# Model of fractional heat conduction in a thermoelastic thin slim strip with temperature-dependent thermal conductivity and thermal shock

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January 14, 2020

**Abstract**

In this work, the fractional order thermoelasticity theory is used to investigate the thermoelastic problem of a thin slim strip considering the thermal conductivity is to be variable. The theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative of order $\alpha$ is used. The surface of the strip is subjected to a thermal shock and assumed to be traction free. By using the Laplace transform and numerical Laplace inversion, the governing equations are solved. The inverse of the Laplace transform is done numerically using a method based on Fourier expansion techniques. Numerical calculations for the considered variables are performed and the results obtained have been presented graphically. The effects of fractional order parameter and the variation of thermal conductivity on temperature, stress, and displacement are investigated.

**Keywords:**Thermoelasticity, Non-Fourier heat conduction, Fractional derivative, Variable thermal conductivity.

## 1 Introduction

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms. Second, the heat equation

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of a parabolic type, predicting infinite speeds of propagation for heat waves. Biot [1] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

At present, there are different theories of the generalized thermoelasticity, the first was developed by Lord and Shulman [2], the second was developed by Green and Lindsay [3] and the third is due to Green and Naghdi [4]. The Lord and Shulman (**LS**) theory is based on the modified Fourier's law of heat conduction, and admits one relaxation time. The Green and Lindsay (**GL**) theory modifies both the energy equation and the Duhamel–Neumann relation, and allows two relaxation times. Green and Naghdi (**GN**) proposed a new generalized theromoelasticity theory by including the 'thermal-displacement gradient' among the independent constitutive variables. An important feature of this theory, which is not present in other thermoelasticity theories, is that this theory does not accommodate dissipation of thermal energy.

Modern structural elements are often subjected to temperature changes of such magnitude that their material properties may no longer be regarded as having constant values even in an approximate sense. It is usual to assume in thermal stress calculations that material properties are in dependent of temperature. Significant variations do however occur over the working temperature range of the "engineering ceramics," particularly in the coefficient of thermal conductivity "$K$". Godfrey has reported decreases of up to 45 percent in the thermal conductivity of various samples of silicon nitride between $1^o$ and $400C^o$. At high temperature the material characteristics such as the modulus of elasticity, Poisson's ratio, the coefficient of thermal expansion and the thermal conductivity are no longer constants [5]. In recent years and due to the progress in various fields in science and technology the necessity of taking into consideration the real behavior of the material characteristics became actual. Temperature-dependent measurements of Young's modulus were performed for the first time on black and transparent bulk material of chemical vapor deposited diamond by a dynamic three point bending method in a temperature range from $150^o$ to 850 $C^o$ [8]. The temperature dependencies of shear elasticity of some liquids have been investigated by Budaev et al. [6]. It was found that the shear modulus decreases with increasing temperature. This decrease may be explained by the increase of the fluctuation free volume [6]. The dynamic resonance method was used by Rishin et al. [7] to determine the temperature dependence of the modulus of elasticity of some plasma-sprayed materials. The rise in test temperature was found to cause a monotonic decrease in the modulus of elasticity.

Modern structural elements are often subjected to temperature changes of such magnitude that their material properties may no longer be regarded as having constant values even in an approximate sense. The thermal and mechanical properties of materials vary with temperature, so that the temperature depends on material properties must be taken into consideration in the thermal stress analysis of these elements. Especially, in the case of large temperature differences and variable thermal conductivity has a strong effect on performance of such a surface.

The calculus of fractional derivatives and fractional differential equations has been used recently to solve a range of problems in physics, chemistry, biology, mechanical engineering, signal processing, systems identification, electrical engineering, control theory, finance, and fractional dynamics [10]. Fractional calculus which is a calculus of derivatives and integrals of any order is a good candidate to describe the dynamics of nonlocal complex systems. Time-fractional diffusion equation arises by replacing the standard time partial derivative in the diffusion equation with a time-fractional partial derivative. It is usually used to describe anomalous diffusion (superdiffusion, non-Gaussian diffusion, subdiffusion) which is not consistent with the classical Fick (or Fourier) law [11, 12]. Recently, numerical experiments show that in many one-dimensional systems with total momentum conservation, the heat conduction does not obey the Fourier law and the heat conductivity depends on the system size [13].

