tau_k^{\alpha-\gamma_{k_2}-4} - \tau_k^{\alpha-\gamma_{k_2}-\sigma-3} \big)}{1-d} \right) B_q(1-\sigma,\alpha-\gamma_{k_2}-3) \right. \\
&\left. - B_q(1-\sigma,\alpha-\gamma_{k_2}) \right].
\end{aligned}
$$

We can prove next theorem by applying some calculations.

Theorem 3.2. *Let $\sigma \in J$, $4 < \alpha - \gamma_{k_2} \leq 5$, $d \neq 1$ and real-valued function H define on $(0,1]$ be a continuous such that $\lim_{t\to 0^+} H(t) = \infty$. The functions $F_1(t) = \int_0^1 G_1(t,qs) H(s)\, d_q s$ and*

$$
F_2(t) = \frac{t^{\alpha-\gamma_{k_2}-1} d_0}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} \int_0^1 G_2(\tau_k,qs) H(s)\, d_q s
$$

are continuous on $\overline{J}$ whenever $t^\sigma H(t)$ is a continuous function on $\overline{J}$.

You can find the following results in [16, 41]. Consider a real Banach space $(\mathcal{X}, \|.\|)$ which has a partially order by using a cone $P \subset \mathcal{X}$. A nonempty closed convex set $P \subset \mathcal{X}$ is called cone, if for each $r \in P$ and all $\lambda \geq 0$ implies λr belongs to P and $P \cap (-P) = \{0\}$. Each cone P defines the order $\leq$ on $\mathcal{X}$ by $r \leq s$ if and only if $s - r \in P$. We say a cone P is solid, normal whenever interior of P is nonempty, there exists a constant $\xi > 0$ such that $\vartheta \leq r \leq s$ implies $\|r\| \leq \xi \|s\|$, respectively. In this case, least number ξ is called the normal constant of P. Define $r \sim s$ whenever there exist $\xi_1, \xi_2 > 0$ such that $\xi_1 r \leq s \leq \xi_2 r$. Then $\sim$ is an equivalence relation on $\mathcal{X}$. For each $k \geq \vartheta$ with $k \neq \vartheta$, define $P_k = \{r \in \mathcal{X} : r \sim k\}$. It is easy to check that $P_k \subset P$ for almost all $k \in P$.

Theorem 3.3. *Let $w_1, w_2 : (0,1] \times \mathbb{R}^{k_2+2} \to [0, \infty)$ are continuous maps such that $w_i(t, ., ., \dots, .)$ and $w_2(t, ., ., \dots, .)$ tend to infty as $t \to 0^+$. Also, for all $x_i, y_j \in [0, \infty)$ ($i \in N_{k_1+1}, j \in N_{k_2+2}$) and $\sigma, \lambda \in J$, consider the following assumptions:*

1) *The maps $t^\sigma w_1(t, y_1, y_2, \dots, y_{k_2+2})$ and $t^\sigma w_2(t, y_1, y_2, \dots, y_{k_2+2})$ define on $\overline{J} \times \mathbb{R}^{k_2+2}$ are continuous and increasing with respect their components on $\mathbb{R}^{\geq 0}$ for each fixed t in $\overline{J}$ such that $t^\sigma w_2(t, 0, 0, \dots, 0) \not\equiv 0$ and*
$$t^\sigma w_2(t, \lambda y_1, \lambda y_2, \dots, \lambda y_{k_2+2}) \geq \lambda t^\sigma w_2(t, y_1, y_2, \dots, y_{k_2+2}).$$

2) *There exists the map $f(t, s, ., \dots, .) : \overline{J}^2 \times \mathbb{R}^{k_1+1} \to [0, \infty)$ such that*
$$f(t, s, \lambda x_1, \lambda x_2, \dots, \lambda x_{k_1+1}) \geq \lambda f(t, s, x_1, x_2, \dots, x_{k_1+1}).$$

3) *There exists a constant $r \in [0, 1)$ such that*
$$t^\sigma w_1(t, \lambda y_1, \lambda y_2, \dots, \lambda y_{k_2+2}) \geq \lambda^r t^\sigma w_1(t, y_1, y_2, \dots, y_{k_2+2}).$$

4) *There exists $\delta_0 > 0$ such that*
$$t^\sigma w_1(t, y_1, y_2, \dots, y_{k_2+2}) \geq \delta_0 t^\sigma w_2(t, y_1, y_2, \dots, y_{k_2+2}),$$
for all $t \in \overline{J}$.

