Homotopy perturbation method to a half-space in generalized thermoelastic for two models

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Abstract. In this paper, we considered a one-dimensional problem for a half-space in generalized thermoelastic for two models; Lord-Shulman (L-S) and the dualphase-lag (DPL) theories. The surface of the half-space is assumed to be traction free and subjected to the effects of a heat source varying exponentially with time at the boundary. The homotopy perturbation method is applied to obtain the approximate solution of thermoelastic interactions with boundary condition. The numerical results obtained are displayed graphically to show the influences of the new parameters.The effects of the heat source varying with time and zero traction force are studied on displacement, temperature and stress.

Keywords: Generalized thermoelastic, homotopy perturbation method, Lord-Shulman, dual phase-lag.

1. Introduction

Lord and Shulman [1] investigated ageneralized dynamical theory ofthermo-elasticity. Green and Lindsay [2] proposed a theory of generalized thermo-elasticity with two relaxation time parameters and modified both theenergy equation and constitutive equations. Chandrasekharaiah [3] discussed one-dimensional wave propagation in the linear theory of thermoelasticity with energy dissipation. Dhaliwal and Singh [4] studied dynamic coupled thermo-elasticity. Hetnarski and Ignaczak [5] developed other thermoelasticity theories. Roychoudhuri [6] discussed one-dimensional thermoelastic waves in elastic half-space with dual-phase-lag effects. Abouelregal [7] studied Rayleigh waves in a thermoelastic solid half space using dual-phase-lag model. Aboueregal and Abo-Dahab [8] used dual-phase-model on magneto-thermoelasticity infinite non homogeneous solid having a spherical cavity. Mukhopadhyay et al. [9]showed the representation of solutions for the theory of generalized thermo-elasticity with three phase lags. Chandrasekharaiah and Srinath [10] illustrated one dimensional wave in a thermoelastic half-space without energy dissipation. Chandrasekharaiah [11] studied the hyperbolic theories of thermoelasticity for example extended thermoelasticity and the temperature-rate dependent thermoelasticity. Yadav and Kumar [12] used homotopy analysis approach to thermoelastic interactions under the boundary condition: heat source varying exponentially with time and zero stress. Rashidi, and Pour [13] investigated analytic approximate solutions for unsteady boundary layer flow and heat transfer due to a stretching sheet by homotopy analysis method. Kuppalapalle [14] studied homotopy analysis method for a magneto hydrodynamic viscoelastic fluid flow and heat transfer in a channel with a stretching wall. Abd-Alla et al. [15] studied harmonic wave generation in nonlinear thermoelasticity. Mohyud-Din and Noor [16] and [17] used homotopy perturbation method for solving fourth-order boundary value problems to partial differential equations. He [18] and [19] investigatedapproximate solution of nonlinear differential equations with convolution product nonlinearities. Liao [20] used on the homotopy analysis method for nonlinear problems. . Roul [21] used the numerical solution of singular two-point boundary value problems: A domain decomposition homotopy perturbation approach.

In this paper, we used homotopy perturbation method to obtain the approximate solution of thermoelastic interactions with boundary condition under two models of thermoelasticity; Lord-Shulman (L-S) and the dual phase-lag (DPL) theories. The results clear the quality of the proposed method.

2. Introduction to the basic idea of homotopy Perturbation Method

Weconsider general equation of type

$$L(u)=0, \tag{1}$$

Where L is an integral or differential operator. We choose a convex homotopy

$$H(u,p)=(1-p)F(u)+pL(u) \tag{2}$$

$\mathrm{F}(\mathrm{u})$ functional operator with known solution u_0,which can be easily obtained. It is clear that

$$H(u,p)=0. \tag{3}$$

From which we have $H(u,0)$=$F(u)$ and $H(u,1)=L(u)$.This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(u,1)$. The embedding parameter increases monotonically from zero to unity as the problem $F(u)=0$ continuously deforms the original problem $L(u)=0$. Theembedding parameter can be considered as an expanding parameter. The HPM uses the homotopy parameter $p\in[0,1]$ as an expanding parameter to obtain

$$u=\sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots \tag{4}$$

If $p\to 1$, then equation (4) corresponds to (2) and becomes the approximate solution of the form

$$u=\lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i \tag{5}$$

It is well know that the series (5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$.