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property.

A survey of many emerging applications of the fractional calculus in area of science and engineering can be found in the recent text by Podlubny [14]. A brief history of the development of fractional calculus can be found in Ross [15] and Miller and Ross [16]. Youssef and Al-Lehaibi [17] construct a mathematical model of an elastic material with constant parameters fills the half-space in the context of the fractional order generalized thermoelasticity theory. Sherief et al. [18] derived a new theory of thermoelasticity using the methodology of fractional calculus. Povstenko [19] studied a two-dimensional axisymmetric stresses exerted by instantaneous pulses and sources of diffusion in an infinite space in a case of time-fractional diffusion equation. Povstenko [20] used the theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative to investigate thermal stresses in an infinite body with a circular cylindrical hole. Allam et al. [21] applied the model of generalized thermoelasticity proposed by Green and Naghdi, to study the electromagneto–thermoelastic interactions in an infinite perfectly conducting body with a spherical cavity. The modulus of elasticity are taking as linear function of temperature. Abouelregal [22] investigate the fractional order generalized thermo-piezoelectric semi-infinite medium with temperature-dependent properties subjected to a ramp-type heating.

In this paper, we used the fractional order thermoelasticity model to deal with a boundary value problem of one dimension a thin slim strip with its left boundary subjected to a sudden heat considering the thermal conductivity to be variable. When this plate is initially at rest and having a uniform temperature, is suddenly heated at the free surfaces, a heat flow occurs in the plate, and change in thermal and the mechanical fields is brought about. Laplace transform techniques are used. The analytical solutions to stress, displacement and temperature distributions are obtained. The inverses Laplace transforms are obtained numerically and results have been presented graphically.

## 2 The governing equations

Considering the problem of a thermoelastic isotropic homogeneous thin slim strip, the generalized thermoelastic governing differential equations in the context of fractional order thermoelasticity consist of:

(i) The equations of motion in the absence of body forces

$$\rho \ddot{u}_i = (\lambda + \mu)\, u_{j,ji} + \mu u_{i,jj} - \gamma T_i. \tag{1}$$

(ii) The constitutive equations

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma T \delta_{ij} \tag{2}$$

where, $\sigma_{ij}$ is the stress tensor, $\lambda$ and $\mu$ are the Lamé constants, $T$ is the temperature, $T_0$ is the reference temperature, $e_{ij}$ are the components of strain tensor, $u_i$ are the components of displacement vector, $\gamma = \alpha_t(3\lambda + 2\mu)$, $\alpha_t$ is the thermal expansion coefficient, and $\rho$ is the density of the medium. In the above equations, a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time.

(iii) The time-nonlocal dependence between the heat flux vector and the temperature gradient applying the new fractional Taylor's series of time-fractional order $\alpha$ [23] can be interpreted in terms of fractional integrals and derivatives

$$q_i + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} q_i = -K T_{,i} \tag{3}$$

where, $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative, $\alpha\,(0 < \alpha \leq 1)$ is the fractional order parameter, $K$ is the thermal conductivity and $q_i$ is the heat flux vector.

(iv) Then the time-fractional heat conduction equation take the form [23]

$$(K T_{,i})_{,i} = \left(\delta + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial e_{kk}}{\partial t}\right) \tag{4}$$

where, $C_E$ is the specific heat per unit mass at constant strain.

Equation (4) is the generalized energy equation with fractional derivatives and taking into account the relaxation time $\tau_0$. Some theories of heat conduction law follow as limit cases for different values of the parameters $\alpha$ and $\tau_0$. The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time and the generalized theory without energy dissipation follow as limited cases depending on the value of $\delta$, $\tau_0$ and $\alpha$.

The heat conduction Eq. (4), in the limiting case $\alpha \to 0$ and $\delta = 1$ transforms to:

$$(K T_{,i})_{,i} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \gamma T_0 e) \tag{5}$$

which is the same equation obtained by the generalized theory with one relaxation time.

In the limiting case, when $\alpha \to 0,\ \tau_0 = 1$ and $\delta = 0$, the heat conduction Eq. (4), transforms to

$$(KT_{,i})_{,i} = \rho C_E \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2 e}{\partial t^2} \tag{6}$$

which is the same equation of the generalized theory without energy dissipation introduced by Green and Naghdi.

