On new analytical and semi–analytical wave solutions of the quadratic–cubic fractional nonlinear Schrödinger equation

Mostafa M. A. Khater[1]*, Raghda A. M. Attia[1,2], and Dumitru Baleanu[3,4]
[1] Department of Mathematics, Faculty of Science, Jiangsu University, China.
[2] Department of Basic Science, Higher technological institute 10^{th} of Ramadan city, Egypt.
[3]Department of Mathematics, Cankaya University, Ankara, Turkey.
[4]Institute of Space Sciences, Magurele-Bucharest, Romania.

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Abstract This research paper discusses the analytical and semi–analytical solutions of the quadratic–cubic fractional nonlinear Schrödinger (NLS) equation. By applying a new fractional operator we transform the fractional formula of the model to integer–order, which allows applying the analytical and numerical methods on it. The analytical solutions are obtained by the implementation of two distinct systematic schemes and the reported solutions are used in applying the Adomian decomposition method to get the semi–analytical wave solutions of this model. These solutions are used to characterize the changes over time of a physical system in which case of quantum influence, such as wave-particle duality. The comparison between the analytical and semi–analytical solutions are given to explain the accuracy of the obtained solutions.

Keywords: Quadratic–cubic fractional nonlinear Schrödinger equation; New fractional operator; Analytical and semi–analytical schemes; Exact and approximate solutions.

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1 Introduction

Many natural phenomena been representing by nonlinear partial differential equations (NLPDEs). Based on these models, certain studies are applied to find the approximate and exact traveling wave solutions. These solutions help to discover new characterizes of these models since the physical properties of each model play an important role in its applications. These applications extend to many fields (nuclear science, atomic science, engineering, biological science, chemistry, and so on). The nonlinear partial differential equation has two types of formulas; the first type is a nonlinear partial differential equation with integer derivative order, while the second type uses the fractional derivative order where the order of the derivative is a fractional number. The second type of nonlinear partial differential equation is recently discussed. There exist many fractional definitions that investigate and study this kind of nonlinear partial differential equations such conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, Caputo–Fabrizio definition, and recently fractional derivative [2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 20, 28].
Schrödinger equation is one of these models that has many formulas, and there exist many researchers who interested in mathematics or even in physics tried and did his best to get the closed-form of solutions for this vital model for examples:
In 2010, Yang, Jianke investigated in his book the nonlinear wave in integrable and nonintegrable systems.

*Email: mostafa.khater2024@yahoo.com

Moreover he applied some numerical methods to this equation to get approximate solutions of these models, in 2011, Fujioka, J., E. Cortés, R. Pérez–Pascual, R. F. Rodríguez, A. Espinosa, and B. A. Malomed investigated the chaotic solitons of our model [15], in 2006, Galaktionov, Victor A., and Sergey R. Svirshchevskii applied some methods to this model to get exact and solitary traveling wave solutions [17], in 2017, Triki, Houria, Anjan Biswas, Seithuti P. Moshokoa, and Milivoj Belic obtained optical soliton solutions of our model by implement of utilizing the method of undetermined coefficients [37], in 2009, Khare, Avinash, Avadh Saxena, and Kody JH Law tried to study the mapping between generalized nonlinear Schrödinger equations and neutral scalar and also obtained exactly traveling wave solutions of this model,...and so on.
Through the last five-decade, those in the meantime succeeded in that purpose then located powerful dead techniques to reap closed form of solutions and solitary traveling wave solutions regarding many one-of-a-kind types on nonlinear partial differential equations. [1, 13, 25, 32, 34, 35, 36]
In this paper, we use two methods that are considered as two novel methods in this field, the generalized $\exp(-\phi(\vartheta))$-expansion method and the modified method. The generalized $\exp(-\phi(\vartheta))$-expansion method was discovered by M. G. Hafez, and Dianchen Lu [18] while the Khater method was discovered by Mostafa M. A. Khater [26, 33]. We can see in [21, 22, 23, 24, 26, 30], the MK method is a natural extension of many methods in this field and not only this, but it is one of the most productive methods for different forms of solitary wave solutions. This feature gives strength to the method. The number of solutions enables researchers interested in the physical properties of these models to discover more and more about the properties and applications of these models.
The strategy of this paper is systematized as follows: Section 2 applies the modified Khater method and the generalized exp- $(-\phi\,(\xi))$ expansion method to get the exact solutions of the fractional NLS equation. Section 3 illustrates our solutions and what is the difference between our results and that obtained by using different methods and also what is new in this paper, which makes our paper is suitable for publication. Section 4 gives a conclusion of our paper.