3. Formulation of the problem and the basic equations

Equation of motion

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{6}$$

Heat conduction equation

$$\kappa\left(1+\tau_\Theta \frac{\partial}{\partial t}\right)\mathrm{T}_{,ii} = \left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E \mathrm{T}+\gamma \mathrm{T}_0 e\right) \tag{7}$$

The constitutive equations are given by

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma\left(\mathrm{T}-\mathrm{T}_0\right)\delta_{ij} \tag{8}$$

We consider a homogeneous isotropic and thermoelastic half-space which fills the region subjected to a heat source varying exponentially with time on the boundary plane and the surface x = 0 is assume to be traction free. The governing equation will be written in the context of two models; Lord-Shulman (L-S) and the dual phase-lag (DPL) theories. The displacement component is of the form$u_i = (u,0,0)$ $\quad u_y = u_z = 0$

From equations (6-8) we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial \mathrm{T}}{\partial x} \tag{9}$$

$$\kappa\left(1+\tau_\Theta \frac{\partial}{\partial t}\right)\frac{\partial^2 \mathrm{T}}{\partial x^2} = \left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E \mathrm{T}+\gamma \mathrm{T}_0 e\right) \tag{10}$$

$$\sigma_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} - \gamma\left(\mathrm{T}-\mathrm{T}_0\right) \tag{11}$$

Where $e = \dfrac{\partial u}{\partial x}$

We define the following non-dimensional variables

$$x' = c_1\eta x, \qquad u' = c_1\eta u, \qquad t' = c_1^2\eta t, \qquad \tau' = c_1^2\eta\tau, \qquad \tau'_\Theta = c_1^2\eta\tau_\Theta,$$
$$\theta' = \frac{\mathrm{T}}{\mathrm{T}_0}, \qquad \eta = \frac{\rho C_E}{\kappa}, \qquad \sigma'_{xx} = \frac{\sigma_{xx}}{\lambda+2\mu}, \qquad c_1^2 = \frac{\lambda+2\mu}{\rho}$$

By using non-dimensional variables in to equations (9)-(11) we obtain

$$\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{12}$$

$$\left(1+\tau_\Theta\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t} + \tau\frac{\partial^2\theta}{\partial t^2} + a_2\left[\frac{\partial^2 u}{\partial t\,\partial x} + \tau\frac{\partial^3 u}{\partial t^2\partial x}\right] \tag{13}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} - a_1\theta \tag{14}$$

where $a_1 = \frac{\gamma T_0}{\lambda + 2\mu}$, $a_2 = \frac{\gamma}{\rho C_E}$

4. Solution using the homotopy perturbation method:

From (12) and (14), we get

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} = \frac{\partial^2 \sigma_{xx}}{\partial t^2} + a_1 \frac{\partial^2 \theta}{\partial t^2} \tag{15}$$

From (13) and (14), we get

$$\frac{\partial^2 \theta}{\partial x^2} = (1+\varepsilon)\frac{\partial \theta}{\partial t} + \tau(1+\varepsilon)\frac{\partial^2 \theta}{\partial t^2} - \tau_\Theta \frac{\partial^3 \theta}{\partial t \partial x^2} + a_2\left[\frac{\partial \sigma_{xx}}{\partial t} + \tau \frac{\partial^2 \sigma_{xx}}{\partial t^2}\right] \tag{16}$$

Where $\varepsilon = a_1 a_2$

We assume that the boundary conditions as

$$\theta(t,0) = e^{-t}, \qquad \sigma_{xx}(t,0) = 0 \tag{17}$$

According to the homotopy perturbation, we construct the following homotopy

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + p\left[-\frac{\partial^2 \sigma_{xx}}{\partial t^2} - a_1 \frac{\partial^2 \theta}{\partial t^2}\right] = 0 \tag{18}$$

$$\frac{\partial^2 \theta}{\partial x^2} + p\left[-(1+\varepsilon)\frac{\partial \theta}{\partial t} - \tau(1+\varepsilon)\frac{\partial^2 \theta}{\partial t^2} + \tau_\Theta \frac{\partial^3 \theta}{\partial t \partial x^2} - a_2\left(\frac{\partial \sigma_{xx}}{\partial t} + \tau \frac{\partial^2 \sigma_{xx}}{\partial t^2}\right)\right] = 0 \tag{19}$$

Where $p \in [0,1]$ is an embedding parameter, we use it to expand the solution in the following form:

$$\sigma_{xx}(x,t) = \sigma_{xx}^0 + p\sigma_{xx}^1 + p^2\sigma_{xx}^2 + \ldots \tag{20}$$

$$\theta(x,t) = \theta + p\theta^1 + p^2\theta^2 + p^3\theta^3 + \ldots \tag{21}$$