The coupled theory of thermoelasticity can be obtained from Eq. (4) in the limiting case $\alpha \to 0,\ \delta = 1$ and $\tau_0 \to 0$ as

$$(KT_{,i})_{,i} = \rho c_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t}. \tag{7}$$

Equation (4) describes the whole spectrum from local heat conduction through the standard heat conduction to the ballistic heat conduction.

Variations in mechanical properties due to an imposed temperature field are not the only ones that accompany heating. Similar variations are observed in thermal properties characterized by such coefficients as the coefficients of thermal linear expansion $\alpha$, thermal conductivity $K$ and others. An acceptable approximation in limited temperature interval obtained by considering the thermal conductivity to depend linearly on the change of temperature. If thermal conductivity is assumed to be a linear function of temperature, it becomes as follows:

$$K = K(T) = K_0(1 + K_1 T), \qquad \frac{K}{k} = \rho C_E, \tag{8}$$

where $K(T)$ is temperature-dependent thermal conductivity, $K_1$ a parameter defining the variation of thermal conductivity (usually negative experimental coefficient), $K_0$ is the thermal conductivity when it depends on the temperature, $k$ is the diffusivity (assumed to be constant), $\gamma$ is equal to $(3\lambda + 2\mu)\alpha_t$, $\alpha_t$ is the thermal expansion coefficient, $C_E$ is the specific heat per unit mass at constant strain, $t$ is the time, and $\rho$ is the density of the medium.

Using Eq. (4) with Eq. (8), we get

$$(KT_{,i})_{,i} = \left(\delta + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{K}{k}\frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial^2 e}{\partial t}\right). \tag{9}$$

We will use the mapping

$$\Theta = \frac{1}{K_0}\int_0^T K(\tau)d\tau \tag{10}$$

where, $\Theta$ is a new function expressing the heat conduction.

Differentiating (10) with respect to the coordinates, we get

$$\Theta_{,i} = (1 + K_1 T)T_{,i}. \tag{11}$$

Differentiating again the above equation with respect to the coordinates, we obtain

$$\Theta_{,ii} = \left[(1 + K_1 T) T_{,i}\right]_{,i}. \tag{12}$$

With the same manner, by differentiating the mapping with respect to time, we have

$$\dot{\Theta} = (1 + K_1 T)\dot{T}. \tag{13}$$

Substituting from Eqs. (12) and (13) in the heat equation (9), we obtain

$$\Theta_{,ii} = \frac{\partial}{\partial t}\left(\delta + \frac{\tau_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\frac{\Theta}{k} + \frac{\gamma T_0}{K_0}e\right). \tag{14}$$

From Eqs. (10) and (11), we have

$$\Theta = T + \frac{K_1}{2}T^2. \tag{15}$$

Substituting from Eq. (11) into Eq. (1), we get

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} - \frac{\gamma}{(1 + K_1 T)}\Theta_{,i}; \tag{16}$$

For linearity of the governing partial differential equations of the problem, we have to take into account the condition $\frac{|T-T_0|}{T_0} << 1$, which give us the approximating function of the thermal conductivity $K(T)$.

Then equation (16) takes the form

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} - \gamma\Theta_{,i}. \tag{17}$$

Using Eq. (15) and neglecting the small values of temperature, the constitutive relation reduces to

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\Theta\delta_{ij}. \tag{18}$$

# 3 Formulation of the problem

We shall consider a thin semi-infinite rod occupying the region $x \geq 0$. The coordinate system is so chosen that the $x-$axis is taken perpendicularly to the layer, and the $y-$ and $z-$axes in parallel. We consider that the displacement vector for one-dimensional problem has the components

$$u_x = u(x,t), \qquad u_y = u_z = 0.$$

The strain components are

$$e = e_{xx} = \frac{\partial u}{\partial x}.$$

The heat equation is

$$\frac{\partial^2\Theta}{\partial x^2} = \frac{\partial}{\partial t}\left(\delta + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{1}{k}\Theta + \frac{\gamma T_0}{K_0}e\right). \tag{19}$$