Then the problem (2.2) has a unique solution $x^* \in P_k$ and the sequence
$$\begin{aligned} x_{n+1}(t) = \int_0^1 &\left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\ &\times [\widetilde{w}_1(s, x_n(s)) + \widetilde{w}_2(s, x_n(s))] \, ds \end{aligned}$$
converges to x^* for each initial value $x_0 \in P_k$, where
$$\begin{aligned} \widetilde{w}_1(s, x_n(s)) = w_1 \bigg(t, x_n(t), \int_0^t &\mu(t, qs) f\big(t, s, x_n(s), D_q^{\beta_1} x_n(s), \\ &\dots, D_q^{\beta_{k_1}} x_n(s)\big) \, d_q s, D_q^{\gamma_i} x_n(t), \dots, D_q^{\gamma_{k_2}} x_n(t) \bigg), \end{aligned}$$

$$\widetilde{w}_2(s,x_n(s)) = w_2\Big(t, x_n(t), \int_0^t \mu(t,qs) f\big(t,s,x_n(s), D_q^{\beta_1} x_n(s), \ldots, D_q^{\beta_{k_1}} x_n(s)\big)\, d_qs, D_q^{\gamma_1} x_n(t), \ldots, D_q^{\gamma_{k_2}} x_n(t)\Big)$$

and $k(t) = t^{\alpha-\gamma_{k_2}-1}$.

Proof. First, we define the operators $\Theta_1, \Theta_2 : P \to X$ by

$$\Theta_1 x(t) = \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \times \widetilde{w}_1(s,x(s))\, d_qs,$$

$$\Theta_2 x(t) = \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \times \widetilde{w}_2(s,x(s))\, d_qs$$

for all $t \in \overline{J}$. By simple review, we obvious that necessary and sufficient conditions for x is a solution for the problem (2.2), is $x = \Theta_1 x + \Theta_2 x$. By employing Theorem 3.2 and the assumptions, we conclude that the operators Θ_1 and Θ_2 maps P into P. Let $x \geq y$. Then, we have

$$\begin{aligned} \Theta_1 x(t) &= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \\ &\quad \times \widetilde{w}_1(s,x(s))\, d_qs \\ &= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \\ &\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s,x(s))\, d_qs \\ &\geq \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \\ &\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s,y(s))\, d_qs \\ &= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_j, qs)\right] \\ &\quad \times \widetilde{w}_1(s,y(s))\, d_qs \\ &= \Theta_1 y1(t) \end{aligned}$$

for each $t \in \overline{J}$. Hence, $\Theta_1 x \geq \Theta_1 y$. By using similar method, we conclude that $\Theta_2 x \geq \Theta_2 y$. Therefore, Θ_1 and Θ_2 are increasing operators. Let $\lambda \in J$ and $x \in P$. Then,

$$\begin{aligned} \Theta_1(\lambda x)(t) &= \int_0^1 \left[G_q(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \\ &\quad \times \widetilde{w}_1(s,\lambda x(s))\, d_qs \\ &= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs)\right] \end{aligned}$$

$$
\begin{aligned}
&\times s^{-\sigma} s^{\sigma} \widetilde{w}(s, \lambda x(s))\, d_q s \\
&\geq \lambda^r \Bigg(\int_0^1 \left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, x(s))\, d_q s \Bigg) \\
&= \lambda^r \Bigg(\int_0^1 \left[G(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times \widetilde{w}_1(s, x(s))\, d_q s \Bigg) \\
&= \lambda^r \Theta x(t)
\end{aligned}
$$

for $t \in \overline{J}$. Hence, $\Theta_1(\lambda x) \geq \lambda^r \Theta_1 x$ for all $\lambda \in J$ and $x \in P$. Therefore, Θ_1 is a r-concave operator. The same method, implies that the operator Θ_2 is sub-homogeneous. Note that,