2 Application:

This part implements two different analytical methods and one semi–analytical scheme to obtain novel forms of the exact traveling wave solutions and approximate solutions of the quadratic–cubic fractional NLS equation which can be written in the following form :

$$i\,\frac{D^{\alpha}\,Y}{D\,t^{\alpha}}+a\,\frac{D^{2\,\alpha}\,Y}{D\,x^{2\,\alpha}}-h_1\,Y\,\,|Y|+h_2\,Y\,\,|Y|^2=0, \tag{1}$$

where $i=\sqrt{-1}$, $0<\alpha<1$, h_1, h_2 are arbitrary constants. Additionally, $Y=Y(x,t)$ is the dependent variable such that t and x are the independent variables representing the temporal and spatial variables respectively. While, the real-valued constant a represents group velocity dispersion (GVD) , b_1 and b_2 are real–valued constants while non fractional form of Eq. (1) takes the same from of equation when $\alpha=1$ [12, 16, 31, 38]. The chaotic phenomenon of the equation was studied in [19]. In [29], the analytical self-similar wave solutions of the equation were constructed. In [39], the method of undetermined clients was adopted to extract the soliton solutions and the conservation laws of the equation were reported. In [40], the He's semi-inverse variational principle was adopted to study the equation. Applying the Atangana–Baleanu fractional derivative that has the following definition [2, 3, 5, 6, 7, 10, 14**?**]:

$$^{ABR}D^{\alpha}_{a+}\,f(t)=\frac{B(\alpha)}{1-\alpha}\,\frac{d}{d\,t}\int_a^t f(x)\,E_{\alpha}\left(\frac{-\alpha\,(t-\alpha)^{\alpha}}{1-\alpha}\right)dx, \tag{2}$$

where E_{α} is the Mittag–Leffler function which is defined by

$$E_{\alpha}\left(\frac{-\alpha\,(t-\alpha)^{\alpha}}{1-\alpha}\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,(t-x)^{\alpha\,n}}{\Gamma(\alpha\,n+1)}$$

and $B(\alpha)$ being a normalisation function. Thus

$$ {}^{ABR}D_{a+}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}I_a^{\alpha n} f(t), \tag{3} $$

where $a > 0,\ 0 < \alpha < 1$ and α is the order of the derivatives for the function $Y(x,t)$ and implementation of the wave transformation on (1)

$$ Y(x,t) = e^{i\varphi} u(\vartheta),\ \varphi = \frac{(1-\alpha)}{B(\alpha)\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)} \left[-\rho x^{-\alpha n} + \omega t^{-\alpha n}\right], $$

$$ \vartheta = \frac{\eta(1-\alpha)}{B(\alpha)\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)} \left[x^{-\alpha n} + \varrho t^{-\alpha n}\right], $$

where the term $\varphi = \varphi(x,t)$ represents the phase component, η is the frequency, ω represents the wave number, c represents the phase constant, k is the velocity while represents the width of the traveling wave and separating the result into real and imaginary components, a pair of the equation is acquired where the imaginary part yields, $\varrho = 2\rho a$, leads to a real part gives the following form:

$$ a\eta^2 u'' - (a\rho^2 + \omega) u - h_1 u^2 + h_2 u^3 = 0. \tag{4} $$

Balancing the highest order derivative term and nonlinear term in Eq. (4), gets $N = 1$.

2.1 Explicit wave solutions via MK method:

Based to the MK method, the general solution of Eq. (4) is given by:

$$ u(\vartheta) = a_0 + a_1 K^{\Upsilon(\vartheta)} + d_1 K^{-\Upsilon(\vartheta)}, \tag{5} $$

where a_0, a_1, d_1, K are arbitrary constants. Additionally, $\Upsilon(\vartheta)$ is the solution function of the following equation $\left[\Upsilon'(\vartheta) = \frac{\chi + \delta K^{-\Upsilon(\vartheta)} + \varsigma K^{\Upsilon(\vartheta)}}{\ln(K)},\right]$ where χ, δ, ς are arbitrary constants. Substituting Eq. (5) and its derivative into Eq. (4) lead to obtain a system of algebraic equations. Equating the coefficient of $K^{i\Upsilon(\vartheta)}$, where $\{i = 3, 2, 1, 0\}$ to zero, and solving the obtained system by Maple 16, yield

Family I

$$ \left[a_1 \to -\frac{\sqrt{a_0^2\chi^2 - 4\delta a_0^2\varsigma} - a_0\chi}{2\delta}, b_1 \to 0, \omega \to a\left(\eta^2\left(\chi^2 - 4\delta\varsigma\right) - \rho^2\right),\right. $$
$$ \left. h_1 \to -\frac{3a\eta^2\left(\chi\sqrt{a_0^2\left(\chi^2 - 4\delta\varsigma\right)} + a_0\left(\chi^2 - 4\delta\varsigma\right)\right)}{2a_0^2}, h_2 \to -\frac{a\eta^2\left(\chi\sqrt{a_0^2\left(\chi^2 - 4\delta\varsigma\right)} + a_0\left(\chi^2 - 2\delta\varsigma\right)\right)}{a_0^3}\right] $$