The equation of motion is

$$\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial\Theta}{\partial x}. \tag{20}$$

The constitutive relation takes the form

$$\sigma_{xx} = \sigma = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\Theta. \tag{21}$$

# 4 Boundary conditions

For this problem the half-space $x \geq 0$ is assumed to be initially at rest and has reference temperature $T_0$ and zero temperature velocity such that the initial conditions are assumed to be

$$\begin{aligned} u(x,0) &= \frac{\partial u(x,0)}{\partial t} = 0, \\ T(x,0) &= \frac{\partial T(x,0)}{\partial t} = 0 \end{aligned} \tag{22}$$

We consider the half-space $x \geq 0$ at a uniform temperature $T_0$ with its boundary $x = 0$, free of stress and subjected to sudden heating so that the boundary conditions are

1 The thermal boundary conditions:

$$T = T_0 H(t), \quad \text{for } x = 0, \tag{23}$$

$$\Theta = \upsilon_0 H(t), \quad \text{for } x = 0, \tag{24}$$

where $H(t)$ is Heaviside unit step function and

$$\upsilon_0 = T_0\left(1 + \frac{K_1}{2}T_0\right)$$

2 The mechanical boundary conditions.

$$\sigma_{xx} = 0, \qquad \text{for } x = 0 \tag{25}$$

and that

$$\{u(x,t), \quad T(x,t), \Theta(x,t)\} \to 0, \qquad \text{as} \qquad x \to \infty, \quad t > 0.$$

## 5 Solution of the problem

For simplicity, we will use the following non-dimensional variables

$$
\begin{aligned}
x' &= \frac{c_1}{k}x, & u' &= \frac{c_1}{k}u, & t' &= \frac{c_1^2}{k}t, \\
\sigma'_{xx} &= \frac{\sigma_{xx}}{\rho c_1^2}, & & & \tau'_0 &= \frac{c_1^2}{k}\tau_0, \\
\Theta' &= \frac{\gamma}{\rho c_1^2}\Theta, & & & c_1^2 &= \frac{(\lambda+2\mu)}{\rho}.
\end{aligned} \tag{26}
$$

Using these non-dimensional variables, the governing equations take the forms (dropping the primes for convenience)

$$\sigma = \frac{\partial u}{\partial x} - \Theta = e - \Theta, \tag{27}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial \Theta}{\partial x} = De - D\Theta = D\sigma, \tag{28}$$

$$\frac{\partial^2 e}{\partial t^2} = D^2 e - D^2\Theta = D^2\sigma, \tag{29}$$

$$D^2\Theta = \frac{\partial}{\partial t}\left(\delta + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial}{\partial t}\right)(\Theta + \varepsilon e) \tag{30}$$

where, $D = \partial/\partial x$ and $\varepsilon = \gamma^2 T_0 k/\left(\rho c_1^2 K_0\right)$.

## 6 Solution in the Laplace transform domain

If we apply the Laplace transform defined by the formula

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

to Eqs. (27)-(30), and using the initial conditions (22), we obtain

$$\overline{\sigma} = \overline{e} - \overline{\Theta}, \tag{31}$$

$$D^2\overline{\sigma} = s^2\overline{e}, \tag{32}$$

$$(D^2 - g)\overline{\Theta} = g\varepsilon\overline{e}, \qquad g = s\left(\delta + \frac{\tau_0^\alpha}{\alpha!}s^\alpha\right). \tag{33}$$

The boundary conditions (24) and (25) and the regularity condition in the Laplace transform domain may be transformed to

$$\overline{\Theta} = \frac{\upsilon_0}{s} \quad \text{for} \quad x = 0, \tag{34}$$

$$\overline{\sigma} = 0 \quad \text{for} \quad x = 0, \tag{35}$$

$$\{u(x,s), \quad T(x,s)\} \to 0, \qquad \text{as} \qquad x \to \infty.$$

By eliminating $\overline{e}$, we get

$$(D^2 - s^2)\overline{\sigma} = s^2\overline{\Theta}, \tag{36}$$

$$(D^2 - g(1+\varepsilon))\overline{\Theta} = g\varepsilon\overline{\sigma}. \tag{37}$$