The approximate solution can be obtained by setting $p = 1$ in equation (20) and (21). Substituting $\sigma_{xx}(x,t)$ and $\theta(x,t)$ from equations (20), (21) to (18), (19) respectively, we can obtain series of linear equations. Here we write only the first few linear equations:

$$p^0 : \frac{\partial^2 \sigma_{xx}^0}{\partial x^2} = 0 \tag{22}$$

$$p^0 : \frac{\partial^2 \theta^0}{\partial x^2} = 0 \tag{23}$$

$$p^1 : \frac{\partial^2 \sigma_{xx}^1}{\partial x^2} = \frac{\partial^2 \sigma_{xx}^0}{\partial t^2} + a_1 \frac{\partial^2 \theta^0}{\partial t^2} \tag{24}$$

$$p^1 : \frac{\partial^2 \theta^1}{\partial x^2} = (1+\varepsilon)\frac{\partial \theta^0}{\partial t} + \tau(1+\varepsilon)\frac{\partial^2 \theta^0}{\partial t^2} - \tau_\Theta \frac{\partial^3 \theta^0}{\partial t \partial x^2} + a_2 \frac{\partial \sigma_{xx}^0}{\partial t} + a_2 \tau \frac{\partial^2 \sigma_{xx}^0}{\partial t^2} \tag{25}$$

$$p^2 : \frac{\partial^2 \sigma_{xx}^2}{\partial x^2} = \frac{\partial^2 \sigma_{xx}^1}{\partial t^2} + a_1 \frac{\partial^2 \theta^1}{\partial t^2} \tag{26}$$

$$p^2 : \frac{\partial^2 \theta^2}{\partial x^2} = (1+\varepsilon)\frac{\partial \theta^1}{\partial t} + \tau(1+\varepsilon)\frac{\partial^2 \theta^1}{\partial t^2} - \tau_\Theta \frac{\partial^3 \theta^1}{\partial t \partial x^2} + a_2 \frac{\partial \sigma_{xx}^1}{\partial t} + a_2 \tau \frac{\partial^2 \sigma_{xx}^1}{\partial t^2} \tag{27}$$

The solution of equations (22) and (23) can be calculated by using the boundary conditions (17):

$$\sigma_{xx}^0(x,t) = \sigma_{xx}(t,0) = 0, \qquad \theta^0(x,t) = \theta(t,0) = e^{-t} \tag{28}$$

From equations (24) and (25) we can find:

$$\sigma_{xx}^1(x,t) = \int_0^x \int_0^x \frac{\partial^2 \sigma_{xx}^0}{\partial t^2} dx dx + a_1 \int_0^x \int_0^x \frac{\partial^2 \theta^0}{\partial t^2} dx dx$$

$$\sigma_{xx}^1 = a_1 \left(\frac{x^2}{2!}\right) e^{-t} \tag{29}$$

$$\theta^1(x,t) = \int_0^x \int_0^x (1+\varepsilon)\frac{\partial \theta^0}{\partial t} dx dx + \tau \int_0^x \int_0^x (1+\varepsilon)\frac{\partial^2 \theta^0}{\partial t^2} dx dx + a_2 \tau \int_0^x \int_0^x \frac{\partial^2 \sigma_{xx}^0}{\partial t^2} dx dx$$

$$-\tau_\Theta \int_0^x \int_0^x \frac{\partial^3 \theta^0}{\partial t \partial x^2} dx dx$$

$$\theta^1(x,t) = (1+\varepsilon)(\tau - 1)\left(\frac{x^2}{2!}\right) e^{-t} \tag{30}$$

Similarly,

$$\sigma_{xx}^2(x,t) = a_1\left[\tau(1+\varepsilon) - \varepsilon\right]\left(\frac{x^4}{4!}\right) e^{-t} \tag{31}$$

$$\theta^2(x,t) = (1-\tau)\left[(1+\varepsilon)^2 - \tau(1+\varepsilon)^2 - \varepsilon\right]\left(\frac{x^4}{4!}\right) e^{-t} + \tau_\Theta (1+\varepsilon)(\tau-1)\left(\frac{x^2}{2!}\right) e^{-t} \tag{32}$$

$$\begin{aligned} \sigma_{xx}^3(x,t) &= a_1\left[\tau(1+\varepsilon) + (1+\varepsilon)^2 - 2\tau(1+\varepsilon)^2 - \varepsilon(2-\tau) + \tau^2(1+\varepsilon)^2\right]\left(\frac{x^6}{6!}\right) e^{-t} \\ &+ a_1\left[\tau_\Theta (1+\varepsilon)(\tau-1)\right]\left(\frac{x^4}{4!}\right) e^{-t} \end{aligned} \tag{33}$$

$$\begin{aligned} \theta^3(x,t) &= \left\{\begin{array}{l} -(1+\varepsilon)(1-\tau)\left[(1+\varepsilon)^2 - \tau(1+\varepsilon)^2 - \varepsilon\right] \\ +\tau(1+\varepsilon)(1-\tau)\left[(1+\varepsilon)^2 - \tau(1+\varepsilon)^2 - \varepsilon\right] \\ -a_1 a_2\left[\tau(1+\varepsilon) - \varepsilon\right] + a_1 a_2 \tau\left[\tau(1+\varepsilon) - \varepsilon\right] \end{array}\right\}\left(\frac{x^6}{6!}\right) e^{-t} \\ &+ \left\{\tau(1+\varepsilon)^2(\tau-1)\tau_\Theta - (1+\varepsilon)^2(\tau-1)\tau_\Theta\right\}\left(\frac{x^4}{4!}\right) e^{-t} \end{aligned} \tag{34}$$