$$
\begin{aligned}
\Theta_1 k(t) &= \int_0^1 \left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times \widetilde{w}_1(s, k(s)) d_q s \\
&= \int_0^1 \left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, k(s))\, d_q s \\
&\leq \frac{k(t)}{\Gamma_q(\alpha - \gamma_{k_2})} \int_0^1 (1 - qs)^{(\alpha - \gamma_{k_2} - 4)} s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, 1)\, d_q s \\
&\quad + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} \\
&\quad \times \int_0^1 G_2(\tau_k, qs) s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, 1)\, d_q s \\
&= l_1 k(t), \\
\Theta_2 k(t) &= \int_0^1 \left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times \widetilde{w}_1(s, k(s))\, d_q s \\
&= \int_0^1 \left[G_1(t, qs) + \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, k(s))\, d_q s \\
&\geq \frac{k(t)}{\Gamma_q(\alpha - \gamma_{k_2})} \int_0^1 s(s^2 - 3s + 3)(1 - qs)^{(\alpha - \gamma_{k_2} - 4)} s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, 0)\, d_q s \\
&+ \frac{d_0 k(t)}{(\alpha - \gamma_{k_2} - 1)(\alpha - \gamma_{k_2} - 2)(\alpha - \gamma_{k_2} - 3)\,(1-d)}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_0^1 G_2(\tau_k, qs) s^{-\sigma} s^{\sigma} \widetilde{w}_1(s,0)\, d_q s \\
&= l_2 k(t)
\end{aligned}$$

for all $t \in \overline{J}$. By using the assumptions, we get

$$s^{\sigma}\widetilde{w}_1(s,1) \geq s^{\sigma}\widetilde{w}_1(s,0) \geq s^{\sigma} w(s,0,0,\ldots,0) \geq \delta_0 s^{\sigma} w_2(s,0,0,\ldots,0) \geq 0.$$

Since $s^{\sigma} w_2(s,0,0,\ldots,0) \not\equiv 0$,

$$\int_0^1 s^{\sigma}\widetilde{w}_1(s,1)\, ds \geq \int_0^1 s^{\sigma}\widetilde{w}_1(s,0)\, ds \geq \delta_0 \int_0^1 s^{\sigma} w_2(s,0,0,\ldots,0) ds > 0$$

and so $l_1, l_2 \in (0,\infty)$. Thus, $l_1 k(t) \leq \Theta_1 k(t) \leq l_2 k(t)$ for all $t \in \overline{J}$ and so $\Theta_1 k \in P_k$. Again with the same technique, we obtain $\Theta_2 k \in P_k$. On the other hand, for $x \in P$, we get

$$\begin{aligned}
\Theta_1 x(t) &= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times \widetilde{w}_1(s, x(s))\, d_q s \\
&= \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_1(s, x(s))\, d_q s \\
&\geq \delta_0 \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times s^{-\sigma} s^{\sigma} \widetilde{w}_2(s, x(s))\, d_q s \\
&= \delta_0 \int_0^1 \left[G_1(t,qs) + \frac{d_0 k(t)}{(\alpha-\gamma_{k_2}-1)(\alpha-\gamma_{k_2}-2)(\alpha-\gamma_{k_2}-3)\,(1-d)} G_2(\tau_k, qs) \right] \\
&\quad \times \widetilde{w}_2(s, x(s))\, d_q s \\
&= \delta_0 \Theta_2 x(t).
\end{aligned}$$

Hence, $\Theta_1 x \geq \delta_0 \Theta_2 x$ for all $x \in P$. Thus, Theorem 2.5 implies that the operator equation $\Theta_1 x + \Theta_2 x = x$ has a unique solution $x^* \in P_k$. Moreover, the sequence $x_n = \Theta_1 x_{n-1} + \Theta_2 x_{n-1}$ for $n \geq 1$ with initial value $x_0 \in P_k$ converges to x^*. Indeed, the problem (2.2) has the unique positive solution $x^* \in P_k$ and $,I_q^{\gamma_{k_2}} x^*$ is a unique positive solution for the problem (1.1). $\square$

In the sequel, by using different conditions in Theorem 2.4, we show similar result such as Theorem 3.3, which we omit its proof.