Consequently, the solitary traveling wave solutions of Eq. (1) are given by:
When $\chi^2 - 4\delta\varsigma < 0 \,\&\, \varsigma \neq 0$

$$ \begin{aligned} Y_1(x,t) = \exp&\left(\frac{i(1-\delta)\left(at^{-\delta}\left(\eta^2\left(\chi^2-4\delta\varsigma\right)-\rho^2\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right) \times \left[a_0 + \frac{\left(\sqrt{a_0^2\left(\chi^2-4\delta\varsigma\right)}-a_0\chi\right)}{4\delta\varsigma}\right. \\ &\left.\times\left(\chi - \sqrt{4\delta\varsigma-\chi^2}\tan\left(\frac{(1-\delta)\eta\sqrt{4\delta\varsigma-\chi^2}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{2B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\right)\right], \end{aligned} \tag{6} $$

$$ \begin{aligned} Y_2(x,t) = \exp&\left(\frac{i(1-\delta)\left(at^{-\delta}\left(\eta^2\left(\chi^2-4\delta\varsigma\right)-\rho^2\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right) \times \left[a_0 + \frac{\left(\sqrt{a_0^2\left(\chi^2-4\delta\varsigma\right)}-a_0\chi\right)}{4\delta\varsigma}\right. \\ &\left.\times\left(\chi - \sqrt{4\delta\varsigma-\chi^2}\cot\left(\frac{(1-\delta)\eta\sqrt{4\delta\varsigma-\chi^2}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{2B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\right)\right]. \end{aligned} \tag{7} $$

When $\chi^2 - 4\,\delta\,\varsigma > 0\,\&\,\varsigma \neq 0$

$$Y_3(x,t) = \exp\left(\frac{i(1-\delta)\big(at^{-\delta}\big(\eta^2\big(\chi^2-4\delta\varsigma\big)-\rho^2\big)-\rho x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)\times\Bigg[a_0+\frac{\Big(\sqrt{a_0^2(\chi^2-4\delta\varsigma)}-a_0\chi\Big)}{4\delta\varsigma}\left(\sqrt{\chi^2-4\delta\varsigma}\tanh\left(\frac{(1-\delta)\eta\sqrt{\chi^2-4\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{2B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)+\chi\right)\Bigg], \tag{8}$$

$$Y_4(x,t) = \exp\left(\frac{i(1-\delta)\big(at^{-\delta}\big(\eta^2\big(\chi^2-4\delta\varsigma\big)-\rho^2\big)-\rho x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)\times\Bigg[a_0+\frac{\Big(\sqrt{a_0^2(\chi^2-4\delta\varsigma)}-a_0\chi\Big)}{4\delta\varsigma}\times\left(\sqrt{\chi^2-4\delta\varsigma}\coth\left(\frac{(1-\delta)\eta\sqrt{\chi^2-4\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{2B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)+\chi\right)\Bigg]. \tag{9}$$

When $\chi^2 + 4\,\delta^2 < 0\,\&\,\delta = -\varsigma$

$$Y_5(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^2\varsigma+\rho^2\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\left(a_0-\frac{\sqrt{-\delta a_0^2\varsigma}\tan\left(\frac{(1-\delta)\eta\sqrt{\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)}{\sqrt{\delta\varsigma}}\right), \tag{10}$$

$$Y_6(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^2\varsigma+\rho^2\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\left(\frac{\sqrt{-\delta a_0^2\varsigma}\cot\left(\frac{(1-\delta)\eta\sqrt{\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)}{\sqrt{\delta\varsigma}}+a_0\right). \tag{11}$$

When $\chi^2 + 4\,\delta^2 > 0\,\&\,\delta = -\varsigma$

$$Y_7(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^2\varsigma+\rho^2\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\left(a_0-\frac{\sqrt{-\delta a_0^2\varsigma}\tanh\left(\frac{(1-\delta)\eta\sqrt{-\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)}{\sqrt{-\delta\varsigma}}\right), \tag{12}$$

$$Y_8(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^2\varsigma+\rho^2\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^n\Gamma(1-\delta n)}\right)\left(a_0-\frac{\sqrt{-\delta a_0^2\varsigma}\coth\left(\frac{(1-\delta)\eta\sqrt{-\delta\varsigma}\big(\varrho t^{-\delta}+x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)}{\sqrt{-\delta\varsigma}}\right). \tag{13}$$

When $\chi = 0\,\&\,\delta = -\varsigma$

$$Y_9(x,t) = \frac{\exp\left(\frac{i(1-\delta)\big(at^{-\delta}\big(4\delta^2\eta^2-\rho^2\big)-\rho x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)\left(\delta a_0-\sqrt{\delta^2 a_0^2}\coth\left(\frac{(1-\delta)\delta\eta\big(\varrho t^{-\delta}+x^{-\delta}\big)}{B(\delta)\sum_{n=0}^{\infty}\big(-\frac{\delta}{1-\delta}\big)^n\Gamma(1-\delta n)}\right)\right)}{\delta}, \tag{14}$$

When $\chi = \frac{\delta}{2} = \kappa\,\&\,\varsigma = 0$

$$Y_{10}(x,t) = \exp\left(\frac{i(1-2\kappa)\big(at^{-2\kappa}(\eta\kappa-\rho)(\eta\kappa+\rho)-\rho x^{-2\kappa}\big)}{B(2\kappa)\sum_{n=0}^{\infty}2^n\big(-\frac{\kappa}{1-2\kappa}\big)^n\Gamma(1-2\kappa n)}\right)\Bigg[a_0+\frac{\Big(a_0\kappa-\sqrt{a_0^2\kappa^2}\Big)}{4\kappa}\times\left(\exp\left(\frac{\eta(1-2\kappa)\kappa\big(\varrho t^{-2\kappa}+x^{-2\kappa}\big)}{B(2\kappa)\sum_{n=0}^{\infty}2^n\big(-\frac{\kappa}{1-2\kappa}\big)^n\Gamma(1-2\kappa n)}\right)-2\right)\Bigg]. \tag{15}$$