Eliminating $\overline{\sigma}$, we get

$$[D^4 - (s^2 + g(1+\varepsilon))D^2 + g\varepsilon s^2]\overline{\Theta} = 0. \tag{38}$$

In a similar manner, we can show that $\overline{\sigma}$ satisfies the equation

$$[D^4 - LD^2 + M]\overline{\sigma} = 0 \tag{39}$$

where

$$L = (s^2 + g(1+\varepsilon)), \qquad M = \varepsilon s^2 g.$$

The solution of Eqs. (38) and (39) takes the form

$$\overline{\sigma} = A_1 s^2 \exp(-m_1 x) + A_2 s^2 \exp(-m_2 x), \tag{40}$$

$$\overline{\Theta} = A_1\left(m_1^2 - s^2\right)\exp(m_1 x) + A_2\left(m_2^2 - s^2\right)\exp(m_2 x) \tag{41}$$

where, the parameters $m_1$ and $m_2$ satisfy the equation

$$m^4 - Lm^2 + M = 0.$$

We can get the displacement by using Eq. (28), such that

$$\overline{u} = \frac{1}{s^2} D\overline{\sigma}$$

so, we obtain

$$\overline{u} = A_1 m_1 \exp(m_1 x) + A_2 m_2 \exp(m_2 x). \tag{42}$$

The temperature increment $\overline{T}$ can be obtained by solving (15) to give

$$\overline{T} = \frac{-1 + \sqrt{1 + 2K_1\overline{\Theta}}}{K_1}. \tag{43}$$

We shall now use the boundary conditions of the problem to evaluate the unknown parameters of the problem, namely $A_1$ and $A_2$. Equations (34) and (35) together with Eqs. (40) and (41) immediately give

$$A_1 + A_2 = 0, \tag{44}$$

$$A_1\left(m_1^2 - s^2\right) + A_2\left(m_2^2 - s^2\right)\exp(m_2 x) = \frac{\overline{v}_0}{s}. \tag{45}$$

Solution of the above system of linear equations gives the unknown parameters $A_1$ and $A_2$ in the form

$$A_1 = -\frac{\overline{v}_0}{s\left(m_2^2 - m_1^2\right)}, \qquad A_1 = \frac{\overline{v}_0}{s\left(m_2^2 - m_1^2\right)}. \tag{46}$$

Hence

$$\overline{\sigma} = -\frac{\overline{v}_0 s}{\left(m_2^2 - m_1^2\right)}\exp(-m_1 x) + \frac{\overline{v}_0 s}{\left(m_2^2 - m_1^2\right)}\exp(-m_2 x), \tag{47}$$

$$\overline{\Theta} = -\frac{\overline{v}_0\left(m_1^2 - s^2\right)}{s\left(m_2^2 - m_1^2\right)}\exp(m_1 x) + \frac{\overline{v}_0\left(m_2^2 - s^2\right)}{s\left(m_2^2 - m_1^2\right)}\exp(m_2 x), \tag{48}$$

$$\overline{u} = -\frac{\overline{v}_0 m_1}{s\left(m_2^2 - m_1^2\right)}\exp(m_1 x) + \frac{\overline{v}_0 m_2}{s\left(m_2^2 - m_1^2\right)}\exp(m_2 x). \tag{49}$$

This completes the solution of the problem in the transformed domain.

# 7 Inversion of the Laplace transform

It is difficult to find the analytical inverse Laplace transform of the complicated solutions for the displacement, temperature, stress and strain in Laplace transform domain. We will now outline the numerical inversion method to obtain the solution of the problem in the physical domain. Durbin [24] derived the approximation formula

$$f(t) = \frac{2e^{st}}{t_1}\left(-\frac{1}{2}\operatorname{Re}[F(s)] + \operatorname{Re}\sum_{n=0}^{N}\left[\begin{array}{c}\operatorname{Re}\left(F\left(s + \frac{2in\pi}{t_1}\right)\right)\cos\left(\frac{2n\pi}{t_1}\right) \\ -\operatorname{Im}\left(F\left(s + \frac{2in\pi}{t_1}\right)\right)\sin\left(\frac{2n\pi}{t_1}\right)\end{array}\right]\right) \tag{50}$$

It should be noted that a good choice of the free parameters $N$ and $st_1$ is not only important for the accuracy of the results, but also for the application of the Korrecktur method and the methods for the acceleration of convergence. The values of all parameters in Eq. (50) are defined as $t_1 = 20$, $s = 0.25$ and $N = 1000$ in this paper.