Theorem 3.4. *Assume that the positive real-valued functions w_1 and $t^{\sigma} w_1$ define on $(0,1] \times \mathbb{R}^{k_2+2}$ and $\overline{J} \times \mathbb{R}^{k_2+2}$, respectively, are continuous such that $\lim_{t\to 0^+} w_1(t,.,.,\ldots,.) = +\infty$. Also, suppose that $w_2 = 0$, $\beta_0 = 0$, $\sigma \in J$ and consider positive constants $\eta_1, \ldots, \eta_{k_1}$ such that*

$$0 \leq f\big(t,s,x_0,x_1,\ldots,x_{k_1}\big) - f\big(t,s,y_0,y_1,\ldots,y_{k_1}\big) \leq \sum_{j=0}^{k_1} \eta_j (x_j - y_j)$$

for each $t,s \in \overline{J}$ and $x_j, y_j \in [0,\infty)$ with $x_j \geq y_j$, here $0 \leq j \leq k_1$. Then the problem (1.1) has a unique positive solution whenever the following assumptions hold.

1) There exist positive constants $p_1, \dots, p_{k_2+2}$ *such that* $\Lambda\left(\sum_{i=1}^{k_2+2} p_i\right) \le 1$.

2) For each $i \in N_{k_2+2}$, *there exist* $b_i > 0$, *defined as* $b_1 \le \Gamma(\gamma_{k_2}+1)$,

$$b_2 \le \left(r_0 \sum_{j=0}^{k_1} \eta_j \frac{1}{\Gamma_q(\gamma_{k_2} - \beta_j + 1)}\right)^{-1},$$

$b_{i+2} \le \Gamma_q(\gamma_{k_2} - \gamma_i + 1)$, *for* $1 \le i \le k_2 - 1$ *and* $0 < b_{k_2+2} \le 1$ *such that*

$$0 \le t^\sigma \Big(w_1\big(t, x_1, x_2, \dots, x_{k_2+2}\big) - w_1\big(t, y_1, y_2, \dots, y_{k_2+2}\big)\Big) \le \sum_{i=1}^{k_2+2} p_i \varphi(z_i(x_i - y_i)),$$

for $t \in \overline{J}$ *and* $x_i, y_i \in [0,\infty)$ *with* $x_i \ge y_i$ $(1 \le i \le k_2+2)$, *where positive real-valued continuous map* φ *define on* $[0,\infty)$ *is nondecreasing such that* $\phi\,[0,\infty) \to [0,\infty)$ *is nondecreasing,* $\phi(0) = 0$ *and* ϕ *is positive on* $(0,\infty)$ *in Theorem 2.4. Here,* $\phi(t) = t - \varphi(t)$.

Here, we provide some examples to illustrate our main results. In this way, we give a computational technique for checking the problem (1.1). We need to present a simplified analysis could be executed values of the q-Gamma function. To this aim, we consider a pseudo-code description of the method for calculation of the q-Gamma function of order n in Algorithm 2 (for more details, see the link `https://en.wikipedia.org/wiki/Q-gamma_function`).

Example 3.5. *We consider a similar example of the problem* (1.1) *to form of*

$$\begin{aligned} D_q^{\frac{36}{5}} x(t) = \frac{1}{\sqrt{t}} \Bigg[& |x(t)|^{\frac{1}{2}} + r_1(t) \left(\frac{|(\Omega x)(t) + x(t)|}{1 + |(\Omega x)(t) + x(t)|} \right)^{\frac{1}{3}} \\ & + r_2(t) \left(\arctan \left| D_q^{\frac{1}{4}} u(t) \right| \right)^{\frac{1}{4}} + \left(\left| D_q^{\frac{10}{9}} x(t) \right|^3 + \left| D_q^{\frac{31}{10}} x(t) \right| \right)^{\frac{1}{12}} \\ & + \arctan \left(\left| x(t) + D_q^{\frac{1}{4}} x(t) \right| \right) + r_3(t) \frac{|(\Omega x)(t)|}{1 + |(\Omega x)(t)|} \\ & + r_4(t) \frac{\left| D_q^{\frac{1}{4}} x(t) + D_q^{\frac{10}{9}} x(t) \right|}{1 + \left| D_q^{\frac{1}{4}} x(t) + D_q^{\frac{10}{9}} x(t) \right|} \\ & + \ln \left(1 + \left(\left| D_q^{\frac{10}{9}} x(t) \right|^3 + \left| D_q^{\frac{31}{10}} x(t) \right| \right)^{\frac{1}{12}} \right) + r_5(t) + \frac{\pi}{2} + \zeta_1 \Bigg] \end{aligned} \tag{3.3}$$