When $\chi=0\,\&\,\delta=\varsigma$

$$Y_{11}(x,t)=\exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta^{2}\eta^{2}+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\Bigg[a_{0}-\frac{\sqrt{-\delta^{2}a_{0}^{2}}}{\delta}\times\tan\left(\frac{(1-\delta)\delta\eta\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}+C\right)\Bigg], \tag{16}$$

When $\varsigma=0\,\&\,\chi\neq0\,\&\,\delta\neq0$

$$Y_{12}(x,t)=\exp\left(\frac{i(1-\delta)\left(at^{-\delta}(\chi\eta-\rho)(\chi\eta+\rho)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\Bigg[a_{0}-\frac{\left(\sqrt{a_{0}^{2}\chi^{2}}-a_{0}\chi\right)}{2\delta}\times\left(\exp\left(\frac{(1-\delta)\chi\eta\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)-\frac{\delta}{\chi}\right)\Bigg]. \tag{17}$$

When $\chi^{2}-4\delta\varsigma=0$

$$Y_{13}(x,t)=\frac{2a_{0}B(\delta)t^{\delta}x^{\delta}\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)\exp\left(\frac{i(\delta-1)\rho t^{-\delta}x^{-\delta}\left(a\rho x^{\delta}+t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}{(\delta-1)\chi\eta\left(t^{\delta}+\varrho x^{\delta}\right)}. \tag{18}$$

Family II:

$$\Bigg[a_{1}\to0,b_{1}\to-\frac{\sqrt{a_{0}^{2}\chi^{2}-4\delta a_{0}^{2}\varsigma}-a_{0}\chi}{2\varsigma},\omega\to a\left(\eta^{2}\left(\chi^{2}-4\delta\varsigma\right)-\rho^{2}\right),$$
$$h_{1}\to-\frac{3a\eta^{2}\left(\chi\sqrt{a_{0}^{2}\left(\chi^{2}-4\delta\varsigma\right)}+a_{0}\left(\chi^{2}-4\delta\varsigma\right)\right)}{2a_{0}^{2}},h_{2}\to-\frac{a\eta^{2}\left(\chi\sqrt{a_{0}^{2}\left(\chi^{2}-4\delta\varsigma\right)}+a_{0}\left(\chi^{2}-2\delta\varsigma\right)\right)}{a_{0}^{3}}\Bigg]$$

Consequently, the solitary traveling wave solutions of Eq. (1) are given by:
When, $\chi^{2}-4\,\delta\,\varsigma<0,\ \varsigma\neq0$

$$Y_{14}(x,t)=\exp\left(\frac{(i(1-\delta))\left(\left(at^{-\delta}\right)\left(\eta^{2}\left(\chi^{2}-(4\delta)\varsigma\right)-\rho^{2}\right)-x^{-\delta}\rho\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-n\delta)}\right)\times\Bigg[a_{0}+\frac{\sqrt{\left(\chi^{2}-(4\delta)\varsigma\right)a_{0}^{2}}-\chi a_{0}}{\chi-\sqrt{(4\delta)\varsigma-\chi^{2}}\tan\left(\frac{(((1-\delta)\eta)\left(\varrho t^{-\delta}+x^{-\delta}\right))\sqrt{(4\delta)\varsigma-\chi^{2}}}{2\left(B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-n\delta)\right)}\right)}\Bigg], \tag{19}$$

$$Y_{15}(x,t)=\exp\left(\frac{i(1-\delta)\left(at^{-\delta}\left(\eta^{2}\left(\chi^{2}-4\delta\varsigma\right)-\rho^{2}\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\Bigg[a_{0}+\frac{\sqrt{a_{0}^{2}\left(\chi^{2}-4\delta\varsigma\right)}-a_{0}\chi}{\chi-\sqrt{4\delta\varsigma-\chi^{2}}\cot\left(\frac{(1-\delta)\eta\sqrt{4\delta\varsigma-\chi^{2}}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{2B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}\Bigg]. \tag{20}$$

When, $\chi^{2}-4\,\delta\,\varsigma>0,\ \varsigma\neq0$

$$Y_{16}(x,t)=\exp\left(\frac{i(1-\delta)\left(at^{-\delta}\left(\eta^{2}\left(\chi^{2}-4\delta\varsigma\right)-\rho^{2}\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\Bigg[a_{0}+\frac{\sqrt{a_{0}^{2}\left(\chi^{2}-4\delta\varsigma\right)}-a_{0}\chi}{\sqrt{\chi^{2}-4\delta\varsigma}\tanh\left(\frac{(1-\delta)\eta\sqrt{\chi^{2}-4\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{2B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)+\chi}\Bigg], \tag{21}$$

$$Y_{17}(x,t)=\exp\left(\frac{i(1-\delta)\left(at^{-\delta}\left(\eta^{2}\left(\chi^{2}-4\delta\varsigma\right)-\rho^{2}\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\Bigg[a_{0}+\frac{\sqrt{a_{0}^{2}\left(\chi^{2}-4\delta\varsigma\right)}-a_{0}\chi}{\sqrt{\chi^{2}-4\delta\varsigma}\coth\left(\frac{(1-\delta)\eta\sqrt{\chi^{2}-4\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{2B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)+\chi}\Bigg]. \tag{22}$$