Consequently, we have the following solution in a series form

$$\sigma_{xx}(x,t) = \sum_{i=0}^{3} \sigma_{xx}^i(x,t) = \left[m_1 \frac{x^2}{2!} + m_2 \frac{x^4}{4!} + m_3 \frac{x^6}{6!}\right] e^{-t} \tag{35}$$

$$\theta(x,t)=\sum_{i=0}^{3}\theta^{i}(x,t)=\left[1+n_1\frac{x^2}{2!}+n_2\frac{x^4}{4!}+n_3\frac{x^6}{6!}\right]e^{-t} \tag{36}$$

From equation (12), we find that

$$\frac{\partial u}{\partial x}=\left[a_1+(a_1n_1+m_1)\frac{x^2}{2!}+(a_1n_2+m_2)\frac{x^4}{4!}+(a_1n_3+m_3)\frac{x^6}{6!}\right]e^{-t} \tag{37}$$

By integrating equation (35) with respect to x, we get

$$u(x,t)=\left[a_1\frac{x}{1!}+(a_1n_1+m_1)\frac{x^3}{3!}+(a_1n_2+m_2)\frac{x^5}{5!}+(a_1n_3+m_3)\frac{x^7}{7!}\right]e^{-t} \tag{38}$$

Where

$m_1=a_1,\quad m_2=a_1\left[\tau(1+\varepsilon)-\varepsilon\right]+a_1\left[\tau_\Theta(1+\varepsilon)(\tau-1)\right],$

$m_3=a_1\left[\tau(1+\varepsilon)+(1+\varepsilon)^2-2\tau(1+\varepsilon)^2-2\varepsilon+\tau\varepsilon+\tau^2(1+\varepsilon)^2\right],$

$n_1=(1+\varepsilon)(\tau-1)+\tau_\Theta(1+\varepsilon)(\tau-1),\ n_2=(1-\tau)\left[(1+\varepsilon)^2-\tau(1+\varepsilon)^2-\varepsilon\right]+(1+\varepsilon)^2(\tau-1)^2\tau_\Theta,$

$n_3=[-(1+\varepsilon)^3+3\tau(1+\varepsilon)^3+2\varepsilon(1+\varepsilon)-3\tau^2(1+\varepsilon)^3-4\tau\varepsilon(1+\varepsilon)+\tau^3(1+\varepsilon)$

$\quad +2\tau^2\varepsilon(1+\varepsilon)-\varepsilon+\tau\varepsilon],$

5. Numerical results and discussion

We choose the copper material for purposes of numerical evaluations. The physical data which given as

$\lambda=7.76\times10^{10}Nm^{-2},\quad \mu=3.86\times10^{10}Kg.m^{-1}S^{-2},\quad \rho=8954kgm^{-3},$

$\alpha_t=1.78\times10^{-5}k^{-1},\quad \kappa=8886.73sm^{-3},\quad C_E=383.1,\quad T_0=293^\circ k$

Figs. (1) and (4): illustrate variation of displacement u at $t=0.2$ under L-S and DPL theories. It noticed that the distribution of u is increase with increase space variable x to (L-S) theory, but the distribution of u is decrease with increase space variable x to (DPL) theory.

Figs. (2) and (5): display the variation of temperature θ at $t=0.2$ under L-S and DPL theories. The distribution of temperature θ decreases gradually and finally gets zero value after travelling a distance. We notice that effect of $LS>DPL$ in all interval of variable x .

Figs. (3) and (6): explain variation of stress σ_{xx} at $t=0.2$ under L-S and DPL theories. The distribution of stress σ_{xx} is increase with increase space variable x to (L-S) theory, but the distribution of u is decrease with increase space variable x to (DPL) theory.

6.Conclusion

According to the above results, we can conclude that:

1.We found that, the parameters τ and τ_Θ have significant effects on all the fields.

2.The comparison of different theories of thermoelasticity; Lord and Shulman (LS) theory and Chandrasekharaiah and Tzou (DPL) theory is very clear.

3.All the physical quantities satisfy the boundary conditions.

4.Homotopy perturbation method used to derive displacement, temperature and stress.

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