where $\alpha = \frac{36}{5}$, $k_1 = k_2 = 3$, $\beta_1 = \frac{1}{7}$, $\beta_2 = \frac{4}{3}$, $\beta_3 = \frac{21}{10}$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{10}{9}$, $\gamma_3 = \frac{31}{10}$, $\zeta > 0$ *is a constant,* $r_i : \overline{J} \to [0,\infty)$ *are continuous, with boundary conditions*

$$\begin{aligned} D_q^{\frac{1}{4}} x(0) &= D_q^{\frac{10}{9}} x(0) = D_q^{\frac{31}{10}} x(0) = D_q^{\frac{41}{10}} x(0) = D_q^{\frac{51}{10}} x(0) \\ &= D_q^{\frac{61}{10}} x(0) = D_q^{\frac{1}{7}} x(0) = D_q^{\frac{4}{3}} x(0) = D_q^{\frac{21}{10}} x(0) = 0, \end{aligned}$$

$$D_q^{\frac{61}{10}} x(1) = \frac{\pi}{4} D_q^{\frac{61}{10}} x(\frac{1}{13}) + \frac{1}{17} D_q^{\frac{61}{10}} x(\frac{1}{7}) + \frac{e^{-3}}{2} D_q^{\frac{61}{10}} x(\frac{3}{4}) + \frac{3}{19} D_q^{\frac{61}{10}} x(\frac{5}{6}) + \frac{4}{21} D_q^{\frac{61}{10}} x(\frac{7}{8}),$$

where $\eta_1 = \frac{\pi}{4}$, $\eta_2 = \frac{1}{17}$, $\eta_3 = \frac{e^{-3}}{2}$, $\eta_4 = \frac{3}{19}$, $\eta_5 = \frac{4}{21}$, $\tau_1 = \frac{1}{13}$, $\tau_2 = \frac{1}{7}$, $\tau_3 = \frac{3}{4}$, $\tau_4 = \frac{5}{6}$, $\tau_5 = \frac{7}{8}$, *and*

$$\begin{aligned}(\Omega x)(t) = \int_0^t \frac{(t-qs)^{(5)}}{\sqrt{1+s^2}} \Bigg[\frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} + \frac{3+\cos(s^3)}{1+s^2} \\ \times \Bigg(\ln(1+|x(s)|) + |x(s)|^{\alpha_1} \left| D_q^{\frac{1}{7}} x(s) \right|^{\alpha_2} \left| D_q^{\frac{4}{3}} x(s) \right|^{\alpha_3} \left| D_q^{\frac{21}{10}} x(s) \right|^{\alpha_4} \\ + \left(b_1 |x(s)|^p + b_2 \left| D_q^{\frac{1}{7}} x(s) \right|^p + b_3 \left| D_q^{\frac{4}{3}} x(s) \right|^p + b_4 \left| D_q^{\frac{21}{10}} x(s) \right|^p \right)^{\frac{1}{p}} \Bigg) \Bigg] d_q s,\end{aligned}$$

here α_i *and* $b_i \in [0, \infty)$ *for* $i \in N_4$, $\sum_{i=1}^4 \alpha_i \leq 1$, $p > 0$. *We take*

$$\begin{aligned}w_1(t, x_1, x_2, x_3, x_4, x_5) = \frac{1}{\sqrt{t}} \Bigg[|x_1|^{\frac{1}{2}} + r_1(t) \left(\frac{|x_2 + x_1|}{1 + |x_2 + x_1|} \right)^{\frac{1}{3}} + r_2(t) (\arctan |x_3|)^{\frac{1}{4}} \\ + \left(|x_4|^3 + |x_5| \right)^{\frac{1}{12}} + r_3(t) + \frac{\pi}{2} + \zeta_2 \Bigg], \\ w_2(t, x_1, x_2, x_3, x_4, x_5) = \frac{1}{\sqrt{t}} \Bigg[\arctan(|x_1 + x_3|) + r_3(t) \frac{|x_2|}{1+|x_2|} + r_4(t) \frac{|x_3 + x_4|}{1 + |x_3 + x_4|} \\ + \ln \left(1 + \left(|x_4|^3 + |x_5| \right)^{\frac{1}{12}} \right) + \zeta_1 - \zeta_2 \Bigg]\end{aligned}$$