When, $\delta\varsigma > 0 \,\&\, \varsigma \neq 0 \,\&\, \delta \neq 0 \,\&\, \chi = 0$

$$Y_{18}(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^{2}\varsigma+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\left[a_0 - \frac{\sqrt{-\delta a_0^2\varsigma}\cot\left(\frac{(1-\delta)\eta\sqrt{\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}{\sqrt{\delta\varsigma}}\right], \tag{23}$$

$$Y_{19}(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^{2}\varsigma+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\left[a_0 + \frac{\sqrt{-\delta a_0^2\varsigma}\tan\left(\frac{(1-\delta)\eta\sqrt{\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}{\sqrt{\delta\varsigma}}\right]. \tag{24}$$

When, $\delta\varsigma < 0 \,\&\, \varsigma \neq 0 \,\&\, \delta \neq 0 \,\&\, \chi = 0$

$$Y_{20}(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^{2}\varsigma+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\left[a_0 + \frac{\sqrt{-\delta a_0^2\varsigma}\coth\left(\frac{(1-\delta)\eta\sqrt{-\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}{\sqrt{-\delta\varsigma}}\right], \tag{25}$$

$$Y_{21}(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta\eta^{2}\varsigma+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\left[a_0 + \frac{\sqrt{-\delta a_0^2\varsigma}\tanh\left(\frac{(1-\delta)\eta\sqrt{-\delta\varsigma}\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)}{\sqrt{-\delta\varsigma}}\right]. \tag{26}$$

When, $\chi = 0 \,\&\, \delta = -\varsigma$

$$Y_{22}(x,t) = \frac{\exp\left(\frac{i(1-\delta)\left(at^{-\delta}\left(4\delta^{2}\eta^{2}-\rho^{2}\right)-\rho x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\left(\sqrt{\delta^{2}a_0^{2}}\tanh\left(\frac{(1-\delta)\delta\eta\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)+\delta a_0\right)}{\delta}. \tag{27}$$

When $\chi = 0 \,\&\, \delta = \varsigma$

$$Y_{23}(x,t) = \exp\left(\frac{i(\delta-1)t^{-\delta}x^{-\delta}\left(ax^{\delta}\left(4\delta^{2}\eta^{2}+\rho^{2}\right)+\rho t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\times\left[a_0 - \frac{\sqrt{-\delta^{2}a_0^{2}}\cot\left(\frac{(1-\delta)\delta\eta\left(\varrho t^{-\delta}+x^{-\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}+C\right)}{\delta}\right]. \tag{28}$$

When, $\chi^{2} - 4\delta\varsigma = 0$

$$Y_{24}(x,t) = \exp\left(\frac{i(\delta-1)\rho t^{-\delta}x^{-\delta}\left(a\rho x^{\delta}+t^{\delta}\right)}{B(\delta)\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)}\right)\left[a_0 - \frac{(\delta-1)a_0\chi^{3}\eta\left(t^{\delta}+\varrho x^{\delta}\right)}{4\delta\varsigma\left((\delta-1)\chi\eta\left(t^{\delta}+\varrho x^{\delta}\right)-2B(\delta)t^{\delta}x^{\delta}\sum_{n=0}^{\infty}\left(-\frac{\delta}{1-\delta}\right)^{n}\Gamma(1-\delta n)\right)}\right]. \tag{29}$$

2.2 Explicit wave solutions via generalized exp $(-\phi(\vartheta))$- expansion method:

Based to the generalized exp- $(-\phi(\vartheta))$ expansion method, the general solution of Eq. (4) takes the following formate:

$$u(\varphi) = a_0 + a_1 e^{-\phi(\vartheta)}, \tag{30}$$

where a_0, a_1 are arbitrary constants. Additionally, $\phi(\varphi)$ is the solution function of the following equation $\left[\phi'(\vartheta) = \mathcal{L}_1 + \mathcal{L}_2 e^{\phi(\vartheta)} + \frac{\mathcal{L}_3}{e^{\phi(\vartheta)}}\right]$, where $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are arbitrary constants. Substituting Eq. (30) and its derivative into Eq. (4), collection the coefficient of the $e^{-i\phi(\vartheta)}$, where $\{i = 0, 1, 2, 3\}$ and equating them by zero, give a system of algebraic equations. Solving this system of equations by any computer software, yields:

$$\Bigg[a_1 \to \frac{a_0\mathcal{L}_1 - a_0\sqrt{\mathcal{L}_1^2 - 4\mathcal{L}_2\mathcal{L}_3}}{2\mathcal{L}_2}, \omega \to a\eta^2\left(\mathcal{L}_1^2 - 4\mathcal{L}_2\mathcal{L}_3\right) - a\rho^2,$$
$$h_1 \to -\frac{3a\eta^2\left(\mathcal{L}_1\left(\sqrt{\mathcal{L}_1^2 - 4\mathcal{L}_2\mathcal{L}_3} + \mathcal{L}_1\right) - 4\mathcal{L}_2\mathcal{L}_3\right)}{2a_0}, h_2 \to -\frac{a\eta^2\left(\mathcal{L}_1\left(\sqrt{\mathcal{L}_1^2 - 4\mathcal{L}_2\mathcal{L}_3} + \mathcal{L}_1\right) - 2\mathcal{L}_2\mathcal{L}_3\right)}{a_0^2}\Bigg]$$