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# 8 Numerical results

For the sake of illustrating the analytical procedure presented, we consider a numerical example. The results depicts the variations of the dimensionless values of displacement, temperature and thermal stresses. For this purpose, we take the following values material constants (Copper material and the type 316):

$$\alpha_t = 17.8 \times 10^{-6}\mathrm{k}^{-1}, \quad \rho = 8954\mathrm{kgm}^{-3}, \quad C_E = 383.1\mathrm{m}^2\mathrm{k}^{-1}\mathrm{s}^{-2},$$
$$K_0 = 386\mathrm{kgmk}^{-1}\mathrm{s}^{-3}, \quad T_0 = 293\mathrm{k}, \quad \mu = 0.497425\lambda,$$
$$\beta = -2654.53, \quad K_1 = -0.1, \quad \varepsilon = 0.0150$$

The computations were carried out for a value of time $t = 0.15$. The numerical technique outlined previously was used to obtain the temperature $T$, displacement $u$, and stress $\sigma_{xx}$ distributions. The computations were carried out for wide range of $x(0 \leq x \leq 5)$, for different values of the parameter $\alpha$ with wide range $(0 < \alpha \leq 1)$ which cover the two cases of the conductivity; $(0 < \alpha < 1)$ for weak conductivity, $\alpha = 1$ for normal conductivity. Here the numerical results are not listed while the results are displayed graphically in Figs. $1 - 3$.

It should be pointed that, the increasing of the value of the parameter $\alpha$ causes decreasing in the speed of the waves propagation of the stress and the temperature distributions, whereas the distribution of the displacement increasing. We have noticed that, the value of $\alpha$ has a significant effect on all distributions. From these figures, the stress at the surface is zero as shown, which agrees with the boundary condition prescribed.

In Figures $4-6$, we display the stress, the temperature and the displacement respectively with the different values of $K_1$, and we notice that the parameter has a significant effect on all fields. Physically, we can say that, when $K$ is variable with linear function of temperature with negative values of $K_1$, the values of the thermal conductivity decreasing with increasing temperature and then the distance between the particles will increase which makes the speed of waves progress of all the fields will be more slow and hence the values of all that fields will be decreasing.

The numerical values of the physical field variables are computed and are plotted in figures $7 - 9$ against $x$ in order to observe the nature of variations of the field for different thermoelastic models. From the different figures it is observed that the nature of variations of all the field variables is nearly the same for **L-S** model and **G-N** model however for **C-D** model their behaviors are significantly different. The phenomenon of finite speeds of propagation is manifested in all these figures for **L-S** and **G-N** models. This is not the case when using the coupled equation of heat conduction (**C-D** model), where the thermal and mechanical effects extend to fill the whole space.

# 9 Conclusions

The main observations from these figures are organized as:

1. The Laplace transform technique is used to derive displacement, stress and temperature distribution due to mechanical and thermal shock temperature.

2. We found that, the parameter $\alpha$ has a significant effects on all the fields.

3. According to this new theory, we have to construct a new classification for materials according to their fractional parameter $\alpha$ where this parameter becomes a new indicator of its ability to conduct heat in conducting medium.

4. The result provides a motivation to investigate conducting thermoelastic materials as a new class of applicable thermoelectric materials.

5. The curves, demonstrate that the effect of the thermal conductivity on all the fields and in different materials is clear and we have to take it into account in any analysis of heat conduction. The field quantities, temperature, stresses and displacement depend not only on the state and space variables $t$ and $x$ but also depend on the value of $K_1$. It has been observed that, $K_1$ plays a vital role on the development of all the fields.

6. The comparison of different theories of thermoelasticity, i.e. Lord and Shulman, Green and Naghdi and classical dynamical coupled theories is carried out.

7. The fact that in generalized thermoelasticity, the waves propagate with finite speeds is evident in all these figures. This is not the case in coupled thermoelasticity, where the considered function have non-vanishing values for all values of $x$ due to the infinite speed of propagation of heat waves.

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