and

$$\begin{aligned}f(t, s, y_1, y_2, y_3, y_4) = \frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} + \frac{3 + \cos\left(s^3\right)}{1+s^2} \Bigg[\ln(1 + |y_1|) \\ + \prod_{j=1}^4 |y_j|^{\alpha_j} + \left(\sum_{j=1}^4 b_j |y_j|^p \right)^{\frac{1}{p}} \Bigg]\end{aligned}$$

for each $t, s \in \overline{J}$ *and* x_i, y_j *belong to* $\mathbb{R}$, *here* $i \in N_5$, $j \in N_4$. *We take* $\sigma = c = \frac{1}{2}$, *and* $r_{\max}$, $s_{\max}$ *are maximum of* $r_3(t)$, $r_4(t)$, , *respectively, for* $t \in \overline{J}$, *with* $0 < \zeta_2 \leq \frac{\zeta_1}{2}$. *One can easy to review that the maps* $t^\sigma w_1(t, ., ., \dots, .)$, $t^\sigma w_2(t, ., ., \dots, .)$ *and* $f(t, s, ., ., \dots, .)$ *are increasing with respect to their components on* $[0, \infty)$ *for each* $t, s \in \overline{J}$ *and*

$$t^\sigma w_2(t, 0, 0, \dots, 0) = \zeta_1 - \zeta_2 > 0.$$

Also, we have

$$t^\sigma w_2(t, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \arctan(\lambda(x_1 + x_3))$$

$$
\begin{aligned}
&+ r_3(t)\frac{\lambda x_2}{1+\lambda x_2} + r_4(t)\frac{\lambda(x_3+x_4)}{1+\lambda(x_3+x_4)}\\
&+ \ln\left(1+\left(\lambda^3 x_4^3+\lambda x_5\right)^{\frac{1}{12}}\right) + \zeta_1 - \zeta_2\\
&\geq \lambda\Bigg(\arctan(x_1+x_3) + r_3(t)\frac{x_2}{1+x_2}\\
&+ r_4(t)\frac{x_3+x_4}{1+x_3+x_4}\\
&+ \ln\left(1+\left(x_4^3+x_5\right)^{\frac{1}{12}}\right) + \zeta_1 - \zeta_2\Bigg)\\
&= \lambda t^\sigma w_2(t,x_1,x_2,x_3,x_4,x_5),
\end{aligned}
$$

$$
\begin{aligned}
t^\sigma f(t,\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4,\lambda x_5) &= \lambda^{\frac{1}{2}} x_1^{\frac{1}{2}} + r_1(t)\left(\frac{\lambda(x_2+x_1)}{1+\lambda(x_2+x_1)}\right)^{\frac{1}{3}}\\
&+ r_2(t)\left(\arctan(\lambda x_3)\right)^{\frac{1}{4}}\\
&+ \left(\lambda^3 x_4^3+\lambda x_5\right)^{\frac{1}{12}} + r_5(t) + \frac{\pi}{2} + \zeta_2\\
&\geq \lambda^{\frac{1}{2}}\Bigg(x_1^{\frac{1}{2}} + r_1(t)\left(\frac{x_2+x_1}{1+x_2+x_1}\right)^{\frac{1}{3}}\\
&+ r_2(t)\left(\arctan(x_3)\right)^{\frac{1}{4}}\\
&+ \left(x_4^3+x_5\right)^{\frac{1}{12}} + r_5(t) + \frac{\pi}{2} + \zeta_2\Bigg)\\
&= \lambda^\gamma t^\sigma w_1(t,x_1,x_2,x_3,x_4,x_5)
\end{aligned}
$$

and

$$
\begin{aligned}
f(t,s,\lambda y_1,\lambda y_2,\lambda y_3,\lambda y_4) &= \frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}} + \frac{3+\cos(s^3)}{1+s^2}\Bigg(\ln(1+\lambda y_1))\\
&+ \lambda^{(\sum_{i=1}^4 \alpha_i)} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4}\\
&+ \lambda\left(b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p\right)^{\frac{1}{p}}\Bigg)\\
&\geq \lambda\Bigg(\frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}} + \frac{3+\cos(s^3)}{1+s^2}\Bigg[\ln(1+y_1) + y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4}\\
&+ \left(b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p\right)^{\frac{1}{p}}\Bigg]\Bigg)\\
&= \lambda f(t,s,y_1,y_2,y_3,y_4)
\end{aligned}
$$