Consequently, the solitary traveling wave solutions of Eq. (1) have the following form:
Case I. $\Rightarrow$ $(\mathcal{L}_3 = 1)$
When, $\mathcal{L}_1^2 - 4\,\mathcal{L}_2 > 0\,\&\,\mathcal{L}_2 \neq 0$:

$$Y_{25}(x,t) = a_0 \exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\frac{\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}-\mathcal{L}_1}{\mathcal{L}_1-\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}\tanh\left(\frac{1}{2}\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)}+1\right), \tag{31}$$

$$Y_{26}(x,t) = a_0 \exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\frac{\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}-\mathcal{L}_1}{\mathcal{L}_1-\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}\coth\left(\frac{1}{2}\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)}+1\right). \tag{32}$$

When $\mathcal{L}_1^2 - 4\,\mathcal{L}_2 = 0\,\&\,\mathcal{L}_2 \neq 0$

$$Y_{27}(x,t) = \frac{1}{2}a_0 \exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\frac{\left(\mathcal{L}_1-\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}\right)\mathcal{L}_1}{\mathcal{L}_2\left(\exp\left(\mathcal{L}_1\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)-1\right)}+2\right). \tag{33}$$

When $\mathcal{L}_1^2 - 4\,\mathcal{L}_2 = 0\,\&\,\mathcal{L}_1 \neq 0\,\&\,\mathcal{L}_2 \neq 0$

$$Y_{28}(x,t) = \frac{1}{4}a_0\left(\frac{\left(\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}-\mathcal{L}_1\right)\mathcal{L}_1^2\left((\alpha-1)\eta(t^\alpha+\varrho x^\alpha)-\varphi B(\alpha)t^\alpha x^\alpha\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)\right)}{\mathcal{L}_2\left(x^\alpha\left((\alpha-1)\eta\mathcal{L}_1\varrho-B(\alpha)t^\alpha(\mathcal{L}_1\varphi+2)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)\right)+(\alpha-1)\eta\mathcal{L}_1 t^\alpha\right)}+4\right)$$
$$\times\exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right). \tag{34}$$

When $\mathcal{L}_1^2 - 4\,\mathcal{L}_2 = 0\,\&\,\mathcal{L}_1 = 0\,\&\,\mathcal{L}_2 = 0$

$$Y_{29}(x,t) = \frac{1}{2}a_0\left(\frac{\mathcal{L}_1-\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}}{\mathcal{L}_2\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)}+2\right)$$
$$\times\exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right). \tag{35}$$

When $\mathcal{L}_1^2 - 4\,\mathcal{L}_2 < 0\,\&\,\mathcal{L}_2 \neq 0$

$$Y_{30}(x,t) = a_0\exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\frac{\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}-\mathcal{L}_1}{\mathcal{L}_1-\sqrt{4\mathcal{L}_2-\mathcal{L}_1^2}\tan\left(\frac{1}{2}\sqrt{4\mathcal{L}_2-\mathcal{L}_1^2}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)}+1\right), \tag{36}$$

$$Y_{31}(x,t) = a_0\exp\left(\frac{i(1-\alpha)\left(t^{-\alpha}\left(a\eta^2\left(\mathcal{L}_1^2-4\mathcal{L}_2\right)-a\rho^2\right)-\rho x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\frac{\sqrt{\mathcal{L}_1^2-4\mathcal{L}_2}-\mathcal{L}_1}{\mathcal{L}_1-\sqrt{4\mathcal{L}_2-\mathcal{L}_1^2}\cot\left(\frac{1}{2}\sqrt{4\mathcal{L}_2-\mathcal{L}_1^2}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)}+1\right). \tag{37}$$

Case II. $\Rightarrow (\mathcal{L}_1 = 0)$
When, $\mathcal{L}_3\,\&\,\mathcal{L}_2 > 0$:

$$Y_{32}(x,t) = \frac{1}{\mathcal{L}_3}\Bigg[a_0\exp\left(\frac{i(\alpha-1)t^{-\alpha}x^{-\alpha}\left(ax^{\alpha}\left(\rho^2+4\eta^2\mathcal{L}_2\mathcal{L}_3\right)+\rho t^{\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\mathcal{L}_3-\sqrt{\frac{\mathcal{L}_3}{\mathcal{L}_2}}\sqrt{-\mathcal{L}_2\mathcal{L}_3}\cot\left(\sqrt{\mathcal{L}_2\mathcal{L}_3}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)\right)\Bigg], \tag{38}$$

$$Y_{33}(x,t) = \frac{1}{\mathcal{L}_3}\Bigg[a_0\exp\left(\frac{i(\alpha-1)t^{-\alpha}x^{-\alpha}\left(ax^{\alpha}\left(\rho^2+4\eta^2\mathcal{L}_2\mathcal{L}_3\right)+\rho t^{\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\mathcal{L}_3-\sqrt{\frac{\mathcal{L}_3}{\mathcal{L}_2}}\sqrt{-\mathcal{L}_2\mathcal{L}_3}\tan\left(\sqrt{\mathcal{L}_2\mathcal{L}_3}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)\right)\Bigg]. \tag{39}$$