for each t, s belong to $\overline{J}$, $\lambda \in J$ and $x_i, y_j \in [0,\infty)$, here $i \in N_5$, $j \in N_4$. If

$$
\delta_0 \in \left(0, \frac{c}{r_{\max} + s_{\max} + \zeta_1 - \zeta_2}\right],
$$

then we obtain

$$\begin{aligned}
t^\sigma w_1(t, x_1, x_2, x_3, x_4, x_5) &= x_1^{\frac{1}{2}} + r_1(t)\left(\frac{x_2 + x_1}{1 + x_2 + x_1}\right)^{\frac{1}{3}} \\
&\quad + r_2(t)\left(\arctan(x_3)\right)^{\frac{1}{4}} \\
&\quad + \left(x_4^3 + x_5\right)^{\frac{1}{12}} + r_5(t) + \frac{\pi}{2} + \zeta_2 \\
&\geq \frac{\pi}{2} + (x_4^3 + x_5)^{\frac{1}{12}} + \zeta_2 \\
&\geq \arctan(x_1 + x_3) + \ln\left(1 + \left(x_4^3 + x_5\right)^{\frac{1}{12}}\right) \\
&\quad + \frac{\zeta_2(r_{\max} + s_{\max} + \zeta_1 - \zeta_2)}{r_{\max} + s_{\max} + \zeta_1 - \zeta_2} \\
&\geq \delta_0\Bigg(\arctan\left(x_1 + x_3\right) + r_3(t)\frac{x_2}{1 + x_2} \\
&\quad + r_4(t)\frac{x_3 + x_4}{1 + x_3 + x_4} \\
&\quad + \ln\left(1 + \left(x_4^3 + x_5\right)^{\frac{1}{12}}\right) + \zeta_1 - \zeta_2\Bigg) \\
&= \delta_0 t^\sigma w_2(t, x_1, x_2, x_3, x_4, x_5).
\end{aligned}$$

Therefor, Theorem 3.3 implies that the problem (3.3) *has a unique solution.*

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q
$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^{\alpha} * s$
end if
Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: n, $q \in (0,1)$, $x \in \mathbb{R}\backslash\{0, -1, 2, \cdots\}$
$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1 - q^{k+1})(1 - q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$
Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1)$, $f(x)$, x
syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q * z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q * x))/((1-q)x)$
end if
Output: $(D_q f)(x)$

Algorithm 4 The proposed method for calculated $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x
$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1 - q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow (x^{\alpha} * (1-q) * s)/(\Gamma_q(x))$
Output: $(I_q^{\alpha} f)(x)$

Table 1: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}$ which is constant, for $x = 9.5, 65, 110, 780$ in Algorithm 2.

n	$x = 9.5$	$x = 65$	$x = 110$	$x = 780$
1	2.679786	4432.545834	1804225.634753	$1.29090809480473E + 45$
2	2.674552	4423.888518	1800701.756560	$1.28838678993206E + 45$
3	2.673899	4422.808467	1800262.132108	$1.28807224237593E + 45$
4	2.673818	4422.673494	1800207.192468	$1.28803293353064E + 45$
5	2.673808	4422.656623	1800200.325222	$1.28802802007493E + 45$
6	$\underline{2.673806}$	4422.654514	1800199.466820	$1.28802740589531E + 45$
7	2.673806	4422.654250	1800199.359519	$1.28802732912289E + 45$
8	2.673806	4422.654217	1800199.346107	$1.28802731952634E + 45$
9	2.673806	4422.654213	1800199.344430	$1.28802731832677E + 45$
10	2.673806	4422.654213	1800199.344221	$1.28802731817683E + 45$
11	2.673806	$\underline{4422.654212}$	1800199.344195	$1.28802731815808E + 45$
12	2.673806	4422.654212	$\underline{1800199.344191}$	$1.28802731815574E + 45$
13	2.673806	4422.654212	1800199.344191	$1.28802731815545E + 45$
14	2.673806	4422.654212	1800199.344191	$\underline{1.28802731815541E + 45}$
15	2.673806	4422.654212	1800199.344191	$1.28802731815541E + 45$
16	2.673806	4422.654212	1800199.344191	$1.28802731815541E + 45$
17	2.673806	4422.654212	1800199.344191	$1.28802731815541E + 45$
18	2.673806	4422.654212	1800199.344191	$1.28802731815541E + 45$
19	2.673806	4422.654212	1800199.344191	$1.28802731815541E + 45$

Table 2: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 9.5$ of Algorithm 2.

n	$q = \frac{1}{8}$	$q = \frac{1}{2}$	$q = \frac{4}{5}$	$q = \frac{8}{9}$
1	2.679786	136.046206	79062.138227	6301918.338883
2	2.674552	119.081545	41793.335091	2528395.395827
3	2.673899	111.658224	26290.733638	1232715.590371
4	2.673818	108.178242	18589.881264	689176.848061
5	2.673808	106.492553	14278.326587	426538.394173
6	$\underline{2.673806}$	105.662861	11650.586796	285518.687713
7	2.673806	105.251251	9946.3508930	203363.796571
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
26	2.673806	104.841780	5522.283831	25842.863721
27	2.673806	104.841780	5513.202433	25230.371788
28	2.673806	$\underline{104.841779}$	5505.949683	24699.649904
29	2.673806	104.841779	5500.155385	24238.446645
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
106	2.673806	104.841779	5477.048235	20879.606269
107	2.673806	104.841779	$\underline{5477.048234}$	20879.566792
108	2.673806	104.841779	5477.048234	20879.531702
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
118	2.673806	104.841779	5477.048234	20879.337427
119	2.673806	104.841779	5477.048234	20879.327822
120	2.673806	104.841779	5477.048234	$\underline{20879.319284}$

Table 3: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 110$ of Algorithm 2.

n	$q=\frac{1}{8}$	$q=\frac{1}{2}$	$q=\frac{4}{5}$	$q=\frac{8}{9}$
1	1804225.634753	$2.4338891524382E+32$	$1.10933564801075E+75$	$2.3996994906237E+102$
2	1800701.75656	$2.12965300838343E+32$	$5.41355796236824E+74$	$7.1431517307455E+101$
3	1800262.132108	$1.99654969535946E+32$	$3.196164621018E+74$	$2.6837217226512E+101$
4	1800207.192468	$1.93415751737948E+32$	$2.14884539802207E+74$	$1.1944485864825E+101$
5	1800200.325222	$1.90393630617042E+32$	$1.58553847001434E+74$	$6.0526350536381E+100$
6	1800199.46682	$1.88906180377847E+32$	$1.25302695267477E+74$	$3.3987862057282E+100$
7	1800199.359519	$1.88168265610746E+32$	$1.04280391429109E+74$	$2.0741306563269E+100$
8	1800199.346107	$1.87800749466975E+32$	$9.02841142168746E+73$	$1.3555712905453E+100$
9	1800199.34443	$1.87617350297573E+32$	$8.05899312693661E+73$	$9.3812910130705E+99$
10	1800199.344221	$1.87525740263248E+32$	$7.36673088857628E+73$	$6.8133560326577E+99$
11	1800199.344195	$1.87479957611817E+32$	$6.86049299667128E+73$	$5.1555644082141E+99$
12	$\underline{1800199.344191}$	$1.87457071874804E+32$	$6.48333340557523E+73$	$4.0405190844465E+99$
⋮	⋮	⋮	⋮	⋮
48	1800199.344191	$\underline{1.87434189862553E+32}$	$5.18960499065178E+73$	$6.66324790738213E+98$
⋮	⋮	⋮	⋮	⋮
90	1800199.344191	$1.87434189862553E+32$	$\underline{5.18923469131315E+73}$	$6.5002587652483E+98$
91	1800199.344191	$1.87434189862553E+32$	$5.18923468501255E+73$	$6.50013085733126E+98$
92	1800199.344191	$1.87434189862553E+32$	$5.18923467997207E+73$	$6.50001716364224E+98$
93	1800199.344191	$1.87434189862553E+32$	$5.18923467593968E+73$	$6.499916104353E+98$
⋮	⋮	⋮	⋮	⋮
118	1800199.344191	$1.87434189862553E+32$	$5.18923465987107E+73$	$6.4991502295767E+98$
119	1800199.344191	$1.87434189862553E+32$	$5.18923465985889E+73$	$6.4991455029345E+98$
120	1800199.344191	$1.87434189862553E+32$	$5.18923465984914E+73$	$\underline{6.49914130147782E+98}$