When, $\mathcal{L}_2\,\mathcal{L}_3 < 0$:

$$Y_{34}(x,t) = \frac{1}{\mathcal{L}_2}\Bigg[a_0\exp\left(\frac{i(\alpha-1)t^{-\alpha}x^{-\alpha}\left(ax^{\alpha}\left(\rho^2+4\eta^2\mathcal{L}_2\mathcal{L}_3\right)+\rho t^{\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\sqrt{-\frac{\mathcal{L}_2}{\mathcal{L}_3}}\sqrt{-\mathcal{L}_2\mathcal{L}_3}\tanh\left(\sqrt{-\mathcal{L}_2\mathcal{L}_3}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)+\mathcal{L}_2\right)\Bigg], \tag{40}$$

When, $[\delta\,\chi < 0]$:

$$Y_{35}(x,t) = \frac{1}{\mathcal{L}_2}\Bigg[a_0\exp\left(\frac{i(\alpha-1)t^{-\alpha}x^{-\alpha}\left(ax^{\alpha}\left(\rho^2+4\eta^2\mathcal{L}_2\mathcal{L}_3\right)+\rho t^{\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)$$
$$\times\left(\sqrt{-\frac{\mathcal{L}_2}{\mathcal{L}_3}}\sqrt{-\mathcal{L}_2\mathcal{L}_3}\coth\left(\sqrt{-\mathcal{L}_2\mathcal{L}_3}\left(\frac{(1-\alpha)\eta\left(\varrho t^{-\alpha}+x^{-\alpha}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}+\varphi\right)\right)+\mathcal{L}_2\right)\Bigg]. \tag{41}$$

2.3 Numerical Simulation

Applying the Adomian decomposition method to Eq. (4) enables rewriting it to take the following form:

$$\mathcal{L}\, u(\vartheta) + \mathcal{R}\, u(\vartheta) + \mathcal{N}\, u(\vartheta) = 0, \tag{42}$$

where $(\mathcal{L}\,\&\,\mathcal{R}\,\&\,\mathcal{N})$ represent a differential operator, a linear operator and nonlinear term, respectively. Using the inverse operator $\mathcal{L}^{-1}$ on (42), gets

$$\sum_{i=0}^{\infty} u_i(\vartheta) = u(0) + u'(0)\,\vartheta + \frac{a\,k^2 + \omega}{a\,\eta^2}\,\mathcal{L}^{-1}\left(\sum_{i=0}^{\infty} u_i\right) + \frac{b_1}{a\,\eta^2}\,\mathcal{L}^{-1}\left(\sum_{i=0}^{\infty} A_i\right) - \frac{b_2}{a\,\eta^2}\,\mathcal{L}^{-1}\left(\sum_{i=0}^{\infty} A_i\right). \tag{43}$$

Under the following condition:

Table 1: Initial conditions for both analytical methods

−−−−−−−	**Modified Khater method**	**Generalized exp $(-\phi\,(\vartheta))-$ method**
Sol. Num.	Eq. (8)	Eq. (40)
Arbitrary Constants	$\left[a_0 = -1\,\&\,\chi = 5\,\&\,\delta = 6\,\&\,\varsigma = 1\right]$	$\left[a_0 = -1\,\&\,\mathcal{L}_2 = -4\,\&\,\mathcal{L}_3 = 1\,\&\,\mathcal{L}_1 = 0\right]$
Exact solution	$\left[\frac{-1}{24} + \frac{5}{24}\tanh\left(\frac{\vartheta}{2}\right)\right]$	$\left[\tanh(2\,\vartheta) + 1\right]$

Using above conditions and applying the Adomian decomposition method on Eq. (4), lead to the following data in table 2

Table 2: Absolute Error between semi–analytical and exact solution which obtained by using the modified Khater method and generalized exp $(-\phi\,(\vartheta))-$ method

Value of ϑ	**Abs. Error of MK method**	**Abs. Error of the generalized exp $(-\phi\,(\vartheta))-$ method**
0.001	2.34793×10^{-8}	4.44799×10^{-6}
0.002	9.37949×10^{-8}	1.78061×10^{-5}
0.003	2.10764×10^{-7}	4.00955×10^{-5}
0.004	3.74203×10^{-7}	7.13372×10^{-5}
0.005	5.83929×10^{-7}	0.000111552
0.006	8.3976×10^{-7}	0.00016076
0.007	1.14151×10^{-6}	0.000218983
0.008	1.48901×10^{-6}	0.000286241
0.009	1.88206×10^{-6}	0.000362554
0.01	2.32048×10^{-6}	0.000447941

3 Discussion

Necessary steps of the methods and the relation between these two methods and the Riccati equation:

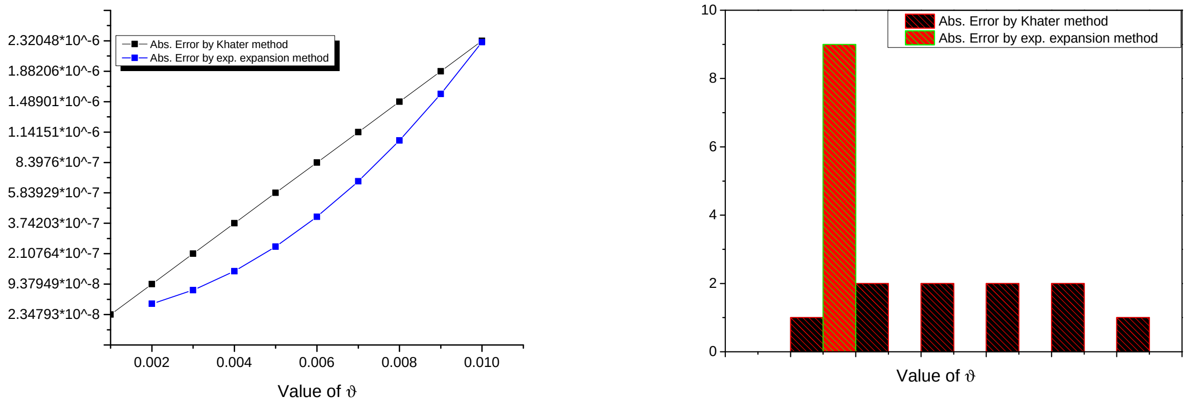

Figure 1: Relation between absolute error between Adomian decomposition method and the two used analytical schemes based on table 2. It shows the superiority of modified Khater method on the another used method.

- Basic steps of the methods:
  In this paper, two analytical techniques were employed to find novel traveling wave solutions formulas of the quadratic-cubic fractional NLS equation. The main idea of these methods (the modified Khater method and generalized exp $(-\phi(\vartheta))$ expansion method) is to convert the nonlinear partial differential nonlinear PDE to ordinary nonlinear differential nonlinear ODE by using the traveling wave transformation then using the homogeneous balance rule between the highest derivative term and nonlinear term in obtained nonlinear ODE then using the general solution that suggested by methods themselves as following:

$$Y(\vartheta) = \begin{cases} \sum_{i=0}^{N} a_i\, k^{i\,\Upsilon(\vartheta)} \Rightarrow \text{The modified Khater method,} \\ \sum_{i=0}^{N} a_i\, exp\,(-i\,\phi(\vartheta)) \Rightarrow \;\text{Generalized exp } (-\phi\,(\vartheta)) \text{ expansion method.} \end{cases}$$

  These solutions depend on the following auxiliary equations, respectively:

$$\begin{cases} f'(\vartheta) = \dfrac{1}{ln(K)}\left(\delta\, k^{-\Upsilon(\vartheta)} + \chi + \varsigma\, k^{\Upsilon(\vartheta)}\right), \\ \phi'(\vartheta) = \delta\, exp(-\phi(\vartheta)) + \chi\, exp(\phi(\vartheta)) + \varsigma. \end{cases}$$

  Using the solutions of these auxiliary equations under specific conditions and submit these solutions into the exact traveling wave solution lead to the solitary traveling wave solutions of the suggested model.

- The similarity between both methods:
  The solitary solutions of both methods depending on the solutions of auxiliary equations for each of them. By careful look for both equations, we can find that both equations are same when $\Big[K = e = 2.7183,\ \Upsilon(\vartheta) = \phi(\vartheta)\Big]$ that leads to same solutions. The exp-function properties are used for both methods to get many forms of solutions that help many researchers who do not have backgrounds in mathematics. This similarity not limited to these three ways, but it applies to most of the schemes in this area [19],[27].

- Comparison between our solutions and that obtained in previous work:
  We give a comparison between our solutions and that obtained by Aslan, Ebru Cavlak, and Mustafa Inc In [4] as follows:
  Aslan, Ebru Cavlak, and Mustafa Inc applied the Jacobi elliptic functions to the quadratic-cubic NLS equation when ($\delta = 1$). They obtained bright and dark optical soliton solutions (15), (28). We obtain many different forms of solutions that are completely different from that obtained in [4] that makes our solutions are novel and considerable for publication.

- Numerical solutions of quadratic-cubic fractional NLS equation:
  The Adomian decomposition method applied to this model under the specific boundary conditions obtained by using the result solutions of the analytical used methods. Fig. 1 shows the difference between absolute error for both used analytical schemes (modified Khater method and generalized exp $(-\phi(\vartheta))$ expansion method). This shows the accuracy of solutions which obtained by the modified Khater method than the second method.

4 Conclusion

In this research, We succeeded in the implementation of the modified Khater method and generalized exp-$(-\phi(\xi))$-expansion method for the quadratic-cubic fractional NLS equation. We obtained different formulas of solitary traveling wave solutions of this model. The modified Khater method has been considered as one of the few generalization methods to get exact and solitary method as it can cover most of solitary traveling wave solutions that obtained by some of the methods. We gave a comparison between our solutions and that obtained by another researcher who used a different method [4]. We also gave a numerical study of our obtained to show the accuracy of our exact solutions. We showed this convergence between exact and numerical solutions in Figure 1.

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